# STOKES PROBLEM OF A CONVECTIVE FLOW PAST A VERTICAL INFINITE PLATE IN A ROTATING SYSTEM IN PRESENCE OF VARIABLE MAGNETIC FIELD

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# Stokes problem of a Convective Flow Past a Vertical Infinite Plate in a Rotating System in Presence of Variable Magnetic Field

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## DECLARATION

This thesis is my original work and has not been presented for a degree in any other University.

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This thesis has been submitted for examination with our approval as University Supervisors.

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# DEDICATION

This thesis is dedicated to my loving wife Rebecca, my three wonderful children Emmanuel, Albanus and Daniel.

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# LIST OF ABBREVIATIONS

| | |
|---|---|
| **MHD** | Magnetohydrodynamics |
| **FDM** | Finite Différence Method |
| **HOT** | Higher Order Terms |

# NOMENCLATURE

| ROMAN SYMBOL | QUANTITY |
|---|---|
| $\vec{E}$ | Electric intensity vector (V/m) |
| $\vec{F}$ | Body force vector (N) |
| $e$ | Unit charge (C) |
| $L$ | Characteristic length (m) |
| $\vec{J}$ | Current density vector ($Am^{-2}$) |
| $P$ | Pressure force vector ($Nm^{-2}$) |
| **U** | Characteristic velocity ($ms^{-1}$) |
| $t^*$ | Dimensional Time (S) |
| $\kappa$ | Thermal conductivity ($Wm^{-1}\ k^{-1}$) |
| $\vec{q}$ | Velocity vector ($ms^{-1}$) |
| $\vec{B}$ | Magnetic field vector ($Wbm^{-2}$) |
| $\vec{D}$ | Electric displacement vector ($cm^{-2}$) |
| $\vec{H}$ | Magnetic field intensity vector ($Wbm^{-2}$) |
| $\vec{i}, \vec{j}, \vec{k}$ | Unit vectors in the x, y and z directions respectively |
| $u, v, w$ | Components of velocity vector q |
| $\vec{F}_e$ | Electromagnetic force ($kgm^{-2}$) |

| | |
|---|---|
| $g$ | Acceleration due to gravity ($ms^{-2}$) |
| $\frac{D}{Dt}\left(=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)$ | Material derivative |
| $Q$ | Amount of heat added to the system (Nm) |
| $h$ | Dimensional distance between vertical Plates |
| $T$ | General fluid temperature |
| $C_p$ | Specific heat at constant pressure ($Jkg^{-1}\,K^{-1}$) |
| $U_\infty$ | Free stream fluid velocity ($ms^{-1}$) |
| $T_\infty$ | Characteristic free stream temperature (K) |
| $m$ | Hall parameter |

## DIMENSIONLESS QUANTITIES

| | |
|---|---|
| $\theta$ | Dimensionless fluid temperature |
| $U, V, W$ | Dimensionless fluid velocity |
| $x, y, z$ | Dimensionless Cartesian coordinates |
| $t$ | Dimensionless time |

| | |
|---|---|
| $E_c$ | Eckert number $\left\{= \dfrac{U^2}{C_p(T - T_\infty)}\right\}$ |
| Pr | Prandtl number $\left(= \dfrac{\mu Cp}{k}\right)$ |
| $Rm$ | Magnetic Reynolds number $(= \sigma \mu_c Lu)$ |
| $Nu$ | Nusselt number $\left(= \dfrac{hL}{\kappa}\right)$ |
| $S$ | Magnetic force number $\left(= \dfrac{Ho}{L} \dfrac{\sqrt{\mu c}}{\rho}\right)$ |
| $M$ | Magnetic Parameter $\left(= \sqrt{\dfrac{\sigma Ho^2 \nu}{\mu U_m^2 / \nu}}\right)$ |
| $Gr$ | Grashof number $\left(= \dfrac{g\beta \Delta T l^3}{\nu^2}\right)$ |

| GREEK SYMBOLS | QUANTITY |
|---|---|
| $\mu$ | Coefficient of Viscosity, Kg/ms. |
| $\nu$ | Kinematic Viscosity $m^2s^{-1}$ |
| $\rho$ | Fluid density, $kg/m^3$. |
| $\rho_e$ | Electrical charge density ($cm^{-2}$) |
| $\sigma$ | Electrical conductivity ($\Omega^{-1}m^{-1}$) |
| $\mu_e$ | Magnetic permeability ($Hm^{-1}$) |
| $\Delta t, \Delta y, \Delta z$ | Time and distance intervals respectively(s, m) |
| $\Delta T$ | Temperature change (K) |

| | |
|---|---|
| $\nabla$ | Gradient operator $\left(= i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)$ |
| $\nabla^2$ | Laplacian operator $\left(= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$ |
| $\phi$ | Viscous dissipation function ($s^2$) |
| $\alpha$ | Electrical conductivity |
| $\beta$ | Coefficient of thermal expansion, $K^1 \left[\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p\right]$ |

# ABSTRACT

In this study, stokes problem of a convective flow past a vertical infinite plate in a rotating system in presence of variable magnetic field is considered. The fluid flow is unsteady and a variable magnetic field is transversely applied to the plate. Evaluation of velocity gradients and temperature gradients across the plate is done. Observations and discussions of the effects of various parameters on flow variables is done. The non-dimensional parameters observed and discussed are Hall parameter, m; Magnetic number, $M^2$; Eckert number, Ec and Rotational parameter, Er. The velocity profiles and temperature profiles are presented graphically for both free convectional heating and free convectional cooling of the plate. The skin friction and rate of heat transfer values are obtained and presented in tables. For free convectional heating and cooling of the plate, the Grashof number is taken as constants -0.5 and 0.5 respectively. Prandtl number is 0.71 which corresponds to air. The variation of the parameters mentioned above is noted to increase or decrease or had no effect on the skin friction, rate of heat transfer, the velocity profiles and temperature profiles.

# CHAPTER ONE

# 1.0 INTRODUCTION AND LITERATURE REVIEW

## 1.1 OVERVIEW

In this chapter, definitions of the main terms used in the thesis are done. Literature review related to the present work is also given. Objectives and justifications of the study are also stated and finally at the end of the chapter, an outline of the thesis is presented.

## 1.2 INTRODUCTION AND LITERATURE REVIEW

A fluid is defined as a substance that deforms continuously when acted on by a shearing stress of any magnitude. A shearing stress (force per unit area) is created whenever a tangential force acts on a surface. Common fluids such as water, oil, and air satisfy the definition of a fluid, that is, they will flow when acted on by a shearing stress. In this thesis it will be assumed that all the fluid characteristics of interest (pressure, velocity, e.t.c) vary continuously throughout the fluid, that is, we treat the fluid as a continuum. This concept will certainly be valid for all the circumstances considered in the thesis.

Fluid Mechanics is the study of fluid motion and forces that cause the motion. Fluid Mechanics is categorized into two namely: Fluid kinematics and fluid dynamics. Fluid kinematics involves forces which induces or causes motion of fluid. i.e. a branch of fluid mechanics that is concerned with the forces that cause the fluid motion. Fluid dynamics is the study of fluid motion.

The experimental observation that the fluid “sticks” to the solid boundaries is a very important one in fluid mechanics and is usually referred to as the no-slip condition. Many fluids, both liquids and gases, satisfy this condition.

Fluids fall under two categories: Incompressible and compressible fluids. Incompressible fluids are those fluids that do not change significantly in volume when subjected to change in pressure and temperature. A fluid is termed as compressible when there is significant change in the pressure and temperature that are sufficiently large to cause density changes of the fluid.

The shearing stress and rate of shearing strain (velocity gradient) is given by a relationship of the form: (Currie, 1974)

$$\tau = \mu \frac{du}{dy} \tag{1.1}$$

where the constant of proportionality is designated by the Greek symbol $\mu$ (mu) , referred to as the viscosity of the fluid. Fluids in which the shearing stress is linearly related to the rate of shearing strain (also referred to as the rate of angular deformation) are designated as Newtonian fluids. Fluids for which the shearing stress is not linearly related to the rate of shearing strain are designated as non-Newtonian fluids. This study considers a Newtonian fluid. As such fluids can be classified according to the relation between shear $\tau$ and rate of angular deformation:

$$\tau = 0 \qquad \text{Ideal fluids} \tag{1.2}$$

$$\tau = \mu \frac{du}{dy}$$ Newtonian fluids (1.3)

$$\tau = Const + \mu \left( \frac{du}{dy} \right)$$ Ideal plastics or Bingham plastics (1.4)

$$\tau = Const + \mu \left( \frac{du}{dy} \right)^n$$ Thyxotropic fluids (1.5)

$$\tau = \mu \left( \frac{du}{dy} \right)^n$$ Non-Newtonian fluids (1.6)

For non-Newtonian fluids, if n is less than unity, they are called Pseudo-plastics while fluids in which n is greater than unity are known as dilatants.

If F is a flow or fluid property such as velocity, pressure, mass, density or temperature, then the following types of flows have been defined:

$$\left.\begin{aligned} &Steady\,flow: && \left( \frac{\partial F}{\partial t} \right)_{at\,a\,point\,or\,section} = 0 \\ &Unsteady\,flow: && \left( \frac{\partial F}{\partial t} \right)_{at\,a\,point\,or\,section} \neq 0 \end{aligned}\right\} \quad (1.7)$$

$$\left.\begin{aligned} &Uniform\,flow: && \left( \frac{\partial F}{\partial t} \right)_{t=to} = 0 \\ &Non-uniform\,flow: && \left( \frac{\partial F}{\partial t} \right)_{t=to} \neq 0 \end{aligned}\right\} \quad (1.8)$$

Where

$$\left.\begin{array}{ll} One\dim ensional: & F = F(x,t) or F(s,t) \\ Two\dim ensional: & F = F(x,y,t) \\ Three\dim ensional & F = F(x,y,z,t) \end{array}\right\} \quad (1.9)$$

Magnetohydrodynamics (MHD) is a branch of science of the dynamics of fluid flowing in presence of electromagnetic field, especially where induced currents in the fluid by induction modify the field, so that the field and dynamics equations are coupled. MHD treats; in particular, certain conducting fluids, whether liquid or gaseous, in which some phenomenon are accepted. The phenomenon is, Maxwell displacement current is neglected and the fluid is treated as a continuum, without mean-free path effects (Calvert, 2002).

The study of rotating fluids has had considerable progress in the last few decades. For instance, the effect of an applied variable magnetic field on unsteady free convection flow along a vertical plate has been given considerable interest because of its application in the cooling of nuclear reactors or in the study of the structures of stars and planets. Important engineering applications in which the study of MHD flows of rotating fluids with variable magnetic field poises includes: power generators, heat exchangers, reactors and MHD accelerators among other devices.

Many investigators have considered the flow problem of MHD natural convection past an infinite or semi infinite vertical moving plate with uniform magnetic field (Ram and Kinyanjui, 1995; Aldoss *et al*, 2005; Takhar *et al*, 1995; Dorch, 2007).

**1.2.1 Unsteady and steady flow.**

For steady flow the velocity at any given point in space does not vary with time, $\frac{\partial v}{\partial t} = 0$.

In reality almost all flows are unsteady in some sense, that is, the velocity varies with time. An example of a non-periodic, unsteady flow is that produced by turning off a faucet to stop the flow of water. In other flows the unsteady effects may be periodic, occurring time after time in basically the same manner. The periodic injection of the air-gasoline mixture into the cylinder of an automobile engine is such an example. In many situations the unsteady character of a flow is quite random, that is, there is no repeated regular variation to the unsteadiness. This behavior occurs in turbulent flow and is absent in laminar flow. The "smooth" flow of highly viscous syrup onto a pancake represents a "deterministic" laminar flow. It is quite different from the turbulent flow observed in the "irregular" splashing of water from a faucet onto the sink below it. The "irregular" gustiness of the wind represents another random turbulent flow. In an unsteady flow, the flow variables such as velocity and the thermodynamic properties at every point in space vary with respect to time. On the other hand, in steady flows none of the fluid variables will vary with time.

### 1.2.2 Hydromagnetic

Hydrodynamics is the study of the motion of fluid when forces are applied. The interaction between electric and magnetic fields is referred to as electromagnetism. Hydromagnetics is a branch of science in which hydrodynamics and electromagnetism interact. It is also referred to as magnetohydrodynamics (MHD). Therefore, in hydromagnetic flows there are three non-linear terms in the governing equation while in hydrodynamic flows there is only one. The flow of an electrically conducting fluid such as mercury under a magnetic field in general gives rise to an induced electric current that yields mechanical forces that are exerted by the magnetic force. The induced electric current also produces induced magnetic field thus the original magnetic field is changed.

### 1.2.3 Free convection flow

Due to the varied range of application in engineering and universe, MHD free convection flow has become significant. A fluid flow in which the motion is as a result of body force acting on the fluid in which there are density gradients is called a free convection flow. Temperature or concentration gradients existing in the fluid yields density gradients while the gravitational force yields the body force. Thus the action of the body force on the fluid amounts to buoyancy force that eventually induces free convection current.

### 1.2.4 Heat transfer

This constitutes the study of energy transfer that takes place between bodies due to temperature difference. The temperature difference may be as a result of a fluid dissipating heat or introduction of heat to the flow field. Heat transfer can be either by conduction, radiation or convection. Conduction is heat transfer that takes place when a temperature gradient exists in a stationary medium which may be solid or fluid. The mode of heat transfer that takes place between a surface and a fluid in motion at different temperatures is termed as convection. The motion of the fluid is as a result of imbalance on forces acting on the fluid flow. Free or natural convection is the mode of heat transfer in which the flow is as a result of density gradient created by temperature variation while forced convection occurs when the flow is caused by some external forces. Radiation is type of heat transfer in which there is a net heat transmission due to electromagnetic wave propagation that takes place in a vacuum as well as in a medium. In our study heat transfer by free convection is considered.

### 1.2.5 Mass transfer

The relative motion of a mixture's species as a result of concentration gradients is termed as mass transfer. Thus, this is mass in transit caused by concentration difference of the species in a mixture. Modes of heat transfer that are similar to convection and conduction do exist.

Mass transfer by free convection will be studied in this study. In natural convection, external forces are not required since effects of buoyancy and the force of gravity induce the motion thereby resulting in the heat transfer. Thus both heat transfer and fluid flow due to convection rely on the fundamental principles of heat transfer and fluid flow. Significant laws of convection include: conservation of mass, momentum conservation and energy conservation law.

### 1.2.6 Boundary layer

A thin layer of fluid near the surface of a body or solid in which the flow is affected by viscous forces is called Boundary layer. In analyzing flow problems that involve transfer by convection, boundary layer theory plays a significant role.

Three boundary layers may exist when a fluid flows on a surface. These are thermal, concentration and velocity boundary layer. A zero velocity is assumed by fluid particles when they come into contact with a surface (no-slip condition). Velocity boundary layer is the region in which the velocity gradient is large.
These fluid particles attain a thermal equilibrium state when they come into contact with an isothermal plate on its surface temperature. Thermal boundary layer is the region on the fluid in which temperature gradients exist.

A concentration boundary layer is the layer on the fluid in which concentration gradient is present, i.e., it is region that develops if the concentration of species at the surface differs from that in the free stream. In this study, the three boundary layers are considered and investigation of convection of heat transfer and the skin friction is done.

If the fluid were non-viscous, the streamlines would be parallel to the plate and nothing very interesting happens. For a viscous fluid, however, we must apply the no-slip boundary condition on the surface of the plate. The thickness of the boundary layer, which is denoted by $\delta$, is the distance required for the velocity profile to approach its free stream value (see Figure 1.1). Considering that the viscosity is a measure of the diffusion of velocity (or vorticity), the thickness of the boundary layer after a time $t$ is approximately given by

$$\delta \sim \sqrt{\nu t}. \tag{1.10}$$

Now in a time $t$ an element of fluid which starts at the leading edge of the plate will have moved a distance $x \sim Ut$, so that the boundary layer thickness a distance $x$ from the leading edge is

$$\delta \sim \sqrt{\frac{\nu x}{U}}. \tag{1.11}$$

Therefore, the boundary layer thickness at the trailing edge of the plate, measured relative to the length of the plate itself, is

$$\frac{\delta_x}{l_x} \sim \mathrm{Re}_x^{-1/2} \tag{1.12}$$

where, it has been observed that the boundary layer thickness decreases with increasing Reynolds number (for an assumed laminar flow).

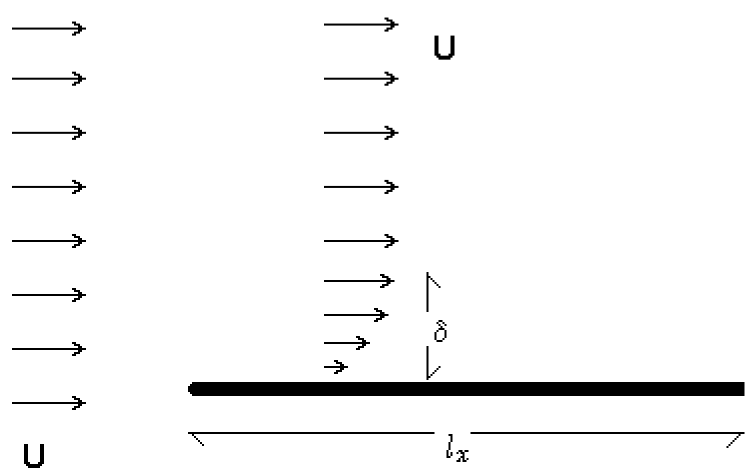


**Figure 1.1: Geometry for viscous flow past a thin plate**

In the present study, a laminar flow is considered, hence possesses the following characteristics:

i. "No-slip" at the boundary, i.e. because of viscosity, velocity of fluid at $y = 0$ is zero if boundary is stationary or is equal to velocity of the boundary if it is in motion.

ii. Because of viscosity there is shear between fluid layers which is given by $\tau = \mu \frac{du}{dy}$ (since the fluid is Newtonian) for flow in x-direction.

iii. The flow is rotational.

iv. There is continuous dissipation of energy due to viscous shear and energy must be supplied externally to maintain the flow.

v. There is no mixing between different fluid layers except by molecular motion, which is very small.

vi. Flow remains laminar as long as $\frac{Ul\rho}{\mu}$ is less than what is known as the critical value of Reynolds number.

vii. Energy loss is proportional to first power of velocity and first power of viscosity.

### 1.2.7 Skin friction

Friction (sometimes called skin friction) is a resistance to motion created by two objects rubbing against one another. When a fluid flows past an object, the amount of friction is determined by: the viscosity of the fluid; and the smoothness of the surface of the object. Viscosity mu $\mu$ is a measure of how much a fluid will resist flowing. e.g. honey versus water.

Quite often viscosity appears in fluid flow problems combined with the density in the form

$$\upsilon = \frac{\mu}{\rho}$$

This ratio is referred to as the Kinematic viscosity and is denoted by the Greek symbol $\upsilon$ (nu).

The boundary layer produces a drag on the plate due to the viscous stresses which are developed at the wall. This drag is often referred to as skin friction, and is due to the viscous stresses acting on the surface of the plate. If the boundary layer remain attached to the body (which it may not), then this is the sole source of drag on a body. The turbulent mixing of the fluid near the surface of a solid body leads to more efficient momentum transport away from the body, increasing the gradient of the velocity profile at the surface and therefore the viscous stress on the plate. If the boundary layer can be persuaded to remain laminar, then boundary layers which remain attached to a body the drag due to skin friction can be reduced. At high Reynolds number, the flow will be turbulent. In this thesis $\mu$ is taken as the Coefficient of Viscosity, $\upsilon$ as the Kinematic Viscosity and $\eta$ as the Dynamic Viscosity.

### 1.2.8 Inviscid flow

Flow fields in which the shearing stresses are assumed to be negligible are said to be inviscid, non-viscous or frictionless. These terms are used interchangeably.

### 1.2.9 Kinematic similarity

Kinematic similarity is similarity of motion. If at the corresponding (or homologous) points in the model and in the prototype, the velocity or acceleration ratios are the same, the velocity or acceleration vectors point in the same direction, the two flows are said to be kinematically similar.

### 1.2.10 Dynamic similarity

Dynamic similarity is the similarity of forces. The flows in the model and in the prototype are dynamically similar if at all the corresponding points, identical types of forces are parallel and bear the same ratio. In dynamic similarity, the force polygons of the two flows can be superimposed by change in force scale.

### 1.2.11 Hall and Ion-slip Currents

The electrical current density $\vec{J}$ represents the relative motion of charged particles in a fluid. The equation of electric current density may be derived from the diffusion velocities of the charged particles. When electric field $\vec{E}$ is applied, there will be an electrical current in the direction of $\vec{E}$. If the magnetic field $\vec{H}$ is perpendicular to $\vec{E}$, there will be an electromagnetic force $\vec{J}\times\vec{B}$ which is perpendicular to both $\vec{E}$ and $\vec{H}$. Thus, there is a new component of electric current density in the direction perpendicular to both $\vec{E}$ and $\vec{H}$,which is known as Hall Current. For the same electromagnetic force, the motion of ions is different from that of electrons, when the electromagnetic force is very large (such as in a very strong magnetic field) the diffusion velocity of ions cannot be neglected. If the diffusion velocity of ions is considered then this results to Ion-slip current. In our present investigation, the effect of Hall currents neglecting the Ion-slip current on the flow field is studied.

## 1.3 LITERATURE REVIEW

The concept of MHD was first introduced by Hartman (1938) when he studied the effects of a conductor in an electrically conducting fluid. In the last two decades, considerable progress has been made in the general theory of rotating fluids. It was proved that in a rotating fluid near a flat plate, an Ekman layer exists. Some of the researchers who investigated on such a layer are Raptis *et al* (1983) and Gupta (1975). The important point is that the flow of electrically conducting fluid such as mercury under a magnetic field in general gives rise to an induced electric current. Much work in MHD was done by Alfven (1942) who established transverse waves in electrically conducting fluids and explained many astrophysical phenomena with it.

Linguistics (1952) showed that the interaction between the two branches (Electromagnetic and Hydrodynamics) is significant if the non-dimensional number $\sqrt{}BL\ (\sigma\mu_{e/\rho})^{1/2}>1$ where B is the magnetic field, L- characteristic length, σ- electrical conductivity, $\mu_e$ is the magnetic permeability and ρ is the density of the fluid.

An analytical study of the flow past an impulsively started semi-infinite plate was done by Stewardsone (1951). Kinyanjui *et al* (1998) studied stokes problem of convective flow from a vertical infinite plate in a rotating fluid. Rossow (1958) studied Stokes problem under a transversely applied magnetic field. A software tool using finite elements for the solution of fluid flow problems was investigated by Naroua *et al* (2004). Plaut (2003) studied the non linear dynamics of traveling waves in rotating

Rayleigh-Bernard convection in which he examined the effects of the boundary conditions and of the topology.

Stokes (1851) studied the flow of an incompressible viscous fluid past an impulsively started infinite horizontal plate. Ogulu and Prakash (2004) studied the effect of slip velocity on oscillatory MHD flow with radiative heat transfer and variable suction.
Ram *et al* (1990) analyzed the effects of Hall current and wall temperature oscillation on convective flow in a rotating fluid through porous medium bounded by an infinite vertical limiting surface. The effects of various parameters on the velocity and shear stress were determined. Ram *et al* (1991) employed the finite difference method to analyze the MHD Stokes problem for a vertical plate with Hall and Ion-slip currents. Ram *et al* (1995) discussed MHD Stokes problem of a convective flow of a vertical infinite plate in a dissipative rotating fluid with Hall current, an analysis of the effects of various parameters on the concentration, velocity and temperature profiles was done.

Aldoss *et al* (2005) studied the transient hydrodynamics and thermal behaviour of free connective flow over an isothermal vertical flat plate while hydromagnetic convective flow heat generating fluid past a vertical plate with Hall current and heat flux through a porous medium was studied by Takhar *et al* (1995).

Sachdeva (1994) studied the fundamentals of engineering heat and mass transfer. Kinyanjui *et al* (2001) studied MHD free convective heat and mass transfer of a heat generating fluid past an impulsively started infinite vertical porous plate with Hall current and radiation absorptions. MHD stokes problem for a vertical infinite plate in a dissipative rotating fluid with Hall current was studied by Chaturvedi (1998). Kwanza *et al* (2003) analyzed MHD stokes free convection flow past an infinite vertical porous plate subjected to constant heat flux with ion-slip current and radiation absorption. Seth (1986) studied unsteady hydromagnetic flow in a rotating channel in the presence of inclined magnetic field. Soundalgekar (1973) studied hydromagnetic free convention flow past a vertical infinite porous plate in a rotating fluid. Soundalgekar (1980) dwelled with the effects of free convective current on the oscillatory flow past a vertical semi-infinite plate. Chaudhary (2006) studied combined heat and mass transfer effects on MHD free convection flow past an oscillating plate embedded in porous medium. Muthucumaraswamy *et al (*2001) investigated mass diffusion effects on flow past a vertical surface. Ghaddar (1998) studied an analytical model of induced electric current from a free convection loop placed in a transverse magnetic field. Error analysis due to two dimensional approximations in heat transfer analysis of welds was analyzed by Kamala (1993).

Chamkha (2000) studied hydromagnetic combined heat and mass transfer by natural convection from a permeable surface embedded in a fluid saturated porous medium. Chamkha and Khaled (2001) investigated coupled heat and mass transfer in MHD free convection flow from an inclined plate in the presence of internal heat generation or absorption. Aboeldahab and Elbarbary (2001) took into account the Hall current effect on MHD free convection heat and mass transfer over a vertical surface upon which the flow is subjected to a strong external magnetic field. Emad *et al* (2001) studied Hall current effect on magnetohydrodynamic free-convection flow past a semi-infinite vertical plate with mass transfer. They discussed the effects of magnetic parameter, Hall parameter and the relative buoyancy force effect between species and thermal diffusion on the velocity, temperature, and concentration. Sahoo (2003) investigated Magnetohydrodynamic unsteady free convection flow past an infinite vertical plate with constant suction and heat sink. Researchers Camargo *et al* (1996) conducted a numerical study of the natural convective cooling of a vertical plate. An analysis of the MHD flow of a conducting fluid past a plate in presence of radiation has been studied by Rapits and Massalas (1997). Ogulu *et al* (2002) studied the unsteady free convection and mass transfer of a fluid past an infinite plate in presence of thermal diffusion.

Feng-Chen *et al* (2005) investigated MHD effect on the flow structure and heat transfer characteristics. This was studied numerically for a liquid-gas annular flow under a transverse magnetic field. The results showed that temperature distribution in the liquid film and the Nusselt number distribution in the angular direction were influenced by the flow structures with the side walls.

Hang and Shi-Jun (2005) presented the unsteady magnetohydrodynamic viscous flows of non-Newtonian fluids caused by an impulsive stretching plate. Chamkha (2004) considered unsteady, two-dimensional, laminar, boundary-layer flow of a viscous, incompressible, electrically conducting and heat-absorbing fluid along a semi-infinite vertical permeable moving plate in the presence of a uniform transverse magnetic field. In the same year he studied unsteady heat and mass transfer by mixed convection flow over a vertical permeable cone rotating in an ambient fluid with a time dependent angular velocity in the presence of a magnetic field and heat generation or absorption effects.

Zakari (2004) analyzed heat transfer from a non-isothermal stretching sheet in the presence of a transverse magnetic field by means of the successive approximation method. In the same year Chien (2004) analyzed the problem of combined heat and mass transfer of an electrically conducting fluid in MHD free convection adjacent to a vertical surface taking into account the effects of Ohmic heating. Emad *et al* (2005) studied the effects of viscous dissipation and joule heating on MHD free convection flow past a semi-infinite vertical plate in the presence of combined effect of Hall and Ion-slip currents for the case of power-law variation of the wall temperature. They found that the magnetic field acts as a retarding force on the tangential flow but has a propelling effect on the induced lateral flow. The skin friction factor for the tangential flow and the Nusselt number decreases but the skin friction factor for the lateral flow increases as the magnetic field increases. The skin friction factor for the tangential and lateral flows is increased while the local Nusselt number is decreased if the effect of viscous dissipation joule heating and heat generation are considered.

Chaturvedi (2006) has also considered the flow of a polar fluid past an infinite plate with constant suction. Alam and Rahman (2005) studied local similarity solutions for unsteady MHD free convection and mass transfer flow past an impulsively started vertical plate with Dufour and Soret Effects. Singha and Deka (2006) analyzed Skin-friction for unsteady free convection MHD flow between two heated parallel plates. Seddeek and Abdelmeguid (2004) investigated Hall and Ion-Slip effects on magneto-micropolar fluid with combined forced and free convection in boundary layer flow over a horizontal plate. When the strength of the variable magnetic field is high, hall currents become significant and have to be considered in the analysis.

Jordan (2007) analyzed the effects of thermal radiation and viscous dissipation on MHD free-convection flow over a semi-infinite vertical porous plate. The network simulation method is used to solve the boundary layer equations based on the finite difference formulation. It was found that an increase in viscous dissipation leads to an increase in both velocity and temperature profiles, an increase in the magnetic parameter leads to an increase in the temperature profiles and a decrease in the velocity profiles finally an increase in the suction parameter leads to an increase in the local skin friction and Nusselt number. Osalusi *et al* (2007) studied the effects of ohmic heating and viscous dissipation on unsteady MHD and slip flow over a porous rotating disk with variable properties in the presence of Hall and Ion-slip currents. Stokes (1951) studied the flow of an incompressible viscous fluid past an impulsively started horizontal plate.

From the literature review, much has not been done on Stokes problem of a convective flow past an infinite vertical plate in a rotating system in presence of a variable magnetic field. The present research therefore seeks to study Stokes problem of a convective flow past an infinite vertical plate in a rotating system in presence of a variable magnetic field. The present investigation is to study the effects of a variable magnetic field resulting to Hall currents on MHD stokes problem for a vertical infinite plate in a rotating system. When there is a variable magnetic field, motion of the fluid is decelerated and Hall currents, Hartman numbers become significant and hence their consideration in the analysis is important.

## 1.4 STATEMENT OF THE PROBLEM

When an electrically conducting fluid flows past a vertical infinite plate in a rotating system in presence of a variable magnetic field, the motion of the fluid is retarded and thus velocity and temperature changes are observed.

This study therefore intends to obtain an approximate solution to the shape of the velocity and temperature profiles. A study of a variable magnetic field, H, which is assumed to be applied transversely to the direction of the flow as shown in Figure 1.2 below. A sketch diagram of the research problem is shown in Figure 1.2.

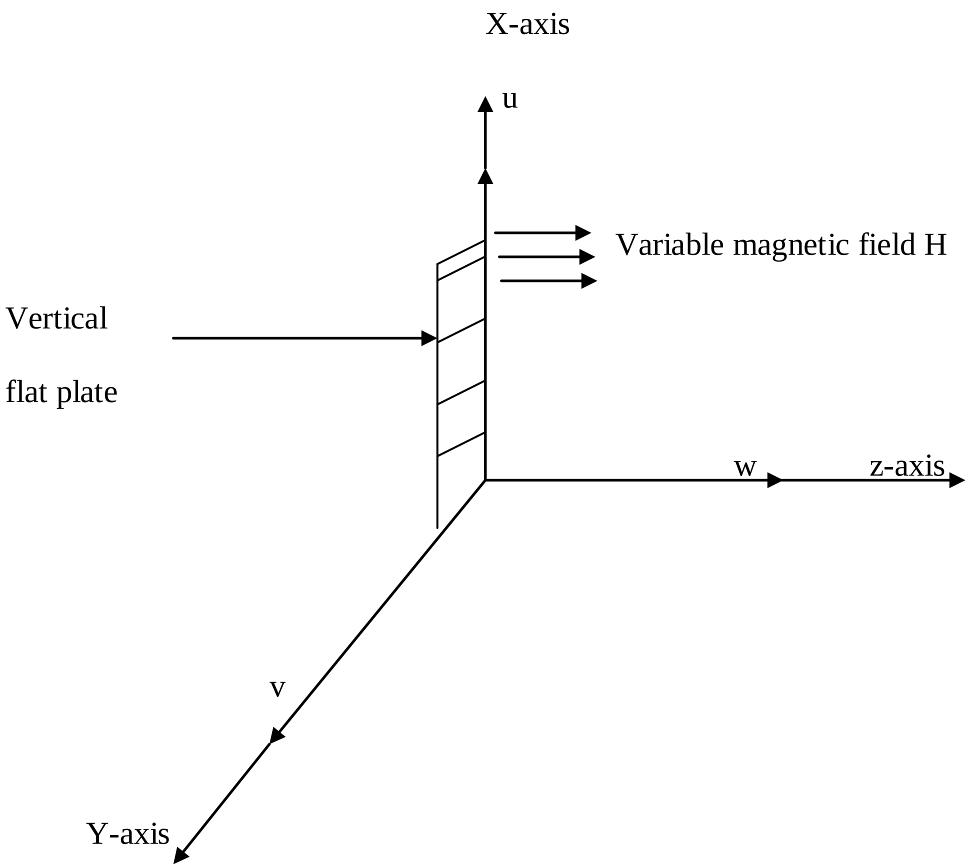


**Fig 1.2: The flow configuration with the co-ordinate system of Stokes Problem of a Convective Flow Past a Vertical Infinite Plate in a Rotating System in Presence of Variable Magnetic Field.**

## 1.5 JUSTIFICATION

For the sake of application, we consider a magnetohydrodynamic Stokes problem of convective flow for a vertical infinite plate in a rotating system in presence of a variable magnetic field. Magnetohydrodynamic (MHD) convection flow has many important engineering applications in the design of power generators, heat exchangers, pumps and flow meters, in solving space vehicle propulsion, control and re-entry problems; in designing communications and radar system; in creating novel power generating systems; in developing confinement schemes for controlled fusion and in design of nuclear cooling reactors and MHD accelerators.

## 1.6 RESEARCH OBJECTIVES

The objectives of our study are:

1) To determine both the velocity and temperature distribution for electrically conducting fluid flowing past a vertical infinite plate in a rotating system subjected to variable magnetic field.
2) To investigate the effect of the various parameters (Hall parameter, Eckert number, Grashof number, magnetic parameter) on the flow field.
3) To analyze the skin friction and rate of heat transfer.

## 1.7 OUTLINE OF THESIS

Chapter one constitutes introduction in which the main terms used are defined. Literature review, research objectives and applications are also outlined in chapter one.
Equations governing the flow including mass conservation equation, momentum and energy equations are given in chapter two. Chapter three presents the numerical method employed in solving the non linear equations.

Chapter four considers Stokes problem of a convective flow past a vertical infinite plate. The vertical infinite plate is subject to strong transverse variable magnetic field that is applied perpendicularly. The fluid considered is viscous and in a rotating system. Finite difference method has been used to obtain the velocity and temperature profiles as the governing equations obtained are non-linear. The results of velocity and temperature profiles are presented using graphs. To calculate the skin friction, numerical differentiation using Newton interpolation formula has been used. Nusselt number is employed in the computation of the rate of heat transfer. Results of rate of heat transfer are presented by using tables. At the end of the chapter, discussion of the effects various parameters on the velocity and temperature profiles are given.

Chapter six is a summary of the conclusions and recommendations. References have also been provided with their names arranged alphabetically (Harvard Referencing system). The equations of an electrically conducting incompressible, viscous fluid flow past an impulsively started infinite vertical plate in presence of transverse variable magnetic field are outlined in their general form in the next chapter.

# CHAPTER TWO

# 2.0 GOVERNING EQUATIONS

## 2.1 INTRODUCTION

This chapter considers the governing equations of magneto-hydrodynamics that are obtained from a combination of electromagnetic theory and fluid mechanics. The approximations made in this particular flow problem are also considered. Conservation equations namely conservation of mass, momentum and energy are stated. Electromagnetic equations such as Maxwell's Equation and Ohms' Law are considered. In chapter 2 section 2.3, the general dimensional form of the equations governing the fluid flow is presented. Non-dimensional numbers are then defined. Non-dimensionalization of the governing equations is then done by selecting certain characteristic quantities. Finite difference method is then used to solve the final set of non-dimensionalized equations.

## 2.2 ASSUMPTIONS AND APPROXIMATIONS

In order to reduce complexity and achieve the outlined objectives in the previous chapter, the following assumptions and approximations were made.

i) Assume the ratio of the square of the fluid velocity $v$ and that of the square of the velocity of light $C$ are too small, i.e. $\frac{v^2}{C^2} \leq 1$

ii) The fluid flow is restricted to a laminar domain.

iii) The fluid is incompressible hence the density of the fluid is assumed to remain constant.

iv) There no chemical reactions taking place in the fluid.

v) There is no externally applied electric current thus the Lorenz Force is given by $\vec{J} \times \vec{B}$. Thus the force $\rho_e \vec{E}$ due to electric field (induced) is negligible i.e. $\vec{E} = 0$.

vi) The induced magnetic field due to the fluid motion and presence of magnetic field does not affect the original externally applied magnetic field and if this happens, it’s negligible.

vii) The induced magnetic field is assumed negligible which is justified for very small Reynolds number.

viii) The plate is non-conducting

ix) The following properties are assumed to be isotropic; permittivity, permeability and conductivity. Thus $\vec{D}$ and $\vec{J}$ have the same direction as $\vec{E}$, and $\vec{B}$ has the same direction as $\vec{H}$ and we write $\vec{B} = \mu_e \vec{H}$ in any frame of reference.

x) Ohm’s law is given by $\vec{J} = \sigma \vec{E}' = \sigma\left(\vec{E} + \vec{V} \times \vec{B}\right)$,and $\vec{J}' \approx \vec{J}$ since $\rho_e \vec{U}$ is negligible compared with $\sigma\left(\vec{E} + \vec{V} \times \vec{B}\right)$

xi) For high conductivity i.e. $\sigma \to \infty$, ohm's law indicates that for finite $\vec{J}$ then $\vec{E}' = 0$ and $\vec{E} = -\vec{V} \times \vec{B}$.The current is then determined by $\nabla \times \vec{H} = \vec{J}$ and not ohm's law.

## 2.3 THE GOVERNING EQUATIONS

We consider the flow of a homogeneous, isotropic, viscous, electrically conducting incompressible fluid with constant density $\rho$, constant conductivity $\sigma$ and constant coefficient of viscosity $\mu$, with velocity vector $\vec{q}$, having components $u, v, w$ or $U_j$ in the direction $X_j$ $j = 1,2,3$ and pressure $\mathrm{P}$. Further let $\vec{H}$ denote the magnetic field strength with components $H_x, H_y$ and $H_z$ or $H_j$ ,electric field $\vec{E}$ with components $E_x, E_y$ and $E_z$ or $E_j$ ,the electrical current density $\vec{J}$ with components $J_x, J_y, J_z$ or $J_j$ and $\rho_e$ be the excess electrical charge. Classical thermodynamics postulates that the thermal state of a fluid is determined by only two independent thermodynamic properties. A third thermodynamic property is related to two independent properties by the equation of state of the fluid for example $\rho = \rho(\mathrm{P}, \mathrm{T})$ where the physical quantities have their usual meaning as given in the nomenclature.

The fundamental equations of fluid dynamics are based on the following universal laws of conservation namely: Conservation of mass, momentum and energy.

### 2.3.1 Equation of continuity

Conservation of mass equation (also called the equation of continuity) can be expressed as (Hughes and Gaylord, 1964)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho U_j\right) = 0 \qquad (2.1)$$

The equation of continuity (conservation of mass equation) is based on two fundamental principles namely:

1. The fluid is neither created nor destroyed in the field of flow i.e. the fluid mass is conserved.
2. There are no empty spaces between particles that were in contact and that the fluid volume is not affected by an increase in pressure i.e. the flow is continuous. This is the so called continuum hypothesis.

### 2.3.2 Equation of conservation of momentum (Equation of motion)

Equation of conservation of momentum is derived from Newton's second law of motion which states that the sum of resultant forces is equal to the rate of change of momentum of the flow i.e. the net rate of momentum must be equal to the net sum of forces acting on the fluid. The momentum of a body is defined as the product of its mass and velocity. Thus, when a force is applied to an incompressible fluid of any given mass, its velocity changes. This equation may be expressed mathematically and in tensor form as follows (Donald et al, 1997):

$$\rho\left(\frac{\partial u_j}{\partial t}+U_j\frac{\partial u_i}{\partial x_j}\right)=\rho F_i+\frac{\partial \sigma_{ji}}{\partial x_j} \tag{2.2}$$

Since this study involves viscous fluids and hence from Newton's constitutive law we have

$$\sigma_{ji}=-\left(P+\frac{2}{3}\eta\, div\, u\right)\delta_{ij}+\eta\left(\frac{\partial u_j}{\partial x_j}+\frac{\partial u_i}{\partial x_i}\right) \tag{2.3}$$

In which case $\sigma_{ji}, \delta_{ij}$ and $\eta$ are the stress tensor, kronecker delta and dynamic viscosity respectively. For incompressible flow, Equation (2.3) reduces to

$$\sigma_{ji}=-P\delta_{ij}+\eta\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_i}{\partial x_j}\right) \tag{2.4}$$

On substitution of equation (2.4) into (2.2) and using the fact that the flow is incompressible with invariant viscosity, equation (2.2) (equation of conservation of momentum) yields:

$$\rho\left(\frac{\partial u_j}{\partial t}+U_j\frac{\partial u_i}{\partial x_j}\right)=-\frac{\partial P}{\partial x_i}+\rho\upsilon\nabla^2 u_i+\rho F_i \tag{2.5}$$

Both gravitation force $g$ and electromotive force (Laplace force) influences the fluid under consideration so that the volume of density of the external forces is given by the equation (Holman, 1992)

$$\rho\vec{F}=\rho g+\vec{J}\times\vec{B} \tag{2.6}$$

Substituting (2.6) in (2.5) we obtain

$$\frac{\partial\vec{u}}{\partial t}+(\vec{u}\cdot\nabla)\vec{u}=-\frac{1}{\rho}\nabla P+\sigma\nabla^2\vec{u}+\rho g+\frac{1}{\rho}\vec{J}\times\vec{B} \tag{2.7}$$

Equation (2.7) can be written as

$$\rho\frac{D\vec{q}}{Dt}=\rho\left(\frac{\partial\vec{q}}{\partial t}+(\vec{q}\cdot\nabla)\vec{q}\right)=-\nabla P+v\nabla^{2}\vec{q}+\nabla\vec{i}_{o}+\vec{F}_{e}+\vec{F}_{g} \tag{2.8}$$

In which case, $\vec{i}_o$ is the stress tensor, $\vec{F}_g$ is the non-electric force per unit volume for instance, gravitational force $\vec{F}$ and $\vec{F}_e$ is the electromagnetic force which mathematically can be expressed as

$$\vec{F}_e=\rho_e\vec{E}+\vec{J}\times\vec{B} \tag{2.9}$$

Where $\rho_e\vec{E}$ is the electrostatic force and the second term is $\vec{J}\times\vec{B}$ is the pondermotive force which is well known as the driving force of an electric motor. This pondermotive force is defined as the vector product of the electric current density $\vec{J}$ and the magnetic induction, $\vec{B}=\mu_e\vec{H}$ where $\mu_e$ is the magnetic permeability. The interaction between the magnetic field and the flow field has been considered as the only important term in the dynamics of conducting fluid in this study. The pondermotive force $\vec{F}_e$ is in the direction perpendicular to both the magnetic field $\vec{H}$ and the electrical current density $\vec{J}$. Therefore the direction of the magnetic field $\vec{H}$ will have significant influence on the flow field.

If equation (2.8) is divided both sides by the density, $\rho$, then the first term on the L.H.S represents the temporal acceleration and the second term is the convective acceleration. The convective acceleration is responsible for the acceleration even when the flow is steady.

On the R.H.S, the first term represents pressure gradient force, second term is the viscous force, third term is the stress tensor force, the fourth term is the Lorentz force (electromagnetic force) and the fifth term is the gravitational force. The combination of the gravitational force and the Lorentz force takes care of the body force. Since the flow is along the vertical plane and the gravitational force acts in a vertical direction downwards, the gravitational force term cannot be neglected. Thus, the equation of momentum in simplified form incorporating all the forces under consideration can be written as:

$$\rho\left(\frac{\partial v}{\partial t}+v\cdot\nabla v\right)=-\nabla p+\mu\nabla^2 v+f \qquad (2.10)$$

where $\rho\left(\frac{\partial v}{\partial t}+v\cdot\nabla v\right)$ is the inertia term, $\frac{\partial v}{\partial t}$ is the unsteady acceleration, $v\cdot\nabla v$ is the convective acceleration, $-\nabla p$ is the Pressure gradient, $\mu\nabla^2 v$ is the Viscous forces and $f$ represents "other" body forces (forces per unit volume), such as gravity, Lorentz force or centrifugal force.

It should be noted that only the convective terms are non linear for incompressible Newtonian flow. The convective acceleration is an acceleration caused by a (possibly steady) change in velocity over position, for example the speeding up of fluid entering a converging nozzle. Though individual fluid particles are being accelerated and thus are under unsteady motion, the flow field will not necessarily be time dependent.

### 2.3.3 Equation of conservation of thermal energy (Energy Equation)

This equation results from the first law of thermodynamics which states that the amount of heat added to a system $dQ$ equals to the change in internal energy $dE$ plus the work done $dW$ i.e. $dQ = dE + dW$. In tensor form, the energy equation can be expressed mathematically as:

$$\rho \frac{\partial h}{\partial t} + \frac{\partial}{\partial x_j} \rho u_j h = \frac{\partial P}{\partial t} + \frac{\partial}{\partial x_j} u_j P - \frac{\partial \vec{q}_j^{\,*}}{\partial x_j} + \phi \tag{2.11}$$

Where $\phi = i_{ij} \frac{\partial u_i}{\partial x_j}$ (2.12)

is known as the dissipation function, $i_{ij}$ is the viscous stress tensor is the specific enthalpy and $q^*{}_j$ is the local rate of heat transfer per unit area. In equation (2.11), heat produced by external forces has been neglected. For an incompressible fluid flow in two dimensions, we have

$$\phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2$$

From the Fourier law the conduction can be obtained as

$$q^*{}_j = -k \frac{\partial T}{\partial x_j} \tag{2.13}$$

where in this case k is the thermal conductivity. Equation (2.11) can be simplified using the definition of h which is given as

$$h = e + \frac{P}{\rho} \tag{2.14}$$

in which e is the specific internal energy. In differential form equation (2.14) can be written as

$$dh = de + \frac{1}{\rho}dp + pd\left(\frac{1}{\rho}\right) \tag{2.15}$$

Applying the first and second laws of thermodynamics to equation (2.14) yields

$$de = Tds - Pd\left(\frac{1}{\rho}\right) \tag{2.16}$$

In this case s is the specific enthalpy. Substituting equation (2.16) into (2.15) yields

$$dh = Tds + \frac{1}{\rho}dP \tag{2.17}$$

Since enthalpy is a property it can be expressed as $s = s(p,T)$so that on differentiating both sides of this equation yields

$$ds = \left(\frac{\partial s}{\partial p}\right)_T dp + \left(\frac{\partial s}{\partial T}\right)_p dT \tag{2.18}$$

On application of the following generalized thermodynamic relations

$$\left(\frac{\partial s}{\partial p}\right)_T = \frac{\beta *}{\rho} \text{ and} \left(\frac{\partial s}{\partial T}\right)_p = \frac{C_p}{T}, \left(\frac{\partial\left(\frac{1}{\rho}\right)}{\partial T}\right)_p = -\frac{\beta}{\rho}$$

to equation (2.18) yields

$$ds = -\frac{\beta *}{\rho}dp + \frac{C_p}{T}dT \tag{2.19}$$

In which $\beta^*$ is the volumetric coefficient of expansion and $C_p$ is the specific heat at constant pressure. Substituting equation (2.19) into (2.17) we obtain

$$dh = C_p dT + \frac{1}{\rho}(1-\beta T)dp \tag{2.20}$$

Making use of equation (2.20) and (2.13) and further substituting into equation (2.11), the energy equation can be expressed mathematically as

$$\frac{\partial \rho C_p T}{\partial t} + \frac{\partial}{\partial t}(\rho C_p U_j T) = \frac{\partial}{\partial x_j}\left(\lambda \frac{\partial T}{\partial x_j}\right) + \beta T\left(\frac{\partial \rho}{\partial t} + \frac{\partial u_i P}{\partial x_j}\right) + \phi \tag{2.21}$$

In this study, it is assumed that the hydromagnetic flow of a viscous incompressible and undilatable electrically conducting fluids have constant physical properties .Thus equation (2.21) is altered by the presence of electrical dissipation which is the heat energy produced by work done by the electrical currents. The work done by electrical currents is equal to $\vec{j}.\vec{E}'$ where $\vec{E} = \vec{E} + \vec{V} \times \vec{H}$ is called the effective electric field. $\sigma^{-1} j^2$ is the dissipative heat due to electrical current. On substitution of this dissipative heat into equation (2.21), it yields

$$\rho C_p \frac{DT}{Dt} = k\nabla^2 T + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 + \frac{1}{\sigma} j^2 + \phi \tag{2.22}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla$ and $\nabla$ is the grad operator.

On neglecting the energy dissipated into heat due to joule effect then equation (2.22) yields

$$\rho Cp\frac{DT}{Dt}=k\nabla^2T+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)^2+\phi \tag{2.23}$$

In vector form equation (2.23) can be expressed as

$$\frac{\partial T}{\partial t}+(\vec{q}.\nabla)T=\frac{k}{\rho C_p}\nabla^2T+\phi \tag{2.24}$$

where $\vec{q}$ is the velocity vector.

To determine the velocity and temperature distributions of a hydromagnetic flow, equations (1.13), (2.8) and (2.24) together with appropriate boundary conditions and Maxwell equations is used iteratively by use of a computer program.

### 2.3.4 Electromagnetic equations

To analyze and describe the action of charged particles (electrons and ions) on each other, the concept of point charge is useful. However in electrodynamics of continuous media we must accept that electric charge is a continuously distributed quantity. Maxwell equations are the governing equations in electromagnetic fields. The Maxwell equations relate the field of the vectors, $\vec{E}$, $\vec{B}$, $\vec{D}$, $\vec{H}$, $\vec{J}$ and the charge density $Q$ independently on the properties of the matter. Since these equations represent mathematical equations of certain experimental results, it may not be easy to prove them though their applicability to any situation can be verified.

The Maxwell's equations for time- varying magnetic are expressed as (Moreau, 1990):

$$Curl\vec{E} = \nabla\times\vec{E} = -\left(\frac{\partial\vec{B}}{\partial t}\right) \tag{2.25a}$$

$$\left.\begin{aligned} div\vec{B} = \nabla\cdot\vec{B} = 0 \\ div\vec{D} = \nabla\cdot\vec{D} = Q \end{aligned}\right\} \tag{2.25b}$$

$$Curl\vec{H} = \nabla\times\vec{H} = \vec{J} + \left(\frac{\partial\vec{D}}{\partial t}\right) \tag{2.25c}$$

In which $\vec{E}$, $\vec{B}$ , $\vec{D}$ , $Q$ and $\vec{H}$ are the electric field, magnetic field, electric induction, density of electric charge and magnetic induction respectively. In our study, we are interested in electrically conducting fluids which are sufficiently conducting, thus the charge relaxation time is much shorter than the transit time of electromagnetic waves. Equation (2.25 b) has been obtained owing to the fact that the magnetic field is considered as divergenceless i.e. using the assumption that there are no magnetic flux sources and sinks within the flow field. Thus the Maxwell equations in our study reduces to

$$div\vec{B} = 0 \tag{2.26a}$$

$$Curl\vec{E} = -\left(\frac{\partial\vec{B}}{\partial t}\right) \tag{2.26b}$$

$$Curl\vec{H} = \vec{J} \tag{2.26c}$$

Equation (2.26b) represents the Faraday's law. The equation expresses the postulate for electromagnetic induction which asserts that the electric field intensity in a region of time- varying magnetic flux density is non-conservative and cannot be expressed as gradient or scalar potential. On the other hand, equation (2.26 c) represents the Amperes' law.

The physical conservation laws of electric charge and equations of electrical current density are required in describing MHD phenomenon mathematically.

### 2.3.4.1 Equations of electrical current density (Ohm's law)

This law (Ohm's law) is a characteristic feature of the ability of a material to transport electric charge under the influence of an applied magnetic field. In reference to laboratory results, it had been realized that, in a metal at constant temperature the current density is linearly proportional to the electric field. In other words, for an electrically conducting fluid at rest, the electric current density can be written mathematically as:

$$\vec{J} = \sigma\vec{E} \tag{2.27}$$

where $\sigma$ is the conductivity of the material. When a fluid moves, the magnetic field induces a current in the conductor which is of magnitude $J \times B$.As a result therefore, in this motion two types of force exist: electric force $\vec{E}$ (due to electric field) and magnetic force, $\vec{U} \times \vec{B}$(due to magnetic field).The sum of these two forces give the Lorentz force which is the total electromagnetic force $\vec{F}_e$ on a unit electric charge which in mathematical form is given as $\vec{F}_e = \vec{E} + \vec{U} \times \vec{B}$.The Lorentz force acts on the fluid particles. Equation (2.27) represents a constitutive law (Ohm's) which characterizes the ability of the material to transport electric charge under the influence of an applied electrical field. Ohm's law also holds in an electrically conducting fluid in which the electric current density is due to the regulating motion of charged particles in the field. For an observer attached to the particle, this property of matter holds true for fluids in motion. Let $\vec{J}'$ and $\vec{E}'$ be respectively the current density and electric field as seen by this observer (Principle of special relativity). To retain the symbols $\vec{J}$ and $\vec{E}$ for these quantities in the laboratory frame of reference then equation (2.27) can be expressed as

$$\vec{J}' = \sigma \vec{E}' \tag{2.28}$$

By the Lorentz transformation

$$\left.\begin{aligned} &\vec{J}' = \vec{j}' - q\vec{u} \\ &and \\ &\vec{E}' = \vec{E} + \vec{U} \times \vec{B} \end{aligned}\right\} \qquad \text{(2.29a, b)}$$

in which $\vec{U}$ and $\vec{B}$ are the fluid velocity and magnetic field respectively. The transport of electric charge by convection in equation 2.29 (a) is denoted by the term $\vec{q}'u$.In our study, this term has been neglected as compared to the transport by conduction which is proportional to $\sigma$. On substitution of equation 2.29 (a, b) in equation (2.28) and consideration this approximation we obtain

$$\vec{J} = \sigma\left(\vec{E} + \vec{U} \times \vec{B}\right) \tag{2.30}$$

Equation (2.30) is referred to as the generalized ohm's law. In the present study, the flow of an electrically conducting fluid in presence of a strong variable magnetic field is considered. This equation will be handled in the next chapter when the Hall currents are incorporated and neglecting the Ion-slip.

### 2.3.4.2 The principle of conservation of electric field

(Equation of conservation of electric charge)

This principle is analogous to the conservation of mass and it must be satisfied at all times. It states that if $Q*$ is the charge of a given quantity of matter, its particular derivative $\frac{dQ*}{dt}$ must be zero. In mathematical form, the law can be expressed as (Shercliff, 1965)

$$\frac{dQ*}{dt} = \int_D \left[ \frac{\partial Q}{\partial t} + div\{Q(U+V)\} \right] dv = 0 \qquad (2.31)$$

In which $Q$ is the volume density of electric charge, $U$ is the local velocity of the matter in the given frame and $V$ is the relative velocity of the charge carriers with respect to the matter. For equation (2.31) to be true for any choice of $D$, then at any point the following relations should be true

$$\frac{dQ*}{dt} + div\vec{J} = 0 \qquad (2.32)$$

The charge $Q$ of a domain $D$, is defined as

$$Q* = \int_D Q dv \qquad (2.33)$$

From equation (2.33), the particular derivative of the volume integral can be expressed as:

$$\begin{aligned}\frac{dQ^*}{dt} &= \int_D \frac{\partial q}{\partial t} dv + \oiint_s (U+V).nds \\ &= \int_D \left[ \frac{\partial q}{\partial t} + \nabla.\{Q(U+V)\} \right] dv\end{aligned} \tag{2.34}$$

In which S is the closed surface which bounds the domain $D$.

Since this study deals with electrically conducting fluids, the charge density becomes negligible, and the equation of conservation of charge i.e. by definition of $div\vec{J}$ equation (2.32) yields

$$div\vec{J} = \nabla.\vec{J} \tag{2.35}$$

In this equation, it can be deduced that $\vec{J}$ belongs to the class of conservative vector fields and the RH.S shows that the rate of change of charge density with respect to time vanishes. This is the equation of charge conservation and it results from the equation of continuity.

#### 2.3.4.3 Induction equation

The main objective in MHD is to study velocity and magnetic field distributions and their interactions. In order to study the transport of plasma and magnetic field lines quantitatively, consider the fundamental induction equation, i.e. Faraday‘s law in combination with the simple phenomenological Ohm‘s law, relating the electric field in the plasma frame with its current:

$$j = \sigma_0 \left( \vec{E} + \vec{V} \times \vec{B} \right) \tag{2.36}$$

Using Ampere‘s law for slow time variations, without the displacement current and the fact that the field is free of divergence $(\nabla . B = 0)$, yields the induction equation (with conductivity $\sigma_0$) (Polovin, 1990):

$$\frac{\partial H}{\partial t} = \nabla \times (V \times H) + \frac{1}{\mu_o \sigma_o} \nabla^2 H \tag{2.37}$$

where $\nabla \times (V \times H)$ is the convection and $\frac{1}{\mu_o \sigma_o} \nabla^2 H$ is the diffusion.

In an ideal collisionless plasma in motion with infinite conductivity the induction equation becomes

$$\frac{\partial H}{\partial t} = \nabla \times (V \times H) \tag{2.38}$$

The field lines are constrained to move with the plasma frozen-in field. If plasma patches on different sections of a bundle of field lines move oppositely, then the lines will be deformed accordingly. Electric field in plasma frame, E' = 0, voltage drop around closed loop is equal to zero. Assuming the plasma streams at bulk speed *V*, then the induction equation can be written in simple dimensional form as:

$$\frac{B}{\tau} = \frac{VB}{L_B} + \frac{B}{\tau_d} \tag{2.39}$$

The ratio of the first to the second term gives the so-called magnetic Reynolds number, $R_m = \mu_0 \sigma_0 LBV$ which is useful to decide whether plasma is diffusion or convection dominated.

From the generalized ohm's law

$$\vec{J} = \sigma\left(\vec{E} + \vec{V} \times \vec{B}\right) + \rho_e \vec{V} \tag{2.40}$$

But $\vec{B} = \mu_e \vec{H}$ and $\eta = \sigma \mu_e$

Thus $\vec{J} = \sigma\left[E + \vec{V} \times \mu_e H\right]$ (2.41)

On opening the brackets it yields

$$\vec{J} = \sigma E + \sigma \vec{V} \times \mu_e \sigma H + \rho_{e\vec{V}} \tag{2.42}$$

Taking the curl of the above equation and eliminating $E$ we have

$$Curl\,\vec{J} = Curl\,\sigma\vec{E} + Curl\left[\sigma\left(\vec{V} \times \mu_e \vec{H}\right)\right] + Curl\,\rho_e \vec{V} \tag{2.43}$$

But $Curl\vec{J} = \nabla \times \vec{J}$ ,thus

$$\nabla \times \vec{J} = \nabla \times \sigma\vec{E} + \sigma\mu_e Curl\left(\vec{V} \times \vec{H}\right) + \nabla \times \rho_e V \tag{2.44}$$

$$\nabla \times \vec{J} = \nabla \times \sigma\vec{E} + \eta Curl\left(\vec{V} \times \vec{H}\right) + \nabla \times \rho_e \vec{V} \tag{2.45}$$

But $CurlE = \nabla \times E = -\left(\frac{\partial B}{\partial t}\right)$ (2.46)

Hence

$$\frac{\nabla \times \vec{J}}{\eta} = Curl\left(V \times H\right) + \frac{1}{\eta}\nabla \times \sigma\vec{E} + \frac{1}{\eta}\nabla \times \rho_e V \tag{2.47}$$

$$\frac{\nabla \times \vec{J}}{\eta} = Curl\left(\vec{V} \times \vec{H}\right) - \frac{1}{\eta}\sigma\left(\frac{\partial B}{\partial t}\right) + \frac{1}{\eta}\nabla \times \rho_e \vec{V} \tag{2.48}$$

Thus rearranging the equation yields

$$\frac{\partial B}{\partial t} = \frac{\eta}{\sigma} Curl(V \times H) + \frac{1}{\sigma} \nabla \times \rho e V - \frac{\nabla \times \vec{J}}{\sigma} \tag{2.49}$$

$$\frac{\partial}{\partial t}(\mu_e H) = \frac{\eta}{\sigma} Curl(V \times H) + \frac{1}{\sigma} \nabla \times \rho_e V - \frac{\nabla \times \vec{J}}{\sigma} \tag{2.50}$$

$$\mu_e \frac{\partial H}{\partial t} = \frac{\eta}{\sigma} Curl(V \times H) + \frac{1}{\sigma} \nabla \times \rho e V - \frac{\nabla \times \vec{J}}{\sigma} \tag{2.51}$$

Dividing both sides by $\mu_e$ yields

$$\frac{\partial H}{\partial t} = \frac{\eta}{\sigma \mu_e} Curl(V \times H) + \frac{1}{\sigma \mu_e} \nabla \times \rho e V - \frac{\nabla \times \vec{J}}{\sigma \mu_e} \tag{2.52}$$

$$\frac{\partial H}{\partial t} = Curl(V \times H) + \frac{1}{\eta} \nabla \times \rho e V - \frac{\nabla \times \vec{J}}{\eta} \tag{2.53}$$

But from laboratory results, $\frac{\nabla \times \vec{J}}{\eta} = 0$ hence the equation yields

$$\frac{\partial H}{\partial t} = Curl(V \times H) + \frac{1}{\eta} \nabla \times \rho e V \tag{2.54}$$

Expanding $\nabla \times \rho_e V$ using the cross product we have;

$$\nabla \times \rho_e V = \rho e \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \rho_e \left[ i\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - j\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + k\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\right]$$

Substituting this expansion in equation (2.54) and simplifying yields

$$\frac{\partial H}{\partial t} = Curl(V \times H) + \frac{1}{\eta}\nabla^2 H \tag{2.56}$$

Where $\eta = \sigma\mu_e$ is called the electrical diffusivity of the fluid and the equation is known as the induction equation.

Further, recalling that $\nabla \times \vec{H} = \vec{J}$ ,then

$$\nabla \times \vec{J} = \nabla \times \left(\nabla \times \vec{H}\right) = \left(\nabla \cdot \vec{H}\right)\nabla - (\nabla \cdot \nabla)\vec{H} = \frac{\partial \vec{H}}{\partial t} \tag{2.57}$$

This equation reduces to

$$\frac{\partial \vec{H}}{\partial t} + (\vec{u} \cdot \nabla)\vec{H} - \left(\vec{H} \cdot \nabla\right)\vec{u} = \nu_H \nabla^2 \vec{H} \tag{2.58}$$

where $\mu$ and $\nu_H$ are assumed to be constants.

For large electrical conductivity, then $\nu_H = 0$ since $\frac{1}{\sigma\mu_e} = 0$ as $\mu_e \rightarrow \infty$. Hence equation (2.58) becomes

$$\frac{\partial \vec{H}}{\partial t} + (\vec{u} \cdot \nabla)\vec{H} - \left(\vec{H} \cdot \nabla\right)\vec{u} = 0 \tag{2.59}$$

**2.3.5 DESCRIPTION OF THE FLOW**

In the present study Stokes flow past an infinite vertical plate in rotating system in presence of a variable magnetic field is considered. The magnetic field is applied transversely along the z-axis and perpendicular to the vertical plate. The plate is non-conducting and the fluid is electrically conducting.

At $t = 0$, the vertical plate is set into impulsive motion in its own plane (x-axis direction) at a constant velocity $U$. The transverse inhomogeneous magnetic field is in the z-direction. The vertical plate is kept at a lower temperature than the fluid i.e. $T_\infty > T_w$. Fluid flow is assumed incompressible, Newtonian, electrically conducting and the density fluctuations are Boussinesq approximated. The Boussinesq approximations means that the density differences are confined to the buoyancy term, without violating the assumption of incompressibility and that the effect of the pressure on the fluid density is negligible. The fluid flow being studied is free convectional and takes place along the x-axis under the action of transverse variable magnetic field.

### 2.3.5.1 Flow conditions

The flow conditions for this problem are

$$\left.\begin{array}{l} u = U_o u^*(x^*),\ v = 0,\ \ w = 0 \\ H_z = H_z(x,t),\ H_z = H_o H^*{}_z(x^*, t^*),\ H_x = 0,\ H_y = 0 \\ P = \rho U_o{}^2 P^*(z^*, t^*),\ \dfrac{\partial(\ )}{\partial t} = \dfrac{\partial(\ )}{\partial y} = 0 \end{array}\right\} \qquad (2.60)$$

Where $U_o$ and $H_o$ are the characteristic velocity and magnetic field respectively and $\rho$ is the constant fluid density. The fluid velocity in x-direction depends on z and t only, the magnetic field intensity is a function of x and t only and the pressure depend on z and t only. The partial derivatives with respect to z vanish since the flow field is infinite in extent in the Z-direction or in other words it is unbounded in this direction. The vertical plate is electrically insulated and the velocity of the fluid particles in contact with the boundaries is equal to that of the plates due to the no-slip condition.

## 2.4 FINAL SET OF EQUATIONS

In the present study, MHD Stokes problem of free convection flows of electrically conducting fluids past an impulsively started infinite plate (or vertical channel) to which a variable strong magnetic field is applied in the normal direction to the plate is studied (Fig 2.1). The motion of the fluid is impeded by the variable magnetic field which exerts a restraining force. The system is rotating and hence this makes the problem more complicated than the same problem in either electromagnetic theory or in fluid mechanics.

Owing to the fact that the flow under consideration is due to density differences, these differences are very small such that the velocities are also small. Hence as in the case of forced convection the assumption of incompressibility i.e. $\rho$ = a constant (Boussinesq approximation) is justified. From now onwards, dimensional quantities are denoted by star (superscript).The flow considered would be governed by the equations outlined as:

$$\frac{\partial (u_j *)}{\partial x_j *} = 0 \tag{2.61}$$

Because of the no-slip condition, the velocity is equal to zero at wall and it increases to a maximum then decreases to zero at the edge of the boundary layer as the free stream conditions are at rest in the convection system (Fig 2.1).

In this case, we choose $x_j*$ to be the coordinate along the plate and $x_i*$ coordinate perpendicular to the plate. For the flow along a vertical infinite plate all the variables except pressure of the fluid are functions of $x_i*$ and $t*$.The pressure gradient $\frac{\partial p}{\partial x_j*}$ in equation(2.7), in $x_j*$ direction results from the elevation up the plate, and hence

$$\frac{\partial p}{\partial x_j*} = -\rho_\infty g \tag{2.62}$$

That is, change in pressure over a height $dx_j*$ equal to the weight per unit area of the fluid elements. The weight is the same as the force in this fluid element i.e. $\rho g$. On substitution of this weight and equation (2.62) in equation (2.7) we obtain the equation of motion for the problem under consideration in mathematical form as

$$\rho\left[\frac{\partial u_j*}{\partial t} + u_j*\frac{\partial u_j*}{\partial x_j*} + ui*\frac{\partial u_j*}{\partial x_i*}\right] = +\rho_\infty g - \rho g + \mu\frac{\partial^2 u_j*}{\partial x_j*^2} + \vec{J}\times\vec{B} \tag{2.63}$$

Or

$$\rho\left[\frac{\partial u_j*}{\partial t*} + u_j*\frac{\partial u_j*}{\partial x_j*} + ui*\frac{\partial u_j*}{\partial x_i*}\right] = g(\rho_\infty - \rho) + \mu\frac{\partial^2 u_j*}{\partial x_i*^2} + \vec{J}\times\vec{B} \tag{2.64}$$

The density differences $\rho_\infty - \rho$ may be expressed in terms of volume coefficient of expansion $\beta'$ defined by

$$\beta' = \frac{1}{V*}\left[\frac{\partial V*}{\partial T*}\right] = \frac{1}{V*}\left[\frac{V*-V_\infty*}{T*-T_\infty*}\right] = \frac{\rho_\infty - \rho}{\rho(T*-T_\infty*)} \tag{2.65}$$

Or

$$\beta'\rho(T*-T_\infty*) = \rho_\infty - \rho$$

On substitution of (2.65) into (2.64) yields

$$\rho\left[\frac{\partial u_j *}{\partial t *}+u_j *\frac{\partial u_j *}{\partial x_j *}+u_i *\frac{\partial u_j *}{\partial x_i *}\right]=g\rho\beta *\left(T *-T_\infty *\right)+\mu\frac{\partial^2 u_j *}{\partial x_j *^2}+\vec{J}\times\vec{B} \qquad (2.66)$$

The energy equation remains the same as equation (2.24).Thus the equation becomes:

$$\rho C_p\left[\frac{\partial T *}{\partial t *}+u_j\frac{\partial T *}{\partial x_j *}+u_i\frac{\partial T *}{\partial x_j *}\right]=k\frac{\partial^2 T *}{\partial x_j *^2}\mu\left(\frac{\partial u_i *}{\partial x_j *}+\frac{\partial u_j *}{\partial x_i *}\right)^2+\phi \qquad (2.67)$$

In which the energy dissipated as heat due to joule effect has been neglected from equation (2.67).

The induction equation becomes

$$\frac{\partial H_y^*}{\partial t^*}+u^*\frac{\partial H_y^*}{\partial x^*}-H_y^*\frac{\partial u^*}{\partial y^*}=\frac{1}{R_m}\frac{\partial^2 H_y^*}{\partial x^{*2}} \qquad (2.68)$$

But for large $R_m$ then $\frac{1}{R_m}\frac{\partial^2 H_y^*}{\partial x^{*2}}=0$.Thus the equation becomes

$$\frac{\partial H_y^*}{\partial t^*}+u^*\frac{\partial H_y^*}{\partial x^*}-H_y^*\frac{\partial u^*}{\partial y^*}=0 \qquad (2.69)$$

Equation (2.66) and (2.67) are non linear equations and hence to solve them some initial and boundary conditions are imposed.

## 2.5 NON-DIMENSIONALIZATION

A dimensionless group is the ratio of two similar physical quantities. The dimensionless groups are useful means of defining the conditions, which exist in a physical system, and indicating which properties are of importance. Dimensional groups are useful in our present investigation since;

(i) The analysis of these dimensionless groups helps in experimental investigation of reducing the number of variables in the problem. The result of the analysis is to replace an unknown relation between n variables by a relationship between a smaller numbers, n-m, of dimensionless groups. Any reduction in the number of variables greatly reduces the labor of experimental investigation.

(ii) Dimensionless presentation of experimental data is independent of the units employed and should, therefore, be internationally intelligible and convenient to use.

There are four fundamental primary dimensions namely mass ($m$), temperature ($\mathrm{T}$), time ($t$) and length ($L$). The four basic dimensions form the basis for all other physical variables of any phenomenon that can be obtained from these basic dimensions.

Thus the non-dimensionalization process is important so that the results obtained can be applied to a surface experiencing the set of conditions to a geometrically similar surface.

A suitable non-dimensional, scheme is therefore necessary in order to normalize the boundary layer equations. The method by which the number of independent variables in the problem can be reduced into dimensionless groups such as Grashof number, Nusselt number is referred to as dimensional analysis.

A non-dimensionalization scheme normalizes the boundary layer equations. This process is necessary in fluid mechanics problem analysis so that the results obtained can be applied to a surface experiencing the set of conditions to a geometrically similar surface. The nature of fluid velocity or size of the surface determines the variations of the conditions. Efficiency and boundedness of experimental and analytical results can be achieved through selection of an appropriate scheme. Since the velocity scale is not imposed by the boundary condition, the choice of the velocity scale in free convection is difficult. Often there exists two or more velocity scales in different regions of the flow. In our case we choose the characteristic velocity as the free stream velocity $U_o$. All other physical properties are made dimensionless by their respective values at a reference temperature. The non-dimensionalization in our study is based on the following variables:

$$\left.\begin{aligned} t &= \frac{t^* U_o^2}{v},\ X_i = \frac{X_i^* U_o}{v},\ u_j = \frac{u_j^*}{U_o} \\ u_i &= \frac{u_i^*}{U_o},\ X_j = \frac{X_j^* U_o}{v},\ \theta = \frac{T^* - T_\infty^*}{T_s^* - T_\infty^*} \end{aligned}\right\} \quad (2.69)$$

In which $U_0$ is the free-stream velocity, $T_s^* - T_\infty^*$ is the temperature difference between the surface and free-stream temperature. $T_s^*$ is the convenient temperature which will result in $\theta$ being bounded in the solution. The dimensional variables are denoted by the superscript (*) star.

On substitution of the non-dimensional variables (2.69) into continuity equation (2.61) the momentum equation (2.66) and energy equation (2.67) we obtain the following equations

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{2.70}$$

$$\rho\left[\frac{\partial u_j}{\partial t} + u_j \frac{\partial u_j}{\partial x_j} + u_i \frac{\partial u_j}{\partial x_j}\right] = \frac{\partial^2 u_j}{\partial x_i^{\,2}} + \frac{\nu}{\rho u_0^{\,3}}\left(\vec{J} \times \vec{B}\right) + \frac{g\nu\beta'}{u_o^{\,3}}\left(\mathrm{T}^* - \mathrm{T}_\infty{}^*\right) \tag{2.71}$$

and

$$\frac{\mu C_p}{\kappa} \cdot \frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x_i^{\,2}} + \frac{\mu U^2}{P\kappa\left(\mathrm{T}_s - \mathrm{T}_\infty\right)}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]^2 + \phi \tag{2.72}$$

In vector form, equation (2.70) and (2.71) yields

$$\nabla.\vec{q} = 0 \tag{2.73}$$

$$\frac{\partial \vec{q}}{\partial t} + \left(\vec{q}.\nabla\right)\vec{q} = \nabla^2.\vec{q} + \frac{\nu}{\rho u^3}\left(\vec{J} \times \vec{B}\right) + \frac{g\nu\rho\beta}{u_0^{\,3}}\left(\mathrm{T}^* - \mathrm{T}_\infty{}^*\right) \tag{2.74}$$

in which case $\vec{q}$ is the velocity vector with components $u$ , $v$ and $w$ in three dimensions. All variables and equations referenced will be assumed to be non-dimensional. . The boundary conditions have been converted to conditions in the non-dimensional variables. Equation (2.70)-(2.72) above are written in their general form since a number of non-dimensional schemes are possible.

### 2.5.1 DIMENSIONLESS PARAMETERS AND THEIR SIGNIFICANCE

In this study, the fundamental equations of MHD include all the terms of fundamental equations of hydrodynamics. In this section hydrodynamic parameters and other extra parameters which result due to the interaction between the hydrodynamics variables and the electromagnetic variables are outlined. Using the scale variables in equation (2.69) and introducing extra non-dimensional variables commonly used in MHD, we now let $U_o, P_o, L, t_o, H_0$, and $E_o$ respectively be the representative values of velocity, pressure, length, time, magnetic field and electric field, then the following parameters relevant and significant to the flow can be defined.

#### 2.5.1.1 Time Parameter $R_t$

The time parameter

$$R_t = t_o \frac{U_o}{L}$$

characterizes the time scale of the problem with respect to flow velocity. In our investigation we only consider case where $R_t$ is of the order unity or greater.

#### 2.5.1.2 Mach number

Mach number is the measure of the compressibility of a fluid due to high flow velocity and is defined as the ratio of flow velocity $u$ to the speed of sound $a_o$.The ratio of the variation of density of the fluid to the variation of velocity is, to the first approximation, proportional to the square of the Mach number of the flow. Thus, for a very small mach number, the variation of density i.e. the compressibility effect due to variation of velocity of the flow field is negligible, and the fluid may be considered as incompressible.

#### 2.5.1.3 The Pressure Parameter $R_p$

For incompressible fluid flow, the pressure number

$$R_p = \frac{P_o}{\rho U_o^2}$$

is usually of the order of unity. It should be noted that the fundamental equations of MHD are applicable to flow of an ionized gas if the Mach number is small. $R_p$ is inversely proportional to the square of the reference Mach number.

#### 2.5.1.4 The Reynolds number $\mathrm{Re}$

The Reynolds number is given by

$$\mathrm{Re} = \frac{\rho U_o L}{\mu} = \frac{U_o L}{\upsilon}$$

For forced viscous flow, this number is one of the important parameters. When $\mathrm{Re}$ of the system is small, the viscous force is predominant and the effect of viscosity is important in the whole flow field, on the other hand if $\mathrm{Re}$ is large, the inertial force is predominant and the effect of viscosity is important only in the narrow boundary layer region near the solid boundary or in any other region of large variation in velocity such as inside of shockwave.

### 2.5.1.5 Magnetic Reynolds number $Rm$

This number is given as

$$R_m = \sigma \mu_e U_o L = \frac{U_o L}{\nu_H}$$

and is similar to ordinary Reynolds number, with magnetic diffusivity $\nu_H$ in place of kinematics viscosity. Physically, magnetic Reynolds number $R_m$ is also similar to Reynolds number $\mathrm{Re}$. It represents the diffusion of magnetic field while $\mathrm{Re}$ represents the diffusion of momentum. If $R_m$ is negligibly small, the magnetic field is practically unaffected by the flow field. On the other hand ,if $R_m$ is very large, then magnetic field will stay with the so called "frozen" in fields , and it will be greatly influenced by the motion of the fluid.

### 2.5.1.6 Magnetic Pressure number $R_H$

This number is given by

$$R_H = \frac{\mu_e H_o^2}{\rho U_o^2} = \frac{\nu^2{}_H}{U_o^2}$$

in which $\nu_H$ is the speed of Alfrens' wave. Alfren (1942) showed that if there is a homogeneous magnetic field $H_o$ in an incompressible fluid and inviscid fluid of density $\rho$ and of infinite electrical conductivity $\sigma = \infty$, the disturbance in this liquid will propagate a wave in the direction of $H_o$ with speed of $\nu_H$ -Alfren's wave speed. In this study the interest is free-convection MHD flow with a variable magnetic field. Hence, the need to introduce more non-dimensional parameters that are significant in this type of flow.

### 2.5.1.7 Prandtl number $\Pr$

This is a non-dimensional parameter which represents the ratio of momentum diffusivity $\nu$ to thermal diffusivity, $\kappa$ this number is given by

Prandtl Number = $\dfrac{\text{Momentum diffusivity}}{\text{Thermal diffusivity}}$

Mathematically, it is expressed as

$$\Pr = \frac{\nu\rho C_p}{k} = \frac{\mu C_p}{k}$$

As defined, $\Pr$ then provides a measure of relative effectiveness of momentum and energy transport of diffusion in the velocity and thermal boundary layers respectively, e.g. incase of gases $\Pr$ is nearly equal to unity therefore energy and momentum transfer by diffusion are comparable whereas for liquid metals $\Pr < 1$ and energy diffusion rate greatly exceeds the momentum diffusion rate. On the other hand incase of oils $\Pr < 1$. From this interpretation it applies that value of $\Pr$ influences the growth of the velocity and thermal boundary layer. Thus the Prandtl number acts as the conducting link between the velocity field and the temperature field since it involves momentum transfer that consequently yields heat transfer.

### 2.5.1.8 Grashof number $Gr$

This is a non-dimensional parameter that occurs in natural convection problems .Grashof number is given by

$$Gr = \frac{\nu g \beta' \left(T_s * - T_\infty *\right)}{U_o^3}$$

$Gr$ Provides a measure of the ratio of buoyancy forces to viscous forces in the velocity boundary layer, its role in free-convection is much the same as the Reynolds number in forced convection.

### 2.5.1.9 Eckert number $Ec$

Eckert number is used in momentum and heat transfer in general and compressible flow calculations in particular. This number is given by

$$Ec = \frac{U_o^2}{C_p \left(T_s * - T_\infty *\right)}$$

which is the measure of the kinetic energy of the flow relative to the enthalpy difference across the boundary layer, this number plays an important role in high speed flows for which dissipation is significant.

#### 2.5.1.10 Hartmann number $M$

It is obtained from the ratio of the magnetic force to viscous force and is defined as

$$M = \mu_e H_o U_o \left(\frac{\sigma}{\mu}\right)^{\frac{1}{2}} = \sqrt{\frac{\sigma \mu_e^2 H_o^2 U_o}{\mu U / U_o^2}} = \sqrt{\frac{magnetic force}{viscous force}}$$

Hartmann was the first scientist to use this number in the study of flow in channels where the important forces are the magnetic force and viscous force.

#### 2.5.1.11 Magnetic Parameter $M_1$

This number is obtained from the ratio of electromagnetic force to the inertial force and is defined mathematically as

$$M_1 = \mu_e H_o U_o \left(\frac{\sigma u_o}{\rho u}\right)^{\frac{1}{2}} = \sqrt{\frac{\sigma \mu_e^2 H_o^2 U}{\mu U^2 / U_o}} = \sqrt{\frac{magnetic force}{inertial force}}$$

The section that follow presents the final set of laminar boundary layer equations (in non-dimensional form) governing the flow of an electrically conducting viscous incompressible Newtonian fluid, past a vertical infinite plate in presence of a strong transverse variable magnetic field.

## 2.6 FINAL SET OF GOVERNING EQUATIONS IN NON-DIMENSIONAL FORM

To come up with the governing equations described in the previous sections, the following simplification which involves the Boussinesq's approximations is made:

i. All the transport properties except for the density $\rho$ ,are treated as constants

ii. The variation in density is negligible except when it directly causes buoyancy forces.

iii. The density varies linearly with temperature and the deviation from a reference value $(\rho_o)$ is small.

The use of the Boussinesqs approximations allows the buoyancy effect to be handled without added complication of having to consider a fully compressible fluid. On application of these assumptions and the use of dimensionless parameters in the conservation equations (equation (2.70) and equation (2.71)) and equation of energy we have the final form of the governing equations for an incompressible hydromagnetic laminar flow as

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{2.75}$$

$$\left[\frac{\partial u_j}{\partial t} + u_j\frac{\partial u_j}{\partial x_j} + u_i\frac{\partial u_j}{\partial x_j}\right] = \frac{\partial^2 u_j}{\partial {x_i}^2} + Gr\theta + \frac{\nu}{\rho {u_o}^2}\left(\vec{J}\times\vec{B}\right) \tag{2.76}$$

$$\Pr\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial {x_j}^2} + \Pr Ec\left[\frac{\partial u_i}{\partial x_j} + u_i\frac{\partial u_j}{\partial x_i}\right]^2 \tag{2.77}$$

The last term in equation (2.76) represents the electromagnetic force; the final form of this term depends on the problem considered.

In the present flow problem the displacement current is neglected. The variable transverse magnetic field induces a current given by

$$\vec{J} = \nabla \times \vec{H} = \nabla \times \frac{\vec{B}}{\mu_e} \tag{2.78}$$

Applying the vector cross product rule equation (2.78) simplifies to

$$\nabla \times \mu_e \vec{H} = \mu_e \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_z \end{vmatrix} = \mu_e \left( i \frac{\partial (H_z)}{\partial y} - j \frac{\partial H_z}{\partial x} \right) = -j \mu_e \frac{\partial H_z}{\partial x} \tag{2.79}$$

The component of the current in the x-direction vanishes since any derivative with respect to y is equal to zero i.e. $J_x = 0$. The resultant induced current which is in the y-direction can be expressed as

$$J_y = \frac{\partial H_z}{\partial x} \tag{2.80}$$

The Lorentz force is obtained as

$$\vec{J} \times \vec{B} = \vec{J} \times \mu e \vec{H} = \begin{vmatrix} i & j & k \\ 0 & J_y & 0 \\ 0 & 0 & H_z \end{vmatrix} = i \vec{H}_z J_y \tag{2.81}$$

Substituting $J_y$ from equation (2.80) into equation (2.81) yields

$$\vec{J} \times \vec{B} = \vec{J} \times \mu_e \vec{H} = H_z \frac{\partial H_z}{\partial x} i \tag{2.82}$$

This force which acts on the fluid particles is in the negative x-direction and therefore trying to oppose the flow. From Ohm's law we have;

$$\left.\begin{aligned} J &= \sigma\left(\vec{V}\times\vec{B}\right) = \sigma\left(u\vec{i} + v\vec{j}\right)\times B\vec{k} \\ &= \sigma\left(-Bu\vec{j} + Bv\vec{i}\right) \end{aligned}\right\} \tag{2.83}$$

$$J = \sigma B\left(-u\vec{j} + v\vec{i}\right) \tag{2.84}$$

Thus

$$\left.\begin{aligned} J\times B &= \sigma B\left(-u\vec{j} + v\vec{i}\right)\times B\vec{k} \\ &= -\sigma B^2 u\vec{i} - \sigma B^2 v\vec{j} \end{aligned}\right\} \tag{2.85}$$

$$J\times B = -\sigma B^2\left(u\vec{i} + v\vec{j}\right) \tag{2.86}$$

Since $\vec{B} = \mu_e\vec{H}$

Then

$$J\times B = -\sigma\mu_e^2 H^2\left(u\vec{i} + v\vec{j}\right) \tag{2.87}$$

Or in component form we have

$$\left.\begin{aligned} \left(J\times B\right)_x &= -\sigma\mu_e^2 H^2 u \\ \left(J\times B\right)_y &= -\sigma\mu_e^2 H^2 v \end{aligned}\right\} \tag{2.88}$$

Substituting the values of the Lorentz force obtained above in component form gives the momentum equation to be used in the next section.

The induction equation (2.59) is modified by substituting the magnetic field intensity H with the magnetic induction vector B. We consider that B is in the direction of H and $\vec{B} = \mu_e\vec{H}$ yielding

$$\frac{\partial \vec{B}}{\partial t} + (\vec{u} \cdot \nabla)\vec{B} - (\vec{B} \cdot \nabla)\vec{u} = \frac{1}{\sigma \mu_e} \nabla^2 \vec{B} \tag{2.89}$$

Substituting $\vec{B} = \mu_e \vec{H}$ in (2.89) yields

$$\frac{\partial \mu_e \vec{H}}{\partial t} + (\vec{u} \cdot \nabla)\mu_e \vec{H} - (\mu_e \vec{H} \cdot \nabla)\vec{u} = \frac{1}{\sigma \mu_e} \nabla^2 \mu_e \vec{H} \tag{2.90}$$

$$\mu_e \frac{\partial \vec{H}}{\partial t} + \mu_e (\vec{u} \cdot \nabla)\vec{H} - \mu_e (H \cdot \nabla)\vec{u} = \frac{1}{\sigma} \nabla^2 \vec{H} \tag{2.91}$$

$$\mu_e \left[ \frac{\partial \vec{H}}{\partial t} + (\vec{u} \cdot \nabla)\vec{H} - (\vec{H} \cdot \nabla)\vec{u} \right] = \frac{1}{\sigma} \nabla^2 \vec{H} \tag{2.92}$$

$$\frac{\partial \vec{H}}{\partial t} + (\vec{u} \cdot \nabla)\vec{H} - (H \cdot \nabla)\vec{u} = \frac{1}{\sigma \mu_e} \nabla^2 \vec{H}. \tag{2.93}$$

This equation can be expanded as follows:

$$\left.\begin{aligned} & i\frac{\partial H_x}{\partial t} + j\frac{\partial H_y}{\partial t} + k\frac{\partial H_z}{\partial t} + u\frac{\partial H_x}{\partial x} + v\frac{\partial H_x}{\partial y} + w\frac{\partial H_x}{\partial z} + u\frac{\partial H_y}{\partial x} + v\frac{\partial H_y}{\partial y} + w\frac{\partial H_y}{\partial z} + u\frac{\partial H_z}{\partial x} + v\frac{\partial H_z}{\partial y} \\ & + w\frac{\partial H_z}{\partial z} - H_x\frac{\partial u}{\partial x} - H_y\frac{\partial u}{\partial y} - H_z\frac{\partial u}{\partial z} - H_x\frac{\partial v}{\partial x} - H_y\frac{\partial v}{\partial y} - H_z\frac{\partial v}{\partial z} - H_x\frac{\partial w}{\partial x} - H_y\frac{\partial w}{\partial y} - H_z\frac{\partial w}{\partial z} \\ & = \frac{1}{\sigma \mu_e}\left[ i\left( \frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} \right) + j\left( \frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + \frac{\partial^2 H_y}{\partial z^2} \right) + k\left( \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} \right) \right] \end{aligned}\right\} (2.94)$$

Using the fact that $H = (H_x, H_y, H_z) = (0,0,H_z)$ and that the fluid flow depends on z and t only while the applied magnetic field depends on x and t, then $u = w = 0$. Then the above equation reduces to

$$\frac{\partial H_z}{\partial t} + u\frac{\partial H_z}{\partial x} = \frac{1}{\sigma \mu_e}\frac{\partial^2 H_z}{\partial x^2} \tag{2.95}$$

The next section of the thesis presents the numerical method used to solve equations (2.75)-(2.77) together with the corresponding initial and boundary conditions, which depends on the problem under consideration.

# CHAPTER THREE

# 3.0 NUMERICAL METHODS

## 3.1 Overview

Many real life problems generally do not have "analytical" solutions. Mathematics being one of the scientific research disciplines that lead to real life situations requires numerical techniques to accomplish non-analytical solutions. The part of numerical analysis which has been most changed so far, is the solution of partial differential equations by difference methods. This is owing to the fact that second-order partial differential equations govern many of the real-life physical phenomena. Such equations include Maxwell's equations, heat and momentum equations and Newton's laws of motion. A very powerful and quite a general method of dealing with most second-order (partial) differential equations is the finite difference method.

## 3.2 FINITE DIFFERENCE METHOD

This is a numerical method that makes use of finite difference codes/solvers that take low computational memory and is easy to program and modify, hence more advantageous to use in electrical problems. Before numerical computations are made, there are three important properties of finite difference equations that must be considered, namely;

- *Convergence*: A finite difference equation is convergent if the solution of finite difference equation approaches the exact solution of the partial differential equation as the mesh sizes approaches zero.
- *Consistency*: When a truncation error goes to zero, a finite difference equation is said to be consistent or compatible with a partial differential equation.
- *Stability:* The difference between a partial differential equation and the equivalent finite difference expression is referred to as truncation error. A numerical process is said to be stable if it limits amplification of all components of the initial conditions.

The governing equations described in the previous section are approximated by the application of finite difference techniques. The use of the finite difference techniques for the solution of partial differential equation is a three step process namely:

1) The partial differential equations are approximated by a set of linear equations relating to the values of the functions at each mesh point.
2) The set of the algebraic equations, generated in (1) must be solved and
3) An iteration procedure has to be developed which takes into account the non-linear character of the equation.

The solution of the finite difference equation (FDE) requires a suitable technique to advance the transient fluid motion through time. If the transient solutions are not required then the transient terms in the equations are dropped and the problem is simplified and reduced to just determining the steady state solution. In this study the partial differential equations governing the flow are replaced by a set of difference equations which are solved by successive over-relation method (SOR) to converge at each time interval. On the other hand the governing equations (2.75)-(2.77) together with initial and boundary conditions imposed (depending on the problem considered) are properly posed (i.e. their solution exists, is unique and depends on the given conditions) thus any finite difference set of equations to them which satisfies consistency conditions and is stable ensure that the method is convergent. In order to solve the system of finite difference equations, a computer program will be used for the iterative scheme. In order to approximate the difference equation (2.76)-(2.77) by a set of finite difference equation we first define a suitable mesh in the next section.

## 3.3 DEFINITION OF MESH

In order to give an explicit relation between the partial derivatives in equations (2.75)-(2.77) and the function values at the adjacent nodal points, we use a uniform mesh. This type of mesh involves subdividing the rectangular region of interest (Figure 3.1) into uniform rectangular elements, centered about mesh point whose coordinates are denoted by integer variables.

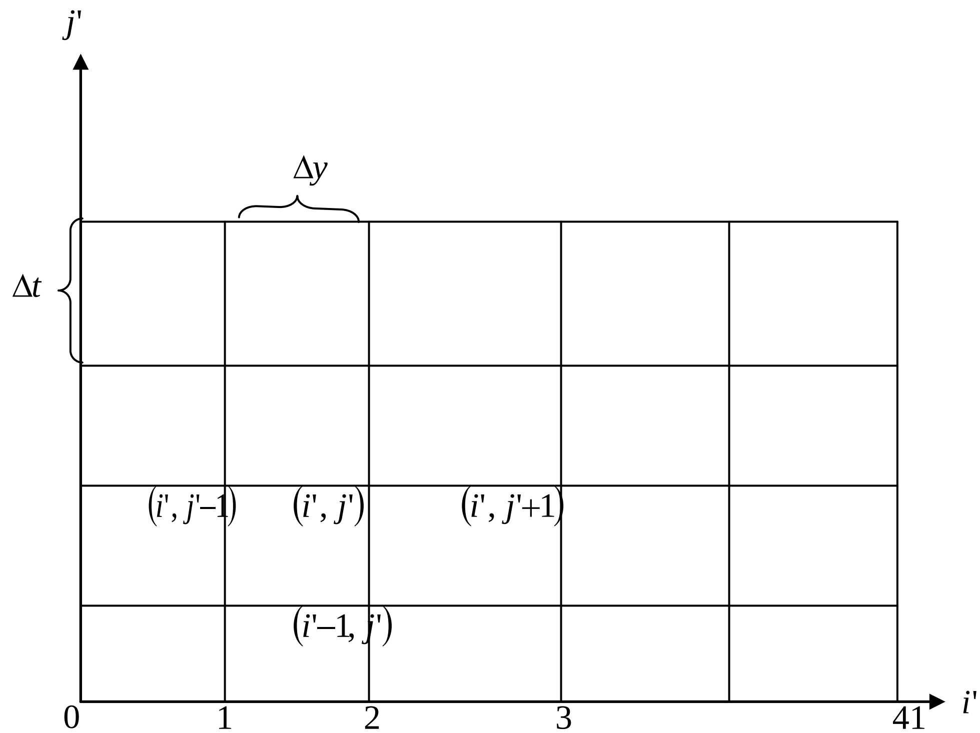


**Figure 3.1 Mesh notation of the computation domain**

$i'$ and $j'$ i.e. from fig.1, indices $i'$ and $j'$ refers to y and t respectively. If we let $\Delta t$ represent increment in t and $\Delta y$ represent increment in y then $t = j'\Delta t$ and $y = i'\Delta y$.

In viscous flow, the problem arises where the solution varies rapidly over a small domain but over the rest of the domain they change very slowly. At very large Reynolds number viscous fluid flow pattern changes rapidly in narrow boundary layers close to the wall where the fluid is brought to rest. In some cases e.g. turbulent shear drives flow, it is possible to drive asymptotic expansion applicable in the boundary layers which can be matched easily to a numerical or analytical solution in the interior. However this is usually not possible and the boundary layer has been solved numerically by ensuring that several mesh point fall with them.

The calculation of the first and second derivatives by applying either central differences or forward differences or backward differences on a uniform mesh gives approximations, which are $0(\Delta t^2)$ or $0(\Delta y^2)$, accurate in the mesh interval. In order to apply the finite difference approximations to the partial differential equations, mesh point variable are typically denoted by

$$\phi(i', j') = \phi((i'-1)\Delta y, (j'-1)\Delta t) \tag{3.1}$$

The notation is rendered more compact by omitting arguments when they have default values $i'$ or $j'$, thus

$$\phi = (i', j')$$

$$\phi(i'-1) = \phi(i'-1, j')$$

Using the Taylor's series expansion of $\phi$ about point 0 and 2 (see figure 3.2) results in

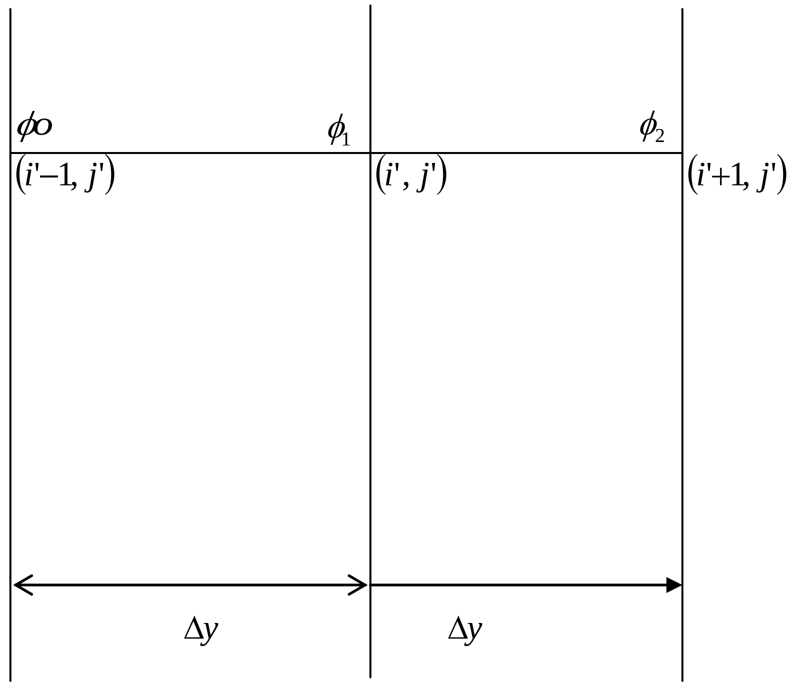


**Fig 3.2 Mesh points for expression of $\phi = (i'+1)$**

$$\left.\begin{aligned} \phi(i'+1) &= \phi_2 = \phi_1 + \Delta y \phi'_1 + \frac{\Delta y^2}{2}\phi''_1 + \frac{\Delta y^3}{6}\phi'''_1 + \ldots\ldots\ldots\ldots\ldots \\ \phi(i'-1) &= \phi_0 = \phi_1 - \Delta y \phi'_1 + \frac{\Delta y^2}{2}\phi''_1 - \frac{\Delta y^3}{6}\phi'''_1 + \ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{3.2}$$

On eliminating $\phi''$ from (3.2) we have:

$$\left.\begin{aligned} &\phi(i+1) - \phi(i-1) = 2\Delta y \phi'_1 + HOT \\ &or \\ &\phi'_1 = \frac{\phi(i+1) - \phi(i-1)}{2\Delta y} + HOT \end{aligned}\right\} \tag{3.3}$$

in which *HOT* is an abbreviation for higher order terms (this denotes the truncation error of $0(\Delta y^2)$ which can be minimized by choosing a very small value of $\Delta y$. The finite difference approximation to the second order partial derivative is obtained by eliminating $\phi'$ from (3.2) which yields

$$\phi''_1 = \frac{\phi(i+1) - 2\phi(i) + \phi(i-1)}{\Delta y} + HOT \tag{3.4}$$

Equations (3.2) and (3.3) are the central difference formulae for the first and second order partial derivatives of $\phi_1$ (with respect to y) respectively. Similarly, the central difference formulae for the first and second partial derivatives with respect to t can be written as

$$\left.\begin{aligned} &\phi'_1 = \frac{\phi(i'+1) - \phi'(i'-1)}{2\Delta t} + HOT \\ &and \\ &\phi''_1 = \frac{\phi(i+1) - 2\phi(i') + \phi(i-1)}{(\Delta t)^2} + HOT \end{aligned}\right\} \tag{3.5}$$

In forward difference from equations form (3.2) to (3.5) in their general form can be written as

$$\left.\begin{aligned} &\phi' = \frac{\phi(i+1) - 2\phi(i)}{\Delta y} + HOT \\ &\phi'' = \frac{\phi(i+1) - 2\phi(i) + \phi(i-1)}{(\Delta y)^2} + HOT \\ &\phi' = \frac{\phi(i, j+1) - \phi(i, j)}{\Delta t} + HOT \\ &\phi'' = \frac{\phi(i, j+1) - 2\phi(i, j) + \phi(i, j-1)}{(\Delta t)^2} + HOT \end{aligned}\right\} \tag{3.6}$$

The finite difference equations used in this study were obtained by direct use of both the first and second forward difference approximation for the partial derivatives. Substituting equations (2.67) in equations (2.75) to (2.77), then the finite difference form of the equations governing the flows considered in this thesis becomes

$$\frac{u(i+1)-u(i)}{\Delta y}=0 \tag{3.7}$$

$$\frac{u_j(i'+1)-u_j(i')}{\Delta t}+u_j(i)\frac{u_j(i'+1)-u_j(i')}{\Delta y}+u_j(i)\frac{u_j(i'+1)-u_j(i')}{\Delta x}=\frac{u_j(i'+1)-2u_j(i')+u_j(i'-1)}{(\Delta x)^2} +Gr\theta(i')+\frac{\nu}{\rho u_o{}^2}\left(\vec{J}\times\mu e\vec{H}\right) \tag{3.8}$$

$$\frac{\Pr\theta(i'+1)-\theta(i)}{\Delta t}=\frac{\theta(i'+1)-2\theta(i')+\theta(i'-1)}{(\Delta y)^2}+\Pr Ec\left[\frac{u_j(i'+1)-u_j(i')}{\Delta y}+\frac{u_i(i'+1)-u_i(i')}{\Delta x}\right]^2 \tag{3.9}$$

We lastly express the induction equation (2.95) in implicit finite difference form via time step $H^{i+1}$considering that the magnetic flux is dependent on x and t yielding

$$\frac{H^{i+1}(i,j)-H_z^i(i,j)}{\Delta t}+u^{i+1}\frac{H^i(i+1,j)-H^i(i-1,j)}{2\Delta x}=\frac{1}{R_m}\frac{H^{i+1}(i+1,j)-2H^{i+1}(i,j)+H^{i+1}(i-1,j)}{(\Delta x)^2} \tag{3.10}$$

Multiplying through by $R_m$ we obtain

$$\left[\frac{H^{i+1}(i,j)-H_y^i(i,j)}{\Delta t}+u^{i+1}\frac{H^i(i+1,j)-H^i(i-1,j)}{2\Delta x}\right]R_m=\frac{H^{i+1}(i+1,j)-2H^{i+1}(i,j)+H^{i+1}(i-1,j)}{(\Delta x)^2} \tag{3.11}$$

The order of the local truncation error in equation (2.76) is $0(\Delta y)$, in (3.8) the order is $0\left(\Delta t+\Delta x+(\Delta x)^2\right)$and in (3.9) the order is $0\left(\Delta t+(\Delta y)^2+\Delta t\right)$.In our computation, the overall truncation error is neglected. In the later chapters of this thesis depending on the initial and boundary condition imposed on the problem, the numerical differentiation using Newton's' interpolation formulae is used to compute both the skin friction and average rate of heat transfer at the plate. The chapters which follow present various results of the velocity, temperature, skin friction and rate of heat transfer followed by discussion of the results in terms of the various parameters.

# CHAPTER FOUR

# 4.0 STOKES PROBLEM OF A FREE CONVECTIVE FLOW PAST A VERTICAL INFINITE PLATE IN A ROTATING FLUID WITH HALL CURRENTS IN PRESENCE OF A VARIABLE MAGNETIC FIELD.

## 4.1 INTRODUCTION

The two dimensional Newtonian electrically conducting fluid, convective flow problem past an impulsively started vertical infinite non-conducting plate, in the presence of a strong transverse variable magnetic field is considered. The coupled non-linear equations are solved by explicit finite difference method. Expressions for velocity, temperature, skin friction and rate of heat transfer at the plate have been obtained in dimensionless forms putting into consideration the effects of the hall currents, rotation and Grashof number, on the flow field.

In this case the flow field for the fluid is obtained, the results are discussed in terms of the parameters considered and are compared with those of Ram (1990) for water.

However in practical application another situation arises, in which the system is in a state of rigid relation in presence of a strong magnetic field, such that the effects of the Hall currents, ion-slips currents and rotation affects of the flow field significantly. Hence the purpose of this chapter is to study the effects of Hall currents, ion-slip currents and mutation on MHD stokes problem for a vertical infinite plate in a rotating fluid system.

## 4.2 MATHEMATICAL ANALYSIS

Consider the flow of an electrically conducting fluid past an impulsively started vertical infinite plate. A strong magnetic field of variable strength is assumed to be applied transversely to the direction normal to the flow as shown in Figure 1.2 (Chapter 1).

Let the fluid and the plate be in a state of rigid rotation with uniform angular velocity $\Omega$ about the z* axis taken normal to the plate. In this study the plate is taken to be of infinite length, thus all variables are functions of z* and t* only. Initially the temperature of the fluid and the plate are assumed to be the same. At time t* >O, the plate starts moving impulsively in its own plane with a constant velocity $U_o$ and its temperature is instantaneously raised or lowered to $T_w*$ which is maintained constant. At a later stage it is assumed that the induced magnetic field is negligible so that $H = (0,0,H_z)$, an assumption which is justified when the magnetic Reynolds is very small (Moreau (1990)). The equation of electric charge (i.e. equation (2.34)) $\nabla \cdot \vec{J} = 0$ (since the displacement current is neglected), gives $J_z* = Constant$ ,where $\vec{J} = (\vec{J}_x*, \vec{J}_y*, \vec{J}_z*)$. This constant is assumed to be zero, since $J_z* = 0$ at the plate which is electrically non-conducting, thus $J_z* = 0$ everywhere in the flow. The generalized ohm's law (i.e. equation (2.29)) including the effects of Hall currents and variable magnetic field (Cowling (1957)) is

$$\vec{J} + \frac{\omega_e \tau_e}{H_o} \vec{J} \times \vec{H}_y = \sigma \left[ \vec{E} + \mu_e \vec{q} \times \vec{H}_y + \frac{1}{e\eta_e} \nabla \cdot P_e \right] \tag{4.1}$$

Where $\sigma$ is the electrical conductivity, $\mu_e$ the magnetic permeability, $\omega_e$ the cyclotron frequency, $\tau_e$ the collision time, $e$ the electric charge, $\eta_e$ the number density of electron and $P_e$ the electron pressure respectively. In equation (3.11), $\vec{q}$ denotes the fluid velocity with components $u*$, $v*$ and $w*$ in the $x*, y*$ and $z*$-axis directions respectively. In equation (4.1) the effects of ion-slip and thermoelectric are neglected. In our study we only consider a short circuit problem in which the applied electric field $\vec{E} = 0$. Under these assumptions expanding equation (4.1) we have

$$\left(J_x*, J_y*\right) + \frac{\omega_e \tau_e}{H_o}\left(J_y * H_o, - J_x * H_o\right) = \sigma\mu_e\left(v * H_o, -u * H_o\right) \tag{4.2}$$

Equating the $x*$ and $y*$ components equation (4.2) yields

$$\left.\begin{aligned} J_x * + \omega_e \tau_e J_y * &= \sigma\mu_e v * H_o \\ J_y * - \omega_e \tau_e J_x * &= -\sigma\mu_e u * H_o \end{aligned}\right\} \tag{4.3}$$

Or

$$\left.\begin{aligned} J_x^* + mJ_y^* &= \sigma\mu_e v^* H_o \\ J_y^* - mJ_x^* &= -\sigma\mu_e u^* H_o \end{aligned}\right\} \tag{4.4}$$

where $m = \omega_e \tau_e$ is the Hall parameter. Eliminating $J_x^*$ from (4.4) we have

$$J_y^* = \frac{\mu_e H_o}{1+m^2}\left(mv^* - u^*\right) \tag{4.5}$$

Similarly on eliminating $J_y^*$ from (4.4) we have

$$J_x^* = \frac{\mu_e H_o}{1+m^2}\left(mu^* + v^*\right) \tag{4.6}$$

In equation (4.3) and (4.4) electron pressure has been neglected.

In rotating frame of reference the equation of motion i.e. equation (2.66) including the Coriolis force (i.e. $-2\Omega \times q$) Greenspan (1963), in components form become:

$$\frac{\partial u^*}{\partial t^*} - 2\Omega v^* = \upsilon \frac{\partial^2 u^*}{\partial z^{*2}} + g\beta^*\left(T^* - T_\infty^*\right) + \frac{\mu_e J_y^* H_z}{\rho} \tag{4.7}$$

$$\frac{\partial u^*}{\partial t^*} + 2\Omega u^* = \upsilon \frac{\partial^2 u^*}{\partial z^{*2}} - \frac{\mu_e J_x^* H_z}{\rho} \tag{4.8}$$

Where $g$ is the acceleration due to gravity, $\beta^*$ the coefficient of volume expansion, $\nu$ kinematics viscosity and $\rho$ the density of the fluid. On neglecting the energy dissipated as heat, energy equation (2.67) becomes

$$\frac{\partial T^*}{\partial t^*} = \frac{\kappa}{\rho C_p}\frac{\partial^2 T^*}{\partial z^{*2}} \tag{4.9}$$

Where $\kappa$ is the thermal conductivity, $C_p$ the specific heat of the gas at constant pressure. In (4.7) $T^*$ and $T_\infty^*$ denote the temperature in the boundary layer and free stream respectively, $J_x^*$ and $J_y^*$ are the current density components and $u^*$, $v^*$ are the velocity components in the $x^*$ and $y^*$ directions respectively.

This thesis considers a fully developed flow, thus in equations (4.7) to (4.9) the inertia terms have been neglected (Hermann Schlichting, 1968). As a result of this the solution obtained will be true for a short time after the motion started and temperature jump at the wall (i.e. the results are true in the boundary layer).

The interest in this study is free convection flows only thus together with the condition of no-slip of the fluid at the wall the boundary and the initial conditions of this problem are:

$$\left.\begin{aligned} &t^* \le 0,\ u^* = 0,\ T^* = T_\infty^*,\ at \quad z^* = 0 \\ &t^* > 0,\ u^* = u_o,\ v^* = 0,\ T^* = T_\infty^*,\ at \quad z^* = 0 \\ &t^* > o,\ u^* \to 0,\ v^* \to 0,\ T^* \to T_\infty^*,\ at \quad z^* \to \infty \end{aligned}\right\} \tag{4.10}$$

In order to normalize equations (4.7) to (4.9) the following non-dimensional variables are introduced:

$$\left.\begin{aligned}
&t=\frac{t^{*}u_o^{2}}{\nu},\; z=\frac{z^{*}u_o}{\nu},\; u=\frac{u^{*}}{u_o}\\
&v=\frac{v^{*}}{u_o^{2}},\; X_j=\frac{X_j^{*}u_o}{\nu},\; \theta=\frac{T^{*}-T_\infty^{*}}{T_w^{*}-T_\infty^{*}}\\
&Gr=\upsilon g\beta'\frac{\left(T_w^{*}-T_\infty^{*}\right)}{u_o^{3}},\; \Pr=\frac{\mu C_p}{k}\\
&M1^2=\frac{\sigma H_o^{2}\mu_e^{2}\nu}{\rho u_o^{2}},\qquad Er=\frac{\Omega\nu}{u_o^{2}}
\end{aligned}\right\} \tag{4.11 a, b, c, d}$$

Using (129a) the partial derivatives in equations (4.7) becomes

$$\left.\begin{aligned}
&\frac{\partial u^{*}}{\partial t^{*}}=\frac{\partial u}{\partial t^{*}}\cdot\frac{\partial u^{*}}{\partial u}\cdot\frac{\partial t^{*}}{\partial t}=\frac{u_o^{2}}{\nu}\cdot u_o\frac{\partial u}{\partial t}=\frac{u_o^{3}}{\nu}\cdot\frac{\partial u}{\partial t}\\
&\frac{\partial u^{*}}{\partial z^{*}}=\frac{\partial u}{\partial z^{*}}\cdot\frac{\partial u^{*}}{\partial u}\cdot\frac{\partial z^{*}}{\partial z}=\frac{u_o}{\nu}\cdot u_o\frac{\partial u}{\partial z}=\frac{u_o^{2}}{\nu}\cdot\frac{\partial u}{\partial z}\\
&\frac{\partial^2 u^{*}}{\partial z^{*2}}=\frac{\partial}{\partial z^{*}}\left(\frac{\partial u^{*}}{\partial z^{*}}\right)=\frac{\partial}{\partial z^{*}}\left(\frac{u_o^{2}}{\nu}\cdot\frac{\partial u}{\partial z}\right)=\frac{\partial}{\partial z}\left(\frac{u_o^{2}}{\nu}\cdot\frac{\partial u}{\partial z}\right)\frac{\partial z}{\partial z^{*}}=\frac{u_o^{3}}{\nu^{2}}\cdot\frac{\partial^2 u}{\partial z^{2}}
\end{aligned}\right\} \tag{4.12 a, b, c}$$

Similarly partial derivatives in (4.8) becomes

$$\left.\begin{aligned}
&\frac{\partial v^{*}}{\partial t^{*}}=\frac{u_o^{3}}{\nu}\cdot\frac{\partial v}{\partial t}\\
&\frac{\partial^2 v^{*}}{\partial z^{*2}}=\frac{u_o^{3}}{\nu^{2}}\cdot\frac{\partial^2 v}{\partial t^{2}}
\end{aligned}\right\} \tag{4.13}$$

Substituting (4.5), (4.6), (4.12) and (4.11a, b, c, d, e, f) in equation (4.7) and (4.8) we have;

$$\left.\begin{aligned}
&\frac{u_o^{3}}{\nu}\cdot\frac{\partial u}{\partial t}-2\Omega u_o\nu=\frac{u_o^{3}}{\nu}\frac{\partial^2 u}{\partial z^{2}}+g\beta'\theta\left(T_w^{*}-T_\infty^{*}\right)+\frac{\mu_e^{2}H_o^{2}H_z^{2}u_o}{\rho\left(1+m^{2}\right)}\left(mv-u\right)\\
&or\\
&\frac{\partial u}{\partial t}-\frac{2\Omega u_o\nu}{u_o^{2}}=\frac{\partial^2 u}{\partial z^{2}}+\frac{g\beta'\theta\left(T_w^{*}-T_\infty^{*}\right)}{u_o^{3}}+\frac{\mu_e^{2}H_o^{2}H_z^{2}u_o}{\rho u_o^{2}\left(1+m^{2}\right)}\left(mv-u\right)\\
&\frac{\partial u}{\partial t}-2E_r\nu=\frac{\partial^2 u}{\partial z^{2}}+G_r\theta+\frac{M_1^{2}H_z^{2}}{\left(1+m^{2}\right)}\left(mv-u\right)
\end{aligned}\right\} \tag{4.14}$$

$$\left.\begin{array}{l}\frac{u_o^{3}}{\nu}\cdot\frac{\partial v}{\partial t}-2\Omega u_{ou}=\frac{u_o^{3}}{\nu}\frac{\partial^2 v}{\partial z^2}-\frac{\mu_e^{2}H_o^{2}H_z^{2}u_o}{\rho\left(1+m^2\right)}\left(mu+v\right)\\ or\\ \frac{\partial v}{\partial t}+2E_{ru}=\frac{\partial^2 v}{\partial z^2}-\frac{M_1^{2}H_z^{2}}{\left(1+m^2\right)}\left(mu+v\right)\end{array}\right\} \quad (4.15)$$

Multiplying equation (4.15) by $i$ ($i$ is the complex number given by $i=\sqrt{-1}$) and adding to (4.14) we have:

$$\frac{\partial}{\partial t}\left(u+iv\right)-\left(u+iv\right)=\frac{\partial^2}{\partial z^2}\left(u+iv\right)+\frac{m_1^{2}}{1+m^2}\left(-u-iv+m\left(v-iu\right)\right)+Gr$$

$$or\,\frac{\partial \vec{q}}{\partial t}+Er\vec{q}i=\frac{\partial^2 q}{\partial z^2}+\frac{M_1^{2}H_z^{2}}{1+m^2}\left(-\vec{q}-mi\vec{q}\right)+Gr\theta$$

$$or\,\frac{\partial \vec{q}}{\partial t}=\frac{\partial^2 \vec{q}}{\partial z^2}-\frac{M_1^{2}H_z^{2}}{1+m^2}\vec{q}-2Er\vec{q}i+Gr\theta$$

$$or\,\frac{\partial \vec{q}}{\partial t}=\frac{\partial^2 \vec{q}}{\partial z^2}-\frac{m_1^{2}}{1+mi}\vec{q}-2Er\vec{q}i=\frac{\partial^2 q}{\partial z^2}-\left(\frac{m_1^{2}H_z^{2}}{1+mi}+2Eri\right)q$$

This can be written as

$$\frac{\partial \vec{q}}{\partial t}=\frac{\partial^2 \vec{q}}{\partial z^2}+Gr\theta-M_2\vec{q} \quad (4.16)$$

Where $M_2=\left[2Eri+\frac{m_1^{2}}{1+mi}\right]$ and $\vec{q}=u+vi$ is the complex velocity of the fluid. Equation (4.16) is the momentum equation in non-dimensional form of the flow considered in this section. Similarly using the non-dimensional quantities (4.11a, b, c, e, f) the partial derivatives (4.9) are

$$\frac{\partial T^*}{\partial t}=\frac{\partial \theta}{\partial t}\cdot\frac{\partial T^*}{\partial \theta}\cdot\frac{\partial t}{\partial t^*}=\frac{u_o^{3}}{\nu}\left(T_w^*-T_\infty^*\right)\frac{\partial \theta}{\partial t} \quad (4.17)$$

$$\left.\begin{aligned} \frac{\partial T^*}{\partial z^*} &= \frac{\partial \theta}{\partial z}\cdot\frac{\partial T^*}{\partial z}\cdot\frac{\partial z}{\partial z^*} = \frac{u_o}{v}\left(T_w^* - T_\infty^*\right)\frac{\partial \theta}{\partial z} \\ \frac{\partial^2 T^*}{\partial z^{*2}} &= \frac{\partial}{\partial z^*}\cdot\frac{\partial T^*}{\partial z^*} = \frac{\partial}{\partial z^*}\left[\frac{u_o}{v}\left(T_w^* - T_\infty^*\right)\frac{\partial \theta}{\partial z}\right] \\ &= \frac{u_o^{\,2}}{v^2}\left(T_w^* - T_\infty^*\right)\frac{\partial^2 \theta}{\partial z^2} \end{aligned}\right\} \tag{4.18}$$

Substituting (4.17), (4.11b) and (4.18) in equation (4.9) we have

$$\frac{u_o^{\,2}}{v}\left(T_w^* - T_\infty^*\right)\frac{\partial \theta}{\partial t} = \frac{k}{\rho C_p}\frac{u_o^{\,2}}{v^2}\left(T_w^* - T_\infty^*\right)\frac{\partial^2 \theta}{\partial z^2}$$

*or*

$$\frac{v\rho C_p}{k}\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial z^2} \tag{4.19}$$

*or*

$$\Pr\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial z^2}$$

This is the equation of energy for this analysis.

Non-dimensionalising induction equation $\frac{\partial H_z}{\partial t} + u\frac{\partial H_z}{\partial x} = \frac{1}{\sigma\mu_e}\frac{\partial^2 H_z}{\partial x^2}$ and substituting the magnetic Reynolds number yields

$$\frac{\partial\left(H_o H_z\right)}{\partial\left(\frac{t^* U_o^{\,2}}{v}\right)} + \frac{u^*}{U_o}\frac{\partial\left(H_o H_z\right)}{\partial\left(\frac{x^* U_o}{v}\right)} = \frac{1}{\sigma\mu_e}\frac{\partial^2\left(H_o H_z^*\right)}{\partial\left(\frac{x^{*2} U_o^2}{v^2}\right)}$$

$$\frac{H_o}{\frac{U_o^2}{v}}\frac{\partial H_z^*}{\partial t^*} + \frac{u^*}{U_o}\frac{H_o}{\frac{U_o}{v}}\frac{\partial H_z^*}{\partial x^*} = \frac{1}{\sigma\mu_e}\frac{H_o^2}{\frac{U_o^2}{v^2}}\frac{\partial^2 H_z^*}{\partial x^{*2}}$$

Which on simplification yields

$$\frac{\partial H_z^*}{\partial t^*} + u^* \frac{\partial H_z^*}{\partial x^*} = \frac{1}{\sigma \mu_e L U_o} \frac{\partial^2 H_z^*}{\partial x^{*2}}$$

$$\frac{\partial H_z^*}{\partial t^*} + u^* \frac{\partial H_z^*}{\partial x^*} = \frac{1}{R_m} \frac{\partial^2 H_z^*}{\partial x^{*2}} \qquad (4.20)$$

Where $\sigma \mu_e L U_o = R_m$, the Magnetic Reynolds number.

According to Calvert (2002), the electrical conductivity is 'infinite' when $R_m$ is large and magnetic effect may be expected to be prominent. If $R_m$ is small the magnetic field is not changed appreciably by the flow thus induced magnetic field can be taken to be zero

The boundary conditions (4.10) in non-dimensional form reduces to

$$\left.\begin{array}{l} t \leq 0,\ \ q(z,t) = 0,\ \ \theta(z,t) = 0 \\ t > 0,\ \ q(0,t) = 1,\ \ \theta(0,t) = 1 \\ t > 0,\ \ q(\infty,t) = 0,\ \ \theta(\infty,t) = 0 \end{array}\right\} \qquad (4.21\ a, b, c)$$

## 4.3 METHOD OF SOLUTION

To solve equation (3.7), (3.8), (3.9) and (3.11) together with the boundary conditions (4.21) the finite difference method is applied. The mesh system considered here is shown in chapter 3 (figure 3.1).

Substituting the finite difference form of the partial derivatives (i.e. from equation (3.7) in equation (4.16) and (4.19) respectively) we obtain the following system

$$\frac{q(i',j'+1)-q(i',j')}{\Delta t}=\frac{q(i'+1,j')-2q(i',j')+q(i'-1,j')}{(\Delta z)^2}+Gr\theta(i',j')-m_2q(i',j') \qquad (4.22)$$

And

$$\Pr\frac{\theta(i',j'+1)-\theta(i',j')}{\Delta t}=\frac{\theta(i'+1,j')-2\theta(i',j')+\theta(i'-1,j')}{(\Delta z)^2} \qquad (4.23)$$

In equations (4.22) and (4.23) the index $i'$ refers to $z$ and $j'$ to time. The mesh system in this case is divided by taking $\Delta z = 0.1$ and $\Delta t = 0.00125$.From equation (4.20 a) the initial conditions at $z = 0$ takes the finite form

$$q(0,0)=1,\ \theta(0,0)=1,\ q(i',0)=\theta(i',0)=0, \text{ For all except } i'=0 \qquad (4.24)$$

In finite difference form, the boundary condition (4.21b) takes the form

$$q(0,j')=1,\ \theta(0,j')=1 \text{ For all } j' \qquad (4.25)$$

Though the boundary condition (4.21c) applies at $z=\infty$, we take $z=4.1$ as corresponding to $z=41$, since both the values of $q$ and $\theta$ tend to zero as $z \to 4$. Therefore in this section we set $q(41,j')=\theta(41,j')=0$ for all $j'$. From (4.22) we note that the velocity at the end of time step $q(i',j'+1), i'=1,2,\ldots\ldots\ldots,40$ is computed in terms of velocities and temperatures at points on earlier time step. Similarly, $\theta(i',j'+1)$ is computed from equation (4.23). The procedure is repeated till $j'=400$ i.e. up to time $t=0.5$.During the computation, to test the convergence and stability of the finite difference scheme, computations were done with smaller values of $\Delta t$, viz $\Delta t = 0.0009, 0.001$ and $0.0002$. In this analysis it is noted that increasing the number of mesh points by using smaller values of $\Delta t$ does not have a significant effect in the result, thus the finite difference scheme used is stable and convergent.

In order to get the physical understanding of this problem and for the purpose of discussing the results, the numerical calculations have been carried out as explained above for both velocity and temperature. In these calculations the Prandtl number is taken to be equal to $0 \cdot 71$ which corresponds to air and magnetic parameter $M_1^2 = 5.0$ which signifies strong magnetic field. The calculations were carried out for both $Gr > 0(= 0.5)$ in the presence of cooling of the plate by free convection currents) and $Gr < 0(= -0.5)$ in the presence of heating of the plate by free-convection currents). Now the results obtained for the unsteady flow for various parameters are shown in Figures 5.1 to 5.6. In the next section, the numerical method employed in computing the skin friction and the rate of heat transfer at the plate is presented.

## 4.4 CALCULATION OF THE SKIN FRICTION AND RATE OF HEAT TRANSFER

After obtaining the velocity and temperature distributions of the flow as explained in the previous section we now compute the skin friction given by (Holman (1989)) and Soundalgekar *et al* (1985))

$$\tau = -\left.\frac{\partial q}{\partial z}\right|_{z=0} \qquad (4.26)$$

where $\tau = \frac{\tau^*}{\rho u_o^2}$. On the other hand the heat flux $\dot{q}$ at the wall is given by

$$\dot{q} = -\left.\frac{\partial \theta}{\partial z}\right|_{z=0} \tag{4.27}$$

In order to solve equations (4.26) and (4.27) we apply a second-order least squares correlation used over the gradients of the first ten points, Georgion *et al* (1986). Using this method to solve equation (4.26), we first fit the first ten values for $q$ obtained in the previous section to the model

$$q(z,t) = \beta_o + \beta_1 z + \beta_2 t + \beta_{11} z^2 + \beta_{22} t^2 + \beta_{12} zt \tag{4.28}$$

Where $\beta_o, \beta_1, \beta_2, \beta_{11}, \beta_{12}, and\, \beta_{22}$ are constants to be determined such that the aggregate error between the approximate value $q'(z,t)$ and $q(z,t)$ when squared is minimum, Jain *et al* (1987) i.e.

$$\begin{aligned} I(\beta_o \beta_1 \beta_2 \beta_{11} \beta_{22} \beta_{12}) &= \int_{0.10}^{1.0} \int_{0.25}^{0.5} \left[q(z,t) - q'(z,t)\right]^2 \partial t \partial z \\ &= Minimum \end{aligned} \tag{4.29}$$

The least squares estimators for the constants $\beta_o, \beta_1, \beta_2, \beta_{11}, \beta_{22}, and\, \beta_{12}$ which will minimize equation (4.29) is given by (Montgomery 1991)

$$\hat{\beta} = \begin{bmatrix} \beta_o \\ \beta_1 \\ \beta_2 \\ \beta_{11} \\ \beta_{22} \\ \beta_{12} \end{bmatrix} = \left[x^1 x\right]^{-1} x^1 q \tag{4.30}$$

Where $\boldsymbol{x}$ is a $6\times6$ matrix whose columns are the coefficients of the constants $\beta_0,\beta_1,\beta_2,\beta_{11},\beta_{22}, and\, \beta_{12}$ in the model (4.28) which are constant $\boldsymbol{1, z, t, z^2, t^2,}$ and $zt$ values obtained from the ten values of $q$ chosen respectively. Similarly $x'$ denotes the transpose of $x$ and $[x'x]^{-1}$ denotes the inverse matrix of $[x'x]$. The constant obtained are substituted in equation (4.28) and $\tau$ is obtained as

$$\tau = -\left.\frac{\partial q}{\partial z}\right|_{z=0} = -\left(\beta_1 + 2\beta_{11}z + \beta_{12}t\big|_{z=0}\right) = -\left(\beta_1 + \beta_{12}t\right) \tag{4.31}$$

Similarly $\theta$ is fitted to

$$\theta'(z,t) = \gamma_0 + \gamma_1 z + \gamma_2 t + \gamma_{11}z^2 + \gamma_{22}t^2 + \gamma_{12}zt$$

Then

$$\dot{q} = -\left.\frac{\partial \theta}{\partial z}\right|_{z=0} = -\left(\gamma_1 + 2\gamma_{11}z + \gamma_{12}t\big|_{z=0}\right) = -\left(\gamma_1 + \gamma_{12}t\right) \tag{4.32}$$

In our case we used the regression line fitting package to obtain the values of $\beta_0\, \beta_1\, \beta_2\, \beta_{11}\, \beta_{22}\, \beta_{12}, \gamma_1\, \gamma_2\, \gamma_{11}\, \gamma_{22}$ and $\gamma_{12}$, and the values obtained are substituted in (4.31) and (4.32) to obtain the values of both $\tau$ and $\dot{q}$.These values for various parameters for both $Gr > 0(=0.5)$ and $Gr < 0(=-0.5)$ are represented in tables 2 to 4.

# CHAPTER FIVE

## 5.0 RESULTS AND DISCUSSION

We now discuss the behaviour of velocity (both primary u and secondary v), the temperature distribution, the skin friction (average shear stress $\tau_{uo}$ due to primary velocity as well as the shear stress $\tau_{vo}$ due to secondary velocity), for different parameters involved in the flow problem solved in this thesis.

In order to get physical insight into the problem under study, the velocity field, temperature field, skin-friction and rate of heat transfer are discussed by assigning numerical values to the parameters encountered into the corresponding equations. The values of Eckert number $Ec = 0.02$ to $Ec = 0.5$ are used. The value of Prandtl number is chosen as $\Pr = 0.71$ that corresponds to air. Grashof number for heat transfer is chosen as $Gr = -0.5$ to 0.5. The values $Gr > 0$ correspond to correspond to cooling of the plate while the values $Gr < 0$ correspond to heating of the plate. The values of the magnetic parameter $M = 1$ to 1.5 and Rotation parameter ( $Er = 0.05$ to 0.5) are chosen arbitrarily.

A program was written and run for various values of velocities and temperatures for the finite difference equations (3.11), (4.22) and (4.23) using different values of Ec, Er, m, t and M. the velocities were classified as Primary velocity (u) and Secondary velocity (v) along the x and y axes respectively. The analysis of the data obtained for Pr=0.71 corresponding to air was done and the resultant results are represented graphically on

Figures 5.1 to 5.6 and on Tables 2 to 5. The graphs represent the general trends of the velocities and temperature along the axis of rotation of the flow field. On the other hand, the variation in rate of heat transfer and skin friction on the thermal and velocity boundary layers are depicted using tables 2 to 5. A consideration of two cases of heat changes is done. These two cases are cooling at the plate and heating at the plate with constant heat being supplied to or withdrawn from the plate. Using Table 1 as a reference for both free convectional cooling and heating at the plate the results obtained are presented in Figures 5.1 to 5.6.

**Case 1: Cooling at the plate**

In this case, the Grashof number Gr>0. Hence the plate is at higher temperature than the surrounding and so Gr=0.5.

**(a) Primary velocity (u) profiles**

From Figure 5.1;

(i) An increase in the rotation parameter Er, magnetic parameter $M_2$ and Eckert number Ec leads to a decrease in the velocity profiles. This is because the presence of the transverse magnetic field creates a resistive force similar to the drag force that acts in the opposite direction of the fluid; thus causing the velocity of the fluid to decrease. An increase in the Magnetic parameter leads to a decrease in the primary velocity and an increase in the secondary velocity profiles. Due the Lorentz force, there is a resistive force along the x-axis and this reduces the primary velocity but the secondary velocity profile increases since it is in the direction of the induced force. An increase in Eckert number means an increase in kinetic energy of the fluid particles and for this reason both primary and secondary velocity profiles.

(ii) An increase in the Hall parameter m leads to an increase in the velocity profiles. This can be attributed to the fact that when the Hall parameter is increased the induced current along x-axis increases and this translates to an increase in the primary velocity.

**(b) Secondary Velocity profiles**

From Figure 5.2 it is noted that;

(i) An increase in the rotation parameter and magnetic parameter $M_2$ leads to increase in the velocity profiles. This is attributed to the fact that the wall moves in opposite direction to that of the free stream, it tends to retard the flow. Similarly, the convectional currents due to rotation cause the fluid to retard in motion. Increasing the rotation means increasing the rate at which the ions are rotating thereby overcoming the resistive force created by the magnetic field consequently increasing the velocity of the fluid.

(ii) An increase in the Hall parameter and Eckert number leads to a decrease in the velocity profile. The Hall parameter increases with the magnetic field strength. Physically, the trajectories of electrons are curved by the Lorentz force. When the Hall parameter is low, the motion between the two (trajectories of electrons) encounters with heavy particles (neutral or ion) is almost linear. But if the Hall parameter it is high, the electron movements are highly curved. This can be attributed to Hall parameter decreases the resistive force imposed by the magnetic field due to its effect in reducing the effective conductivity ( $\frac{\sigma}{1+m^2}$ ).

**(c) Temperature profiles**

From Figure 5.3;

(i) An increase in the magnetic parameter $M_2$ and an increase in the Eckert number Ec lead an increase in the temperature profile. Increasing the Eckert number causes the fluid to become warmer and therefore increase its temperature. This is attributed to the viscous dissipation.

(ii) An increase in the Hall parameter m leads to a decrease in temperature profiles. This is because increase in Hall parameter means an increase of ion collisions which translates to more thermal generation hence increasing the rate at which heat is being lost. As a consequence there is a decrease in temperature. However, as the distance from the plate increases these profiles remain constant.

(iii) An increase in the rotation parameter Er results in no significant change in temperature. Rotation has been achieved by a transfer of angular momentum. Once this is drastically reduced, the rate at which the particles move and collide is too small such that the change is insignificant. Thus thermal generation is too small an indication of insignificant change in temperature.

**(d) Rate of heat transfer**

From Table 2;

i. An increase in the rotation parameter Er, Eckert number Ec and magnetic parameter $M_2$leads to an increase in the rate of heat transfer. This is due to the assumption that the Joule dissipation is neglected since viscous dissipation effects are neglected. Due the presence of the Lorentz force and the gravitational force rotating at very low speeds, a friction factor is realized that results in thermal dispersion thereby increasing the rate of heat transfer.

ii. An increase in the Hall parameter m leads to a decrease in the rate heat transfer. This reduction is due to the increase in the momentum, thermal and magnetic boundary layer thickness which in turn are caused by the deceleration of the magnetic field.

**(e) Skin Friction $(\tau_x)$ along the x axis and $(\tau_y)$ along the y-axis**

From Table 3

i. An increase in the rotation parameter Er leads to an increase in $(\tau_x)$ but a decrease in $(\tau_y)$**.** This reduction is due to the increase in the momentum, thermal and magnetic boundary layer thickness which in turn are caused by the deceleration of the magnetic field.

ii. An increase in the Hall parameter m leads to a decrease in both $(\tau_x)$ and $(\tau_y)$. The magnetic field gives rise to a resistive force and slows down the movement of the fluid.

iii. An increase in the Eckert number Ec leads to an increase in $(\tau_x)$ but a decrease in $(\tau_y)$. Increasing Ec can lead to a situation that the viscous dissipation becomes a significant hence increasing the temperature. This increment causes an increase in the Skin friction along the x-axis and a decrease in Skin friction along the y-axis.

iv. An increase in magnetic parameter $M_2$ leads to a decrease in both $(\tau_x)$ and $(\tau_y)$. This reduction is due to the increase in the momentum, thermal and magnetic boundary layer thickness which in turn are caused by the deceleration of the fluid by the application of the magnetic field. Since the wall moves in opposite direction to that of the free stream, it tends to retard the flow field. $(\tau_y)$. Further the application of the externally variable magnetic field reduces the velocity vectors and since velocity is inversely proportional to frictional force then this means that $(\tau_x)$ increases and $(\tau_y)$ decreases.

**Case 2: Heating at the Plate (Gr<0)**

In this case the Grashof number Gr<0. In this case the plate is at a lower temperature than the surrounding and Gr=-0.5.

**(a) Primary Velocity (u) Profiles**

From Figure 5.4,

i. An increase in the Hall parameter m leads to an increase in the velocity profiles. Increasing Hall parameter decreases the resistive force imposed by the magnetic field due to its effect in reducing the effective conductivity $\frac{\sigma}{1+m^2}$.

ii. An increase in the rotation parameter Er, Eckert number Ec and magnetic parameter $M_2$ leads to a decrease in the velocity profiles. Since the wall moves in opposite direction to that of the free stream, it tends to retard the flow. Similarly, the convectional currents due to rotation cause the fluid to retard in motion. An increase in Ec increases convectional currents which cause a slight decrease in the primary velocity.

**(b) Secondary Velocity (v) Profiles**

From Figure 5.5, we note that;

i. An increase in the rotation parameter Er and magnetic parameter $M_2$ lead to an increase in the velocity profiles.

ii. An increase in the Hall parameter m and an increase in the Eckert number Ec leads to a decrease in the velocity profiles. Inclusion of Hall parameter decreases the resistive force imposed by the magnetic field due to its effect in reducing the effect conductivity.

**(c) Temperature Profiles**

From Figure 5.6,

i. An increase in the magnetic parameter M and an increase in Eckert number Ec lead to an increase in the temperature profiles. The increase in the fluid temperature induces more flow in the boundary layer causing the velocity of the fluid there to increase. The magnetic field produces a huge increment in the magnitude of the temperature. This can be explained physically as follow: it is well known that a magnetic field imparts some rigidity to the conducting fluid. Thus, with increase in the magnetic field, greater effort will be necessary to maintain the rotation of the plate and this implies an increase in temperature with an increase of the parameter M. Increasing Ec can lead to a situation that the viscous dissipation becomes a significant hence increasing the temperature.

ii. An increase in the Hall parameter m leads to a decrease in the temperature profiles. As the distance from the plate increases, these profiles increase. Increasing the magnetic field decreases the velocity and the micro rotation, while increasing the Hall parameter increases the velocity and the magnitude of micro rotation thereby decreasing the temperature.

iii. An increase in the rotation parameter Er has no effect on the change of temperature profiles. The rotation causes the circulation of induced currents at the surface of the fluid. i.e. the increase of the temperature affects the current distribution which is a fixed ratio. Rotation leads up to additional transport; this contribution is a consequence of the decrease of the ion rotation. Viscous dissipation would immediately lead to an increase of ion-temperature. The ratio of ion momentum to thermal transport is the same hence there is no change in temperature.

**(d) Rate of Heat Transfer**

From Table 4, we note that

i. An increase in the rotation parameter Er, Eckert number Ec and magnetic parameter $M_2$ leads to an increase in the rate of heat transfer. Rotation has been achieved by a transfer of angular momentum. Once this is drastically reduced, the rate at which the particles move and collide is too small thus the rate of heat transfer increases. Increase in Eckert number leads to increase in kinetic energy and as a consequence, the rates of heat transfer increases. The application of the externally variable magnetic field reduces the velocity vectors and leads to an increase in the rate of heat transfer in a free convectional heating.

ii. An increase in the Hall parameter m leads to a decrease in the rate of heat transfer. The magnetic field gives rise to a resistive force and slows down the movement of the fluid. This reduces the rate at which heat is being transferred.

**(e) Skin Friction along x-axis and along the y-axis**

From Table 5, we note that

i. An increase in the rotation parameter Er leads to an increase in $(\tau_x)$ and a decrease in $(\tau_y)$. Due the presence of the Lorentz force and the gravitational force rotating at very low speeds, a friction factor is realized and hence an increase in $(\tau_x)$ and a decrease in $(\tau_y)$.

ii. An increase in the Eckert number Ec leads to an increase in both $(\tau_x)$ and $(\tau_y)$. The increase of Eckert number Ec means reduction of the particles which were causing collisions and this increases skin friction along both x-axis and y-axis.

iii. An increase in the Hall parameter m leads to a decrease in both $(\tau_x)$ and $(\tau_y)$. The skin friction in the y-direction is negative since it is in the opposite direction to that of gravitational force.

iv. An increase in magnetic parameter $M_2$ leads to a decrease in $(\tau_x)$ and an increase in $(\tau_y)$. Hall currents due to the magnetic field give rise to a cross flow making the flow to possess a resistive force that increases in the x- axis and decreases in the y-axis. It is observed that the primary effect of the magnetic field is to decelerate the flow.

**Table 1: Variation of m, H, $M_2$, Ec and Ec for both free convectional cooling (Gr=0.5) and heating (Gr=-0.5) at the Plate.**

| RESULTS | m | H | $M_2$ | Ec | Er |
|---|---|---|---|---|---|
| TEST | 1.0 | 2 | 24 | 0.02 | 0.05 |
| 1 | 2.0 | 2 | 24 | 0.02 | 0.05 |
| 2 | 1.0 | 4 | 96 | 0.02 | 0.05 |
| 3 | 1.0 | 2 | 24 | 0.50 | 0.05 |
| 4 | 1.0 | 2 | 24 | 0.02 | 0.50 |

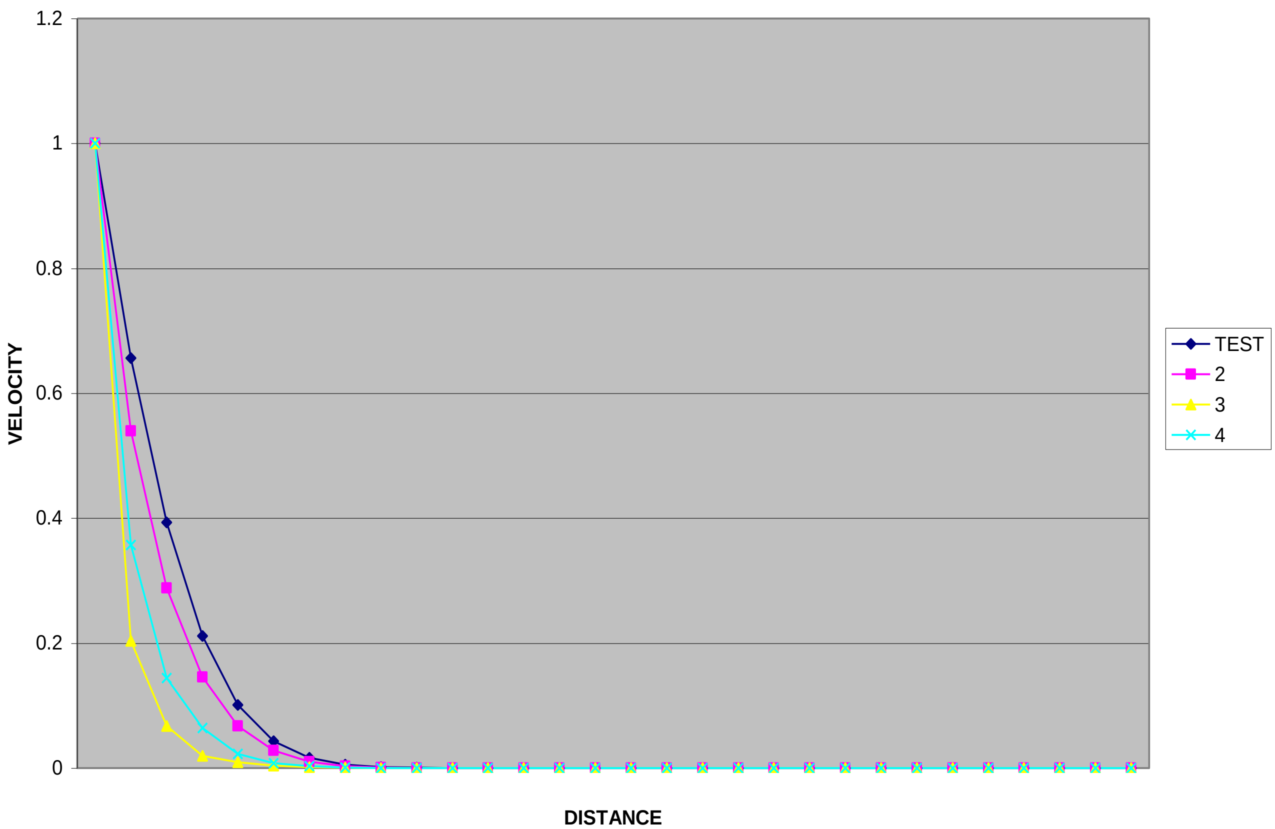


**Figure 5.1: Primary Velocity profiles (Free Convectional cooling at the plate)**

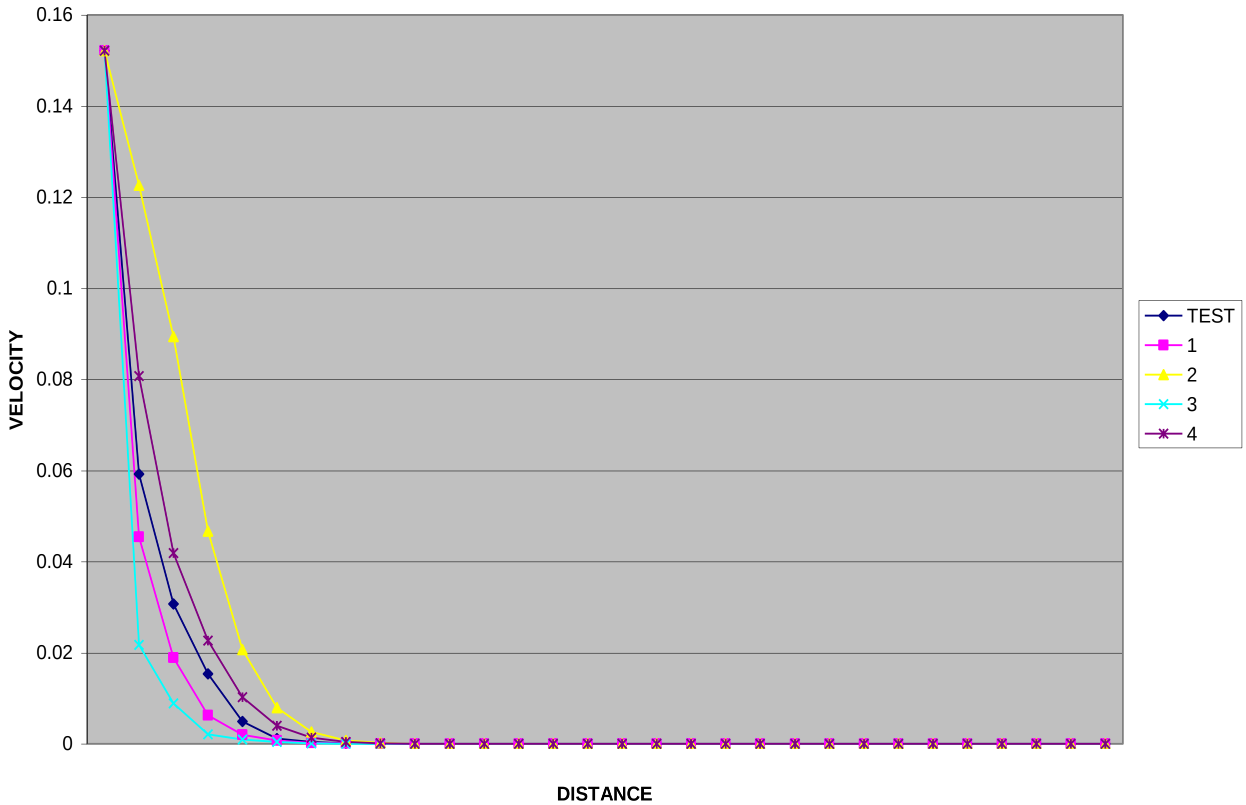


**Figure 5.2: Secondary Velocity profiles (Free Convectional cooling at the plate)**

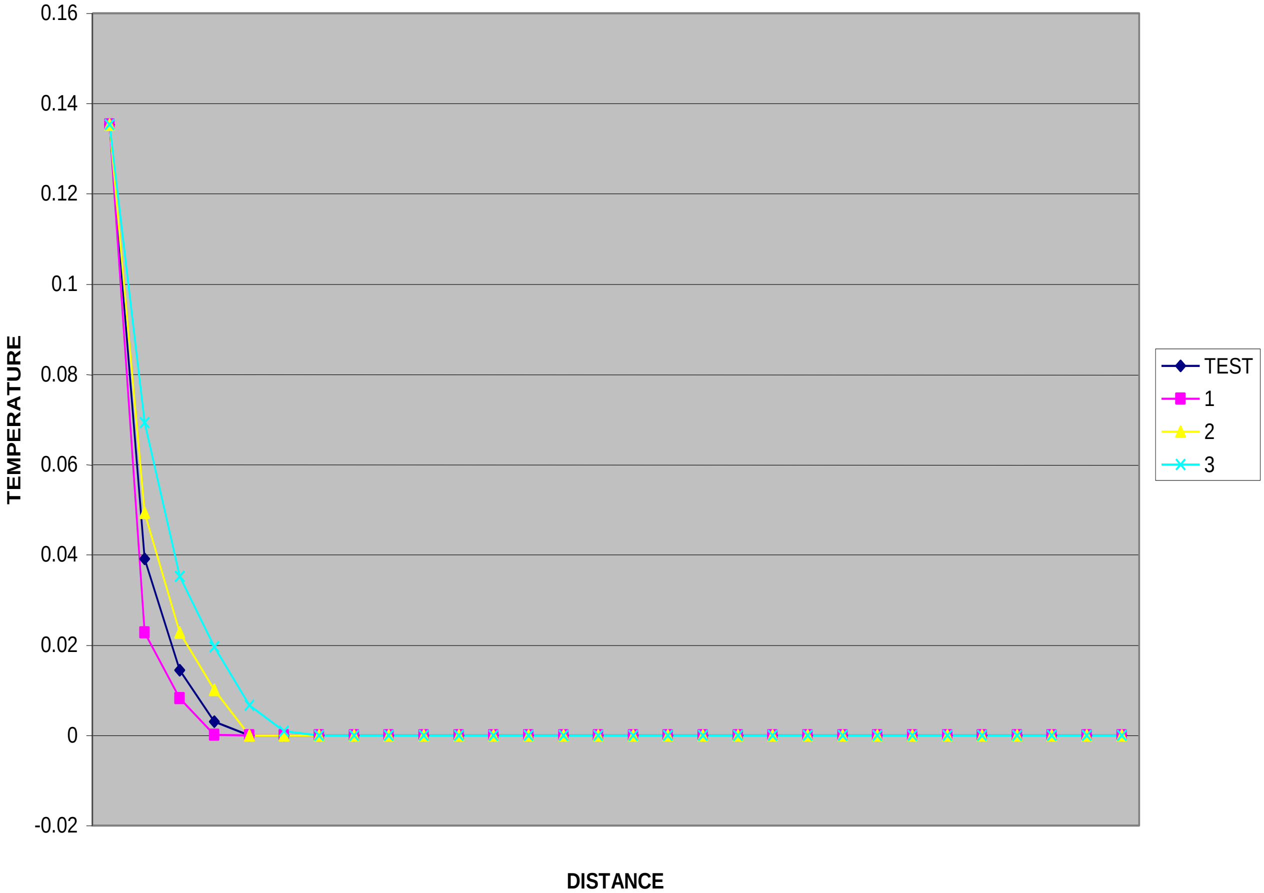


**Figure 5.3: Temperature profiles (Free Convectional cooling at the plate)**

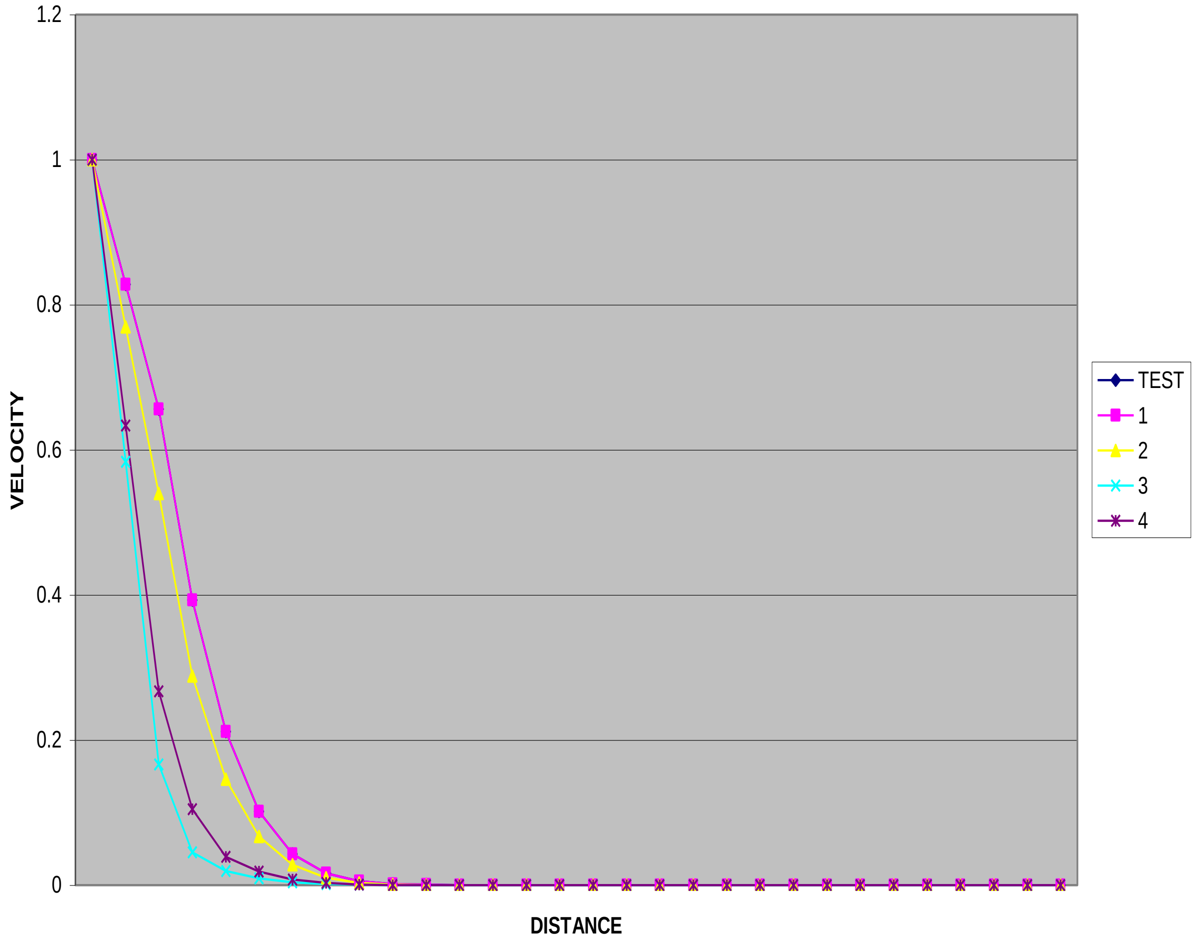


**Figure 5.4: Primary Velocity profiles (Free Convectional heating at the plate)**

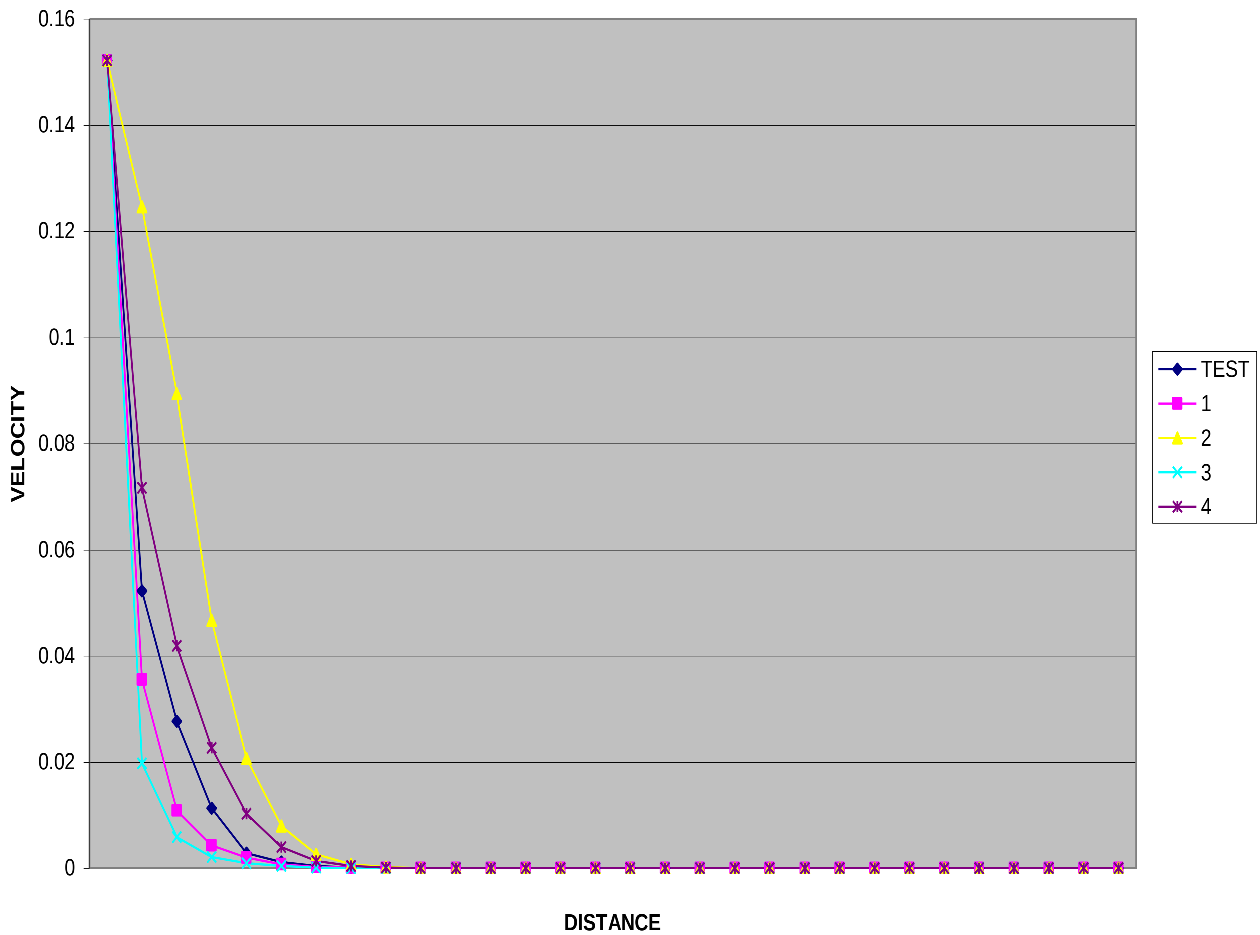


**Figure 5.5: Secondary Velocity profiles (Free Convectional heating at the plate)**

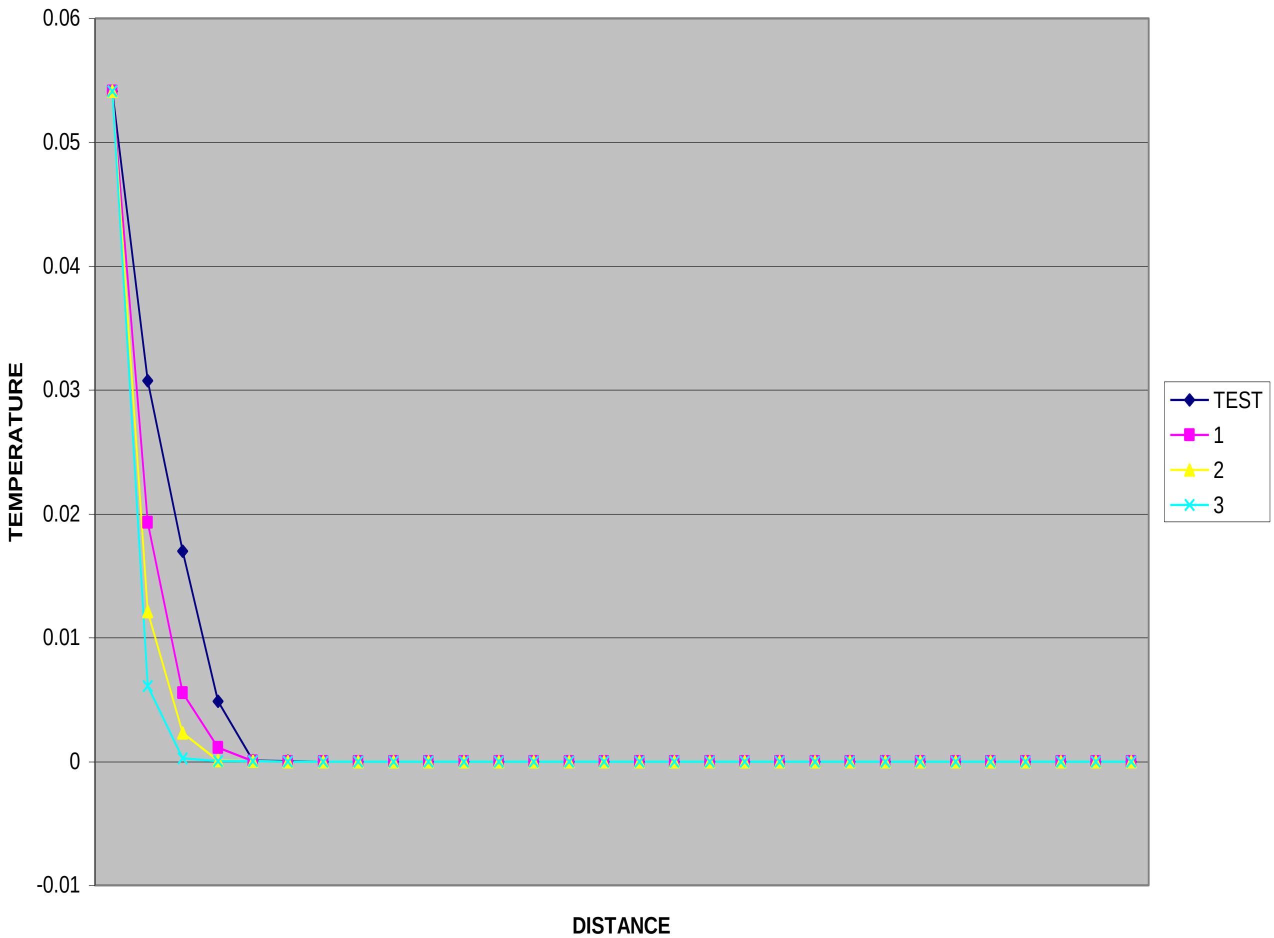


**Figure 5.6: Temperature profiles (Free Convectional heating at the plate)**

**Table 2: Rate of heat transfer with cooling at the plate for Pr=0.71**

| m | M2 | Ec | Er | Nu |
|---|---|---|---|---|
| 1.0 | 24 | 0.02 | 0.05 | 1.263899959 |
| 2.0 | 24 | 0.02 | 0.05 | 1.263461355 |
| 1.0 | 96 | 0.02 | 0.05 | 1.265935278 |
| 1.0 | 24 | 0.5 | 0.05 | 1.275017313 |
| 1.0 | 24 | 0.02 | 0.5 | 1.263900993 |

**Table 3: Skin friction $\tau_x$ and $\tau_y$ with cooling at the plate for** Pr=0.71

| m | M2 | Ec | Er | $\tau_x$ | $\tau_y$ |
|---|---|---|---|---|---|
| 1.0 | 24 | 0.02 | 0.05 | 3.14224995 | -0.139999317 |
| 2.0 | 24 | 0.02 | 0.05 | 3.096609455 | -0.169103684 |
| 1.0 | 96 | 0.02 | 0.05 | 3.013413321 | 0.00174227625 |
| 1.0 | 24 | 0.5 | 0.05 | 3.155085951 | -0.140024952 |
| 1.0 | 24 | 0.02 | 0.5 | 3.14696724 | -0.151650365 |

**Table 4: Rate of heat transfer with heating at the plate for Pr=0.71**

| m | M2 | Ec | Er | Nu |
|---|---|---|---|---|
| 1.0 | 24 | 0.02 | 0.05 | 1.263899959 |
| 2.0 | 24 | 0.02 | 0.05 | 1.263572678 |
| 1.0 | 96 | 0.02 | 0.05 | 1.266032283 |
| 1.0 | 24 | 0.5 | 0.05 | 1.275138535 |
| 1.0 | 24 | 0.02 | 0.5 | 1.263900825 |

**Table 5: Skin friction $\tau_x$ and $\tau_y$ with heating at the plate for Pr=0.71**

| m | M2 | Ec | Er | $\tau_x$ | $\tau_y$ |
|---|---|---|---|---|---|
| 1.0 | 24 | 0.02 | 0.05 | 3.151748888 | -0.13965205 |
| 2.0 | 24 | 0.02 | 0.05 | 3.093358238 | -0.168728416 |
| 1.0 | 96 | 0.02 | 0.05 | 3.010100625 | 0.00203710375 |
| 1.0 | 24 | 0.5 | 0.05 | 3.151910738 | -0.139646217 |
| 1.0 | 24 | 0.02 | 0.5 | 3.143599675 | -0.151287697 |

# CHAPTER SIX

# 6.0 CONCLUSIONS AND RECOMMENDATIONS

## 6.1 Introduction

An analysis of the effects of various parameters on hydromagnetic flows of a rotating fluid past an impulsively started infinite plate has been carried out. In the case under consideration, the applied magnetic field is assumed to be of variable strength along the z axis; hence in this study the laminar boundary layer is considered. The results are therefore true in a layer of greater thickness than that of similar problems in ordinary fluid mechanics. This is owing to the fact the uniform variable magnetic field on the flow of an electrically conducting fluid generally yields greater stability and delay the appearance of turbulence (i.e. increase in the laminar boundary layer thickness).

In this study the results for both Gr>0 and Gr<0 has been obtained, for which from the definition of Gr (i.e. 2.88) it is noted that when the temperature of the plate is greater than that of the fluid in the free stream region Gr>0 which implies that heat will be transferred from the plate to the fluid, a phenomena referred to as cooling of the plate by free convection currents, otherwise heat will be transferred from the fluid to the plate i.e. heating of the plate by free convectional currents. The value of the magnetic parameter $M_1^2$ in our analysis is chosen as equal to 5, which signifies strong magnetic field and as a result it was noted that the Hall currents affected the flow significantly. Once this parameter is neglected, inaccurate results will be obtained. The induced magnetic field is too small owing to the fact that the flow problem being analyzed is unbounded and involves liquids of low magnetic Reynolds number; hence the induced magnetic field can be neglected.

In the analysis of the flow problem, it has been noted that, in free convection, the flow region consists of two regions namely, the boundary layer region in which both the velocity and temperature gradients exits thus the flow field is affected by the parameters described in our study and the other region is the free stream region in which there exist no velocity or temperature gradient. It follows that the velocity and temperature in these region are fixed. The results therefore are for the behaviour of both the velocity and temperature profiles in the boundary layer where the flow is fully developed, thus the inertia terms are neglected in the flow problem.

The results that were obtained and presented in chapter five leads to important conclusions that are made in this chapter. These are presented according to the various parameters including rotation parameter, Hall parameter, Grashof number, Eckert number, magnetic parameter and the time parameter. Some or all of these parameters affect the primary velocity, secondary velocity and temperature. Consequently their effect alters the rate of heat transfer and skin friction along the x and y axes. We thus proceed to consider these parameters for Pr=0.71 which corresponds to air.

### 6.2 (a) Rotation Parameter Er

An increase in Er leads to an increase in the temperature profiles for both free convection cooling of the plate and free convection heating of the plate. An increase in Er further leads to an increase in the rate of heat transfer.

An increase in Er leads to a decrease in the primary velocity profiles for both free convection cooling and heating of the plate and hence an increase in the skin friction along the x-axis. This however, leads to an increase in the secondary velocity profiles for both free convection cooling and heating and consequently an increase in the skin friction along the y-axis.

### 6.2 (b) Hall Parameter m

Increasing m leads to a decrease in the temperature profiles for both free convection heating and cooling of the plate, which increase with an increase in the distance from the plate. An increase in m leads to a decrease in the rate of heat transfer.

An increase in m leads to an increase in the primary velocity profiles for both free convection cooling and heating at the plate but there is a decrease in the secondary velocity profiles for both heating and cooling of the plate. This means that the Hall parameter is very useful as a means to measure either the carrier density or the magnetic field.

### 6.2 (c) Eckert number Ec

Increase in Ec leads to an increase in temperature profiles for both free convection cooling and heating. Further there is a decrease in the rate of heat transfer.

An increase in Ec leads to an increase in the primary and secondary velocity profiles for cooling of the plate but a decrease in the primary and secondary velocity profiles for free convectional heating of the plate. Increasing Ec leads to a decrease in for cooling and an increase in for heating. Increasing Ec leads to an increase in the rate of heat transfer for both free convectional heating and cooling of the plate.

### 6.2 (d) Magnetic parameter M

An increase in M leads to an increase in the temperature profiles for both cooling and heating of the plate as well as the secondary velocity profiles for both cooling and heating of the plate while the primary velocity profiles decreases for both the free convection cooling and heating of the plate. There is an increase in the rate of heat transfer in increasing M.

In conclusion, the finite difference method has led to the study of a wide range of free convection problems in which both the plate and the fluid are in a state of motion.

## 6.3 Recommendations

Our study of the effects of various parameters on the skin friction and rate of heat transfer at the laminar boundary layer has included part of a wide area of study involving MHD fluid flows. The study of the effect of variable magnetic field is very important especially in duct flow problems in which the Hartman number is of order greater than unity. From this study, there are areas that arise for further analysis and development. These may be theoretical or experimental and specific areas of study include:

a) Flow involving variable magnetic field applied at an angle, variable suction/injection, variable viscosity and thermal conductivity.
b) An extension of this study to turbulent hydromagnetic flow.
c) Flow of a fluid in a semi-infinite region.
d) Flow of fluid which is compressible.
e) Study of hydromagnetic flows that are bounded.
f) Solve hydromagnetic flow problem in three dimensions.
g) Incorporating viscous and Ohmic dissipations.
h) An extension of the difference method to problems involving mass transfer.
i) Analysis of the overall computation error in the results obtained.

The intention is to carry out further studies on some of the open problems mentioned above.

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