RELATIONSHIP BETWEEN PRIME FACTORS OF A NUMBER AND ITS CATEGORIZATION AS EITHER FRIENDLY OR SOLITARY

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Relationship Between Prime Factors of a Number and its Categorization as Either Friendly or Solitary

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DECLARATION

This thesis is my original work and has not been presented for a degree in any other University.

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This thesis has been submitted for examination with our approval as University Supervisors.

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DEDICATION

Little did I know of the significance of the day, not only to my life but also to the lives of many others. A new dawn had come; a different kind of light had shone, light that could turn around lives. The truth, however bitter, had finally come; truth that was intended to set the lives of many free. Thursday 12th June, 2003 marked the beginning of a new beginning and unto this day I dedicate this work.

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ABSTRACT

Two distinct positive integers are friendly if they have the same abundancy index. Numbers that are not friendly are said to be solitary. The class of numbers of the form $n = 2p$ for primes $p > 3$ consists of numbers whose categorization as either friendly or solitary is not known. The numbers 26 and 38 belong to this class. The study of friendly and solitary numbers is important in the sense that it aids in the identification of yet unknown prime numbers. Prime numbers are central in the practice of cryptography, and are of great importance in areas such as computer systems security and generation of pseudorandom numbers. In this thesis, properties of abundancy index function are systematically used to include or discriminate various prime numbers as possible factors of the potential friends of 26 and 38. They are also used to compute the greatest lower bounds for the potential friends of the two numbers. The thesis establishes that a friend of 26 must be of the form $m_1 = 13^{2a_1}b_1^2$ where a_1, $b_1 \in \mathbb{N}$, and must be larger than $6.654166091831798 \times 10^{17}$. On the other hand, a friend of 38 must be of the form $m_2 = 19^{2a_2}b_2$ where a_2, $b_2 \in \mathbb{N}$, and b_2 is not a square, and must be larger than $2.884414135676212 \times 10^{20}$. In addition, it is established that in general, if m is a friend of a number of the form $n = 2p$, p is a prime greater than, $I(n) = \frac{r}{s}$ where r, $s \in \mathbb{N}$, $(r, s) = 1$, and $k \in \mathbb{N}$ such that if q is a prime factor of m for which $2^k | (q + 1)$ but $2^k \nmid r$, then the power of q in m must be even.

CHAPTER ONE

1.0 INTRODUCTION AND LITERATURE REVIEW

1.1 Introduction

Natural numbers are mathematical objects used in counting and measuring, and notational symbols which represent natural numbers are called numerals. The number concept and counting process was being used even in most primitive times in counting elapsed time or keeping records of quantities, such as of animals. Today's numerals, called Hindu-Arabic numbers are a combination of just 10 symbols or digits: 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0. In addition to their use in counting and measuring, numerals are often used for numerical representation such as telephone numbers, serial numbers, and construction of codes in cryptography.

Definition 1.1.1 An integer n is said to be divisible by a non-zero integer m or m divides n, denoted $m|n$, if there exists some integer k such that $n = mk$. In this case, m (and also k) is said to be a factor or divisor of n, and n is a multiple of m (and of k). The notation $m \nmid n$ is used to indicate that n is not divisible by m.

Definition 1.1.2 The greatest common divisor of two integers a and b, not both zero, denoted (a, b) is the poistive integer d such that:

- $d|a$ and $d|b$; and
- if $c|a$ and $c|b$, then c$|d$.

Definition 1.1.3 Two integers a and b, not both zero, are said to be relatively prime or coprime if $(a, b) = 1$. Any two consecutive integers are coprime.

Theorem 1.1.1 If p is a prime and $p|mn$, then either $p|m$ or $p|n$, (Dris, 2008). □

Corollary 1.1.1 If $p \mid a_1 a_2 a_3 \ldots . a_n$ where p is prime and $a_1, a_2, \ldots, a_n$ are positive integers then there is an integer i, $1 \le i \le n$ such that $p \mid a_i$, (Rosen, 1993). □

Theorem 1.1.2 Fundamental Theorem of Arithmetic

Every positive integer $n > 1$ is expressible uniquely as a product of primes except possibly for the order of the prime factors, (Rosen, 1993). □

Corollary 1.1.2 Every positive integer $n > 1$ can be written uniquely in the canonical factorization

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r} = \prod_{i=1}^{r} p_i^{\alpha_i}$$

for $i = 1, 2, \ldots, r$, each α_i is a positive integer and each p_i is a prime, with $p_1 < p_2 < \ldots . < p_r$, (Dris, 2008). □

Example 1.1.1 The canonical factorization of the integer 128 is $128 = 2^7$, that of the integer 225 is $225 = 3^2 \times 5^2$, while for 2145 is $2145 = 3^1 \times 5^1 \times 11^1 \times 13^1$, written simply as $2145 = 3 \times 5 \times 11 \times 13$.

Definition 1.1.4 An arithmetic function or a number-theoretic function is one that is defined for all positive integers. A number-theoretic function f is a multiplicative if $\forall$ $x, y \in \mathbb{N}$, $(x, y) = 1$, $f(xy) = f(x)f(y)$. Multiplicative functions are determined by their values at prime powers.

Definition 1.1.5 Let n be a positive integer. The sum of all the positive divisors of n is called the sigma function of n, denoted $\sigma(n)$. The function σ is a number-theoretic function, by definition.

Theorem 1.1.3 If x and y are relatively prime then $\sigma(xy) = \sigma(x)\sigma(y)$, (Rosen, 1993).□

Example 1.1.2 Consider the positive integer $n = 100$. Since the positive divisors of 100 are 1, 2, 4, 5, 10, 20, 25, 50, and 100, then by definition

$$\sigma(100) = \sum_{d|100} d = 1 + 2 + 4 + 5 + 10 + 20 + 25 + 50 + 100 = 217.$$

It follows that $\sigma(1) = 1$, $\sigma(2) = 3$, $\sigma(3) = 4$, $\sigma(4) = 7$, $\sigma(5) = 6$, $\sigma(6) = 12$, and $\sigma(7) = 8$ which illustrates that $\sigma(n)$ is not monotonic.

If p is prime, the only positive divisors of p are 1 and p. So, $\sigma(p) = p + 1$.

If p is a prime and k a positive integer, then the sum of the divisors of p^k is the sum of a GP

$$\sigma(p^k) = \sum_{i=0}^{k} p^i = \frac{p^{k+1} - 1}{p - 1} \qquad (1.1.1)$$

(Ryan, 2009).

Example 1.1.3 Consider the positive integer $n = 729 = 3^6$. Then,

$$\sigma(3^6) = 1 + 3 + 9 + 27 + 81 + 243 + 729 = \frac{3^7 - 1}{3 - 1} = 1093$$

Given a positive integer n*'s* canonical factorization $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ then since σ is a multiplicative function and the fact that the prime powers derived from the canonical factorization of n are pairwise coprime, then $\sigma(n) = \prod_{i=1}^{r} \sigma(p_i^{\alpha_i})$, (Dris, 2008).

Example 1.1.4 Consider the canonical factorization of the positive integer 420.

$420 = 2^2 \times 3 \times 5 \times 7$. Then, $\sigma(420) = \sigma(2^2)\sigma(3)\sigma(5)\sigma(7)$

$$= 7 \times 4 \times 6 \times 8$$

$$= 1344\ .$$

Now, from Equation (1.1.1) and Theorem 1.1.3, if $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ is the canonical factorization of $n > 1$, then

$$\sigma(n) = \prod_{i=1}^{r} \sigma(p_i^{\alpha_i}) = \prod_{i=1}^{r} \left(\frac{p_i^{\alpha_i+1} - 1}{p_i - 1}\right) \tag{1.1.2}$$

(Dris, 2008).

Example 1.1.5 Consider the positive integer $n = 3500 = 2^2 \times 5^3 \times 7$

$$\sigma(3500) = \sigma(2^2)\sigma(5^3)\sigma(7) = \left(\tfrac{2^3-1}{2-1}\right)\left(\tfrac{5^4-1}{5-1}\right)\left(\tfrac{7^2-1}{7-1}\right) = 8736$$

1.2 Perfect Numbers

Definition 1.2.1 A positive integer n is perfect if $\sigma(n) = 2n$.

Example 1.2.1 For $n = 6$, then $\sigma(6) = 1 + 2 + 3 + 6 = 12 = 2(6)$. So, 6 is perfect.

Definition 1.2.2 Numbers of the form $2^k - 1$ for any positive integer k are called Mersenne numbers, named after a 17th century French monk Marin Mersenne. Primes of the form $2^k - 1$ are called Mersenne primes.

Some examples of Mersenne primes are $3 = 2^2 - 1$, $7 = 2^3 - 1$, $31 = 2^5 - 1$, and $127 = 2^7 - 1$.

Definition 1.2.3 A prime number is called Fermat prime if it is of the form $n = 2^{2^{\propto}} + 1$ for some non-negative integer $\propto$.

Example 1.2.2 The number 3 is a Fermat prime since $2^{2^0} + 1 = 2^1 + 1 = 3$. Other Fermat primes include 5, 17, 257, and 65537.

Theorem 1.2.1 (Euclid) If the sum $1 + 2 + 2^2 + \ . \ . \ . + 2^{p-2} + 2^{p-1} = 2^p - 1$ is prime, then $n = 2^{p-1}(2^p - 1)$ is perfect, (Dris, 2008). □

Example 1.2.3

(a) Consider $2^2 - 1 = 3$ which is prime. Then $2^{2-1}(2^2 - 1) = 6$ is perfect.

(b) Consider $2^3 - 1 = 7$ which is prime. Then $2^{3-1}(2^3 - 1) = 28$ is perfect.

Theorem 1.2.2 (Euler) Every even perfect number is of the form $n = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are primes, (Dris, 2008). □

In view of Theorem 1.2.1 and Theorem 1.2.2, the problem of searching for even perfect numbers is thus reduced to looking for primes of the form $2^p - 1$, for some primes p, called Mersenne primes. The theorems essentially say that the even perfect numbers are in one-to-one correspondence with Mersenne primes. The first four perfect numbers are 6, 28, 496, and 8128. According to Weisstein, (2011) there are only 47 known perfect numbers today, the last of which have been found using high-speed computers. However, the question of whether odd perfect numbers exist is still unresolved.

1.3 The Abundancy Index

Definition 1.3.1 The abundancy index of a positive integer n is the rational number $I(n) = \frac{\sigma(n)}{n}$ where $\sigma(n)$ is the sum of the positive divisors of n.

Example 1.3.1 From Examples 1.1.3, 1.1.4 and 1.1.5,

$$I(420) = \frac{\sigma(420)}{420} = \frac{1344}{420} = \frac{16}{5}, \text{ while}$$

$$I(729) = \frac{1093}{729}, \text{ and}$$

$$I(3500) = \frac{8736}{3500} = \frac{312}{125}$$

Theorem 1.3.1 If n is a positive integer, then $I(n) \geq 1$ with equality if and only if $n = 1$, (Ward, 2008). □

Definition 1.3.2 If the abundancy index $I(n) < 2$, then n is said to be deficient, and if $I(n) > 2$, n is said to be abundant.

Example 1.3.2 Considering Example 1.3.1, the numbers 420 and 3500 are abundant since $I(420) = \frac{16}{5} > 2$ and $I(3500) = \frac{312}{125} > 2$ while 729 is deficient since $I(729) = \frac{1093}{729} < 2$. All primes and prime powers are deficient.

Suppose $n = p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k}$ is the prime factorization of a positive integer n. From Equation 1.1.1 and Definition 1.1.4 the formula below for the abundancy index $I(n)$ is obtained;

$$I(n) = \frac{\sigma(n)}{n} = \frac{\sigma(p_1^{r_1})}{p_1^{r_1}} \frac{\sigma(p_2^{r_2})}{p_2^{r_2}} \ldots \frac{\sigma\left(p_k^{r_k}\right)}{p_k^{r_k}}$$

$$= \left(\frac{1+p_1+p_1^2+\ ...\ +p_1^{r_1}}{p_1^{r_1}}\right)\left(\frac{1+p_2+p_2^2+...+p_2^{r_2}}{p_2^{r_2}}\right)...\left(\frac{1+p_k+p_k^2+...+p_k^{r_k}}{p_k^{r_k}}\right)$$

$$= \frac{p_1^{r_1+1}-1}{p_1^{r_1}(p_1-1)}\frac{p_2^{r_2+1}-1}{p_2^{r_2}(p_2-1)}...\frac{p_k^{r_k+1}-1}{p_k^{r_k}(p_k-1)} \qquad (1.3.1)$$

(Laatsch, 1986). □

Example 1.3.3 Consider $n = 3500 = 2^2 \times 5^3 \times 7$. Then,

$$I(3500) = \left(\frac{1+2+4}{2^2}\right)\left(\frac{1+5+25+125}{5^3}\right)\left(\frac{1+7}{7}\right)$$

$$= \left(\frac{2^3-1}{2^2(2-1)}\right)\left(\frac{5^4-1}{5^3(5-1)}\right)\left(\frac{7^2-1}{7(7-1)}\right)$$

$$= \left(\frac{7}{4}\right)\left(\frac{156}{125}\right)\left(\frac{8}{7}\right) = \frac{312}{125}$$

Theorem 1.3.2 If $m|n$, $m > 0$, then $\frac{\sigma(m)}{m} \leq \frac{\sigma(n)}{n}$ with equality if and only if $m = n$, (Weiner, 2000). □

Corollary 1.3.1 Any (non-trivial) multiple of a perfect number is abundant and every proper divisor of a perfect number is deficient, (Dris, 2008). □

Theorem 1.3.3 The abundancy index is a multiplicative number-theoretic function.

Proof: Suppose $n = p_1^{r_1}p_2^{r_2}\ ...\ p_k^{r_k}$ is the prime factorization of a positive integer n. From Theorem 1.1.3 and the definition of abundancy index,

$$I(n) = \frac{\sigma(n)}{n} = \frac{\sigma(p_1^{r_1})}{p_1^{r_1}}\frac{\sigma(p_2^{r_2})}{p_2^{r_2}}...\frac{\sigma(p_k^{r_k})}{p_k^{r_k}} = I(p_1^{r_1})I(p_2^{r_2})\ ...\ I(p_k^{r_k})$$

(1.3.2)

and since $(p_i^{r_i}, p_j^{r_j}) = 1$ whenever $i \neq j$ and $i, j \in \{1, 2, ..., k\}$, then I is multiplicative. □

From Equation (1.3.2) the abundancy index is determined by its values at prime powers p^a.

Now, since the divisors of p^a are in GP,

$$I(p^a) = \frac{\sigma(p^a)}{p^a} = \frac{1}{p^a}\left(\frac{p^{a+1}-1}{p-1}\right) = \frac{p-p^{-a}}{p-1} = \frac{p}{p-1} - \frac{p^{-a}}{p-1} \tag{1.3.3}$$

Theorem 1.3.4 $I(p^a)$ is an increasing function of a when p is fixed, (Laatsch, 1986). □

From Theorem 1.3.4 and Equation (1.3.3), for any fixed prime p and an integer a greater than 1,

$$1 < I(p) < I(p^a) < \frac{p}{p-1} \tag{1.3.4}$$

Corollary 1.3.2 For any integer $n > 1$, then

$$I(n) = \frac{\sigma(n)}{n} < \prod_{p|n} \frac{p}{p-1} = \prod_{p|n} \left(1 + \frac{1}{p-1}\right),$$

(Weiner, 2000). □

Theorem 1.3.5 (Laatsch, 1986) $I(p^a)$ is a decreasing function of p when a is fixed.

Proof: Let p_1 and p_2 be two primes such that $p_1 < p_2$ and let a be a fixed positive integer. Then $p_1^a < p_2^a$.

$$I(p_1^a) = \frac{\sigma(p_1^a)}{p_1^a} = \frac{1+p_1+p_1^2+\cdots+p_1^a}{p_1^a}$$

$$= 1 + \frac{1}{p_1} + \frac{1}{p_1^2} + \ldots + \frac{1}{p_1^{a-2}} + \frac{1}{p_1^{a-1}} + \frac{1}{p_1^a}$$

$$= \sum_{i=0}^{a} \frac{1}{p_1^{a-i}}$$

and

$$I(p_2^a) = \frac{\sigma(p_2^a)}{p_2^a} = \frac{1+p_2+p_2^2+\cdots+p_2^a}{p_2^a}$$

$$= 1 + \frac{1}{p_2} + \frac{1}{p_2^2} + . . . + \frac{1}{p_2^{a-2}} + \frac{1}{p_2^{a-1}} + \frac{1}{p_2^a}$$

$$= \sum_{i=0}^{a} \frac{1}{p_2^{a-i}}$$

Clearly, $\frac{1}{p_1^{a-i}} > \frac{1}{p_2^{a-i}}$ for all $i = 0, 1, 2, . . ., a-1$, implying $\sum_{i=0}^{a} \frac{1}{p_1^{a-i}} > \sum_{i=0}^{a} \frac{1}{p_2^{a-i}}$ and hence $I(p_1^a) > I(p_2^a)$, thus the proof. □

Theorem 1.3.6 If $n = p_1^{r_1} p_2^{r_2} . . . p_k^{r_k}$ and $p_1, p_2, . . ., p_k$ distinct primes, then the least upper bound of $I(n^a)$ is $\frac{p_1}{p_1-1} \frac{p_2}{p_2-1} . . . \frac{p_k}{p_k-1}$, (Laatsch, 1986). □

Theorem 1.3.7 Suppose p and q are primes such that $p > q$. Then $I(q^a) > I(p^e)$ for all positive integers a and e.

Proof: By definition, $I(q) = \frac{q+1}{q}$ and by Inequality (1.3.4), $I(p^e) < \frac{p}{p-1}$ for all $e \in \mathbb{N}$. Since the abundancy index is an increasing function of a when q is fixed, then $I(q^a) \geq I(q)$ for all $a \in \mathbb{N}$, with equality only if $a = 1$. There are two cases to prove: when q is even while p is odd, and when both p and q are odd. Let $q = 2$ and p any odd prime. Then $I(q) = \frac{3}{2}$. Now, $\frac{3}{2} - \frac{p}{p-1} = \frac{p-3}{2(p-1)} = \frac{1}{p-1}\left(\frac{p}{2} - \frac{3}{2}\right) \geq 0$ since $p \geq 3$. Since $I(p^e) < \frac{p}{p-1}$ for all $e \in \mathbb{N}$ then $I(q) - I(p^e) = \frac{3}{2} - I(p^e) > 0$, hence $I(q) > I(p^e)$ for all $e \in \mathbb{N}$ and since $I(q^a) \geq I(q)$ for all $a \in \mathbb{N}$ then $I(q^a) > I(p^e)$ for all $a, e \in \mathbb{N}$ and the proof is done. On the other hand, let both p and q be odd and since $p > q$, then $p > q + 1$ and hence $p - q - 1 > 0$. In this case,

$$I(q) = \frac{q+1}{q} = \frac{(q+1)(p-1)}{q(p-1)} = \frac{qp+p-q-1}{q(p-1)} = \frac{qp+(p-q-1)}{q(p-1)}$$

and

$$I(p^e) < \frac{p}{p-1} = \frac{qp}{q(p-1)} < \frac{qp+(p-q-1)}{q(p-1)} = I(q) \text{ for all } e \in \mathbb{N}, \text{ since}$$

$p - q - 1 > 0$ so that $I(q) > I(p^e)$ for all $e \in \mathbb{N}$ and since $I(q^a) \geq I(q)$ for all $a \in \mathbb{N}$, then $I(q^a) > I(p^e)$ for all $a, e \in \mathbb{N}$. □

Example 1.3.4 For $p = 5, \frac{6}{5} \leq I(5^a) < \frac{5}{4}$ for all $a \in \mathbb{N}$; and for $p = 71, \frac{72}{71} \leq I(71^e) < \frac{71}{70}$ for all $e \in \mathbb{N}$. Hence the contribution of the prime 71 to the abundancy index of a number n having 71 as a factor is limited and does not depend strongly on the number of times the prime occurs as a factor of n. The contribution of still larger prime is proportionally less. Here, $I(71^e) < I(5^a)$ for all positive integers a and e yet $71 > 5$.

Theorem 1.3.8 Every number of the form $n = 2p$ for a prime $p > 3$ is deficient.

Proof: By definition, $I(3) = \frac{4}{3}$ and $I(p) = \frac{p+1}{p}$ and since $p > 3$, then, $\frac{p+1}{p} < \frac{4}{3}$, by Theorem 1.3.7. Now, $I(n) = I(2)I(p) = \frac{3}{2}.\frac{p+1}{p} < \frac{3}{2}.\frac{4}{3} = 2$. So, $I(n) < 2$ and thus n must be deficient. □

Corollary 1.3.3 No perfect number m can be expressed as $m = 2p$ for an odd prime p except $p = 3$. □

Theorem 1.3.9 If $(a, \sigma(a)) = 1$, then the unique solution to $I(n) = I(a)$ is $n = a$, (Dris, 2008).□

1.4 Friendly Numbers and Solitary Numbers

Definition 1.4.1 Let m and n be two distinct positive integers. If m and n satisfy the equation

$I(m) = I(n)$, then $\{m, n\}$ is called a friendly pair. Each member of the pair is called a friendly number, in which case m is called a friend of n and vice versa. Larger clubs of friends such as triples and higher-order tuples also exist. A number is said to be solitary if it is not friendly.

Example 1.4.1 Let m and n be perfect numbers with $m \neq n$, i.e., $\sigma(m) = 2m$, $\sigma(n) = 2n$. Then, $I(m) = \frac{\sigma(m)}{m} = 2 = \frac{\sigma(n)}{n} = I(n)$, implying $\{m, n\}$ is a friendly pair.

Theorem 1.4.1 If $\{m, n\}$ is a friendly pair and k is a positive integer coprime to both m and n, then $\{mk, nk\}$ is also a friendly pair, (Ward, 2008). □

Therefore, multiplying the numbers forming a friendly pair by any positive integer coprime to both friends gives a new friendly pair. Since there exists infinitely many positive integers k coprime to both m and n then there exist infinitely many friendly numbers.

Example 1.4.2 The pair {6, 28} is a friendly pair with common abundancy index 2. The integer 5 is coprime to both 6 and 28. Multiplying each of 6 and 28 by 5, gives 30 and 140 respectively, and now, computing their corresponding abundancy indexes,

$I(30) = I(6)I(5) = 2\left(\frac{6}{5}\right) = \frac{12}{5}$ and $I(140) = I(28)I(5) = 2\left(\frac{6}{5}\right) = \frac{12}{5}$ i.e. $I(30) = I(140) = \frac{12}{5}$ meaning {30, 140} is a friendly pair.

Theorem 1.4.2 All numbers n such that $\left(n, \sigma(n)\right) = 1$ are solitary, (Ward *et al*, 2006).□

Theorem 1.4.3 All primes and prime powers are solitary, (Dris, 2008). □

Remark 1.4.1 Just like friendly numbers there are infinitely many solitary numbers since the number of primes is infinite.

The concept outlined in Theorem 1.4.2 is sufficient but not necessary for a number to be solitary. So, there exist numbers such as $n =$ 18, 45, 48, and 52 which are solitary but for which $\left(n, \sigma(n)\right) \neq 1$.

1.5 Recent Work on Friendly Numbers and Solitary Numbers

The "friendly integer" problem was first stated as an advanced problem in Anderson and Hickerson (1977). The first problem was to show that the density of friendly numbers in the set $N_0 = \{0, 1, 2, 3, ..., n, ...\}$, n a positive integer is unity. That is, if we were to count the number of friendly integers it should be the same size as counting the number of integers in N_0. The second was to show that the density of solitary numbers in N_0 is zero. That is, there should be virtually no solitary numbers compared to the size of N_0, (Ward *et al*, 2006).

Ryan conjectures that $\{80, 200\}$ is the unique friendly pair of the form $\{p^{m_1}q^{n_1}, p^{m_2}q^{n_2}\}$ for distinct primes p and q, and positive integers m_1, n_1, m_2, and n_2, (Ryan, 2006). Every odd perfect square with exactly two distinct prime factors is solitary. In addition, if $p_1, p_2, . . . , p_k$ are distinct Fermat primes and $m_1, m_2, . . . ,$ m_k are even positive integers, then $x = \prod_{j=1}^{k} p_j^{m_j}$ is solitary, (Ryan, 2009).

Ward and others indicated in 2006 that 10 is the smallest number which is not known to be solitary or friendly. Any positive integer less than 10 is a prime (or prime power) except for 6. Since the condition $(\sigma(n), n) = 1$ is only necessary and not sufficient for a positive integer n to be solitary, it is not know whether 10 has a friend at all, (Ward *et al*, 2006).

In 2008, Ward used the properties of the abundancy index function to prove that if n is a friend of 10, then n is a square with at least six distinct prime factors, the smallest being 5. Further, at least one of the prime factors of n must be congruent to 1 modulo 3, and appear in the prime power factorization of n to a power congruent to 2 modulo 6. In addition, if there is only one such prime dividing n, then it appears to a power congruent to 8 modulo 18 in the factorization of n, (Ward, 2008).

In 2006, Ryan established that if $m_1$and n_1 are positive even integers and p and q are fixed distinct odd primes with $max\{p, q\} > 10^{11}$, then $x = p^{m_1}q^{n_1}$ has no friend of the form $p^m q^n$, for any positive integers m and n, (Ryan, 2006).

If $\tau(n)$ denotes the number of positive divisors of n, given positive integers $m_1$and n_1 and distinct primes p and q, then the number of friends of $p^{m_1}q^{n_1}$ of the form $p^m q^n$, for positive integers m and n is less than or equal to $min\,\{\tau(m_1+1), \tau(n_1+1)\}$, (Ryan, 2006).

1.6 Necessary Theorems

Lemma 1.6.1

a) For any positive integer k (whether even or odd), then $\sigma(2^k)$ is always odd.

b) If p is an odd prime and r an odd positive integer, then $\sigma(p^r)$ is even.

c) If q is an odd prime and s is an even positive integer, then $\sigma(q^s)$ is odd.

Proof:

a) $\sigma(2^k) = 1 + 2 + 2^2 + \cdots + 2^k = 1 + 2(1 + 2 + 2^2 + \cdots + 2^{k-1})$ which is odd.

b) $\sigma(p^r) = 1 + p + p^2 + \cdots + p^r$ is a sum of even number of odd terms which is even.

c) $\sigma(q^s) = 1 + q + q^2 + \cdots + q^s$ is a sum of odd number of odd terms which is odd.

Theorem 1.6.1 Let n be a number of the form $n = 2p$ for a prime $p > 3$ and let $I(n) = \frac{r}{s}$ where $(r, s) = 1$. Then,

(i) $s = p$ and hence always odd, with $s > 3$.

(ii) Suppose $I(m) = I(n)$ with $m \neq n$. Then $p|m$ and if r is odd, then m is an odd square number, otherwise, m is an odd non-square number.

Proof:

(i) Since $\frac{\sigma(n)}{n} = \frac{r}{s} = \left(\frac{3}{2}\right)\left(\frac{p+1}{p}\right)$ then $2rp = 3s(p+1)$ and $s|2rp$. Since $s \nmid r$, then $s|2p$ and hence $s|n$. Now, $p \nmid p+1$ and since $p > 3$, then $p \nmid 3$ and so $p \nmid 3(p+1)$. Since $p+1$ is even and $\frac{r}{s}$ is in the most simplified form, and p is prime, then $2|p+1$ and hence $r = 3(p+1)/2$ which in turn implies that $s = p$, which is odd. $s > 3$ follows from $s = p$ and $p > 3$.

(ii) Since $I(m) = I(n)$, then $\frac{\sigma(m)}{m} = \frac{r}{s}$, hence $rm = s\sigma(m)$ which means $s|rm$. Since $(r, s) = 1$, then $s|m$. A similar argument shows that $r|\ \sigma(m)$. Now, m must be odd, otherwise if $2|m$, since $s = p$, $s|m$ and $n = 2p$ then by Theorem 1.3.2 this would mean $I(n) < I(m)$, since $m \neq n$, cancelling possibility of friendship.

Now suppose r is odd. Since m is also odd, then $\sigma(m)$ must be odd and hence for every prime factor q of m, which occurs in the prime factorization of m with an exponent say e, then by Lemma 1.6.1 and the fact that σ is multiplicative, e must be even and therefore m must be a square number.

On the other hand, suppose r is even. Then $2|\sigma(m)$, meaning there exists at least one prime q which appears in the prime factorization of m with an exponent e such that $2|\sigma(q^e)$. By Lemma 1.6.1, e must be odd and, therefore, m is a non-square number. □

1.7 Statement of the Problem

The classification of numbers of the form $n = 2p$ for primes $p > 3$ such as 10, 14, 22, 26, 34 and 38 as either friendly or solitary is undetermined so far, (Anderson, 1977). In this research the numbers 26 and 38, both of the form $2p$ for primes $p = 13$ and $p = 19$ respectively, shall be considered in determination of some compulsory prime factors of potential friends and greatest lower bounds for potential friends of these numbers.

1.8 Justification

Perfect numbers are friendly. There are only 47 known perfect numbers, (Weisstein, 2011) and the search continues. Determination of all friendly numbers would supply all the perfect numbers. Even perfect numbers are in one-to-one correspondence with Mersenne primes and hence determination of more even perfect numbers implies identification of more Mersenne primes. Mersenne primes are used as period lengths in Mersenne Twister, an algorithm which is used to generate high quality pseudorandom numbers which are numbers which appear in a sequence that is deterministic. Pseudorandom numbers are important for simulations, for instance, of physical systems. They are also central in the practice of cryptography and procedural generation.

1.9 Objectives of the Study

1. To determine some compulsory prime factors of potential friends of 26 and potential friends of 38.

2. To compute the greatest lower bounds for potential friends of 26 and potential friends of 38. □

CHAPTER TWO

2.0 METHODOLOGY

2.1 Introduction

In this chapter, methods to be used in determining some necessary conditions for existence of potential friends of the number 26 and those of 38 shall be explored and described. The two numbers belong to the class of numbers of the form $n = 2p, p > 3$, where p is prime. Here $I(26) = \frac{21}{13}$ and $I(38) = \frac{30}{19}$. Hence, from Theorem 1.6.1, the friends of 26 are odd perfect squares while those of 38 are odd non-square numbers.

2.2 Procedure for determination of some compulsory prime factors and greatest lower bound for a friend of 26

Suppose m is a friend of 26. Properties of abundancy index function are used to systematically include or discriminate various prime numbers as possible factors of m and hence determine some compulsory prime factors of m and the greatest lower bound for m.

By definition, $I(m) = \frac{\sigma(m)}{m} = I(26) = \frac{21}{13}$ implying

$$21m = 13\sigma(m) \qquad (2.2.1)$$

and since $(13, 21) = 1$ then $13|m$, from divisibility properties. So, 13 is a compulsory prime factor of m. By Theorem 1.3.2 and Theorem 1.4.3, neither 13 nor its power can be a friend of 26. Therefore, $m > 26$ and $m = 13^{2a}b^2$ for some $a, b \in \mathbb{N}$ such that $(13, b) = 1$, $b > 1$.

2.2.1 Criterion to be used for the task

Let a be as defined above so that $a \in \{1, 2, 3, 4, 5, 6, 7, 8, \ldots\} = \mathbb{N}$. Consecutive values of a shall be considered one at a time. Substituting $m = 13^{2a}b^2$ in Equation (2.2.1) results in the emergence of other compulsory prime factors of m corresponding to the value of a being considered, as follows: from Equation (2.2.1), $\sigma(m)|21m$. Now, if $k|\sigma(m)$ with $(k, 13) = 1$, then $k|21m$, from properties of divisibility. Let $(k, 21) = t$ and $h = \frac{k}{t}$. Then $h|m$ with $h = k$ if $t = 1$, and $(h, 13) = 1$. Since m is defined in terms of the abundancy index, the properties of the abundancy index, and Theorem 1.6.1 shall be used to determine if h is a possible factor of m. If $h|m$, then $h^2|m$ since m is a perfect square. If $h = {p_1}^{e_1}{p_2}^{e_2} \ldots {p_r}^{e_r}$ is the prime factorization of h, then p_1, $p_2, \ldots, p_r$ are emerging compulsory prime factors of m corresponding to the a being considered. Since $13^{2a}h^2$ is a factor of m, then $m > 13^{2a}{p_1}^{2c_1}{p_2}^{2c_2} \ldots {p_r}^{2c_r}$ for some positive integers $c_1, c_2, \ldots, c_r$. Just like for the case of the compulsory prime factor 13, an emergent compulsory prime factor p_i of m shall be considered in order to determine other emerging compulsory prime factors of m corresponding to p_i and its power x_i, where x_i is an even positive integer greater than or equal to $2e_i$. If the emerging compulsory prime factors corresponding to p_i and x_i different from p_j, $j = 1$,

. . ., r are $q_1, q_2, . \ . \ . \ , q_s$ and $a_1, a_2, . . . , a_s$ are their corresponding possible powers (where all a_i are even since m is a square), then

$13^{2a}p_1^{2e_1}p_2^{2e_2}. \ . \ . p_{i-1}^{2e_{i-1}}p_i^{x_i}p_{i+1}^{2e_{i+1}} \ . \ . \ . p_r^{2e_r}q_1^{a_1}q_2^{a_2}. \ . \ . q_s^{a_s}$ shall be a factor of m and so,

$m > 13^{2a}p_1^{2e_1}p_2^{2e_2}. \ . \ . p_{i-1}^{2e_{i-1}}p_i^{x_i}p_{i+1}^{2e_{i+1}} \ . \ . \ . p_r^{2e_r}q_1^{a_1}q_2^{a_2}. \ . \ . q_s^{a_s}$. This gives the greatest lower bound for m corresponding to the value of a, p_i and x_i. Powers of p_i larger than x_i shall also be considered in order to determine the emergent compulsory prime factors of m and the greatest lower bounds for m corresponding to a and the power of p_i being considered.

One difficulty encountered in this task is the determination of the prime factorization of huge numbers which go beyond what the current tools of prime factorization can handle. Another challenge is that it might not be possible to compute the sum of divisors of a given prime beyond a given power, due to the limited capacity of the tools available for the task. In these two cases, limited information can be obtained regarding the emergent compulsory prime factors and the greatest lower bound for m. In this study a computer program in C# was developed. The program computes the prime factorization of a positive integer and hence its abundancy index. Improvement of this program will aid in resolving these problems, which will in turn help in improving on the results of the study.

2.3 Procedure for determination of some compulsory prime factors and greatest lower bound for a friend of 38

Suppose m is a friend of 38. Properties of abundancy index function are used to systematically include or discriminate various prime numbers as possible factors of m and hence determine some compulsory prime factors of m and the greatest lower bound for m.

Now, by definition, $I(m) = \frac{\sigma(m)}{m} = I(38) = \frac{30}{19}$ implying

$$30m = 19\sigma(m) \qquad (2.3.1)$$

and since $(19, 30) = 1$ then $19|m$, from divisibility properties. So, 19 is a mandatory prime factor of m. By Theorem 1.3.2 and Theorem 1.4.3, neither 19 nor its power can be a friend of 38. Hence, $m > 38$.

Theorem 2.3.1 If q is a prime factor of m such that $4|q + 1$, then the power of q must be even.

Proof: Let k be the power of q in m. Suppose k is odd. Then, $k + 1$ is even and by definition,

$\sigma(q^k) = 1 + q + \ldots + q^{k-1} + q^k$, which is the sum of even number of terms. Pairing consecutive summands,

$$\sigma(q^k) = (1 + q) + (q^2 + q^3) + \ldots + (q^{k-1} + q^k)$$

$$= (1+q) + q^2(1+q) + \ldots + q^{k-1}(1+q)$$

$$= (1+q)(1+q^2 + \ldots + q^{k-1})$$

implying that $q+1|\ \sigma(q^k)$. By the multiplicative nature of σ, then $\sigma(q^k)|\ \sigma(m)$ which means $q+1|\ \sigma(m)$ and from Equation (2.3.1), $q+1|30m$ implying $4|30m$ and hence $2|m$, a contradiction since m is odd. This means that k can't be odd. Therefore, the power k of q must be even. □

From Theorem 2.3.1, the power of 19 must be even. Since m is a non-square, then it is of the form $m = 19^{2a}b$ for some positive integers a and b, where b is not a square.

2.3.1 Criterion to be used for the task

Theorem 2.3.1 shall apply. Let a be as defined so that $a \in \{1, 2, 3, 4, 5, 6, 7, 8, \ldots\} = \mathbb{N}$. Consecutive values of a shall be considered one at a time. Substituting $m = 19^{2a}b$ in Equation (2.3.1) results in the emergence of other compulsory prime factors of m corresponding to the value of a being considered, as follows: from Equation (2.3.1), $\sigma(m)|30m$. Now, if $k|\sigma(m)$ with $(k, 19) = 1$, then $k|30m$, from properties of divisibility. If $(k, 30) = t$ and $h = \frac{k}{t}$, then $h|m$ with $h = k$ when $t = 1$, and $(h, 19) = 1$. Since m is defined in terms of the abundancy index, the properties of the abundancy index, and Theorem 1.6.1 shall be used to determine if h is a possible factor of m. If $h = {p_1}^{e_1}{p_2}^{e_2} \ldots {p_r}^{e_r}$ is the prime factorization of h, then $p_1, p_2, \ldots, p_r$ are emerging compulsory prime factors of m corresponding to the a being considered. Since $19^{2a}h$ is a factor of m, then $m > 19^{2a}{p_1}^{e_1}{p_2}^{e_2} \ldots {p_r}^{e_r}$. Just like for the case of the compulsory prime factor 19, an emergent compulsory prime factor p_i of m shall be considered in order to determine other emerging compulsory prime factors of m corresponding to p_i and its power x_i, where x_i is greater than or equal to e_i. If the emerging compulsory prime factors corresponding to p_i and x_i different from p_j, $j = 1$, . . ., r are q_1, q_2, . . . , q_s are q_1, q_2, . . . , q_s and a_1, a_2, . . . , a_s are their corresponding possible powers, then $19^{2a}p_1^{e_1}p_2^{e_2}. . . p_{i-1}^{e_{i-1}}p_i^{x_i}p_{i+1}^{e_{i+1}} . . . p_r^{e_r}q_1^{a_1}q_2^{a_2}. . . q_s^{a_s}$ shall be a factor of m, and so $m > 19^{2a}p_1^{e_1}p_2^{e_2}. . . p_{i-1}^{e_{i-1}}p_i^{x_i}p_{i+1}^{e_{i+1}} . . . p_r^{e_r}q_1^{a_1}q_2^{a_2}. . . q_s^{a_s}$. This gives the greatest lower bound for m corresponding to the value of a, p_i and x_i. Powers of p_i larger than x_i shall also be considered in order to determine the emergent compulsory prime factors of m and the greatest lower bounds for m corresponding to a and the power of p_i being considered.

Similar challenges to those highlighted in section 2.2.1 shall be encountered. Improvement of the program developed in this study will aid in resolving these challenges. □

CHAPTER THREE

3.0 RESULTS

3.1 Introduction

This chapter sets out to determine some compulsory prime factors and greatest computed lower bounds for the potential friends of the number 26 and those of 38. The chapter underscores the usefulness of Theorem 1.6.1 and the properties of the abundancy index in the task.

3.2 On compulsory prime factors and greatest lower bound for potential friends of 26

From section 2.2 it was established that a friend m of 26 is of the form $m = 13^{2a}b^2$ for some natural numbers a and b such that $(13, b) = 1$, $b > 1$. In this section, the value a shall be varied consecutively to be 1, 2, 3, 4, 5, 6, 7, 8, . . . since $a \in \mathbb{N}$, thus determine some of the possible powers of 13 since 13 is a compulsory prime factor of m. By considering different prime power factors of 13, other compulsory prime factors of m may emerge thus contributing to the setting of the greatest lower bound for m.

3.2.1 Case $a = 1$

In this case the power of 13 in m is 2 so that $m = 13^2b^2$. Substituting the value $m = 13^2b^2$ in the Equation (2.2.1), then

$$21m = 13\sigma(13^2)\sigma(b^2) = 13 \times 183 \times \sigma(b^2) = 13 \times (3 \times 61) \times \sigma(b^2).$$

From divisibility properties, this implies that $61|21m$ and since $(61,21) = 1$ then $61|m$. This means that whenever the power of 13 in m is 2, then 61 emerges as a compulsory prime factor of m (which may not be the case if the power of 13 is larger than 2). Since m is a perfect square, then $61^2|m$, which means $m = 13^2 61^{2c} d^2$ where $c, d \in \mathbb{N}$, $(13, d) = (61, d) = 1$.

If $c = 1$, that is the power of 61 in m is 2, then $m = 13^2 61^2 d^2$ and from Equation (2.2.1),

$$21m = 13\sigma(13^2)\sigma(61^2)\sigma(d^2)$$

$$= 13 \times 183 \times 3783 \times \sigma(d^2)$$

$$= 13 \times (3 \times 61) \times (3 \times 13 \times 97) \times \sigma(d^2).$$

Since $3^2|21m$ and $3 \nmid 21$, then $3|m$ and therefore $3^2|m$, from divisibility properties. Similarly, $97|21m$ and hence $97|m$. So, whenever the power of 61 in m is 2, then 3 and 97 emerge as compulsory prime factors of m (which may not be the case if the power of 61 is larger than 2). Now, since m is a perfect square then 3^2 and 97^2 divide m so that,

$$m = 3^{2x} 13^2 61^2 97^{2e} f^2 \text{ where } e, f, x \in \mathbb{N},\ (13, f) = (61, f) = (97, f) = 1.$$

By Theorem 1.3.6, $I(3^2 13^y) < \frac{13}{9}\frac{13}{12} = \frac{169}{108} < \frac{21}{13}\ \forall\ y \in \mathbb{N}$. However, $I(3^4 13^2) = \frac{121}{81}\frac{183}{169} = \frac{7381}{4563} > \frac{21}{13}$. This shows that 3 is a possible factor of m but the power of 3 in m cannot be larger than 2. So, $m = 3^2 13^2 61^2 97^{2e} f^2$ where $e, f \in \mathbb{N}$, $(13, f) = (61, f) = (97, f) = 1$.

If the power of 61 in m is 4, that is $c = 2$, then $m = 13^2 61^4 d^2$ and from Equation (2.2.1),

$$21m = 13\sigma(13^2)\sigma(61^4)\sigma(d^2)$$

$$= 13 \times 183 \times 14076605 \times \sigma(d^2)$$

$$= 13 \times (3 \times 61) \times (5 \times 131 \times 21491) \times \sigma(d^2).$$

From divisibility properties, 5, 131 and 21491 are all factors of $21m$ and since they are all coprime to 21, each of them must be a factor of m. So, whenever the power of 61 is 4, then 5, 131 and 21491 are emergent compulsory prime factors of m (which may not be the case if the power of 61 is different from 4). Since m is a perfect square, then 5^2, 131^2 and 21491^2 divide m. So,

$$m > 5^2 13^2 61^4 131^2 21491^2 = 4.636624150515115 \times 10^{23}.$$

Therefore, if the power of 13 is 2 and that of 61 is 4, then $4.636624150515115 \times 10^{23}$ is the greatest computed lower bound for m.

If the power of 61 in m is 6, that is $c = 3$, then $m = 13^2 61^6 d^2$ and from Equation (2.2.1),

$$21m = 13\sigma(13^2)\sigma(61^6)\sigma(d^2)$$

$$= 13 \times 183 \times 52379047267 \times \sigma(d^2)$$

$$= 13 \times (3 \times 61) \times 52379047267 \times \sigma(d^2).$$

From divisibility properties, $52379047267|21m$. Since $(21, 52379047267) = 1$, then $52379047267|m$ and hence all the prime factors of 52379047267 emerge as compulsory prime factors of m. Since m is a perfect square, then 52379047267^2 divides m. So,

$$m > 13^2 61^6 52379047267^2 = 2.388806125713085 \times 10^{34}.$$

Therefore, if the power of 13 is 2 and that of 61 is 6, then $2.388806125713085 \times 10^{34}$ is the greatest computed lower bound for m.

If the power of 61 in m is 8, that is $c = 4$, then $m = 13^2 61^8 d^2$ and from Equation (2.2.1),

$$\begin{aligned} 21m &= 13\sigma(13^2)\sigma(61^8)\sigma(d^2) \\ &= 13 \times 183 \times 194902434880569 \times \sigma(d^2) \\ &= 13 \times (3 \times 61) \times (3^2 \times 21655826097841) \times \sigma(d^2). \end{aligned}$$

This implies that $3^3|21m$, $3^2|m$, from divisibility properties. So, whenever the power of 61 is 8, then 3 emerge as a compulsory prime factor of m (this may not be the case if the power of 61 is different from 8). Since $(21, 21655826097841) = 1$, then $21655826097841|m$ and hence all the prime factors of 21655826097841 emerge as compulsory prime factors of m. Since m is a square number then 21655826097841^2 divides m. So,

$$m > 3^2 13^2 61^8 21655826097841^2 = 1.367468731918516 \times 10^{44}.$$

Therefore, if the power of 13 is 2 and that of 61 is 8, then $1.367468731918516 \times 10^{44}$ is the greatest computed lower bound for m.

For powers of 61 larger than or equal to 10, that is $c \geq 5$, then $13^2 61^{10}$ is a proper factor of m. In this case,

$$m > 13^2 61^{10} = 1.205549520710272 \times 10^{20}.$$

Therefore, if the power of 13 is 2 and that of 61 is greater than or equal to 10, then $1.205549520710272 \times 10^{20}$ is the greatest computed lower bound for m. Improvement of the computer program developed in this study, so that it can compute the prime factorization and the sum of divisors of an arbitrarily large positive integer such as $13^2 61^{10}$, will help in improving on this lower bound.

Now, recall that if the power of 61 is 2, then 3 and 97 emerge as compulsory prime factors of m and since m is a perfect square, then m is of the form $m = 3^2 13^2 61^2 97^{2e} f^2$ where $e, f \in \mathbb{N}$, $(13, f) = (61, f) = (97, f) = 1$.

If the power of 97 is 2, that is $e = 1$, then $m = 3^2 13^2 61^2 97^2 f^2$, and from Equation (2.2.1),

$$\begin{aligned} 21m &= 13\sigma(3^2)\sigma(13^2)\sigma(61^2)\sigma(97^2)\sigma(f^2) \\ &= 13^2 \times 183 \times 3783 \times 9507 \times \sigma(f^2) \\ &= 13^2 \times (3 \times 61) \times (3 \times 13 \times 97) \times (3 \times 3169) \times \sigma(f^2). \end{aligned}$$

From divisibility properties, $3169|m$. So, whenever the power of 97 is 2, then 3169 emerges as a compulsory prime factor of m (this may not be the case if the power of 97 is larger than 2). Since m is a perfect square, then 3169^2 divides m. So, m is of the form $m = 3^2 13^2 61^2 97^2 3169^{2x} g^2$, x, $g \in \mathbb{N}$, $(3, g) = (13, g) = (61, g) = (97, g) = (3169, g) = 1$.

Suppose the power of 3169 is 2, that is $x = 1$. Then $m = 3^2 13^2 61^2 97^2 3169^2 g^2$ and from Equation (2.2.1),

$$\begin{aligned} 21m &= 13\sigma(3^2)\sigma(13^2)\sigma(61^2)\sigma(97^2)\sigma(3169^2)\sigma(g^2) \\ &= 13^2 \times 183 \times 3783 \times 9507 \times 10045731 \times \sigma(f^2) \\ &= 13^2 \times (3 \times 61) \times (3 \times 13 \times 97) \times (3 \times 3169) \times (3 \times 3348577)\sigma(g^2). \end{aligned}$$

This implies that $3^4|21m$ and therefore $3^3|m$, from divisibility properties. This is not possible since the power of 3 in m cannot go beyond 2. Essentially, this establishes that it's not possible to have every prime factor of m being of power 2. So, at least one prime factor must have a power larger than 2. This means that if the powers of 13, 61, and 97 are all 2, then that of 3169 must be larger than or equal to 4. Hence,

$$m > 3^2 13^2 61^2 97^2 3169^4 = 5.370581473591163 \times 10^{24}.$$

Therefore, if the powers of 13, 61, and 97 are all 2, then $5.370581473591163 \times 10^{24}$ is the greatest computed lower bound for m.

If the power of 97 is 4, that is $e = 2$, then $m = 3^2 13^2 61^2 97^4 f^2$ and from Equation (2.2.1),

$$21m = 13\sigma(3^2)\sigma(13^2)\sigma(61^2)\sigma(97^4)\sigma(f^2)$$

$$= 13^2 \times 183 \times 3783 \times 89451461 \times \sigma(f^2)$$

$$= 13^2 \times (3 \times 61) \times (3 \times 13 \times 97) \times (11 \times 31 \times 262321) \times \sigma(f^2).$$

From divisibility properties, this implies that 11, 31 and 262321 divide m. So, whenever the power of 97 is 4, then 11, 31, and 262321 emerge as compulsory prime factors of m (this may not be the case if the power of 97 is different from 4). Since m is a perfect square, then 11^2, 31^2, and 262321^2 divide m. So,

$$m > 3^2 11^2 13^2 31^2 61^2 97^4 262321^2 = 4.009135157707144 \times 10^{30}.$$

Therefore, if both the powers of 13 and 61 equal 2 and that of 97 is 4, then $4.009135157707144 \times 10^{30}$ is the greatest computed lower bound for m.

If the power of 97 is 6, that is $e = 3$, then $m = 13^2 61^2 97^6 f^2$ and from Equation (2.2.1),

$$21m = 13\sigma(13^2)\sigma(61^2)\sigma(97^6)\sigma(f^2)$$

$$= 13 \times 183 \times 3783 \times 841648796647 \times \sigma(f^2)$$

$$= 13 \times (3 \times 61) \times (3 \times 13 \times 97) \times 841648796647 \times \sigma(f^2)$$

where $(841648796647, 21) = 1$. Hence $841648796647 | m$, from divisibility properties and therefore all the prime factors of 841648796647 are emergent prime factors of m. Since m is a square number, then $841648796647^2|m$. So,

$$m > 3^2 13^2 61^2 97^6 841648796647^2 = 3.339497351124352 \times 10^{42}.$$

Therefore, if the power of 61 is 2 and that of 97 is 6, then $3.339497351124352 \times 10^{42}$ is the greatest computed lower bound for m.

Now, if the power of 97 is larger than or equal to 8, that is $e \geq 4$, then $3^2 13^2 61^2 97^8$ is a proper factor of m and so $m > 3^2 13^2 61^2 97^8 = 4.435706050551322 \times 10^{22}$. Therefore, if both the powers of 13 and 61 equal 2 and the power of 97 is larger than or equal to 8, then $4.435706050551322 \times 10^{22}$ is the greatest computed lower bound for m.

Corresponding to different powers of the emergent prime factors of m are different sets of other emergent prime factors and, consequently, different values of computed greatest lower bounds. All these lower bounds computed indicate that if $a = 1$, then m is a multiple of 13^2 exceeding $1.205549520710272 \times 10^{20}$.

3.2.2 Case $a = 2$

Recall that m is of the form $m = 13^{2a} b^2$, a, $b \in \mathbb{N}$ such that $(13, b) = 1$, $b > 1$. If $a = 2$, then $m = 13^4 b^2$, $(13, b) = 1$, and from Equation (2.2.1),

$$21m = 13\sigma(13^4)\sigma(b^2) = 13 \times 30941 \times \sigma(b^2).$$

This implies that $30941|m$, from divisibility properties. So, whenever the power of 13 in m is 4, then 30941 emerges as a compulsory prime factor of m. So, the emergence of 30941 as a prime factor of m is determined by considering the power of 13 as 4, which may not be the case if the power of 13 is different from 4. Since m is a square number, then 30941^2divides m. So, $m = 13^4 30941^{2x} g^2$, x, $g \in \mathbb{N}$, $(13, g) = (30941, g) = 1$.

If the power of 30941 is 2, that is $x = 1$, then $m = 13^4 30941^2 g^2$, and from Equation (2.2.1),

$$21m = 13\sigma(13^4)\sigma(30941^2)\sigma(g^2)$$

$$= 13 \times 30941 \times 957376423 \times \sigma(g^2)$$

$$= 13 \times 30941 \times (157 \times 433 \times 14083) \times \sigma(g^2).$$

From divisibility properties, this implies that 157, 433, and 14083 divide m. So, whenever the power of 30941 is 2, then 157, 433, and 14083 emerge as compulsory prime factors of m. Since m is a perfect square, then 157^2, 433^2, 14083^2 divide m. So,

$$m > 13^4 157^2 433^2 14083^2 30941^2 = 2.506152860901485 \times 10^{31}.$$

Therefore, if the power of 13 is 4 and that of 30941 is 2, then $2.506152860901485 \times 10^{31}$ is the greatest computed lower bound for m.

If the power of 13 is 4 and that of 30941 is larger than or equal to 4, that is $x \geq 2$, then $13^4 30941^4$ is a proper factor of m and so $m > 13^4 30941^4 = 2.617645267731642 \times$

10^{22}. Hence, if the power of 13 is 4 and that of 30941 is greater than or equal to 4, then $2.617645267731642 \times 10^{22}$ is the greatest computed lower bound for m.

From the different values of the greatest computed lower bounds for m corresponding to different powers of the emerging compulsory prime factors, when $a = 2$, then it can be concluded that m is a multiple of 13^4 exceeding $2.617645267731642 \times 10^{22}$ if $a = 2$.

3.2.3 Case $\boldsymbol{a = 3}$

In this case, the power of 13 is 6 so that $m = 13^6 b^2$, $(13, b) = 1$ and from Equation (2.2.1),

$$21m = 13\sigma(13^6)\sigma(b^2) = 13 \times 5229043 \times \sigma(b^2)$$

From divisibility properties, this implies that $5229043|m$. So, whenever the power of 13 is 6, then 5229043 emerges as a compulsory prime factor of m. Since m is a square number, then 5229043^2 divides m. So, $m = 13^6 5229043^{2i} h^2$ where i, $h \in \mathbb{N}$, $(13, h) = (5229043, h) = 1$.

Since $i \geq 1$, then $13^6 5229043^2$ is a proper factor of m and so, $m > 13^6 5229043^2 = 1.319789108967402 \times 10^{20}$. Therefore, if the power 13 is 6 and that of 5229043 is larger than or equal to 2, then $1.319789108967402 \times 10^{20}$ is the greatest computed lower bound for m.

In conclusion, if $a = 3$, then m is a multiple of 13^6 exceeding $1.319789108967402 \times 10^{20}$.

3.2.4 Case $\boldsymbol{a = 4}$

In this case, the power of 13 is 8 so that $m = 13^8 b^2$, $(13, b) = 1$ and from Equation (2.2.1),

$$21m = 13\sigma(13^8)\sigma(b^2)$$

$$= 13 \times 883708281 \times \sigma(b^2)$$

$$= 13 \times (3^2 \times 61 \times 1609669) \times \sigma(b^2).$$

From divisibility properties, $3^2|21m$, $3|m$ and in addition, 61 and 1609669 divide m. So, if the power of 13 is 8, then 3, 61, and 1609669 emerge as compulsory prime factors of m. Since m is a perfect square, then 3^2, 61^2 and 1609669^2 divide m.

From section 3.2.1, the power of 3 cannot go beyond 2. So, m is of the form $m = 3^2 13^8 61^{2x} 1609669^{2y} z^2$, $x, y, z \in \mathbb{N}$, $(3, z) = (13, z) = (61, z) = (1609669, z) = 1$. Now, different powers of 61 shall be considered and where emergent prime factors identical to those corresponding to various powers of 61, in section 3.2.1, are obtained.

Suppose $x = 1$, then from section 3.2.1, $97^2|m$ and hence m is of the form $m = 3^2 13^8 61^2 97^{2s} 1609669^{2y} t^2$, $s, t, y \in \mathbb{N}$.

If $s = 1$, then from Equation (2.2.1),

$$21m = 13\sigma(3^2)\sigma(13^8)\sigma(61^2)\sigma(97^2)\sigma(1609669^{2y} t^2)$$

$= 13 \times 13 \times 883708281 \times 3783 \times 9507 \times \sigma(1609669^{2y}t^2)$

$= 13^2 \times (3^2 \times 61 \times 1609669) \times (3 \times 13 \times 97) \times (3 \times 3169) \times \sigma(1609669^{2y}t^2)$.

This implies that $3^4|21m$ and hence $3^3|m$, a contradiction since the power of 3 in m cannot be increased beyond 2. This means that $s = 1$ is an impossibility. So, m doesn't exist for $s = 1$, and therefore, it's not possible to have 2 as the power of every prime factor of m distinct from 13.

If $s = 2$, then from section 3.2.1, 11^2, 31^2, 262321^2 divide m so that $m > 3^2 11^2 13^8 31^2 61^2 97^4 262321^2 1609669^2 = 5.01399587013828 \times 10^{49}$.

If $s = 3$, then from section 3.2.1, $841648796647^2|m$ so that $m > 3^2 13^8 61^2 97^6 1609669^2 841648796647^2 = 4.176518193627421 \times 10^{61}$.

If $s \geq 4$, then $3^2 13^8 61^2 97^8 1609669^2$ is a proper factor of m. Hence $m > 3^2 13^8 61^2 97^8 1609669^2 = 5.547483670101881 \times 10^{41}$.

Suppose $x = 2$, then from section 3.2.1, 5^2, 131^2, 21491^2 are factors of m so that $m > 3^2 5^2 13^8 61^4 131^2 21491^2 1609669^2 = 5.218884393966336 \times 10^{43}$.

For $x = 3$, then from section 3.2.1, $52379047267^2|m$ so that $m > 3^2 13^8 61^6 1609669^2 52379047267^2 = 2.68878878360459 \times 10^{54}$.

If $x = 4$, then from section 3.2.1, $m = 3^2 13^8 61^8 1609669^{2y} z^2$. From Equation (2.2.1),

$$21m = 13\sigma(3^2)\sigma(13^8)\sigma(61^8)\sigma(1609669^{2y}z^2)$$

$$= 13 \times 13 \times 883708281 \times 194902434880569 \times \sigma(1609669^{2y} z^2)$$

$$= 13^2 \times (3^2 \times 61 \times 1609669) \times (3^2 \times 21655826097841) \sigma(1609669^{2y} z^2)$$

implying $3^4 | 21m$ and hence $3^3 | m$, a contradiction since the power of 3 in m cannot be increased beyond 2. So, $x = 4$ is an impossibility. Therefore, m doesn't exist if the power of 61 is 8.

If $x \geq 5$, then $3^2 13^8 61^{10} 1609669^2$ is a proper factor of m and so $m > 3^2 13^8 61^{10} 1609669^2 = 1.356940604963517 \times 10^{40}$.

From the different greatest computed lower bounds for m corresponding to different possible powers of the emergent prime factors, then it can be concluded that if $a = 4$, m is a multiple of 13^8 exceeding $1.356940604963517 \times 10^{40}$.

3.2.5 Case $a = 5$

In this case, the power of 13 is 10 so that $m = 13^{10} b^2$, $(13, b) = 1$ and from Equation (2.2.1),

$$21m = 13\sigma(13^{10})\sigma(b^2) = 13 \times 149346699503 \times \sigma(b^2).$$

Neither 3, 7 nor 13 divides 149346699503. From divisibility properties, $149346699503 | m$ and hence all the prime factors of 149346699503 are emergent compulsory prime factors of m, corresponding to $a = 5$. Since m is a square number then $149346699503^2 | m$. So,

$m = 13^{10}149346699503^{2j}t^2$ where $j, t \in \mathbb{N}$, $(13, t) = (149346699503, t) = 1$. Since $j \geq 1$, then $13^{10}149346699503^2$ is a proper factor of m and so, $m > 13^{10}149346699503^2 = 3.074855998446851 \times 10^{33}$. Hence, if the power of 13 is 10, then $3.074855998446851 \times 10^{33}$ is the greatest computed lower bound for m.

In conclusion, if $a = 5$, then m is a multiple of 13^{10} exceeding $3.074855998446851 \times 10^{33}$.

3.2.6 Case $\boldsymbol{a = 6}$

In this case, the power of 13 is 12 so that $m = 13^{12}b^2$, $(13, b) = 1$ and from Equation (2.2.1),

$$21m = 13\sigma(13^{12})\sigma(b^2)$$

$$= 13 \times 25239592216021 \times \sigma(b^2).$$

Here, neither 3, 7 nor 13 divides 25239592216021 and from divisibility properties, $25239592216021 | m$. Hence all the prime factors of m are emergent compulsory prime factors of m. Since m is a perfect square, then 25239592216021^2 divides m. So, $m = 13^{12}25239592216021^{2y}l^2$ where $y, l \in \mathbb{N}$, $(13, l) = (25239592216021, l) = 1$. Since $y \geq 1$, then $13^{12}25239592216021^2$ is a proper factor of m and so, $m > 13^{12}25239592216021^2 = 1.484174260702371 \times 10^{40}$. Hence, if the power of 13 is 12, then $1.484174260702371 \times 10^{40}$ is the greatest computed lower bound for m.

In conclusion, if $a = 6$, then m is a multiple of 13^{12} exceeding $1.484174260702371 \times 10^{40}$.

3.2.7 Case $\boldsymbol{a = 7}$

In this case, power of 13 is 14 so that $m = 13^{14}b^2$, $(13, b) = 1$ and from Equation (2.2.1),

$$
\begin{aligned}
21m &= 13\sigma(13^{14})\sigma(b^2) \\
&= 13 \times 4265491084507563 \times \sigma(b^2) \\
&= 13 \times (3 \times 1421830361502521) \times \sigma(b^2).
\end{aligned}
$$

Neither 3, 7 nor 13 divides 1421830361502521 hence from divisibility properties, $1421830361502521 | m$. So, all the prime factors of 1421830361502521 are emergent compulsory prime factors of m corresponding to $a = 7$. Since m is a perfect square, 1421830361502521^2 divides m. So, $m = 13^{14}1421830361502521^{2r}s^2$ for $r, s \in \mathbb{N}$, $(13, s) = (1421830361502521, s) = 1$.

Since $r \geq 1$, then $13^{14}1421830361502521^2$ is a proper factor of m and so, $m > 13^{14}1421830361502521^2 = 7.959806310140665 \times 10^{45}$. Hence, if the power of 13 is 14, then $7.959806310140665 \times 10^{45}$ is the greatest computed lower bound for m.

In conclusion, if $a = 7$, then m is a multiple of 13^{14} exceeding $7.959806310140665 \times 10^{45}$.

3.2.8 Case $a \geq 8$

If $a \geq 8$, that is the power of 13 is larger than or equal to 16, then 13^{16} is a proper factor of m. So, $m > 13^{16} = 6.654166091831798 \times 10^{17}$ if $a \geq 8$. Hence, if the power of 13 is larger than or equal to 16, then $13^{16} = 6.654166091831798 \times 10^{17}$ is the greatest computed lower bound for m. However, this bound can be improved by considering the sums of divisors of larger powers of 13 so that emerging compulsory prime factors of m corresponding to the values of a being considered can be determined. This is so because the product of 13^{2a}, for $a \geq 8$, and the powers of the emerging compulsory prime factors of m corresponding to the value of a being considered will be much larger than $6.654166091831798 \times 10^{17}$. This may be done by improving the computer program developed in this study, so that it can compute the prime factorization and the sum of divisors of an arbitrarily large positive integer.

In conclusion, if $a \geq 8$, then m is a multiple of 13^{16} exceeding $6.654166091831798 \times 10^{17}$.

3.2.9 Summary

In simple terms, it has been established that for a friend m of 26:

a) m is a square multiple of 13.

b) different values of the power a of 13 give different sets of emergent prime factors of m. Further, for each value of a, different powers of the emergent prime factors give different sets of other emergent prime factors of m.

Consequently, different values of the greatest computed lower bounds for m are obtained for different values of a and different powers of the emergent prime factors of m. After considering various possible values of a, and various powers of the corresponding emergent prime factors, $6.654166091831798 \times 10^{17}$ is the smallest of the various greatest computed lower bounds for different values of a. This means that a friend m of 26 is a square multiple of 13 which must be larger than $6.654166091831798 \times 10^{17}$.

c) if $m = 13^2b^2$, $(13, b) = 1$, then $61|m$; if $m = 13^2 61^2 c^2$, $(13, c) = (61, c) = 1$, then $3, 97|m$; if $m = 13^2 61^2 97^2 d^2$, $(13, d) = (61, d) = (97, d) = 1$, then $3169|m$, in which case 3169 must have a power larger than 2. This means m cannot have every of its prime factors being of power 2.

d) if $m = 13^2 61^2 97^4 d^2$, $(13, d) = (61, d) = (97, d) = 1$, then 11, 31, and 26231 are all factors of m.

e) if $m = 13^4 b^2$, $(13, b) = 1$, then $30941|m$; if $m = 13^4 30941^2 c^2$, $(13, c) = (30941, c) = 1$, then 151, 433 and 14083 are all prime factors of m; if $m = 13^6 b^2$, $(13, b) = 1$, then $5229043|m$; and if $m = 13^8 b^2$, $(13, b) = 1$, then 3, 61 and 1609669 are all prime factors of m, in which case 61 cannot have a power of 8.

3.3 On compulsory prime factors and greatest lower bound for potential friends of 38

From section 2.3 it was established that a friend m of 38 is of the form $m = 19^{2a}b$, a, $b \in \mathbb{N}$ such that $(19, b) = 1$, $b > 1$ and b is not a perfect square. In this section, the value a shall be varied consecutively to be 1, 2, 3, 4, 5, 6, 7, 8, . . . since $a \in \mathbb{N}$, thus determine some of the possible powers of 19 as a compulsory prime power factor of m. Impossible powers of 19 shall also be determined. By considering different prime power factors of 19, other prime factors of m may emerge, thus contributing to the setting of the greatest lower bound of m.

Theorem 3.3.1 m is not divisible by 3

Proof: Suppose $3|m$. Then, by Theorem 1.3.6,

$$I(3^4 19^{e_1}) < \frac{121}{81}\frac{19}{18} = \frac{2299}{1458} < \frac{30}{19}\ \forall\ e_1 \in \mathbb{N}.$$

This indicates that $I(3^4 19^2) < \frac{30}{19}$. However,

$$I(3^5 19^2) = \frac{364}{243}\frac{381}{361} = \frac{46228}{29241} > \frac{30}{19}.$$

So for all $e_2 \geq 2$, then

$$I(3^{e_1} 19^{e_2}) < \frac{30}{19}\ \forall\ e_1 \in \{1, 2, 3, 4\},$$

but

$$I(3^{e_1}19^{e_2}) > \frac{30}{19}\ \forall\ e_1 > 4.$$

From Theorem 2.3.1, the power of 19 must be even. So, the power of 19 is greater than or equal to 2. Hence if $3|m$, the power of 3 is either 1, 2, 3 or 4. By Theorem 2.3.1, the only permissible power of 3 is either 2 or 4.

Now, let $m = 3^x 19^y z$ such that $x, y, z \in \mathbb{N}$, and $(3, z) = (19, z) = 1$. Then, $x \in \{2, 4\}$ and y is an even positive integer. If $x = 2$, then $m = 3^2 19^y z$, and from Equation (2.3.1)

$$30m = 19\sigma(3^2)\sigma(19^y z) = 19.13.\sigma(19^y z)$$

so that $13|m$. But,

$$I(3^2.13.19^2) = \frac{13}{9}\frac{14}{13}\frac{381}{361} = \frac{1778}{1083} > \frac{30}{19}.$$

This means that it's not possible to have every of 3^2, 13 and 19^2 being a factor of m. So, $x \neq 2$. If $x = 4$, then $m = 3^4 19^y z$ and

$$30m = 19\sigma(3^4)\sigma(19^y z) = 19.121.\sigma(19^y z) = 19.11^2.\sigma(19^y z)$$

so that $11^2|m$. But,

$$I(3^4.11^2.19^2) = \frac{121}{81}\frac{133}{121}\frac{381}{361} = \frac{889}{513} > \frac{30}{19}.$$

This means that it's not possible to have every of 3^4, 11^2 and 19^2 being a factor of m. So, $x \neq 4$. This has just shown that neither 2 nor 4 can be a power of 3. But 2 and 4 are the only permissible powers of 3. So, 3 cannot be a factor of m. □

3.3.1 Case $a = 1$

In this case, the power of 19 is 2 so that $m = 19^2 b$, $(19, b) = 1$, and from Equation (2.3.1),

$$\begin{aligned} 30m &= 19\sigma(19^2)\sigma(b) \\ &= 19 \times 381 \times \sigma(b) \\ &= 19 \times (3 \times 127) \times \sigma(b). \end{aligned}$$

From divisibility properties, this implies that $127|30m$ and since $(127, 30) = 1$, then $127|m$. So, whenever the power of 19 is 2, then 127 emerges as a compulsory prime factor of m (this may not be the case if the power of 19 is different from 2). By Theorem 2.3.1 the power of 127 in m must be even. So, $m = 19^2 127^{2r} c$ where r, $c \in \mathbb{N}$, r even and $(19, c) = (127, c) = 1$, and c a non-square.

Suppose the power of 127 is 2, that is $r = 1$, so that $m = 19^2 127^2 c$. From Equation (2.3.1),

$$\begin{aligned} 30m &= 19\sigma(19^2)\sigma(127^2)\sigma(c) \\ &= 19 \times 381 \times 16257 \times \sigma(c) \end{aligned}$$

$$= 19 \times (3 \times 127) \times (3 \times 5419) \times \sigma(c).$$

From divisibility properties, this implies that $3^2|30m$ and hence $3|m$ which is a contradiction, as far as Theorem 3.3.1 is concerned. So, it's not possible to have $r = 1$. Therefore, if the power of 19 is 2, then $127|m$ and the power of 127 cannot be 2 but an even integer greater than 2. This essentially means that it is not possible to have 2 as the power of every prime factor of m which assumes an even power in the prime factorization of m.

Now, consider powers of 127 larger than 2. Suppose the power of 127 is 4, that is $r = 2$. Then $m = 19^2 127^4 c$ and from Equation (2.3.1),

$$30m = 19\sigma(19^2)\sigma(127^4)\sigma(c)$$

$$= 19 \times 381 \times 262209281 \times \sigma(c)$$

$$= 19 \times (3 \times 127) \times 262209281 \times \sigma(c).$$

From divisibility properties, $262209281|m$. So, whenever the power of 127 is 4, then 262209281 emerges as a compulsory prime factor of m (this may not be the case if the power of 127 is different from 4). Thus, m is of the form $m = 19^2 127^4 262209281^k f$ where $f, k \in \mathbb{N}$, $(19, f) = (127, f) = (262209281, f) = 1$.

If $k = 1$, then $m = 19^2 127^4 262209281 f$, and from Equation (2.3.1),

$$30m = 19\sigma(19^2)\sigma(127^4)\sigma(262209281)\sigma(f)$$

$$= 19 \times 381 \times 262209281 \times 262209282 \times \sigma(f)$$

$$= 19 \times (3 \times 127) \times 262209281 \times (2 \times 3 \times 3137 \times 13931) \times \sigma(f).$$

From divisibility properties, this implies that $3^2|30m$ and hence $3|m$, a contradiction. This shows that the power of 262209281 cannot be equal to 1, that is $k \neq 1$ if the power of 127 is 4. So, $k \geq 2$ and hence,

$$m > 19^2 127^4 262209281^2 = 6.456812945395982 \times 10^{27}.$$

Therefore, if the power of 127 is 4, then $6.456812945395982 \times 10^{27}$ is the greatest computed lower bound for m.

If the power of 127 is 6, that is $r = 3$, then $m = 19^2 127^6 c$, and from Equation (2.3.1),

$$30m = 19\sigma(19^2)\sigma(127^6)\sigma(c)$$

$$= 19 \times (3 \times 127) \times 4229173493377 \times \sigma(c)$$

where $(30, 4229173493377) = 1$. From divisibility properties, $4229173493377|m$ and so

$$m > 19^2 127^6 4229173493377 = 6.405971898969618 \times 10^{27}.$$

Therefore, if the power of 127 is 6, then $6.405971898969618 \times 10^{27}$ is the greatest computed lower bound for m.

If the power of 127 is 8, that is $r = 4$, then $m = 19^2 127^8 c$ and from Equation (2.3.1),

$$30m = 19\sigma(19^2)\sigma(127^8)\sigma(c)$$

$$= 19 \times 381 \times 68212339274677761 \times \sigma(c)$$

$$= 19 \times (3 \times 127) \times (3 \times 22737446424892587) \times \sigma(c).$$

From divisibility properties, $3^2|30m$ and hence $3|m$, a contradiction to Theorem 3.3.1. So, $r \neq 4$. Therefore, m doesn't exist for $r = 4$.

If the power of 127 is larger than or equal to 10, that is $r \geq 5$, then $19^2 127^{10}$ is a proper factor of m. So, $m > 19^2 127^{10} = 3.940437209594951 \times 10^{23}$ if $r \geq 5$. Therefore, if the power of 19 is 2 and that of 127 is greater than or equal 10, then $3.940437209594951 \times 10^{23}$ is the greatest computed lower bound.

From the different values of the greatest computed lower bounds for m corresponding to different powers of the emerging compulsory prime factors, when $a = 1$, then it can be concluded that m is a multiple of 19^2 exceeding $3.940437209594951 \times 10^{23}$ if $a = 1$.

3.3.2 Case $a = 2$

In this case, the power of 19 is 4 so that $m = 19^4 b$, $(19, b) = 1$ and from Equation (2.3.1),

$$30m = 19\sigma(19^4)\sigma(b)$$

$$= 19 \times 137561 \times \sigma(b)$$

$$= 19 \times (151 \times 911) \times \sigma(b).$$

From divisibility properties, 151 and 911 divide m. So, whenever the power of 19 is 4, then 151 and 911 emerge as compulsory prime factors of m. Therefore, m is of the form

$m = 19^4 151^x 911^y c$ where $c, x, y \in \mathbb{N}$ and $(19, c) = (151, c) = (911, c) = 1$. By Theorem 2.3.1, both the powers x and y of 151 and 911 respectively must be even.

Let $x = y = 2$. Then $m = 19^4 151^2 911^2 c$ and from Equation (2.3.1),

$$\begin{aligned} 30m &= 19\sigma(19^4)\sigma(151^2)\sigma(911^2)\sigma(c) \\ &= 19 \times 137561 \times 22953 \times 830833 \times \sigma(c) \\ &= 19 \times (151 \times 911) \times (3 \times 7 \times 1093) \times 830833 \times \sigma(c). \end{aligned}$$

From divisibility properties, 7, 1093 and 830833 divide m. So, whenever the powers of both 151 and 911 is 2, then 7, 1093 and 830833 emerge as compulsory prime factors of m. The powers of 1093 and 830833 can be odd. However, by Theorem 2.3.1, the power of 7 must be even. If the power of 7 is 2, then $m = 7^2 19^4 151^2 911^2 f$ and from Equation (2.3.1),

$$\begin{aligned} 30m &= 19\sigma(7^2)\sigma(19^4)\sigma(151^2)\sigma(911^2)\sigma(f) \\ &= 19 \times 57 \times 137561 \times 22953 \times 830833 \times \sigma(f) \\ &= 19 \times (3 \times 19) \times (151 \times 911) \times (3 \times 7 \times 1093) \times 830833 \times \sigma(f), \end{aligned}$$

which implies that $3^2 | 30m$ and hence $3 | m$, a contradiction. So, the power of 7 must be greater than 2. Therefore, the power of 7 must be even and larger than or equal to 4 so that $m > 7^4 \times 19^4 \times 151^2 \times 911^2 \times 1093 \times 830833 = 5.37688951181237 \times 10^{27}$.

So, if the power of 19 is 4, then $5.37688951181237 \times 10^{27}$ is the greatest computed lower bound.

In conclusion, it has been established that if $a = 2$, at least one prime factor of m different from 19 must have an even power larger than 2; that is, at least the power x of 151, which must be even, either equals 2 implying $7|m$ where the power of 7 must be even and larger than 2, or x itself is larger than 2. In addition, m is a multiple of 19^4 exceeding $5.37688951181237 \times 10^{27}$ if $a = 2$.

3.3.3 Case $a = 3$

In this case the power of 19 is 6 so that $m = 19^6 b$, $(19, b) = 1$, and from Equation (2.3.1),

$$\begin{aligned} 30m &= 19\sigma(19^6)\sigma(b) \\ &= 19 \times 49659541 \times \sigma(b) \\ &= 19 \times (701 \times 70841) \times \sigma(b). \end{aligned}$$

From divisibility properties, 701 and 70841 divide m. So, whenever the power of 19 is 6, then 701 and 70841 emerge as compulsory prime factors of m. So, $m = 19^6 701^r 70841^s d$ where $r, s \in \mathbb{N}$ and $(19, d) = (701, d) = (70841, d) = 1$.

The power r of 701 must be larger than 1 since if $r = 1$, then $m = 19^6 \times 701 \times 70841^s \times d$, so that from Equation (2.3.1),

$$30m = 19\sigma(19^6)\sigma(701)\sigma(70841^s d)$$

$$= 19 \times 49659541 \times 702 \times \sigma(70841^s d)$$

$$= 19 \times (701 \times 70841) \times (2 \times 3^3 \times 13) \times \sigma(70841^s d).$$

From divisibility properties, this means $3^3|30m$ and hence $3|m$, a contradiction. So, power of 701 must be larger than 1.

If the power of 701 is 2, that is $r = 2$, $m = 19^6 701^2 70841^s d$, $(19, d) = (701, d) = (70841, d) = 1$, then from Equation (2.3.1),

$$30m = 19\sigma(19^6)\sigma(701^2)\sigma(70841^s d)$$

$$= 19 \times 49659541 \times 492103 \times \sigma(70841^s d)$$

$$= 19 \times (701 \times 70841) \times 492103 \times \sigma(70841^s d).$$

From divisibility properties, this implies that $492103|m$. So, whenever the power of 701 is 2, then 492103 emerges as a compulsory prime factor of m. From Theorem 2.3.1, the power of 492103 is even and hence greater than or equal to 2. So, m must have at least one prime factor distinct from 19 that has an even power. On the hand, 70841 can assume a power 1 or larger. So,

$$m > 19^6 \times 701^2 \times 70841 \times 492103^2 = 3.96601497791775 \times 10^{29}.$$

Hence, if the power of 19 is 6 and that of 701 is 2, then $3.96601497791775 \times 10^{29}$ is the greatest computed lower bound for m.

If $r = 3$, $m = 19^6 701^3 70841^s d$ and from Equation (2.3.1),

$$30m = 19 \times \sigma(19^6)\sigma(701^3)\sigma(70841^s d)$$

$$= 19 \times 49659541 \times 344964204 \times \sigma(70841^s d)$$

$$= 19 \times (701 \times 70841) \times (4 \times 86241051) \times \sigma(70841^s d),$$

implying $4|30m$ and hence $2|m$ which is a contradiction, from Theorem 1.6.1. This means that it is not possible to have $r = 3$. Therefore, m does not exist for $r = 3$.

If $r = 4$, $m = 19^6 701^4 70841^s d$ and from Equation (2.3.1),

$$30m = 19 \times \sigma(19^6)\sigma(701^4)\sigma(70841^s d)$$

$$= 19 \times 49659541 \times 241819907005 \times \sigma(70841^s d)$$

$$= 19 \times (701 \times 70841) \times (5 \times 48363981401) \times \sigma(70841^s d),$$

where $(30, 48363981401) = 1$. From divisibility properties, $48363981401|m$. Therefore, all the prime factors of 48363981401 emerge as compulsory prime factors of m. Since $s \geq 1$ then,

$$m > 19^6 \times 701^4 \times 70841 \times 48363981401 = 3.892247121926787 \times 10^{34}.$$

Hence, if the power of 19 is 6 and that of 701 is 4, then $3.892247121926787 \times 10^{34}$ is the greatest computed lower bound for m.

If $r = 5$, $m = 19^6 701^5 70841^s d$ and from Equation (2.3.1),

$$30m = 19 \times \sigma(19^6)\sigma(701^5)\sigma(70841^s d)$$

$$= 19 \times 49659541 \times 169515754810506 \times \sigma(70841^s d)$$

$$= 19 \times (701 \times 70841) \times (2 \times 3^4 \times 1046393548213) \times \sigma(70841^s d).$$

From divisibility properties, this means $3^4|30m$ and hence $3|m$, a contradiction. So $r \neq 5$, that is, the power of 701 cannot be equal to 5. Therefore, m does not exist for $r = 5$.

If $r \geq 6$, then $19^6 701^6 70841^s$ is a proper factor of m where $s \geq 1$. Hence,

$$m > 19^6 701^6 70841 = 3.954707765069147 \times 10^{29}.$$

Therefore, if the power of 19 is 6 and that of 701 is greater than or equal to 6, then $3.954707765069147 \times 10^{29}$ is the greatest computed lower bound for m.

From the different values of the greatest computed lower bounds for m corresponding to different powers of the emerging compulsory prime factors, when $a = 3$, then it can be concluded that m is a multiple of 19^6 exceeding $3.954707765069147 \times 10^{29}$ if $a = 3$.

3.3.4 Case $a = 4$

In this case, the power of 19 is 8 so that $m = 19^8 b$, $(19, b) = 1$, and from Equation (2.3.1),

$$30m = 19\sigma(19^8)\sigma(b)$$

$$= 19 \times 17927094321 \times \sigma(b)$$

$$= 19 \times (3^2 \times 127 \times 523 \times 29989) \times \sigma(b).$$

From divisibility properties, $3^2|30m$ and hence $3|m$, a contradiction. So, $a \neq 4$, that is 19 cannot have a power of 8 in the prime factorization of m. Therefore, m does not exist for $a = 4$.

3.3.5 Case $a = 5$

In this case, the power of 19 is 10 so that $m = 19^{10}b$, $(19, b) = 1$, and from Equation (2.3.1),

$$30m = 19\sigma(19^{10})\sigma(b)$$

$$= 19 \times 6471681049901 \times \sigma(b).$$

Now, $(30, 6471681049901) = 1$ implying that $6471681049901|m$, from divisibility properties. Therefore, all the prime factors of 6471681049901 emerge as compulsory prime factors of m. So,

$$m > 19^{10} \times 6471681049901 = 3.967830531629817 \times 10^{25}.$$

Hence, if the power of 19 is 10, then $3.967830531629817 \times 10^{25}$ is the greatest computed lower bound for m.

In conclusion, if $a = 5$, then m is a multiple of 19^{10} exceeding $3.967830531629817 \times 10^{25}$.

3.3.6 Case $a = 6$

In this case, the power of 19 is 12 so that $m = 19^{12}b$, $(19, b) = 1$, and from Equation (2.3.1),

$$30m = 19\sigma(19^{12})\sigma(b)$$

$$= 19 \times 2336276859014281 \times \sigma(b).$$

Now, $(30, 2336276859014281) = 1$, implying that $2336276859014281|m$, from divisibility properties. Therefore, all the prime factors of 2336276859014281 emerge as compulsory prime factors of m, corresponding to $a = 6$. So,

$$m > 19^{12} \times 2336276859014281 = 5.170916427125338 \times 10^{30}.$$

Hence, if the power of 19 is 12, then $5.170916427125338 \times 10^{30}$ is the greatest computed lower bound for m.

In conclusion, if $a = 6$, then m is a multiple of 19^{12} exceeding $5.170916427125338 \times 10^{30}$.

3.3.7 Case $a = 7$

In this case, the power of 19 is 14 so that $m = 19^{14}b$, $(19, b) = 1$, and from Equation (2.3.1),

$$30m = 19\sigma(19^{14})\sigma(b)$$

$$= 19 \times 843395946104155461 \times \sigma(b)$$

$$= 19 \times 3 \times 2811319823471847 \times \sigma(b).$$

$(30, 2811319823471847) = 1$ and from divisibility properties, this implies that $2811319823471847|m$. Therefore, all the prime factors of 2811319823471847 emerge as compulsory prime factors of m, corresponding to $a = 7$. So,

$$m > 19^{14} \times 2811319823471847 = 2.246263334827963 \times 10^{33}.$$

Hence, if the power of 19 is 14, then $2.246263334827963 \times 10^{33}$ is the greatest computed lower bound for m.

In conclusion, if $a = 7$, then m is a multiple of 19^{14} exceeding $2.246263334827963 \times 10^{33}$.

3.3.8 Case $a \geq 8$

If $a \geq 8$, that is the power of 19 is larger than or equal to 16, then 19^{16} is a proper factor of m. So, $m > 19^{16} = 2.884414135676212 \times 10^{20}$ if $a \geq 8$. Therefore, if the power of 19 is larger than or equal 16, then $2.884414135676212 \times 10^{20}$ is the greatest computed lower bound for m. However, this bound can be improved by considering the sum of divisors of larger powers of 19 so that emerging compulsory prime factors of m corresponding to the value of a being considered can be determined. This is so because the product of 19^{2a}, for $a \geq 8$, and the powers of the emerging compulsory prime factors of m corresponding to the value of a being considered will be much larger than $2.884414135676212 \times 10^{20}$. This may be done by improving the computer program developed in this study, so that it can compute the prime factorization and the sum of divisors of an arbitrarily large positive integer.

In conclusion, if $a \geq 8$, then m is a multiple of 19^{16} exceeding $2.884414135676212 \times 10^{20}$.

3.3.9 Summary

In simple terms, it has been established that for a friend m of 38:

a) m is a non-square multiple of 19

b) m is not divisible by 3.

c) the powers of all prime factors q of m, $4|q+1$ must be even. Consequently, the power of 19 must be even.

d) different values of a give different sets of emergent prime factors of m. Further, different powers of the emergent prime factors of m give different sets of other emergent prime factors of m. Thus, different values of the greatest lower bounds for m are obtained for different values of a and different powers of the emergent prime factors of m. After considering various possible values of a, and various powers of the corresponding emergent prime factors, $2.884414135676212 \times 10^{20}$ is the smallest of the various greatest computed lower bounds. This means that a friend m of 38 can never be less than or equal to $2.884414135676212 \times 10^{20}$. So, m is a non-square multiple of 19^2 exceeding $2.884414135676212 \times 10^{20}$.

e) if $m = 19^2 b$, $(19, b) = 1$, then $127|m$, and in this case the power of 127 must be even and larger than 2; if $m = 19^4 b$, $(19, b) = 1$, then 151 and 911 must be factors of m and their powers must be even; if $m = 19^6 b$, $(19, b) = 1$, then 701 and 70841 must be factors of m where the power of 701 cannot equal 3; and m does not exist for $a = 4$, that is, 19 cannot have a power 8 in the prime factorization of m.

3.4 Some General Result

Theorem 3.4.1 Suppose m is a friend of a number of the form $n = 2p$, p is a prime greater than 3 and $I(n) = \frac{r}{s}$, $r, s \in \mathbb{N}$, $(r, s) = 1$. If $k \in \mathbb{Z}^+$ and q is a prime factor of m such that $2^k|q + 1$ but $2^k \nmid r$, then the power of q in m must be even.

Proof: Since $\frac{\sigma(m)}{m} = \frac{r}{s}$ then $rm = s\sigma(m)$. From divisibility properties, $\sigma(m)|rm$. Let q be a prime factor of m such that $2^k|q + 1$. Let x be the power of q in m. Since σ is multiplicative, then $\sigma(q^x)|\sigma(m)$. By definition,

$$\sigma(q^x) = 1 + q + q^2 + \ldots + q^{x-1} + q^x.$$

Suppose x is odd. Then $1, q, q^2, \ldots, q^x$ is an even number of terms. So, pairing consecutive summands,

$$\begin{aligned}\sigma(q^x) &= (1 + q) + (q^2 + q^3) + \ldots + (q^{x-1} + q^x) \\ &= (1 + q) + q^2(1 + q) + \ldots + q^{x-1}(1 + q)\end{aligned}$$

$$= (1+q)(1+q^2+\,.\,.\,.+q^{x-1})$$

$$= (q+1)(1+q^2+\,.\,.\,.+q^{x-1})$$

From divisibility properties, $(q+1)|\sigma(q^x)$. Since $\sigma(q^x)|\sigma(m)$, then $q+1|\sigma(m)$, and since $\sigma(m)|rm$, then $q+1|rm$. But, $2^k|q+1$. This means $2^k|rm$. Since $2^k \nmid r$, then $2|m$. This is a contradiction since m is an odd number, by Theorem 1.6.1. So, x cannot be odd and, therefore, x must be even. □

Theorem 3.4.1 defines some necessary condition for existence of a friend of any number of the form $n = 2p$ for a prime $p > 3$. □

CHAPTER FOUR

4.0 DISCUSSION, CONCLUSION AND RECOMMENDATIONS

4.1 Introduction

This chapter discusses the results of this study and also gives some conclusion, and recommendation for further research.

4.2 Discussion

Some of the most stubborn problems in Pure Mathematics arise as a result of the challenge of prime factorization of large numbers. A good example is the RSA code, where very large numbers which are products of two very large primes are involved. This renders the prime factorization of such numbers a problem hence making it extremely hard to break the code. A similar challenge presented itself in this research, thus limiting the determination of emergent compulsory prime factors of the potential friends of the numbers under consideration. The sum of divisors of a number is determined at its prime power factors. The challenge of the prime factorization makes it hard to compute the sum of divisors of powers of large numbers bringing about inability to determine even more emerging compulsory prime factors which could have resulted in much larger values of greatest lower bounds for the potential friends of these numbers. This is the challenge that faces researchers in their efforts to determine the classification of numbers which have not been categorized as either friendly or solitary so far.

The greatest computed lower bound for potential friends of 26 determined in this research is $6.654166091831798 \times 10^{17}$ while that for potential friends of 38 is $2.884414135676212 \times 10^{20}$. This essentially means that if 26 has a friend, then the smallest such friend is larger than $6.654166091831798 \times 10^{17}$. Similarly, if 38 is friendly, then its smallest friend must be larger than $2.884414135676212 \times 10^{20}$. Therefore, potential friends of 26 and those of 38 must be extremely large. This is so compared to known friendly numbers with exactly two distinct prime factors, where the smallest friends of these numbers are fairly small. For instance, the positive integer 6 is friendly where 28, which is fairly close to 6, is its smallest friend. Other examples include the positive integer 12 whose smallest friend is 234, the number 40 which is friendly to 224, and the number 80 which is friendly to 200.

In this thesis, it has also been established that any friend of a number of the form $n = 2p$ for a prime p greater than 3 and such that $4 \nmid p + 1$, is an odd square multiple of p. The numbers in this class include 10, 26, 34, 58, 74, and 82. So, 26 may be seen as a representative of this class. The study of all other numbers in this class will be similar to that of 26. From an observation in section 1.5, (Ryan, 2009), it is apparent that the potential friends of numbers in this class must have more than 2 distinct prime factors, where at least one of the prime factors is not a Fermat prime. On the other hand, if $4|p + 1$, then any friend of n is an odd non-square multiple of p. The numbers in this class include 14, 22, 38, 46, 62, 86, and 94. So, 38 may be seen as a representative of this class and the study all other numbers in the class will be similar to that of 38.

Now, by Corollary 1.3.1 and Theorem 1.3.8, any friend of a number of the form $n = 2p$ for a prime $p > 3$ is not divisible by a perfect number, in general.

Now, let $n = p^x q^y$ where p and q are distinct primes and $x, y \in \mathbb{N}$. Then a friend of n is of the form $m = p^u q^v d$, $(x, y) \neq (u, v)$, $d \in \mathbb{N}$ and $(d, pq) = 1$ where $x \neq u$ and $y \neq v$. In some cases u or v, but not both, may equal zero, (Ryan, 2009). Now, since $I(m) = I(n)$, then

$$I(m) = \frac{\sigma(m)}{m} = \frac{\sigma(p^u)\sigma(q^v)\sigma(d)}{p^u q^v d} = \frac{\sigma(p^x)\sigma(q^y)}{p^x q^y}$$

so that

$$I(d) = \frac{\sigma(d)}{d} = \frac{\sigma(p^x)\sigma(q^y)p^u q^v}{p^x q^y\, \sigma(p^u)\sigma(q^v)} \tag{4.2.1}$$

(Ryan, 2009)

In general, if $p = 2$ and $x = y = 1$ so that n is of the form $n = 2q$, then $u = 0$, as established in this thesis, since m must be odd. In this case,

$$I(d) = \frac{3(q+1)q^v}{2q\, \sigma(q^v)} \tag{4.2.2}$$

Equation (4.2.1) gives the formula for the abundancy index for the unknown factor of m for known values of u and v. Equation (4.2.2) is a specific case of Equation (4.2.1) when $p = 2$ and $x = y = 1$.

The properties of congruences may be incorporated in the determination of possible prime power factors of m corresponding to the values of p, q, x and y. W.L.O.G consider the case where $p = 2$, $x = y = 1$, so that $n = 2q$:

Theorem 4.2.1 Suppose m is a friend of n. Then, m has at least one prime factor $q_i \neq 3$ which appears in the prime factorization of m with an exponent α_i such that

(i) $\alpha_i \equiv 2 \; mod \; 3$ if $q_i \equiv 1 \; mod \; 3$ with $\alpha_i \equiv 2 \; mod \; 6$ if α_i is even.

(ii) $q_i \equiv 2 \; mod \; 3$ only if $4|q + 1$.

Proof: By definition, $I(n) = \frac{\sigma(n)}{n} = \frac{3(q+1)}{2q} = \frac{\sigma(m)}{m} = I(m)$. So, $3m(q + 1) = 2q\sigma(m)$ where $(3, 2q) = 1$. By properties of divisibility, $3|\sigma(m)$. This implies that m must have a prime factor q_i which appears in the prime factorization of m with an exponent α_i such that $3|\sigma(q_i^{\alpha_i})$, that is, $3|(1 + q_i + \cdots + q_i^{\alpha_i})$. Since $(3, \sigma(3^\gamma)) = 1$ for any $\gamma \in \mathbb{N}$, then $q_i \neq 3$.

(i) If $q_i \equiv 1 \; mod \; 3$, then by properties of congruences then $3|\sigma(q_i^{\alpha_i})$ only if $\alpha_i + 1 \equiv 0 \; mod \; 3$ meaning that $\alpha_i \equiv 2 \; mod \; 3$. If $\alpha_i = 2\beta_i$ for some $\beta_i \in \mathbb{N}$, then $2\beta_i + 1 \equiv 0 \; mod \; 3$. This implies that $\beta_i = 3t + 1$, $t = 0, 1, 2, 3,$ ….., and so $\alpha_i = 2\beta_i \equiv 2 \; mod \; 6$.

(ii) Similarly, if $q_i \equiv 2 \; mod \; 3$, then $3|\sigma(q_i^{\alpha_i})$ only if $\alpha_i + 1 \equiv 0 \; mod \; 2$ so that $\alpha_i \equiv 1 \; mod \; 2$, which is possible only if m is a non – square number, that is, only if $4|q + 1$. Consequently, if $4 \nmid q + 1$, then $q_i \not\equiv 2 \; mod \; 3$. □

4.3 Conclusion and Recommendation

The values of the greatest lower bounds for the potential friends of 26 and those of 38 appear to support the belief that the numbers 26 and 38 are solitary. In this thesis, the prime factorization was limited to the currently available prime factorization tools, the most powerful of which can only compute the prime factorization of numbers less than 12.5 billion, yet some of the numbers encountered in this study are larger than this. In addition to the properties of the abundancy index function, the concept of congruences outlined in section 4.2 could be used to further define prime powers that may be included or discriminated as possible factors of the potential friends of 26 and 38. This would help improve on the values of the greatest lower bounds computed and even help researchers get closer to the solution of the problem of determining the categorization of the many numbers whose classification is yet to be determined. Considering numbers whose classification is undetermined, and having exactly two distinct prime factors, some solutions may be arrived at as follows: consider a number of the form $n = p^x q^y$ where p and q are distinct primes and $x, y \in \mathbb{N}$, whose category is undetermined. The method used in this research can be exploited so as to determine all compulsory prime factors of the potential friends of n which emerge as a result of the choice of the possible powers of other compulsory prime factors of these friends. As previously described, a friend of n is of the form $m = p^u q^v d$, $(x, y) \neq (u, v)$, $d \in \mathbb{N}$ and $(d, pq) = 1$ where $x \neq u$ and $y \neq v$. Once a possible pair (u, v) has been determined, then the emerging primes are prime factors of d, corresponding to (u, v). Having determined all possible compulsory prime power factors of d corresponding to (u, v), Equation (4.2.1) can then be used to determine the possibility of the existence of such a d, for the given choice of (u, v). If it does, then m is a friend of n corresponding to u and v. Otherwise, a friend of n doesn't exist for that choice of (u, v). The process should be repeated for all other possible choices of (u, v) to determine if there is a possibility of the existence of such a d. If it does, then a friend of n will have been found. Otherwise, it will be evident that n is solitary. Improvement of the computer program developed in this study will be necessary towards this end. $\square$

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APPENDIX I

Some trivial theorems:

Theorem 1.1.1 Suppose a and b are integers, not both zero. Then (a,b) is the least positive integer that is a linear combination of a and b, (Rosen, 1993). □

Theorem 1.3.2 The abundancy index of a positive integer n is also the sum of the reciprocals of the divisors of n, (Weiner, 2000). □

Theorem 1.3.3 If $k > 1$ and $n > 1$, then $I(kn) > max\,\{I(k), I(n)\}$.

Proof: If $1, a_1, a_2, . . . , a_t, n$ are all the divisors of n, then $1, k, ka_1, ka_2, . . . , ka_t, kn$ is a list of some and not necessarily all the divisors of kn. Thus,

$$I(kn) \geq \frac{1+k+ka_1+ka_2+. . .+kn}{kn}$$

$$= \frac{1}{kn} + \frac{1+a_1+a_2+. . .+n}{n}$$

$$= \frac{1}{kn} + I(n) > I(n)$$

A similar argument would hold if $1, b_1, b_2, . . . , b_s, k$ are all the divisors of k so that $1, n, nb_1, nb_2, . . . , nb_s, nk$ are some and not necessarily all the divisors of nk and $I(nk) > I(k)$.

Now, since $I(kn) > I(n)$ and $I(kn) = I(nk) > I(k)$ then $I(kn) > max\,(I(k), I(n))$, (Laatsch, 1986). □

Theorem 1.3.4 The abundancy index takes on arbitrarily large values, (Weiner, 2000).□

Theorem 1.3.5 The abundancy index function I is not bounded above, (Laatsch, 1986).□

Theorem 1.3.11 If m is an even integer but not a power of 2, then m^n is abundant for some n, (Laatsch, 1986). □

APPENDIX II

Computer Program:

```
namespace ProjectMaker
{
    partial class AbundanceCalculator
    {
        private System.ComponentModel.IContainer components = null;
        protected override void Dispose(bool disposing)
        {
            if (disposing && (components != null))
            {
                components.Dispose();
            }
            base.Dispose(disposing);
        }
        #region Windows Form Designer generated code
        private void InitializeComponent()
        {
            this.txtNumber = new System.Windows.Forms.TextBox();
            this.btnTest = new System.Windows.Forms.Button();
            this.label1 = new System.Windows.Forms.Label();
```

```
this.label2 = new System.Windows.Forms.Label();
this.menuStripMain = new System.Windows.Forms.MenuStrip();
this.fileToolStripMenuItem = new
System.Windows.Forms.ToolStripMenuItem();
this.exitToolStripMenuItem = new
System.Windows.Forms.ToolStripMenuItem();
this.helpToolStripMenuItem = new
System.Windows.Forms.ToolStripMenuItem();
this.aboutToolStripMenuItem = new
System.Windows.Forms.ToolStripMenuItem();
this.statusStripMain = new System.Windows.Forms.StatusStrip();
this.toolStripStatus = new System.Windows.Forms.ToolStripStatusLabel();
this.toolStripStatusLabel1 = new
System.Windows.Forms.ToolStripStatusLabel();
this.groupBox1 = new System.Windows.Forms.GroupBox();
this.lblStatus = new System.Windows.Forms.Label();
this.progressBarTest = new System.Windows.Forms.ProgressBar();
this.groupBox2 = new System.Windows.Forms.GroupBox();
this.label5 = new System.Windows.Forms.Label();
this.label4 = new System.Windows.Forms.Label();
this.label3 = new System.Windows.Forms.Label();
this.label8 = new System.Windows.Forms.Label();
```

```
this.label7 = new System.Windows.Forms.Label();
this.label6 = new System.Windows.Forms.Label();
this.lblRDecomposition = new System.Windows.Forms.Label();
this.label12 = new System.Windows.Forms.Label();
this.menuStripMain.SuspendLayout();
this.statusStripMain.SuspendLayout();
this.groupBox1.SuspendLayout();
this.groupBox2.SuspendLayout();
this.SuspendLayout();
// txtNumber
this.txtNumber.Location = new System.Drawing.Point(22, 43);
this.txtNumber.Name = "txtNumber";
this.txtNumber.Size = new System.Drawing.Size(189, 20);
this.txtNumber.TabIndex = 0;
// btnTest
this.btnTest.Location = new System.Drawing.Point(369, 43);
this.btnTest.Name = "btnTest";
this.btnTest.Size = new System.Drawing.Size(75, 23);
this.btnTest.TabIndex = 2;
this.btnTest.Text = "Test";
this.btnTest.UseVisualStyleBackColor = true;
this.btnTest.Click += new System.EventHandler(this.btnTest_Click);
```

```
// label1
this.label1.AutoSize = true;
this.label1.Location = new System.Drawing.Point(269, 42);
this.label1.Name = "label1";
this.label1.Size = new System.Drawing.Size(10, 13);
this.label1.TabIndex = 9;
this.label1.Text = "-";
// label2
this.label2.AutoSize = true;
this.label2.Location = new System.Drawing.Point(269,73);
this.label2.Name = "label2";
this.label2.Size = new System.Drawing.Size(10, 13);
this.label2.TabIndex = 11;
this.label2.Text = "-";
// menuStripMain
this.menuStripMain.Items.AddRange(new
System.Windows.Forms.ToolStripItem[] {
this.fileToolStripMenuItem,
this.helpToolStripMenuItem});
this.menuStripMain.Location = new System.Drawing.Point(0, 0);
this.menuStripMain.Name = "menuStripMain";
this.menuStripMain.Size = new System.Drawing.Size(890, 24);
```

```
this.menuStripMain.TabIndex = 3;
this.menuStripMain.Text = "menuStrip1";
// fileToolStripMenuItem
this.fileToolStripMenuItem.DropDownItems.AddRange(new
System.Windows.Forms.ToolStripItem[] {
this.exitToolStripMenuItem});
this.fileToolStripMenuItem.Name = "fileToolStripMenuItem";
this.fileToolStripMenuItem.Size = new System.Drawing.Size(37, 20);
this.fileToolStripMenuItem.Text = "&File";
// exitToolStripMenuItem
this.exitToolStripMenuItem.Name = "exitToolStripMenuItem";
this.exitToolStripMenuItem.Size = new System.Drawing.Size(92, 22);
this.exitToolStripMenuItem.Text = "E&xit";
this.exitToolStripMenuItem.Click += new
System.EventHandler(this.exitToolStripMenuItem_Click);
// helpToolStripMenuItem
this.helpToolStripMenuItem.DropDownItems.AddRange(new
System.Windows.Forms.ToolStripItem[] {
this.aboutToolStripMenuItem});
this.helpToolStripMenuItem.Name = "helpToolStripMenuItem";
this.helpToolStripMenuItem.Size = new System.Drawing.Size(44, 20);
this.helpToolStripMenuItem.Text = "&Help";
```

```
// aboutToolStripMenuItem
this.aboutToolStripMenuItem.Name = "aboutToolStripMenuItem";
this.aboutToolStripMenuItem.Size = new System.Drawing.Size(116, 22);
this.aboutToolStripMenuItem.Text = "&About...";
this.aboutToolStripMenuItem.Click += new
System.EventHandler(this.aboutToolStripMenuItem_Click);
// statusStripMain
this.statusStripMain.Items.AddRange(new
System.Windows.Forms.ToolStripItem[] {
this.toolStripStatus,
this.toolStripStatusLabel1});
this.statusStripMain.Location = new System.Drawing.Point(0, 626);
this.statusStripMain.Name = "statusStripMain";
this.statusStripMain.Size = new System.Drawing.Size(890, 22);
this.statusStripMain.TabIndex = 4;
this.statusStripMain.Text = "statusStrip1";
// toolStripStatus
this.toolStripStatus.Name = "toolStripStatus";
this.toolStripStatus.Size = new System.Drawing.Size(39, 17);
this.toolStripStatus.Text = "Ready";
// toolStripStatusLabel1
this.toolStripStatusLabel1.Name = "toolStripStatusLabel1";
```

```
this.toolStripStatusLabel1.Size = new System.Drawing.Size(0, 17);
// groupBox1
this.groupBox1.Controls.Add(this.lblStatus);
this.groupBox1.Controls.Add(this.progressBarTest);
this.groupBox1.Controls.Add(this.txtNumber);
this.groupBox1.Controls.Add(this.btnTest);
this.groupBox1.Location = new System.Drawing.Point(27, 59);
this.groupBox1.Name = "groupBox1";
this.groupBox1.Size = new System.Drawing.Size(484, 116);
this.groupBox1.TabIndex = 4;
this.groupBox1.TabStop = false;
this.groupBox1.Text = "Enter number to test";
// lblStatus
this.lblStatus.AutoSize = true;
this.lblStatus.Location = new System.Drawing.Point(19, 92);
this.lblStatus.Name = "lblStatus";
this.lblStatus.Size = new System.Drawing.Size(38, 13);
this.lblStatus.TabIndex = 7;
this.lblStatus.Text = "Ready";
// progressBarTest
this.progressBarTest.Location = new System.Drawing.Point(22, 79);
this.progressBarTest.Name = "progressBarTest";
```

```
this.progressBarTest.Size = new System.Drawing.Size(418, 10);
this.progressBarTest.Step = 24;
this.progressBarTest.TabIndex = 6;
// groupBox2
this.groupBox2.Controls.Add(this.label5);
this.groupBox2.Controls.Add(this.label4);
this.groupBox2.Controls.Add(this.label3);
this.groupBox2.Controls.Add(this.label1);
this.groupBox2.Controls.Add(this.label8);
this.groupBox2.Controls.Add(this.label7);
this.groupBox2.Controls.Add(this.label2);
this.groupBox2.Location = new System.Drawing.Point(31, 181);
this.groupBox2.Name = "groupBox2";
this.groupBox2.Size = new System.Drawing.Size(480, 184);
this.groupBox2.TabIndex = 5;
this.groupBox2.TabStop = false;
this.groupBox2.Text = "Analysis";
// label5
this.label5.AutoSize = true;
this.label5.Font = new System.Drawing.Font("Microsoft Sans Serif", 9F,
System.Drawing.FontStyle.Bold, System.Drawing.GraphicsUnit.Point,
((byte)(0)));
```

```
this.label5.Location = new System.Drawing.Point(62, 120);
this.label5.Name = "label5";
this.label5.Size = new System.Drawing.Size(113, 15);
this.label5.TabIndex = 13;
this.label5.Text = "Abundace Index:";
// label4
this.label4.AutoSize = true;
this.label4.Font = new System.Drawing.Font("Microsoft Sans Serif", 9F,
System.Drawing.FontStyle.Bold, System.Drawing.GraphicsUnit.Point,
((byte)(0)));
this.label4.Location = new System.Drawing.Point(62, 71);
this.label4.Name = "label4";
this.label4.Size = new System.Drawing.Size(149, 15);
this.label4.TabIndex = 10;
this.label4.Text = "Prime Decomposition:";
// label3
this.label3.AutoSize = true;
this.label3.Font = new System.Drawing.Font("Microsoft Sans Serif", 9F,
System.Drawing.FontStyle.Bold, System.Drawing.GraphicsUnit.Point,
((byte)(0)));
this.label3.Location = new System.Drawing.Point(62, 40);
this.label3.Name = "label3";
```

```
this.label3.Size = new System.Drawing.Size(64, 15);
this.label3.TabIndex = 8;
this.label3.Text = "Is Prime:";
// label8
this.label8.AutoSize = true;
this.label8.Location = new System.Drawing.Point(269, 122);
this.label8.Name = "label8";
this.label8.Size = new System.Drawing.Size(10, 13);
this.label8.TabIndex = 14;
this.label8.Text = "-";
// label7
this.label7.AutoSize = true;
this.label7.Location = new System.Drawing.Point(269, 95);
this.label7.Name = "label7";
this.label7.Size = new System.Drawing.Size(10, 13);
this.label7.TabIndex = 12;
this.label7.Text = "-";
// AbundanceCalculator
this.AcceptButton = this.btnTest;
this.AutoScaleDimensions = new System.Drawing.SizeF(6F, 13F);
this.AutoScaleMode = System.Windows.Forms.AutoScaleMode.Font;
this.ClientSize = new System.Drawing.Size(890, 648);
```

```
    this.Controls.Add(this.groupBox2);
    this.Controls.Add(this.groupBox1);
    this.Controls.Add(this.statusStripMain);
    this.Controls.Add(this.menuStripMain);
    this.MainMenuStrip = this.menuStripMain;
    this.MaximizeBox = false;
    this.Name = "AbundanceCalculator";
    this.Text = "Math Tool";
    this.menuStripMain.ResumeLayout(false);
    this.menuStripMain.PerformLayout();
    this.statusStripMain.ResumeLayout(false);
    this.statusStripMain.PerformLayout();
    this.groupBox1.ResumeLayout(false);
    this.groupBox1.PerformLayout();
    this.groupBox2.ResumeLayout(false);
    this.groupBox2.PerformLayout();
    this.ResumeLayout(false);
    this.PerformLayout();
}
#endregion
private System.Windows.Forms.TextBox txtNumber;
private System.Windows.Forms.Button btnTest;
```

```
private System.Windows.Forms.Label label1;
private System.Windows.Forms.Label label2;
private System.Windows.Forms.MenuStrip menuStripMain;
private System.Windows.Forms.StatusStrip statusStripMain;
private System.Windows.Forms.ToolStripStatusLabel toolStripStatus;
private System.Windows.Forms.ToolStripStatusLabel toolStripStatusLabel1;
private System.Windows.Forms.ToolStripMenuItem fileToolStripMenuItem;
private System.Windows.Forms.ToolStripMenuItem exitToolStripMenuItem;
private System.Windows.Forms.ToolStripMenuItem helpToolStripMenuItem;
private System.Windows.Forms.ToolStripMenuItem aboutToolStripMenuItem;
private System.Windows.Forms.GroupBox groupBox1;
private System.Windows.Forms.GroupBox groupBox2;
private System.Windows.Forms.Label label5;
private System.Windows.Forms.Label label4;
private System.Windows.Forms.Label label3;
private System.Windows.Forms.ProgressBar progressBarTest;
private System.Windows.Forms.Label lblStatus;
private System.Windows.Forms.Label label7;
private System.Windows.Forms.Label label8;
private System.Windows.Forms.Label lblRDecomposition;
private System.Windows.Forms.Label label12;
private System.Windows.Forms.Label label6;
```

```
    }
}
using System;
using System.Collections.Generic;
using System.ComponentModel;
using System.Data;
using System.Drawing;
using System.Linq;
using System.Text;
using System.Windows.Forms;
using System.Collections;
using Mehroz;
namespace ProjectMaker
{
    public partial class AbundanceCalculator : Form
    {
        private Abundance ab = new Abundance();
        private int r = 0;
        public AbundanceCalculator()
        {
            InitializeComponent();
            txtNumber.SelectionStart = 0;
```

```
}
private void btnTest_Click(object sender, EventArgs e)
{
    // progressBarTest.Step
    // label6.Text = ((Convert.ToInt32(txtNumber.Text))/2).ToString();
    try
    {
        if (Convert.ToInt32(txtNumber.Text.ToString()) > 0)
        {
            groupBox2.Text = "Analysis for " + txtNumber.Text;
            progressBarTest.Value = 0;
            lblStatus.Text = "Working... Please wait";
            progressBarTest.PerformStep();
            PrimeClass p = new PrimeClass();
            progressBarTest.PerformStep();
            lblStatus.Text = "Testing if prime... Please wait";
            p.Number = Convert.ToInt32(txtNumber.Text.ToString());
            progressBarTest.PerformStep();
            if (!txtNumber.Text.Equals("1"))
            {
                label1.Text = p.IsPrime.ToString();
                lblStatus.Text = "Decomposing number into prime... Please wait";
```

```
progressBarTest.PerformStep();
if (p.IsPrime && Convert.ToInt32(txtNumber.Text.ToString()) != 1)
{
    label1.ForeColor = Color.Red;
    label2.ForeColor = Color.Red;
    label2.Text = p.Number.ToString();
    progressBarTest.PerformStep();
    lblStatus.Text = "Done!!";
    label7.Text = "";
    label8.Text = (Convert.ToInt32(txtNumber.Text.ToString().Trim()) +
    1).ToString() + "/" + txtNumber.Text.ToString().Trim();
}
else
{
    label1.ForeColor = Color.Black;
    label2.ForeColor = Color.Black;
    label2.Text = p.StringDecomposition.ToString().Remove(0, 2) + " or";
    progressBarTest.PerformStep();
    lblStatus.Text = "Done!!";
    label7.Text =  p.FormattedDecomposition.ToString().Remove(0, 2);
    ArrayList bases = new ArrayList();
    ArrayList powers = new ArrayList();
```

```
            bases = p.PrimeBases;
            powers = p.PrimePower;
            label8.Text = ab.GiveAbundance(ab.CalculateNumerator(bases,
            powers).ToString(), txtNumber.Text.Trim());
          }
        }
        else
        {
          label1.Text = "false";
          label8.Text = "1";
        }
      }
      else
      {
        MessageBox.Show("Please enter a positive integer number greater than
        1.");
      }
    }
    catch (Exception)
    {
      MessageBox.Show("Please enter a positive integer to test.");
    }
```

```
        }
        private void exitToolStripMenuItem_Click(object sender, EventArgs e)
        {
            this.Close();
        }
        private void aboutToolStripMenuItem_Click(object sender, EventArgs e)
        {
            AboutBox ab = new AboutBox();
            ab.Show();
        }
        private void btnClose_Click(object sender, EventArgs e)
        {
            this.Close();
        }
    }
}
using System;
using System.Collections.Generic;
using System.Linq;
using System.Text;
using System.Collections;
namespace ProjectMaker
```

```
{
    public class PrimeClass
    {
        private int number = 0;
        private bool isPrime = true;
        private string stringDecomposition = "", status = "Workin...",
        unFormattedDecomposition = "";
        private ArrayList primesDecomposable = new ArrayList();
        private ArrayList primePower = new ArrayList();
        private Abundance abandance = new Abundance();
        private ArrayList bases = new ArrayList();
        private ArrayList primeBases = new ArrayList();
        private double numerator;
        public string FormattedDecomposition
        {
            get
            {
                //UnFormatDecomposition(unFormattedDecomposition);
                return unFormattedDecomposition;
            }
            set
            {
```

```
            unFormattedDecomposition = value;
        }
    }
    public string StringDecomposition
    {
        get
        {
            DecomposeNum(number);
            return stringDecomposition;
        }
        set
        {
            stringDecomposition = value;
        }
    }
    public bool IsPrime
    {
        get
        {
            NumIsPrime(number);
            return isPrime;
        }
```

```
    set
    {
        isPrime = value;
    }
}
public int Number
{
    get
    {
        return number;
    }
    set
    {
        number = value;
    }
}
public ArrayList PrimePower
{
    get
    {
        return primePower;
    }
```

```
  set
  {
    primePower = value;
  }
}
public ArrayList PrimeBases
{
  get
  {
    return primeBases;
  }
  set
  {
    primeBases = value;
  }
}
public ArrayList Bases
{
  get
  {
    return bases;
  }
```

```
    set
    {
        bases = value;
    }
}
public double Numerator
{
    get
    {
        return numerator;
    }
    set
    {
        numerator = value;
    }
}
private bool NumIsPrime(int num)
{
    isPrime = true;
    ArrayList numbersBeforeNum = new ArrayList();
    int temp = num;
    while (temp > 1)
```

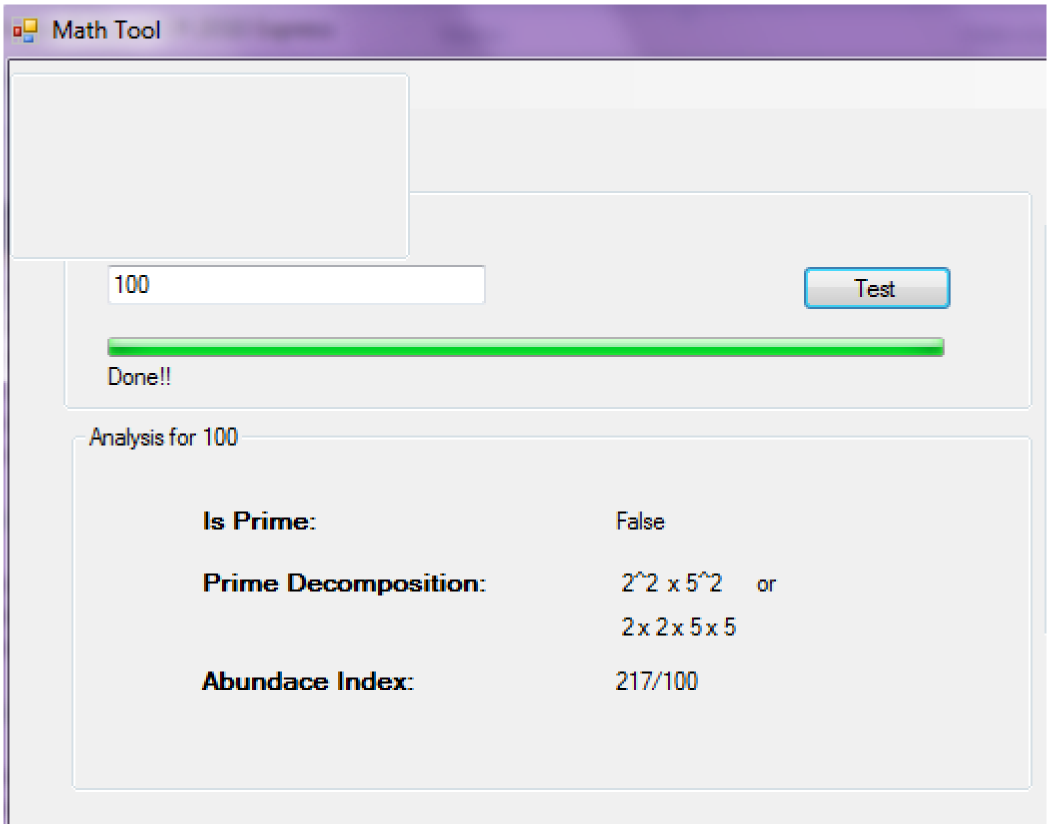
```
    {
        temp--;
        numbersBeforeNum.Add(temp);
    }
    numbersBeforeNum.Remove(1);
    for (int i = 0; i < numbersBeforeNum.Count; i++)
    {
        if (num % (Convert.ToInt32(numbersBeforeNum[i].ToString())) == 0)
        {
            isPrime = false;
        }
    }
    return isPrime;
}
private string DecomposeNum(int num)
{
    string decomposition = "";
    int temp = num;
    ArrayList numbersBeforeNum = new ArrayList();
    ArrayList primesBeforeNum = new ArrayList();
    while (temp > 1)
    {
```

```
            temp--;
            numbersBeforeNum.Add(temp);
        }
        for (int i = 0; i < numbersBeforeNum.Count; i++)
        {
            //if(NumIsPrime(Convert.ToInt32(numbersBeforeNum[i].ToString())))
            if (NumIsPrime(Convert.ToInt32(numbersBeforeNum[i].ToString())))
            {
                primesBeforeNum.Add(numbersBeforeNum[i]);
            }
        }
        primesBeforeNum.Remove(1);
        ArrayList primeDecomposers = new ArrayList();
        for (int i = 0; i < primesBeforeNum.Count; i++)
        {
            if (num % (Convert.ToInt32(primesBeforeNum[i].ToString())) == 0)
            {
                primeDecomposers.Add(primesBeforeNum[i]);
            }
        }
    //  primeDecomposers.Reverse();
        this.primesDecomposable = primeDecomposers;
```

```
for (int i = 0; i < primeDecomposers.Count; i++)
{
    //decomposition = primeDecomposers[i].ToString() + " x " + decomposition;
}
string unFormatedDecomposition = "";
for (int i = 0; i < primeDecomposers.Count; i++)
{
    int count = 0;
    while (num % Convert.ToInt32(primeDecomposers[i].ToString()) == 0)
    {
        count++;
        num = (num / (Convert.ToInt32(primeDecomposers[i].ToString())));
        unFormatedDecomposition = " x " + primeDecomposers[i].ToString() +
        unFormatedDecomposition;
    }
    primePower.Add(count);
    decomposition = " x " + primeDecomposers[i].ToString() +"^" +
    count.ToString() + "  " + decomposition;
}
abandance.PrimePowers = primePower;
UnFormatDecomposition(unFormatedDecomposition);
this.bases = primeDecomposers;
```

```
            this.PrimeBases = primeDecomposers;
            abandance.PrimeBases = primeDecomposers;
            this.stringDecomposition = decomposition;
            this.numerator = abandance.Numerator;
            return decomposition;
        }
        private string UnFormatDecomposition(string uDecomposition)
        {
            this.unFormattedDecomposition = uDecomposition;
            return uDecomposition;
        }
    }
}
```

Computation of the prime factorization and abundancy index of 100



Computation of the prime factorization and abundancy index of 500