RANKS, SUBDEGREES AND SUBORBITAL GRAPHS OF THE SYMMETRIC GROUP S_n ACTING ON UNORDERED r-ELEMENT SUBSETS

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DECLARATION

This thesis is my original work and has not been presented for a degree in any other university

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DEDICATION

To the late Prof. C. Mwathi and my late dad Jacob Nyaga.

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SYMBOLS AND ABBREVIATIONS

- S_n - Symmetric group of degree n and order $n!$
- $X^{(r)}$ - Set of all unordered r-element subsets from $X = \{1, 2, 3, ..., n\}$
- $\text{Stab}_G(x)$ or G_x - Stabilizer of a point x in X
- $G_{\{1,2,3,...,r\}}$ - Stabilizer of $\{1, 2, 3, ..., r\}$
- $|G|$ - Order of a group G
- $|\text{Fix}(g)|$ - Number of elements fixed by $g \in G$
- $\{1, 2, 3, ..., r\}$ - Unordered r-element subset
- Δ - Suborbit of G on X
- Δ^* - The G-suborbit paired with Δ
- O - The suborbital of G on $X \times X$
- Γ - The suborbital graph corresponding to the suborbit Δ
- $\square$ - End of proof
- $\begin{pmatrix} n \\ r \end{pmatrix}$ or ${}_nC_r$ - n combination r
- (u, v) - The greatest common divisor of u and v
- $\hat{\mathbb{Q}}$ - $\mathbb{Q} \cup \{\infty\}$
- PGL (n, q) - The projective general linear group
- PSL (n, q) - The projective special linear group

- $\Gamma_0(N)$ - A subgroup of the inhomogeneneous group $\mathrm{PSL}(2,\mathbb{Z})$ acting on the half plane $H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ via $A(z) = \frac{az+b}{cz+d}, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z})$

- ${}^2E_6(q^2)$ - Steinberg groups from the order 2 automorphism of E_6 acting on a finite field of order q^2

- $GF(q)$ - The Galois field of q elements

- $B_l(q), C_l(q),$ and $D_l(q)$ - Some types of groups of Lie types over K, where $K = GF(q)$

- $rk(D_l)$ - Ranks of the permutation representations of the groups $D_l(q)$ for $1 \leq k \leq l$

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ABSTRACT

The main aim of this research is to determine the ranks, subdegrees, and the suborbital graphs of the symmetric group S_n acting on unordered r-element subsets of $X = \{1, 2, 3, ..., n\}$. These areas have not received much attention, in fact most of the research has been focused on the action of S_n on unordered pairs. In 1970, Higman proved that S_n, $n \geq 4$ acts as a rank 3 group on $X^{(2)}$, with subdegrees 1, $2(n-2)$, $\begin{pmatrix} n-2 \\ 2 \end{pmatrix}$. In this study, it is shown that S_n acts transitively and primitively on $X^{(r)}$(r-element subsets of X). The ranks and suborbits of S_n acting on $X^{(4)}$ and $X^{(5)}$ are determined, after which it is proved that the rank of S_n acting on $X^{(r)}$ is $r+1$ if $n \geq 2r$. The suborbits of S_n acting on $X^{(r)}$ are all self paired is shown. It is also proved that the subdegrees of S_n acting on $X^{(r)}$ are $1, r\begin{pmatrix} n-r \\ r-1 \end{pmatrix}, \begin{pmatrix} r \\ 2 \end{pmatrix}\begin{pmatrix} n-r \\ r-2 \end{pmatrix}, \begin{pmatrix} r \\ 3 \end{pmatrix}\begin{pmatrix} n-r \\ r-3 \end{pmatrix}, ..., \begin{pmatrix} r \\ r-1 \end{pmatrix}\begin{pmatrix} n-r \\ 1 \end{pmatrix}, \begin{pmatrix} n-r \\ r \end{pmatrix}$, after which the subdegrees are arranged in an ascending order. The suborbital graphs corresponding to the suborbits of S_n are then constructed and their properties analysed . It is shown that when S_n acts on $X^{(r)}$, its suborbital graphs are undirected and have girth three if $n \geq 3r$.

CHAPTER 1

INTRODUCTION

Some key and basic concepts of permutation groups and graph theory are defined in this chapter. Some known results and a review of literature on what has been done so far in our field of study are also given.

1.1 Basic Concepts and Preliminary Results

1.1.1 Permutation Groups

Let $X = \{1, 2, ..., n\}$. A **permutation** of X is a one-to-one mapping of X onto itself. The **symmetric group** of degree n is the group of all permutations of X under the binary operation of composition of maps. It is denoted by S_n and is of order $n!$.

1.1.2 Group Actions

Two subgroups H and K of a group G are said to be **conjugate** if $H = gKg^{-1}$ for some $g \in G$.

Let X be a nonempty set and G be a group. We say that G **acts on the left** of X if for each $x \in X$ and $g \in G$ there corresponds a unique element $gx \in X$

such that, for all $x \in X$ and $g_1, g_2 \in G$

(a) $(g_1 g_2)x = g_1(g_2)x$

(b) $1.x = x$, where 1 is the identity in G.

The action of G from the right can be written in a similar way.

Let G act on X. Then X is partitioned into disjoint equivalence classes (with respect to an equivalence relation) called **orbits or transitivity classes** of the action. For each $x \in X$, the orbit containing x is denoted by $Orb_G(x)$. Thus,

$$Orb_G(x) = \{gx | g \in G\}$$

The action of a group G on the set X is said to be **transitive** if for each pair of points $x, y \in X$, there exists $g \in G$ such that $gx = y$; in other words, if the action has only one orbit.

A permutation group G acting on a set X is **k-homogeneous** if it is transitive on unordered k-subsets of X. Let G act on a set X and let $x \in X$.

The **stabilizer** of x in G, denoted by $Stab_G(x)$ or G_x is given by

$$Stab_G(x) = \{g \in G | gx = x\}$$

Suppose that G acts transitively on X. For each subset Y of X and each $g \in G$, let $gY = \{gy | y \in Y\} \subseteq X$. A subset Y of X is said to be a **block** for the action if, for each $g \in G$, either $gY = Y$ or $gY \cap Y = \emptyset$. In particular, $\emptyset, X$ and all 1-element subsets of X are obviously blocks. These are called the trivial blocks. If these are the only blocks, then we say that G acts **primitively** on X. Otherwise, G acts **imprimitively**.

Let G be transitive on X and let G_x be the stabilizer of a point $x \in X$. Suppose $\Delta_0 = \{x\}, \Delta_1, \Delta_2, ..., \Delta_{r-1}$ are the orbits of G_x on X, the **rank** of G is then r. The sizes $n_i = |\Delta_i|, (i = 1, 2, ..., r-1)$, are known as the **subdegrees** of G. The orbits of G_x are also called the suborbits of G.

Theorem 1.1.1 (Orbit-Stabilizer Theorem-Rose, 1978, p.72)

Let G be a group acting on a finite set X and $x \in X$. Then

$$|Orb_G(x)| = |G : Stab_G(x)| \tag{1.1}$$

Let G act on a set X. The set of elements of X fixed by $g \in G$ is called the fixed point set of g denoted by $Fix(g)$. Thus

$$Fix(g) = \{x \in X | gx = x\}$$

If a finite group G acts on a set X with n elements, each $g \in G$ corresponds to a permutation σ of X, which can be written uniquely as a product of disjoint cycles. If σ has α_1 cycles of length 1, α_2 cycles of of length 2,..., α_n cycles of length n, we say that σ and hence g has **cycle type** $(\alpha_1, \alpha_2, ..., \alpha_n)$.

Theorem 1.1.2 (Krishnamurthy, 1985, p.68)

Two permutations in S_n are conjugate if and only if they have the same cycle type; and if $g \in G$ has cycle type $(\alpha_1, \alpha_2, ..., \alpha_n)$, then the number of permutations in S_n conjugate to g is

$$\frac{n!}{\Pi_{i=1}^{n}\alpha_i! i^{\alpha_i}} \tag{1.2}$$

Theorem 1.1.3 (Harary, 1969, p.98)

Let G be a finite group acting on a set X. The number of orbits of G is

$$\frac{1}{|G|}\sum_{g\in G}|Fix(g)| \tag{1.3}$$

This Theorem is referred to as Cauchy-Frobenius Lemma.

Let Δ be an orbit of G_x on X. Define $\Delta^* = \{gx|g \in G, x \in g\Delta\}$, then Δ^* is also an orbit of G_x and is called the G_x-orbit paired with Δ. If $\Delta^* = \Delta$, then Δ is called a self-paired orbit of G_x.

Theorem 1.1.4 (Cameron, 1975 p.422)

If G is primitive, with subdegrees $1 = n_0, n_1, ..., n_{r-1}$ (in increasing order), then $n_1 n_{i-1} \geq n_i$ for $i = 1, ..., r-1$.

1.1.3 Linear Groups

The **Galois field** with q elements, $GF(q)$ is a finite field where q is a power of a prime p.

Given a field K, the **general linear group** $GL(n, K)$ is the group of $n \times n$ invertible matrices with elements in K.

The **projective general linear group** $PGL(n, K)$ is the group obtained from the general linear group $GL(n, K)$ on factoring by the scalar matrices contained in that group.

Given a field K, the **special linear group** $SL(n, K)$ is the group of $n \times n$ matrices with elements in K and determinant 1.

The **projective special linear group** $PSL(n, K)$ is the group obtained from the special linear group $SL(n, K)$ on factoring by the scalar matrices contained in that group.

A **group of Lie type** $G(K)$ is a (not necessarily finite) group of rational points of a reductive linear algebraic group G with values in the field K.

The **Steinberg group** of a ring A is the universal central extension of the commutator subgroup of the stable general linear group.

A subgroup B of a group G is called **Borel** if it is a maximal solvable connected algebraic subgroup of G.

A subgroup P of a group G is called a **parabolic subgroup** if it properly contains a Borel subgroup B of G.

1.1.4 Basic Concepts in Graph Theory

A **(simple) graph** is an ordered pair $H = (V, E)$, where V is a finite, nonempty set of objects called vertices, and E is a (possibly empty) set of 2-subsets of V called edges. The set V is called the **vertex** set of H, and E is called the **edge set** of G. If $e = \{u, v\} \in E(H)$, we say that vertices u and v are **adjacent** in H, and that e joins or connects u and v. The edge e is said to be **incident** with u (and v), and vice versa. The following important facts arise from carefully considering what the definition of graph says.

- E is a set. Therefore two vertices are either adjacent, or not adjacent, period. There can be at most one edge joining any two vertices.
- The elements of E are subsets of V of size 2. Therefore no vertex can be adjacent to itself. Edges join pairs of distinct vertices.

There is no requirement that the edge set be non-empty. Therefore the minimum number of edges a graph can have is zero. If the graph has n vertices, then the maximum number of edges it can have equals the number of two element subsets of V, which is $\binom{n}{2}$. A graph with n vertices has $\binom{n}{2}$ edges if every pair of distinct vertices is an edge. Such a graph is called a **complete graph** on n vertices. We represent graphs by pictures in the plane by associating a point with each vertex and joining points corresponding to adjacent vertices by a (possibly curved) line segment. How the vertices and edges are drawn is unimportant, the same graph can have many pictures. What is important is what the vertices are (i.e., V), and which pairs of vertices are adjacent (i.e., E).

Two graphs are equal if they have the same vertex set and the same edge set. But there are other ways in which two graphs could be regarded the same. For example, one could regard two graphs as being the same if it is possible to rename the vertices of one and obtain the other. If this happens we call the graphs **isomorphic**. (Formally, two graphs J and H are isomorphic if there is a 1-1 correspondence $f : V(J) \to V(H)$ such that $\{x, y\} \in E(J) \Leftrightarrow \{f(x), f(y)\} \in E(H)$.) The relation $\mathfrak{R}$ on the set of all graphs defined by $J\mathfrak{R}H$ if and only if J

and H are isomorphic (i.e., the vertices of J can be renamed so as to obtain H) is an equivalence relation, and the equivalence classes are collections of graphs which are the same in this sense.

The **degree** of a vertex x of a simple graph H is the number of edges that contain x. We use $deg(x)$ to denote the degree of the vertex x. If H is a graph with n vertices, then for any vertex x, $0 \leq deg(x) \leq n-1$. For any graph H, the sum of the degrees of the vertices equals twice the number of edges (i.e.,$\sum_{x \in V} deg(x) = 2|E|$). Notice that this says the sum of the vertex degrees is an even number. The minimum degree of H denoted by $\delta(H)$, is the smallest number of edges incident with a point of H while the maximum degree of H, denoted by $\Delta(H)$, is the largest such number. If $\delta(H) = \Delta(H) = r$, G is called **regular** of degree r.

A **walk** in a simple graph H is a sequence $v_0 v_1 ... v_k$ of vertices such that consecutive vertices in the sequence are adjacent (i.e., $(v_{i-1}, v_i) \in E$ for $i = 1, 2, ..., k$). The integer k is called the **length** of the walk. Thinking of the picture of the graph, it is the number of edges that would be traversed of you started at v_0 and travelled to v_1 along $\{v_0, v_1\}$, then to v_2 along $\{v_1, v_2\}$ and so on until v_n is reached. Observe that a walk is any sequence of consecutive adjacent vertices. It may or may not end where it starts, and may contain the same vertex many times. Also notice that the sequence consisting of a single vertex is a walk (of length zero).

A **path** in a simple graph H is a walk in H that contains no repeated vertices. Notice that every path is a walk, but the converse is false. Also, since every path is a walk, it has a length (as before).

A graph H is called **connected** if for all pairs of vertices u and v there is a walk that starts at u and ends at vertex v; otherwise H is disconnected. A walk in a graph H is called **closed** if its first and last vertex are the same. Since a closed walk is a walk, it has a length as above. Also, notice that a closed walk may or may not contain repeated vertices other than the first and last (which are the same). A closed walk of length at least three in which all vertices are distinct except the first and last is called a **cycle**. The length of the shortest cycle (if any) in H is called the **girth** of H. Every cycle is a closed walk, but not every closed walk is a cycle.

A **tree** is a connected graph that contains no cycles.

A **leaf** of a tree is a vertex of degree one.

1.1.5 Suborbital Graphs

Suppose G acts on X, then G acts on $X \times X$ by $g(x, y) = (gx, gy), g \in G,\ x, y \in X$. If $O \subseteq X \times X$ is a G-orbit, then for a fixed $x \in X$, $\Delta = \{y \in X | (x, y) \in O\}$ is a G_x-orbit. Conversely, if $\Delta \subseteq X$ is a G_x-orbit, then $O = \{(gx, gy) | g \in G, y \in \Delta\}$ is a G-orbit on $X \times X$. We say Δ corresponds to O. The G-orbits on $X \times X$ are called suborbitals.

Let $O_i \subseteq X \times X, (i = 0, 1, 2, ..., r - 1)$ be a suborbital. Then we form a graph Γ_i, by taking X as the points of Γ_i and including a directed line from x to y $(x, y \in X)$ if and only if $(x, y) \in O_i$. Thus each suborbital O_i determines a suborbital graph Γ_i. Now $O_i^* = \{(x, y) | (y, x) \in O_i\}$ is also a G-orbit. Let Γ_i^* be the suborbital graph corresponding to the suborbital O_i^* and let the suborbit

Δ_i $(i = 0, 1, ..., r-1)$ correspond to the suborbital O_i. Then Γ_i is undirected if Δ_i is self-paired and Γ_i is directed if Δ_i is not self-paired.

Theorem 1.1.5 (Sims, 1967)

Let G be transitive on X. Then G is primitive if and only if each suborbital graph $\Gamma_i, i = 1, 2, ..., r$ is connected.

1.2 Literature Review

In relation to ranks, subdegrees, and suborbital graphs of the symmetric group S_n acting on $X^{(r)}$, not much research has been done. The following is a summary of the research done, which is closely related to these areas so far.

In 1964, Wielandt wrote a little monograph on finite permutation groups and graphs. In this monograph a condition for imprimitivity of a group is given in terms of its subdegrees.

Higman (1964) introduced the rank of a group while working on finite permutation groups of rank 3. In 1970, Higman gave a characterization of families of rank 3 permutation groups by the subdegrees. He proved that the symmetric group S_n on $X = \{1, 2, ..., n\}$, $n \geq 4$ acts as a rank 3 group on the set of $\begin{pmatrix} n \\ 2 \end{pmatrix}$ 2-element subsets of X, with subdegrees 1, $2(n-2)$, $\begin{pmatrix} n-2 \\ 2 \end{pmatrix}$.

The idea of suborbital graphs of a permutation group G acting on a set X was introduced by Sims (1967).

Cameron (1972) worked on suborbits of multiply transitive permutation groups. In this paper, it was proved that if G is a primitive permutation group on X which is not 2-transitive, and if the stabilizer G_a of a point a is 2-transitive on an orbit $F(a)$ with $|F(a)| = v > 2$; then G_a has an orbit $A(a)$ with $|A(a)| = w$, where $w > v$ and $w|v(v-1)$ using combinatorial techniques. In 1975, Cameron studied suborbits in transitive permutation groups. In this paper, the combinatorial relations among suborbits and algebraic relations among suborbits is given.

In 1983, Cameron dealt with the orbits of permutation groups on unordered sets. Construction and characterisation of a 3-homogeneous but not 2-primitive permutation group H of countable degree was done in this paper, and it was shown that it has a transitive extension J which is 5-homogeneous but not 3-primitive.

Ivanov *et al.* (1983) gave a method of computing the subdegrees of transitive permutation groups using the table of marks. They gave the sporadic simple group J_1 as an example.

Tchuda (1986) computed the ranks and subdegrees of primitive permutation representations of $PSL(2,q)$. His results were summarized in a paper by Faradžev and Ivanov, which among others include for $q = 2^7$, the subdegrees are 1, $((2^7 - 1)7)^9$ and $2(2^7 - 1)$ for the subgroup $S_{2(q-1)}$. In the same paper, Faradžev and Ivanov showed that if $G = PSL(2,q)$ acts on the cosets of its maximal subgroup H, then the rank is at least $\frac{|G|}{|H|^2}$ and if $q > 100$, the rank is greater than 5.

Kamuti (1992) devised a method for constructing some of the suborbital graphs of $PSL(2,q)$ and $PGL(2,q)$ acting on the cosets of their maximal dihedral subgroups of orders $q-1$ and $2(q-1)$ respectively. This method gave an alternative way of constructing the Coxeter graph which was first constructed by Coxeter in 1986.

In 2006, Kamuti computed the ranks and subdegrees of primitive permutation representations of $PGL(2,q)$. It was shown in this paper that when $PGL(2,q)$ acts on the cosets of its maximal dihedral subgroup of order $2(q-1)$ then its rank is $\frac{1}{2}(q+3)$ if q is odd and $\frac{1}{2}(q+2)$ if q is even.

Korableva (2000) determined the degrees, ranks, subdegrees, and double stabilizers of the permutation representations of ${}^2E_6(q^2)$ on the cosets of the parabolic maximal subgroups of nonminimal index. The results obtained for the representatives P_2 were; the degree

$$n = \frac{(q^{12}-1)(q^2+1)(q^5+1)(q^4+1))}{(q+1)}$$

and the subdegree $n_1 = q(q^2+1)(q-1)$.

The ranks of the permutation representations of the simple groups $B_l(q), C_l(q)$, and $D_l(q)$ on the cosets of the parabolic maximal subgroups was determined by Korableva in 2008 . He found out that the ranks $rk(D_l)$ for $1 \leq k \leq l$ of the permutation representations of the groups $D_l(q)$ for $l \geq 3$ with respect to the parabolic maximal subgroups can be computed by recursion:

$$rk(D_l) = rk(D_{l-1}) + k + 1 \text{ for } 3 \leq k \leq [l/2] + 1,$$

$$rk(D_l) = rk(D_{l-1}) + l - k + 2 \text{ for } [l/2] + 1 < k < l,$$

$$r_1(D_l) = r_2(D_l) = [l/2] + 1, rl(D_l) = 3.$$

Akbas (2001) investigated the suborbital graphs for the modular group. He proved the conjecture by Jones, Singerman and Wicks (1991) that a suborbital graph for the modular group is a forest if and only if it contains no triangles.

In 2010, Besenk *et al.* investigated the conditions for a normalizer to be a forest. He showed that if N has the prime power decomposition as $2^{\alpha}.\ 3^{\beta}\ .\ p_3^{\gamma_3}...p_r^{\gamma_r}$, then the suborbital graphs of the normalizer would be a forest if $\beta \geq 4$.

Guler *et al.* (2008) worked on the suborbital graphs of the congruence subgroup $\Gamma_0(N)$. They showed that the action of $\Gamma_0(N)$ on $\hat{\mathbb{Q}}$ is not transitive and $(\Gamma_0(p), \hat{\mathbb{Q}})$ is an imprimitive permutation group.

In 1996, Akbas and Baskan worked on the suborbital graphs for the normalizer of $\Gamma_0(N)$. They showed that if $E_{u,n} = E_{v,n} = \{1\}$ and if $uv \equiv -1$ mod n, then the suborbital graph $\Delta_{u,n}$ is paired with $\Delta_{v,n}$. They also showed that $\Delta_{u,n}$ is self-paired if and only if there exists $e|N$ such that $N|ne$ and $u^2e = -1$ mod n.

Keskin and Demirturk (2009) worked on the suborbital graphs for the normalizer of $\Gamma_0(N)$. They showed that if m is a square free positive integer and $n > 1$ with $(u, n) = 1$ and if the graph $G(\infty, u/n)$ for $N(\Gamma_0(m))$ contains a triangle, then for any prime divisor p of n greater than 3, we have $p \equiv 1(\text{mod}3)$. They also showed that if $G(\infty, u/n)$ contains a rectangle, then n is an odd natural number and $n \equiv 1(\text{mod}4)$, and if $G(\infty, u/n)$ contains a hexagon, then for any odd prime divisor p of n we have $p \equiv 1(\text{mod}3)$.

1.3 Statement of the Problem

Calculation of the ranks and subdegrees, and constructing suborbital graphs of S_n acting on r-element subsets from X. The problem of ranks and subdegrees of S_n was solved by Higman for 2-element subsets. In this research we extend this work by considering r-element subsets of X for $r \geq 3$.

1.4 Research Objectives

The general objective of this study is to determine the ranks and subdegrees of S_n acting on $X^{(r)}$ and to construct the suborbital graphs of S_n corresponding to the action.

The specific objectives are:

1. To determine the ranks of S_n acting on $X^{(r)}$.
2. To determine the suborbits of S_n acting on $X^{(r)}$.
3. To investigate the properties of the suborbits of S_n.
4. To determine the subdegrees of S_n acting on $X^{(r)}$.
5. To arrange the subdegrees of S_n in increasing order of magnitude.
6. To construct the suborbital graphs of S_n acting on $X^{(r)}$.

7. To investigate the properties of the suborbital graphs of S_n.

CHAPTER 2

RANKS AND SUBORBITS OF S_n ACTING ON $X^{(r)}$

2.1 Introduction

The degree, rank, subdegrees, and structure of the point stabilizer provide sufficiently thorough information on a permutation representation. An attempt has been made to calculate the ranks and subdegrees of all primitive permutation representations of $\mathrm{PSL}(2,q)$ and $\mathrm{PGL}(2,q)$. Also, the rank of the symmetric group S_n acting on unordered 2-element subsets of $X = \{1,2,3,...,n\}$ has been calculated and found to be 3 if $n \geq 3$. This chapter will be devoted to calculating the ranks of the symmetric group S_n acting on unordered r-element subsets of X (i.e $X^{(r)}$) and analysing the properties of this action.

2.2 Order of $\mathbf{Stab}_G\{1,2,3,...,r\}$

Let $G = S_n$ and $\mathrm{Stab}_G\{1,2,3,...,r\}$ be the stabilizer of $\{1,2,...,r\}$, then the following result follows.

Theorem 2.2.1

$$|\text{Stab}_G\{1, 2, 3, ..., r\}| = (n - r)!r! \tag{2.1}$$

Proof

Clearly, the stabilizer of unordered r-element subset $\{1, 2, 3, ..., r\}$ is isomorphic to $S_r \times S_{n-r}$ whose order is $r!(n - r)!$. Thus

$$|\text{Stab}_G\{1, 2, 3, ..., r\}| = (n - r)!r! \qquad \square$$

Some results which will be used extensively in this chapter are given next.

Lemma 2.2.2

Let $g \in S_n$ be a permutation with cycle type $(\alpha_1, \alpha_2, ..., \alpha_n)$. Then the number of permutations in S_n fixing $\{1, 2, 3\} \in X^{(3)}$ and having the same cycle type as g is given by

$$\frac{(n-3)!}{(\alpha_1 - 3)!1^{(\alpha_1 - 3)}\Pi_{i=2}^{n}\alpha_i!i^{\alpha_i}} + \frac{3(n-3)!}{(\alpha_1 - 1)!1^{(\alpha_1 - 1)}(\alpha_2 - 1)!2^{\alpha_2 - 1}\Pi_{i=3}^{n}\alpha_i!i^{\alpha_i}} +$$

$$\frac{2(n-3)!}{\alpha_1!1^{\alpha_1}\alpha_2!2^{\alpha_2}(\alpha_3 - 1)!3^{\alpha_3 - 1}\Pi_{i=4}^{n}\alpha_i!i^{\alpha_i}} \tag{2.2}$$

Proof

Consider a permutation $g \in S_n$ that fixes $\{1, 2, 3\} \in X^{(3)}$. Then g fixes $S = \{1, 2, 3\}$ if either each member of S comes from a 1-cycle of g or if one element of S comes from a 1-cycle and the other two elements come from a 2-cycle in g or if all the three elements of S come from a 3-cycle of g. We consider the three cases as follows:

(a) When each of the members of S comes from a 1-cycle, we apply Theorem 1.1.2 to a permutation of S_{n-3} with cycle type $(\alpha_1 - 3, \alpha_2, \alpha_3, ..., \alpha_n)$ to get

$$\frac{(n-3)!}{(\alpha_1 - 3)! 1^{(\alpha_1 - 3)} \Pi_{i=2}^{n} \alpha_i! i^{\alpha_i}}$$

permutations.

(b) If one of members of S comes from a 1-cycle, and the other two from a 2-cycle, the number of elements coming from the 1-cycle can be chosen in three ways. We apply Theorem 1.1.2 to a permutation of S_{n-3} with cycle type $(\alpha_1 - 1, \alpha_2 - 1, \alpha_3, ..., \alpha_n)$ and considering the three possible ways to get

$$\frac{3(n-3)!}{(\alpha_1 - 1)! 1^{(\alpha_1 - 1)} (\alpha_2 - 1)! 2^{\alpha_2 - 1} \Pi_{i=3}^{n} \alpha_i! i^{\alpha_i}}$$

permutations.

(c) Finally, if all the elements of S come from a 3-cycle, there are two different permutations from a cycle of length three. Applying Theorem 1.1.2 to a permutation of S_{n-3} with cycle type $(\alpha_1, \alpha_2, \alpha_3 - 1, ..., \alpha_n)$

and considering the two cases, we get

$$\frac{2(n-3)!}{\alpha_1!1^{\alpha_1}\alpha_2!2^{\alpha_2}(\alpha_3-1)!3^{\alpha_3-1}\Pi_{i=4}^{n}\alpha_i!i^{\alpha_i}}$$

permutations. Summing up all the three cases (a), (b), and (c) gives the required result. □

Lemma 2.2.3

Let $g \in S_n$ be a permutation with a cycle type $(\alpha_1, \alpha_2, ..., \alpha_n)$ Then the number of permutations in S_n fixing $\{1, 2, 3, 4\} \in X^{(4)}$ and having the same cycle type as g is given by

$$\frac{(n-4)!}{(\alpha_1-4)!\Pi_{i=2}^{n}\alpha_i!i^{\alpha_i}} + \frac{6(n-4)!}{(\alpha_1-2)!(\alpha_2-1)!2^{\alpha_2-1}\Pi_{i=3}^{n}\alpha_1!i^{\alpha_i}} +$$

$$\frac{8(n-4)!}{(\alpha_1-1)!\alpha_2!2^{\alpha_2}(\alpha_3-1)!3^{\alpha_3-1}\Pi_{i=4}^{n}\alpha_i!i^{\alpha_i}} + \frac{3(n-4)!}{\alpha_1!(\alpha_2-2)!2^{\alpha_2-2}\Pi_{i=3}^{n}\alpha_i!i^{\alpha_i}} +$$

$$\frac{6(n-4)!}{\alpha_1!\alpha_2!2^{\alpha_2}\alpha_3!3^{\alpha_3}(\alpha_4-1)!4^{\alpha_4-1}\Pi_{i=5}^{n}\alpha_i!i^{\alpha_i}} \tag{2.3}$$

Proof

The idea of proof of this Lemma is given. We consider the five cases in which g fixes $S = \{a, b, c, d\}$. The first case is when all of the four elements of S come from 1-cycles. Second case is when two of the members of S come from 1-cycles while the other two come from a 2-cycle. Case number three is when one of the members come from a 1-cycle while the other three come from a 3-cycle.

Case number four is when the members of S come from two 2-cycles. Lastly we consider the case where all the members come from a 4-cycle. The proof can be completed by proceeding as in Lemma 2.2.2. □

Lemma 2.2.4

Let $g \in S_n$ be a permutation with cycle type $(\alpha_1, \alpha_2, ..., \alpha_n)$. Then the number of permutations in S_n fixing $\{1, 2, 3, 4, 5\} \in X^{(5)}$ and having the same cycle type as g is given by

$$\frac{(n-5)!}{(\alpha_1-5)!1^{(\alpha_1-5)}\Pi_{i=2}^{n}\alpha_i!i^{\alpha_i}} + \frac{10(n-5)!}{(\alpha_1-3)!1^{(\alpha_1-3)}(\alpha_2-1)!2^{(\alpha_2-1)}\Pi_{i=3}^{n}\alpha_i!i^{\alpha_i}}$$

$$+\frac{20(n-5)!}{(\alpha_1-2)!1^{\alpha_1-2}\alpha_2!2^{\alpha_2}(\alpha_3-1)!3^{(\alpha_3-1)}\Pi_{i=4}^{n}\alpha_i!i^{\alpha_i}} +$$

$$\frac{15(n-5)!}{(\alpha_1-1)!1^{(\alpha_1-1)}(\alpha_2-2)!2^{(\alpha_2-2)}\Pi_{i=3}^{n}\alpha_i!i^{\alpha_i}} +$$

$$\frac{30(n-5)!}{(\alpha_1-1)!1^{(\alpha_1-1)}(\alpha_4-1)!4^{(\alpha_4-1)}\Pi_{i=2}^{3}\alpha_i i^{\alpha_i}\Pi_{i=5}^{n}\alpha_i!i^{\alpha_i}} +$$

$$\frac{20(n-5)!}{\alpha_1!1^{\alpha_1}(\alpha_2-1)!2^{(\alpha_2-1)}(\alpha_3-1)!3^{(\alpha_3-1)}\Pi_{i=4}^{5}\alpha_i!i^{\alpha_i}}$$

$$+\frac{24(n-5)!}{(\alpha_5-1)!5^{(\alpha_5-1)}\Pi_{i=1}^{4}\alpha_i!i^{\alpha_i}\Pi_{i=6}^{n}\alpha_i!i^{\alpha_i}} \tag{2.4}$$

The proof for this Lemma is similar to that of Lemma 2.2.2 and Lemma 2.2.3.

Remark 2.2.5

Any of the summands in Lemmas 2.2.2, 2.2.3, and 2.2 yields a zero whenever $(\alpha_i - j) < 0$. This is because $(\alpha_i - j)!$ does not exist when $(\alpha_i - j) < 0$, consequently

$$\frac{1}{(\alpha_i - j)!} = 0$$

The following result is deduced from Lemmas 2.2.2, 2.2.3, and 2.2.

Proposition 2.2.6

Let $g \in S_n$ be a permutation with cycle type $(\alpha_1, \alpha_2, ..., \alpha_n)$. Then the number of permutations in S_n fixing {1, 2, 3,...,r}$\in X^{(r)}$ and having the same cycle type as g is given by

$\frac{(n-r)!}{(\alpha_1-r)!1^{\alpha_1-r}\Pi_{i=2}^{n}\alpha_i!i^{\alpha_i}}+$

$\frac{\alpha_1 C_1(n-r)!(r-2)!}{(\alpha_1-1)!1^{\alpha_1-1}(\alpha_{r-1}-1)!(r-1)^{\alpha_{r-1}-1}\Pi_{i=2}^{r-2}\alpha_i!i^{\alpha_i}\Pi_{i=r}^{n}\alpha_i!i^{\alpha_i}}+$

$\frac{\alpha_1 C_2(2-1)!(n-r)!(r-3)!}{(\alpha_1-2)!1^{\alpha_1-2}(\alpha_{r-2}-1)!(r-2)^{\alpha_{r-2}-1}\Pi_{i=2}^{r-3}\alpha_i!i^{\alpha_i}\Pi_{i=r-1}^{n}\alpha_i!i^{\alpha_i}}+$

$\frac{\alpha_1 C_3(3-1)!(n-r)!(r-4)!}{(\alpha_1-3)!1^{\alpha_1-3}(\alpha_{r-3}-1)!(r-3)^{\alpha_{r-3}-1}\Pi_{i=2}^{r-4}\alpha_i!i^{\alpha_i}\Pi_{i=r-2}^{n}\alpha_i!i^{\alpha_i}}+...+$

$\frac{\alpha_1 C_k(k-1)!(n-r)!(r-k-1)!}{(\alpha_1-k)!1^{\alpha_1-k}(\alpha_{r-k}-1)!(r-k)^{\alpha_{r-k}-1}\Pi_{i=2}^{r-k-1}\alpha_i!i^{\alpha_i}\Pi_{i=r-k+1}^{n}\alpha_i!i^{\alpha_i}}+$

$\frac{(2-1)!\alpha_2 C_2(n-r)!(r-3)!}{\alpha_1!1^{\alpha_1}(\alpha_2-1)!(2)^{\alpha_2-1}(\alpha_{r-2}-1)!(r-2)^{\alpha_{r-2}-1}\Pi_{i=3}^{r-3}\alpha_i!i^{\alpha_i}\Pi_{i=r-1}^{n}\alpha_i!i^{\alpha_i}}+...+$

$\frac{\alpha_2 C_k(2k-1)!(n-r)!(r-2k-1)!}{\alpha_1!i^{\alpha_1}(\alpha_2-k)!2^{\alpha_2-k}(\alpha_{r-2k}-1)!(r-2k)^{\alpha_{r-2k}-1}\Pi_{i=3}^{r-2k-1}\alpha_i!i^{\alpha_i}\Pi_{i=r-2k+1}^{n}\alpha_i i^{\alpha_i}}+$

$...+\frac{\alpha_k C_q(pq-1)!(n-r)!(r-pq-1)!}{(\alpha_k-q)!k^{\alpha_k-q}(\alpha_{r-pq}-1)!(r-pq)^{\alpha_{r-pq}-1}\Pi_{i=3}^{k-1}\alpha_i i^{\alpha_i}\Pi_{i=k+1}^{r-pq-1}\alpha_i!i^{\alpha_i}\Pi_{i=r-pq+1}^{n}\alpha_i!i^{\alpha_i}}$

$...+...+\frac{(n-r)!(r-1)!}{(\alpha_r-1)!r^{\alpha_r-1}\Pi_{i=1}^{r-1}\alpha_i!i^{\alpha_i}\Pi_{i=r+1}^{n}\alpha_i!i^{\alpha_i}}$

Some examples on the order of $\text{Stab}_G\{1, 2, 3, ..., r\}$ are now given.

Example 2.2.7

Consider $G = S_8$ acting on $X^{(3)}$. Then

$$\begin{aligned} |\text{Stab}_G\{1,2,3\}| &= 5!3! \\ &= 720 \end{aligned}$$

Alternatively, the problem may solved by using Lemma 2.2.2 and coming up with Table 2.1. The second and fourth columns give the number of permutations in S_8 fixing $\{1,2,3\}$ and having the same cycle type, which are obtained by using expression 2.2.

Permutation type	No. fixing {1,2,3}	Permutation type	No. fixing {1,2,3}
I	1	(a b)(c d)(e f)(g h)	0
(a b)	13	(a b)(c d)	45
(a b c)	22	(a b)(c d e)	100
(a b c d)	30	(a b)(c d)(e f)	45
(a b c d e)	24	(a b)(c d)(e f g)	90
(a b c d e f)	0	(a b c d)(e f g h)	0
(a b)(c d)(e f g h)	0	(a b c)(d e f)	40
(a b c d e f g)	0	(a b c)(d e f g)	60
(a b c d e f g h)	0	(a b)(c d e f)	90
(a b)(c d e f g h)	0	(a b c)(d e f g h)	48
(a b)(c d e f g)	72	(a b c)(d e f)(g h)	40
		Total	**720**

Table 2.1: Number of permutations fixing {1,2,3}

From Table 2.1, the order of $\text{Stab}_G\{1,2,3\}$ is 720.

Example 2.2.8

Let $G = S_9$ acting on $X^{(4)}$. Then

$$\begin{aligned} |\text{Stab}_G\{1,2,3,4\}| &= 5!4! \\ &= 2880 \end{aligned} \tag{2.5}$$

Table 2.2 may also be used to find $|\text{Stab}_G\{1,2,3,4\}|$ where the second and fourth columns are obtained by using Lemma 2.2.3

Permutation type	No. fixing {1,2,3,4}	Permutation type	No. fixing {1,2,3,4}
I	1	(a b)	16
(a b c)	28	(a b c e)	36
(a b c d e)	24	(a b c d e f)	0
(a b c d e f g)	0	(a b c d e f g h)	0
(a b c d e f g h i)	0	(a b)(c d e f g h i)	0
(a b)(c d e f g h)	0	(a b)(c d e f g)	144
(a b)(c d e f)	240	(a b)(c d e)	220
(a b)(c d)	78	(a b)(c d)(e f)(g h)	120
(a b)(c d)(e f)	45	(a b)(c d e)(f g h)	60
(a b)(c d)(e f g h i)	300	(a b)(c d)(e f g h)	180
(a b)(c d)(e f g)	72	(a b)(c d e)(f g h i)	0
(a b)(c d)(e f)(g h i)	160	(a b c)(d e f)(g h i)	360
(a b c)(d e f)	192	(a b c)(d e f g)	0
(a b c)(d e f g h)	180	(a b c)(d e f g h i)	144
(a b c d)(e f g h)	120	(a b c d)(e f g h i)	160
		Total	**2880**

Table 2.2: Number of permutations fixing {1,2,3,4}

The total sum entries in the second and fourth columns of Table 2.2 gives the order of $\text{Stab}_G\{1,2,3,4\}$ as equal to 2880.

2.3 Transitivity of S_n

We next show that S_n acts transitively on $X^{(r)}$.

Theorem 2.3.1

Sn acts transitively on $X^{(r)}$

Proof

It suffices to show that

$$|\text{Orb}_G\{1,2,3,...,r\}| = |X^{(r)}| = nC_r$$

By using Orbit-Stabilizer Theorem (Theorem 1.1.1) and Theorem 2.2.1,

$$\begin{aligned}
|\text{Orb}_G\{1,2,3,...,r\}| &= |G : \text{Stab}_G\{1,2,3,...,r\}| \\
&= \frac{|G|}{|\text{Stab}_G\{1,2,3,...,r\}|} \\
&= \frac{n!}{(n-r)!r!} \\
&= nC_r \quad \square
\end{aligned} \tag{2.6}$$

2.4 Number of Fixed Points

Derivation of some formulas for finding the number of fixed elements of $X^{(r)}$ by a permutation $g \in S_n$ is done in this section. The formulas are given in the following results.

Lemma 2.4.1

Let the cycle type of $g \in S_n$ be $(\alpha_1, \alpha_2, ..., \alpha_n)$. Then $|\text{Fix}(g)|$ in $X^{(3)}$ is given by the formula

$$|\text{Fix}(g)| = \binom{\alpha_1}{3} + \alpha_1\alpha_2 + \alpha_3 \tag{2.7}$$

Proof

Let $g \in S_n$ have cycle type $(\alpha_1, \alpha_2, ..., \alpha_n)$ and let $S \in X^{(3)}$. S is fixed by g if each member of S comes from a 1-cycle in g or one of members of S comes from a 1-cycle in g and the other two from a 2-cycle in g or if all members of S come from a 3-cycle in g. From the first case, the number of unordered triples fixed by g is $\binom{\alpha_1}{3}$, while in the second case the number of unordered triples fixed by g is $\alpha_1\alpha_2$ and in the third case the number of unordered triples fixed by g is α_3. Adding up the results from the three cases gives the required result. $\square$

Lemma 2.4.2

Let the cycle type of $g \in S_n$ be $(\alpha_1, \alpha_2, ..., \alpha_n)$. Then $|\text{Fix}(g)|$ in $X^{(4)}$ is given by the formula

$$|\text{Fix}(g)| = \binom{\alpha_1}{4} + \alpha_2 \binom{\alpha_1}{2} + \alpha_1\alpha_3 + \binom{\alpha_2}{2} + \alpha_4 \tag{2.8}$$

Proof

Let $g \in S_n$ be a permutation with cycle type $(\alpha_1, \alpha_2, ..., \alpha_n)$ and let $S \in X^{(4)}$. S is fixed by g if each member of S comes from a 1-cycle in g; if two members of S come from a 1-cycle in g and the other two members from a 2-cycle; if one element of S comes from a 1-cycle in g and the other three from a 3-cycle; if four elements of S come from two 2-cycles in g; if all the elements of S come from a 4-cycle in g. From each of the cases, the number of unordered quadruples fixed by g is $\binom{\alpha_1}{4}$, $\alpha_2 \binom{\alpha_1}{2}$, $\alpha_1\alpha_3$, $\binom{\alpha_2}{2}$, and α_4 respectively. □

Lemma 2.4.3

Let the cycle type of $g \in S_n$ be $(\alpha_1, \alpha_2, ..., \alpha_n)$. Then $|\text{Fix}(g)|$ in $X^{(5)}$ is given by the formula

$$|\text{Fix}(g)| = \binom{\alpha_1}{5} + \alpha_2 \binom{\alpha_1}{3} + \alpha_3 \binom{\alpha_1}{2} + \alpha_1\alpha_4 + \alpha_1 \binom{\alpha_2}{2} + \alpha_2\alpha_3 + \alpha_5 \tag{2.9}$$

The proof of this Lemma is similar to that of Lemma 2.4.1 and 2.4.2.

The following result is deduced from Lemmas 2.4.1, 2.4.2, and 2.4.3.

Proposition 2.4.4

Let the cycle type of $g \in S_n$ be $(\alpha_1, \alpha_2, ..., \alpha_n)$. Then $|\text{Fix}(g)|$ in $X^{(r)}$ is given by

$$|\text{Fix}(g)| = \sum_{(i)} \prod_{k} \begin{pmatrix} \alpha_k \\ i_k \end{pmatrix}, \tag{2.10}$$

where the sum is over all partitions $(i) = (i_1, i_2, ..., i_r)$ of r.

2.5 Ranks and Suborbits of S_n

The results on the ranks of S_n acting on $X^{(r)}$ are proved in this section.

Lemma 2.5.1

Let $G = S_8$ acting on $X^{(4)}$. The number of orbits of $G_{\{1,2,3,4\}}$ acting on $X^{(4)}$ is 5

Proof

Lemma 2.2.3 and Lemma 2.4.2 are applied to get the values in columns two and three in Table 2.3 respectively.

Permutations in $G_{\{1,2,3,4\}}$	Number of permutations	$\lvert\text{Fix}(g)\rvert$in $X^{(4)}$
(1)(2)(3)(4)(5)(6)(7)(8)	1	70
(1)(2)(3)(4)(5)(6)(7 8)	12	30
(1)(2)(3)(4)(5 6)(7 8)	42	14
(1)(2)(3)(4)(5)(6 7 8)	16	10
(1)(2)(3)(4)(5 6 7 8)	12	2
(1)(2)(3 4)(5 6)(7 8)	36	6
(1)(2)(3 4)(5)(6 7 8)	96	6
(1)(2)(3 4)(5 6 7 8)	72	2
(1 2)(3 4)(5 6)(7 8)	9	6
(1 2)(3 4)(5)(6 7 8)	48	2
(1 2)(3 4)(5 6 7 8)	36	2
(1)(2 3 4)(5)(6 7 8)	64	4
(1)(2 3 4)(5 6 7 8)	96	2
(1 2 3 4)(5 6 7 8)	36	2
Total	576	

Table 2.3: Permutations in $G_{\{1,2,3,4\}}$ and the number of fixed points

Applying Cauchy-Frobenius Lemma (Theorem 1.1.3), we get

Number of orbits of $G_{\{1,2,3,4\}}$ acting on $X^{(4)}$ is given by:

$$\begin{aligned}
\frac{1}{|G_{\{1,2,3,4\}}|}\sum_{g\in G_{\{1,2,3,4\}}}|\text{Fix}(g)| &= \frac{1}{576}[1\times 70+12\times 30+42\times 14+16\times 10+ \\
&\quad 12\times 2+36\times 6+96\times 6+72\times 2+9\times 6+ \\
&\quad 48\times 2+36\times 2+64\times 4+96\times 2+36\times 2] \\
&= \frac{1}{576}[70+360+588+160+24+216+576 \\
&\quad +144+54+96+72+256+192+72] \\
&= \frac{2880}{576} \\
&= 5 \quad \square \qquad (2.11)
\end{aligned}$$

The five orbits of $G_{\{1,2,3,4\}}$ are:

(a) $G_{\{1,2,3,4\}}\{1,2,3,4\} = \Delta_0$, the trivial orbit,

(b) $G_{\{1,2,3,4\}}$\{1,5,6,7\}=\{\{1,5,6,7\}, \{1,5,6,8\}, \{1,5,7,8\}, \{1,6,7,8\}, {2,5,6,7}, {2,5,6,8}, {2,5,7,8}, {2,6,7,8}, {3,5,6,7}, {3,5,6,8}, {3,5,7,8}, {3,6,7,8}, {4,5,6,7}, {4,5,6,8}, {4,5,7,8}, {4,6,7,8}}=Δ_1, the orbit containing exactly one of 1, 2, 3, 4.

(c) $G_{\{1,2,3,4\}}${1,2,5,6}={{1,2,5,6}, {1,2,5,7}, {1,2,5,8}, {1,2,6,7}, {1,2,6,8}, {1,2,7,8}, {1,3,5,6}, {1,3,5,7}, {1,3,5,8}, {1,3,6,7}, {1,3,6,8}, {1,3,7,8}, {1,4,5,6}, {1,4,5,7}, {1,4,5,8}, {1,4,6,7}, {1,4,6,8}, {1,4,7,8}, {2,3,5,6}, {2,3,5,7}, {2,3,5,8}, {2,3,6,7}, {2,3,6,8}, {2,3,7,8}, {2,4,5,6}, {2,4,5,7}, {2,4,5,8}, {2,4,6,7}, {2,4,6,8}, {2,4,7,8}, {3,4,5,6}, {3,4,5,7}, {3,4,5,8}, {3,4,6,7}, {3,4,6,8}, {3,4,7,8}}=Δ_2, the orbit containing exactly two of 1, 2, 3, 4.

(d) $G_{\{1,2,3,4\}}\{1,2,3,5\} = \{\{1,2,3,5\},\{1,2,3,6\},\{1,2,3,7\},\{1,2,3,8\},\{1,2,4,5\},\{1,2,4,6\},\{1,2,4,7\},\{1,2,4,8\},\{2,3,4,5\},\{2,3,4,6\},\{2,3,4,7\},\{2,3,4,8\},\{1,3,4,5\},\{1,3,4,6\},\{1,3,4,7\},\{1,3,4,8\}\} = \Delta_3$, the orbit containing exactly three of 1, 2, 3, 4.

(e) $G_{\{1,2,3,4\}}\{1,2,3,4\} = \{5,6,7,8\} = \Delta_4$, the orbit containing none of 1, 2, 3, 4.

Lemma 2.5.2

Let $G = S_{10}$ acting on $X^{(5)}$. The number of orbits of $G_{\{1,2,3,4,5\}}$ acting on $X^{(5)}$ is 6.

Proof

Lemma 2.2 and Lemma 2.4.3 are applied to get the values in columns two and three in Table 2.4 respectively.

Permutations in $G_{\{1,2,3,4,5\}}$	Number of permutations	$\lvert\mathrm{Fix}(g)\rvert$in $X^{(5)}$
(1)(2)(3)(4)(5)(6)(7)(8)(9)(10)	1	252
(1)(2)(3)(4)(5)(6)(7)(8)(9 10)	20	112
(1)(2)(3)(4)(5)(6)(7 8)(9 10)	130	52
(1)(2)(3)(4)(5)(6)(7)(8 9 10)	40	42
(1)(2)(3)(4)(5)(6)(7 8 9 10)	60	12
(1)(2)(3)(4)(5)(6 7)(8 9 10)	440	22
(1)(2)(3)(4)(5)(6 7 8 9 10)	48	2
(1)(2)(3)(4 5)(6)(7 8)(9 10)	300	24
(1)(2)(3)(4 5)(6)(7 8 9 10)	600	8
(1)(2)(3)(4 5)(6 7)(8 9 10)	1000	10
(1)(2)(3)(4 5)(6 7 8 9 10)	480	2
(1)(2 3)(4 5)(6)(7 8)(9 10)	225	12
(1)(2 3)(4 5)(6)(7 8 9 10)	900	4
(1)(2 3)(4 5)(6 7)(8 9 10)	600	6
(1)(2 3)(4 5)(6 7 8 9 10)	720	2
(1)(2)(3 4 5)(6)(7)(8 9 10)	400	12
(1)(2)(3 4 5)(6)(7 8 9 10)	1200	6
(1)(2)(3 4 5)(6 7)(8 9 10)	800	4
(1)(2)(3 4 5)(6 7 8 9 10)	960	2
(1)(2 3 4 5)(6)(7 8 9 10)	900	4
(1)(2 3 4 5)(6 7)(8 9 10)	1200	2
(1)(2 3 4 5)(6 7 8 9 10)	1440	2
(1 2)(3 4 5)(6 7)(8 9 10)	400	4
(1 2)(3 4 5)(6 7 8 9 10)	960	2
(1 2 3 4 5)(6 7 8 9 10)	576	2
Total	14400	

Table 2.4: Permutations in $G_{\{1,2,3,4,5\}}$ and the number of fixed points

Applying Theorem 1.1.3, the number of orbits of $G_{\{1,2,3,4,5\}}$ acting on $X^{(5)}$ is given by:

$$\begin{aligned}
\frac{1}{|G_{\{1,2,3,4,5\}}|}\sum_{g\in G_{\{1,2,3,4,5\}}}|\mathrm{Fix}(g)| &= \frac{1}{14400}[1\times 252+20\times 112+130\times 52+40\times 42\\
&\quad +60\times 12+440\times 22+48\times 2+300\times 24+\\
&\quad 600\times 8+1000\times 10+480\times 2+900\times 4\\
&\quad +225\times 12+600\times 6+720\times 2+400\times 12+\\
&\quad 1200\times 6+800\times 4+960\times 2+900\times 4+\\
&\quad 1200\times 2+1440\times 2+400\times 4+960\times 2+576\times 2]\\
&= \frac{1}{14400}[252+2240+6760+1680+720+9680+96\\
&\quad +7200+4800+10000+960+3600+2700+3600\\
&\quad +1440+4800+7200+3200+1920+3600+2400\\
&\quad +2880+1600+1920+1152]\\
&= \frac{86400}{14400}\\
&= 6 \quad \square \qquad (2.12)
\end{aligned}$$

The six orbits of $G_{\{1,2,3,4,5\}}$ are:

(a) $G_{\{1,2,3,4,5\}}\{1,2,3,4,5\} = \Delta_0$, the trivial orbit,

(b) $G_{\{1,2,3,4,5\}}\{1,6,7,8,9\} = \{\{1,6,7,8,9\},\{1,6,7,8,10\},\{1,6,7,9,10\}, \{1,6,8,9,10\},\{1,7,8,9,10\},\{2,6,7,8,9\},\{2,6,7,8,10\},\{2,6,7,9,10\}, \{2,6,8,9,10\},\{2,7,8,9,10\},\{3,6,7,8,9\},\{3,6,7,8,10\},\{3,6,7,9,10\}, \{3,6,8,9,10\},\{3,7,8,9,10\},\{4,6,7,8,9\},\{4,6,7,8,10\},\{4,6,7,9,10\}, \{4,6,8,9,10\},\{4,7,8,9,10\},\{5,6,7,8,9\},\{5,6,7,8,10\},\{5,6,7,9,10\}, \{5,6,8,9,10\},\{5,7,8,9,10\}\} = \Delta_1$, the orbit containing exactly one of 1, 2, 3, 4, 5.

(c) $G_{\{1,2,3,4,5\}}$\{1,2,6,7,8\}=\{\{1,2,6,7,8\}, \{1,2,6,7,9\}, \{1,2,6,7,10\}, {1,2,6,8,9}, {1,2,6,8,10}, {1,2,6,9,10}, {1,2,7,8,9}, {1,2,7,8,10}, {1,2,7,9,10}, {1,2,8,9,10}, {1,3,6,7,8}, {1,3,6,7,9}, {1,3,6,7,10}, {1,3,6,8,9}, {1,3,6,8,10}, {1,3,6,9,10}, {1,3,7,8,9}, {1,3,7,8,10}, {1,3,7,9,10}, {1,3,8,9,10}, {1,4,6,7,8}, {1,4,6,7,9}, {1,4,6,7,10}, {1,4,6,8,9}, {1,4,6,8,10}, {1,4,6,9,10}, {1,4,7,8,9}, {1,4,7,8,10}, {1,4,7,9,10}, {1,4,8,9,10}, {1,5,6,7,8}, {1,5,6,7,9}, {1,5,6,7,10}, {1,5,6,8,9}, {1,5,6,8,10}, {1,5,6,9,10}, {1,5,7,8,9}, {1,5,7,8,10}, {1,5,7,9,10}, {1,5,8,9,10}, {2,3,6,7,8}, {2,3,6,7,9}, {2,3,6,7,10}, {2,3,6,8,9}, {2,3,6,8,10}, {2,3,6,9,10}, {2,3,7,8,9}, {2,3,7,8,10}, {2,3,7,9,10}, {2,3,8,9,10}, {2,4,6,7,8}, {2,4,6,7,9}, {2,4,6,7,10}, {2,4,6,8,9}, {2,4,6,8,10}, {2,4,6,9,10}, {2,4,7,8,9}, {2,4,7,8,10}, {2,4,7,9,10}, {2,4,8,9,10}, {2,5,6,7,8}, {2,5,6,7,9}, {2,5,6,7,10}, {2,5,6,8,9}, {2,5,6,8,10}, {2,5,6,9,10}, {2,5,7,8,9}, {2,5,7,8,10}, {2,5,7,9,10}, {2,5,8,9,10}, {3,4,6,7,8}, {3,4,6,7,9}, {3,4,6,7,10}, {3,4,6,8,9}, {3,4,6,8,10}, {3,4,6,9,10}, {3,4,7,8,9}, {3,4,7,8,10}, {3,4,7,9,10}, {3,4,8,9,10}, {3,5,6,7,8}, {3,5,6,7,9}, {3,5,6,7,10}, {3,5,6,8,9}, {3,5,6,8,10}, {3,5,6,9,10}, {3,5,7,8,9}, {3,5,7,8,10}, {3,5,7,9,10}, {3,5,8,9,10}, {4,5,6,7,8}, {4,5,6,7,9}, {4,5,6,7,10}, {4,5,6,8,9}, {4,5,6,8,10}, {4,5,6,9,10}, {4,5,7,8,9}, {4,5,7,8,10}, {4,5,7,9,10}, {4,5,8,9,10} }$=\Delta_2$, the orbit containing exactly two of 1, 2, 3, 4, 5.

(d) $G_{\{1,2,3,4,5\}}${1,2,3,6,7}={{1,2,3,6,7}, {1,2,3,6,8}, {1,2,3,6,9}, {1,2,3,6,10}, {1,2,3,7,8}, {1,2,3,7,9}, {1,2,3,7,10}, {1,2,3,8,9}, {1,2,3,8,10}, {1,2,3,9,10}, {1,2,4,6,7}, {1,2,4,6,8}, {1,2,4,6,9}, {1,2,4,6,10}, {1,2,4,7,8}, {1,2,4,7,9}, {1,2,4,7,10}, {1,2,4,8,9}, {1,2,4,8,10}, {1,2,4,9,10}, {1,2,5,6,7}, {1,2,5,6,8}, {1,2,5,6,9}, {1,2,5,6,10},{1,2,5,7,8}, {1,2,5,7,9}, {1,2,5,7,10}, {1,2,5,8,9}, {1,2,5,8,10}, {1,2,5,9,10}, {1,3,4,6,7}, {1,3,4,6,8}, {1,3,4,6,9}, {1,3,4,6,10}, {1,3,4,7,8}, {1,3,4,7,9}, {1,3,4,7,10}, {1,3,4,8,9}, {1,3,4,8,10}, {1,3,4,9,10}, {1,3,5,6,7}, {1,3,5,6,8}, {1,3,5,6,9}, {1,3,5,6,10}, {1,3,5,7,8}, {1,3,5,7,9}, {1,3,5,7,10}, {1,3,5,8,9}, {1,3,5,8,10}, {1,3,5,9,10}, {1,4,5,6,7}, {1,4,5,6,8}, {1,4,5,6,9}, {1,4,5,6,10}, {1,4,5,7,8}, {1,4,5,7,9}, {1,4,5,7,10}, {1,4,5,8,9}, {1,4,5,8,10}, {1,4,5,9,10}, {2,3,4,6,7}, {2,3,4,6,8}, {2,3,4,6,9}, {2,3,4,6,10}, {2,3,4,7,8}, {2,3,4,7,9}, {2,3,4,7,10}, {2,3,4,8,9}, {2,3,4,8,10}, {2,3,4,9,10}, {2,3,5,6,7}, {2,3,5,6,8}, {2,3,5,6,9}, {2,3,5,6,10}, {2,3,5,7,8}, {2,3,5,7,9}, {2,3,5,7,10}, {2,3,5,8,9}, {2,3,5,8,10}, {2,3,5,9,10}, {2,4,5,6,7}, {2,4,5,6,8}, {2,4,5,6,9}, {2,4,5,6,10}, {2,4,5,7,8}, {2,4,5,7,9}, {2,4,5,7,10}, {2,4,5,8,9}, {2,4,5,8,10}, {2,4,5,9,10}, {3,4,5,6,7}, {3,4,5,6,8}, {3,4,5,6,9}, {3,4,5,6,10}, {3,4,5,7,8}, {3,4,5,7,9}, {3,4,5,7,10}, {3,4,5,8,9}, {3,4,5,8,10}, {3,4,5,9,10}}=Δ_3, the orbit containing exactly three of 1, 2, 3, 4, 5.

(e) $G_{\{1,2,3,4,5\}}${1,2,3,4,6}={{1,2,3,4,6}, {1,2,3,4,7}, {1,2,3,4,8}, {1,2,3,4,9}, {1,2,3,4,10}, {1,2,3,5,6}, {1,2,3,5,7}, {1,2,3,5,8}, {1,2,3,5,9}, {1,2,3,5,10}, {1,2,4,5,6}, {1,2,4,5,7}, {1,2,4,5,8}, {1,2,4,5,9}, {1,2,4,5,10}, {1,3,4,5,6}, {1,3,4,5,7}, {1,3,4,5,8}, {1,3,4,5,9}, {1,3,4,5,10}, {2,3,4,5,6}, {2,3,4,5,7}, {2,3,4,5,8}, {2,3,4,5,9}, {2,3,4,5,10}}=Δ_4, the orbit containing exactly four of 1, 2, 3, 4, 5.

(f) $G_{\{1,2,3,4,5\}}\{6,7,8,9,10\} = \{6,7,8,9,10\} = \Delta_5$, the orbit containing none of 1, 2, 3, 4, 5.

From Lemmas 2.5.1 and 2.5.2, we deduce the following result:

Theorem 2.5.3

If $n \geq 2r$, the rank of $G = S_n$ acting on $X^{(r)}$ is $r+1$.

Proof

To start with, $G_{\{1,2,3,\ldots,r\}}$ has $r+1$ orbits. The $r+1$ suborbits are:

$Orb_{G_{\{1,2,3,...,r\}}}\{1, 2, 3, ..., r\} = \Delta_0$, the trivial orbit.

$Orb_{G_{\{1,2,3,...,r\}}}\{1, r+1, r+2, ..., 2r-1\} = \Delta_1$, the orbit containing exactly one of 1, 2, 3,..., r.

$Orb_{G_{\{1,2,3,...,r\}}}\{1, 2, r+1, ..., 2r-2\} = \Delta_2$, the orbit containing exactly two of 1, 2, 3,..., r.

$Orb_{G_{\{1,2,3,...,r\}}}\{1, 2, 3, r+1, ..., 2r-3\} = \Delta_3$, the orbit containing exactly three of 1, 2, 3,..., r.

....

$Orb_{G_{\{1,2,3,...,r\}}}\{1, 2, ..., r-1, r+1\}$ =Δ_{r-1}, the orbit containing exactly $r-1$ of either 1, 2, 3,..., r.

$Orb_{G_{\{1,2,3,...,r\}}}\{r+1, r+2, ..., 2r\} = \Delta_r$, the orbit containing none of 1, 2, 3,..., r. Finally, we prove that this is possible only if $n{\geq}2r$. Suppose $n - r = 0$, then $G_{\{1,2,3,...,r\}}$ has only one orbit, the trivial one; and if $n-r = 1$, $G_{\{1,2,3,...,r\}}$ has two orbits, namely, the trivial orbit and the one containing exactly $r-1$ of 1, 2, 3, ..., and r. If $n-r = 2$, $G_{\{1,2,3,...,r\}}$ has three orbits, namely, the trivial orbit, the one containing exactly $r-1$ of 1, 2, 3, ..., and r, and the one containing exactly $r-2$ of 1, 2, 3, ..., and r. Continuing with this argument, we find that if $n-r = r-1$, then $G_{\{1,2,3,...,r\}}$ has r orbits i.e the trivial one, the one containing exactly $r-1$ of 1, 2, 3,...,and r, the orbit containing exactly $r-2$ of 1, 2, 3, ...,r and so on up to the orbit containing exactly one of 1, 2, 3, ..., r. In a similar manner, if $n-r{\geq}r$, $G_{\{1,2,3,...,r\}}$ will have an additional orbit, the one containing none of 1, 2, 3, ..., and r. This makes it have $r+1$ orbits in total. We can rewrite $n-r{\geq}r$ as $n{\geq}2r$. □

Example 2.5.4

Let $G = S_6$ acting on $X^{(3)}$. Then rank of $G = 4$. The four suborbits of G are:

(a) $G_{\{1,2,3\}}\{1,2,3\}$=$\{\{1,2,3\}\}$=Δ_0, the trivial orbit.

(b) $G_{\{1,2,3\}}$\{1,4,5\}=\{\{1,4,5\},\{1,4,6\},\{1,5,6\},\{2,4,5\},\{2,4,6\}, \{2,5,6\},\{3,4,5\},\{3,4,6\},\{3,5,6\}\}=Δ_1, the orbit containing exactly one of 1, 2, and 3.

(c) $G_{\{1,2,3\}}$\{1,2,4\}=\{\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,3,4\},\{1,3,5\}, \{1,3,6\},\{2,3,4\},\{2,3,5\},\{2,3,6\}\}=Δ_2, the orbit containing exactly two of 1, 2, and 3.

(d) $G_{\{1,2,3\}}\{4,5,6\} = \{\{4,5,6\}\} = \Delta_3$, the orbit containing none of 1, 2, and 3.

Example 2.5.5

Let $G = S_9$ acting on $X^{(4)}$. Then rank of $G = 5$. The five suborbits of G are:

(a) $G_{\{1,2,3,4\}}\{1,2,3,4\} = \Delta_0$, the trivial orbit.

(b) $G_{\{1,2,3,4\}}$\{1,5,6,7\}=\{\{1,5,6,7\}, \{1,5,6,8\}, \{1,5,6,9\}, \{1,5,7,8\}, \{1,5,7,9\}, \{1,5,8,9\}, \{1,6,7,8\}, \{1,6,7,9\}, \{1,6,8,9\}, \{1,7,8,9\}, \{2,5,6,7\}, \{2,5,6,8\}, \{2,5,6,9\}, \{2,5,7,8\}, \{2,5,7,9\},\{2,5,8,9\}, \{2,6,7,8\}, \{2,6,7,9\}, \{2,6,8,9\}, \{2,7,8,9\},\{3,5,6,7\}, \{3,5,6,8\}, \{3,5,6,9\}, \{3,5,7,8\}, \{3,5,7,9\},\{3,5,8,9\}, \{3,6,7,8\}, \{3,6,7,9\}, \{3,6,8,9\}, \{3,7,8,9\}, \{4,5,6,7\}, \{4,5,6,8\}, \{4,5,6,9\}, \{4,5,7,8\}, \{4,5,7,9\},\{4,5,8,9\}, \{4,6,7,8\}, \{4,6,7,9\}, \{4,6,8,9\}, \{4,7,8,9\}\}=Δ_1, the orbit containing exactly one of 1, 2, 3, and 4.

(c) $G_{\{1,2,3,4\}}\{1,2,5,6\}$=\{\{1,2,5,6\}, \{1,2,5,7\}, \{1,2,5,8\}, \{1,2,5,9\},\{1,2,6,7\}, \{1,2,6,8\}, \{1,2,6,9\}, \{1,2,7,8\}, \{1,2,7,9\}, \{1,2,8,9\}, \{1,3,5,6\}, \{1,3,5,7\}, \{1,3,5,8\}, \{1,3,5,9\},\{1,3,6,7\}, \{1,3,6,8\}, \{1,3,6,9\}, \{1,3,7,8\}, \{1,3,7,9\}, \{1,3,8,9\}, \{1,4,5,6\}, \{1,4,5,7\}, \{1,4,5,8\}, \{1,4,5,9\},\{1,4,6,7\}, \{1,4,6,8\}, \{1,4,6,9\}, \{1,4,7,8\}, \{1,4,7,9\}, \{1,4,8,9\}, \{2,3,5,6\}, \{2,3,5,7\}, \{2,3,5,8\}, \{2,3,5,9\},\{2,3,6,7\}, \{2,3,6,8\}, \{2,3,6,9\}, \{2,3,7,8\}, \{2,3,7,9\}, \{2,3,8,9\}, \{2,4,5,6\}, \{2,4,5,7\}, \{2,4,5,8\}, \{2,4,5,9\},\{2,4,6,7\}, \{2,4,6,8\}, \{2,4,6,9\}, \{2,4,7,8\}, \{2,4,7,9\}, \{2,4,8,9\}, \{3,4,5,6\}, \{3,4,5,7\}, \{3,4,5,8\}, \{3,4,5,9\}, {3,4,6,7}, {3,4,6,8}, {3,4,6,9}, {3,4,7,8}, {3,4,7,9}, {3,4,8,9}} $= \Delta_2$, the orbit containing exactly two of 1, 2, 3, and 4.

(d) $G_{\{1,2,3,4\}}\{1,2,3,5\}$={{1,2,3,5}, {1,2,3,6}, {1,2,3,7}, {1,2,3,8},{1,2,3,9}, {1,2,4,5}, {1,2,4,6}, {1,2,4,7}, {1,2,4,8},{1,2,4,9}, {1,3,4,5}, {1,3,4,6}, {1,3,4,7}, {1,3,4,8},{1,3,4,9}, {2,3,4,5}, {2,3,4,6}, {2,3,4,7}, {2,3,4,8}, {2,3,4,9}}$= \Delta_3$, the orbit containing exactly three of 1, 2, 3, and 4.

(e) $G_{\{1,2,3,4\}}\{5,6,7,8\}$={{5,6,7,8}, {5,6,7,9},{5,6,8,9}, {5,7,8,9}, {6,7,8,9}} $= \Delta_4$, the orbit containing none of 1, 2, 3, and 4.

2.6 Self Paired Suborbits of S_n

In the next Theorem we show that all the suborbits of S_n are self paired.

Theorem 2.6.1

The suborbits $\Delta_0, \Delta_1, \Delta_2, ..., \Delta_{r-1}, \Delta_r$ of S_n acting on $X^{(r)}$ are self paired.

Proof

The proof for Δ_0 is trivial. Consider an arbitrary member of Δ_1 say $\{1, r+1, ..., 2r-1\}$. By the definition of a self paired suborbit (see section 1.1.2) if $g\{1, r+1, ..., 2r-1\} = \{1, 2, ..., r\}$, then we can take g to be $(1)(r+1\ 2)(r+2\ 3)...(2r-1\ r)$ and $g\{1, 2, ..., r\} = \{1, r+1, r+3, ..., 2r-1\} \in \Delta_1$. This shows that Δ_1 is self paired.

Similarly consider an arbitrary member of Δ_2 say $\{1, 2, r+1, ..., 2r-2\}$. If $g\{1, 2, r+1, ..., 2r-2\} = \{1, 2, ..., r\}$, then we can take g to be $(1)(2)(r+1\ 3)...(2r-2\ r)$ and $g\{1, 2, ..., r\} = \{1, 2, r+1, ..., 2r-2\} \in \Delta_2$, showing that Δ_2 is also self paired. Using similar arguments, we can show that $\Delta_3, \Delta_4, ..., \Delta_{r-1}$ are self paired. Finally, consider an arbitrary member of Δ_r say $\{r+1, r+2, ..., 2r\}$. If $g\{r+1, r+2, ..., 2r\} = \{1, 2, ..., r\}$, then we can take g to be $(r+1\ 1)(r+2\ 2)...(2r\ r)$ and $g\{1, 2, ..., r\} = \{r+1, r+2..., 2r\} \in \Delta_r$, showing that Δ_r is a self paired. $\square$

Example 2.6.2

Let $G = S_7$ acting on $X^{(3)}$, then Δ_1, Δ_2 and Δ_3 are self paired. This is because {1,4,5} $\in \Delta_1$, and we can take $g = (1)(4\ 2)(5\ 3)$. Consequently $gx = \{1, 4, 5\} \in \Delta_1$. Similarly {1,2,4}$\in \Delta_2$, $g = (1)(2)(4\ 3)$ and $gx = \{1, 2, 4\} \in \Delta_2$. Lastly {4,5,6}$\in \Delta_3$, $g = (4\ 1)(5\ 2)(6\ 3)$ and $gx = \{4, 5, 6\} \in \Delta_3$.

CHAPTER 3

SUBDEGREES OF S_n ACTING ON $X^{(r)}$

3.1 Introduction

The rank and the subdegrees of a group G are closely related in that while the rank is the number of the suborbits of G, the subdegrees are the sizes of these suborbits of G.

3.2 Subdegrees of S_n

The subdegrees of S_n are given by the following result.

Theorem 3.2.1

The subdegrees of S_n acting on $X^{(r)}$ are:

$$1, r\binom{n-r}{r-1}, \binom{r}{2}\binom{n-r}{r-2}, \binom{r}{3}\binom{n-r}{r-3}, \ldots, \binom{r}{r-1}\binom{n-r}{1}, \binom{n-r}{r}$$

Proof

The $r+1$ suborbits of G obtained in Theorem 2.5.3 are considered. The length of $\Delta_0 = 1$. Consider the suborbit Δ_1, which contains exactly one of $1, 2, 3, .., $ and r. One of 1, 2, 3,..., and r may be chosen in r ways while the remaining $r-1$

elements may be chosen from $n-r$ elements of X in $\begin{pmatrix} n-r \\ r-1 \end{pmatrix}$ ways. This makes the total number of selections to be $r\begin{pmatrix} n-r \\ r-1 \end{pmatrix}$ ways. For the suborbit Δ_2, the two elements may be chosen in $\begin{pmatrix} r \\ 2 \end{pmatrix}$ ways while the remaining $r-2$ elements may be chosen in $\begin{pmatrix} n-r \\ r-2 \end{pmatrix}$ ways, making a total of $\begin{pmatrix} r \\ 2 \end{pmatrix}\begin{pmatrix} n-r \\ r-2 \end{pmatrix}$ ways. Continuing with the same argument, in the suborbit Δ_{r-1}, the $r-1$ elements may be chosen in $\begin{pmatrix} r \\ r-1 \end{pmatrix}$ and the remaining 1 element in $\begin{pmatrix} n-r \\ 1 \end{pmatrix}$ ways making a total of $\begin{pmatrix} n-r \\ 1 \end{pmatrix}\begin{pmatrix} r \\ r-1 \end{pmatrix}$ ways. Finally, for Δ_r which does not contain any of elements from {1, 2, 3,..., r}, the r elements from X can be chosen in $\begin{pmatrix} n-r \\ r \end{pmatrix}$ ways. □

Example 3.2.2

If $G = S_7$ acting on $X^{(3)}$, the subdegrees of G are: $1, 3\begin{pmatrix} 4 \\ 2 \end{pmatrix}, 4\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, that is 1, 18, 12, 4. The four suborbits of G are:

(a) Δ_0 = {1,2,3}

(b) Δ_1 = {{1,4,5}, {1,4,6}, {1,4,7}, {1,5,6}, {1,5,7}, {1,5,7}, {2,4,5}, {2,4,6}, {2,4,7}, {2,5,6}, {2,5,7}, {2,6,7}, {3,4,5}, {3,4,6}, {3,4,7}, {3,5,6}, {3,5,7},{3,6,7}}

(c) Δ_2 = {{1,2,4}, {1,2,5}, {1,2,6}, {1,2,7}, {1,3,4}, {1,3,5}, {1,3,6, {1,3,7}, {2,3,4}, {2,3,5}, {2,3,6}, {2,3,7}}

(d) Δ_3 = {{4,5,6}, {4,5,7}, {4,6,7}, {5,6,7}}

$|\Delta_0| = 1, |\Delta_1| = 18, |\Delta_2| = 12, \text{and} |\Delta_3| = 4$. So that the subdegrees are 1, 4, 12, and 18 as expected.

Example 3.2.3

Let $G = S_{10}$ acting on $X^{(5)}$. The subdegrees of G are: $1, 5\begin{pmatrix}5\\4\end{pmatrix}, \begin{pmatrix}5\\2\end{pmatrix}\begin{pmatrix}5\\3\end{pmatrix}, \begin{pmatrix}5\\3\end{pmatrix}\begin{pmatrix}5\\2\end{pmatrix}, \begin{pmatrix}5\\4\end{pmatrix}\begin{pmatrix}5\\1\end{pmatrix}, \begin{pmatrix}5\\5\end{pmatrix}$, that is 1, 25, 100, 100, 25, 1.

Example 3.2.4

Let $G = S_{15}$ acting on $X^{(6)}$. The subdegrees of G are: $1, 6\begin{pmatrix}9\\5\end{pmatrix}, \begin{pmatrix}6\\2\end{pmatrix}\begin{pmatrix}9\\4\end{pmatrix}, \begin{pmatrix}6\\3\end{pmatrix}\begin{pmatrix}9\\3\end{pmatrix}, \begin{pmatrix}6\\4\end{pmatrix}\begin{pmatrix}9\\2\end{pmatrix}, \begin{pmatrix}6\\5\end{pmatrix}\begin{pmatrix}9\\1\end{pmatrix}, \begin{pmatrix}9\\6\end{pmatrix}$

, that is 1,756, 1890, 1680, 540, 54, 84.

3.3 Arrangements of Subdegrees

For practical purposes subdegrees are arranged in order of increasing magnitude. We achieve this using the following results.

Theorem 3.3.1

$\begin{pmatrix}r\\r-i\end{pmatrix} > \begin{pmatrix}r\\r-i-1\end{pmatrix}$, provided $(r-i) \leq \frac{1}{2}r$.

Proof

We use the principle of mathematical induction to prove this theorem.

For $i = 1$, $\begin{pmatrix}r\\r-1\end{pmatrix} = \begin{pmatrix}r\\1\end{pmatrix} = \frac{r!}{(r-1)!} = r$, while $\begin{pmatrix} r \\ r-2 \end{pmatrix} = \begin{pmatrix} r \\ 2 \end{pmatrix} = \frac{r!}{(r-2)!2!} = \frac{1}{2}r(r-1)$

Since $i = 1$, $(r-1) \leq \frac{1}{2}r$ implies that $r \leq 2$, or $r = \{1, 2\}$ since r is a positive integer. And for this set of values of r, $r > \frac{1}{2}r(r-1)$. So the theorem is true for $i = 1$.

If we assume true for $i = k$, then $\begin{pmatrix} r \\ r-k \end{pmatrix} > \begin{pmatrix} r \\ r-k-1 \end{pmatrix}$ which implies that

$$\frac{r!}{(r-k)!k!} > \frac{r!}{(r-k-1)!(k+1)!} \tag{3.1}$$

For $i = k+1$,

$$\begin{aligned} \begin{pmatrix} r \\ r-k-1 \end{pmatrix} &= \begin{pmatrix} r \\ k+1 \end{pmatrix} \\ &= \frac{r!}{(r-k-1)!(k+1)!} \\ &= \frac{r!(r-k)}{(r-k)!(k+1)k!} \\ &= \frac{r-k}{k+1} \times \frac{r!}{(r-k)!k!} \end{aligned} \tag{3.2}$$

$$\begin{aligned} \begin{pmatrix} r \\ r-k-2 \end{pmatrix} &= \begin{pmatrix} r \\ k+2 \end{pmatrix} \\ &= \frac{r!}{(r-k-2)!(k+2)!} \\ &= \frac{r!(r-k-1)}{(r-k-1)!(k+2)(k+1)!} \\ &= \frac{r-k-1}{k+2} \times \frac{r!}{(r-k-1)!(k+1)!} \end{aligned} \tag{3.3}$$

$$\begin{aligned}
\frac{r-k}{k+1} - \frac{r-(k+1)}{k+2} &= \frac{(k+2)(r-k)-(k+1)(r-(k+1))}{(k+1)(k+2)} \\
&= \frac{kr+2r-k^2-2k-kr-r+(k+1)^2}{(k+1)(k+2)} \\
&= \frac{r-k^2+k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{r+2k+1}{(k+1)(k+2)} > 0 \qquad (3.4)
\end{aligned}$$

From 3.4 it follows that,

$$\frac{r-k}{k+1} > \frac{r-k-1}{k+2} \qquad (3.5)$$

From inequality 3.1,

$$\frac{r!}{(r-k)!k!} > \frac{r!}{(r-k-1)!(k+1)!}$$

and since

$$\frac{r-k}{k+1} > \frac{r-k-1}{k+2}$$

it therefore follows that

$$\binom{r}{r-k-1} > \binom{r}{r-k-2}$$

which shows that the Theorem is true for $i = k+1$ whenever true for $i = k$, and so true for all $i \geq 1$. □

Theorem 3.3.2
$\binom{r}{r-i} < \binom{r}{r-i-1}$ whenever $(r-i) \geq \frac{1}{2}r$.

Proof for this theorem is similar to that of Theorem 3.3.1

According to Theorem 3.2.1 the subdegrees of S_n acting on $X^{(r)}$ are given by

$$1, r\binom{n-r}{r-1}, \binom{r}{2}\binom{n-r}{r-2}, \binom{r}{3}\binom{n-r}{r-3}, \ldots, \binom{r}{r-1}\binom{n-r}{1}, \binom{n-r}{r} \tag{3.6}$$

Since

$$\binom{n}{r} = \binom{n}{n-r},$$

if we omit 1 from this sequence, may rewrite expression 3.6 as

$$\binom{n-r}{r}, \binom{r}{r-1}\binom{n-r}{r-1}, \binom{r}{r-2}\binom{n-r}{r-2}, \binom{r}{r-3}\binom{n-r}{r-3}, \ldots, \binom{r}{3}\binom{n-r}{3}, \binom{r}{2}\binom{n-r}{2}, \binom{r}{1}\binom{n-r}{1} \tag{3.7}$$

Using expression 3.7 and Theorems 3.3.1 and 3.3.2 we may arrange the subdegrees in any desired order. As a matter of fact, the terms of expression 3.7 are in an increasing order up to a certain term (which is at the median position of the terms of expression 3.7 when r is an odd number and at $\frac{r+2}{2}$ position of expression 3.7 when r is an even number) after which the terms start decreasing. This is because if we re-write expression 3.7 as in Table 3.1 where the product is done on the terms in the same column in the table, we see that each row in the table

$\binom{n-r}{r}$	$\binom{n-r}{r-1}$	$\binom{n-r}{r-2}$	$\binom{n-r}{r-3}$	...	$\binom{n-r}{3}$	$\binom{n-r}{2}$	$\binom{n-r}{1}$
	$\binom{r}{r-1}$	$\binom{r}{r-2}$	$\binom{r}{r-3}$	...	$\binom{r}{3}$	$\binom{r}{2}$	$\binom{r}{1}$

Table 3.1: Products of expression 3.7

represents a certain row of a Pascal's triangle and so the product of the terms will be in increasing order up to some term and then the terms starts decreasing. Some direct computations shows that the largest term in the sequence is at the median position when r is an odd number and at $\frac{r+2}{2}$ position when r is an even number. The Pascal's triangle is shown in Figure 3.1.

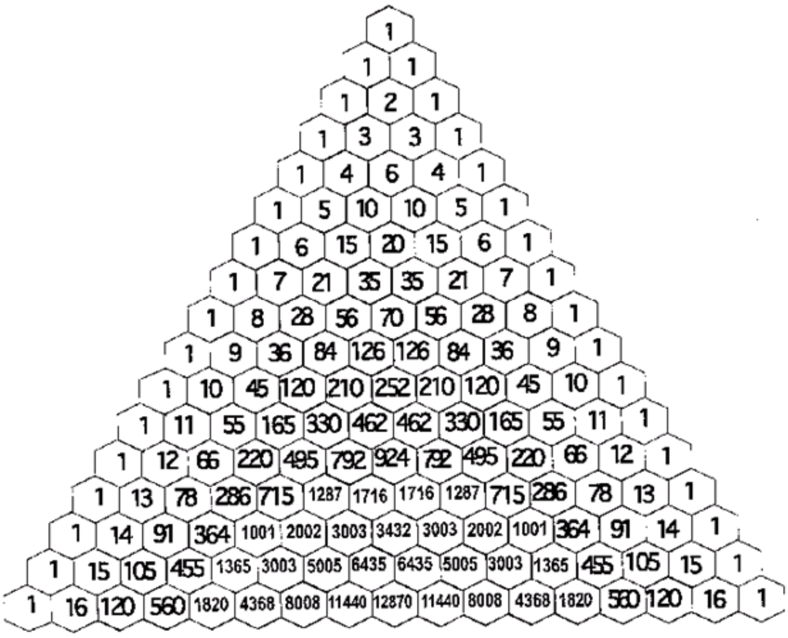

Figure 3.1: Pascal's Triangle

Theorem 3.3.3

If r is an odd number and if $\frac{1}{2}(n-r) \leq i < \frac{1}{2}(r+1),\ i \in \mathbb{N}$, the arrangements of expression 3.7 in an increasing order is

$$\binom{n-r}{r} < \binom{r}{1}\binom{n-r}{1} < \binom{r}{r-1}\binom{n-r}{r-1} < \binom{r}{2}\binom{n-r}{2} <$$
$$\binom{r}{r-2}\binom{n-r}{r-2} < \binom{r}{3}\binom{n-r}{3} < \binom{r}{r-3}\binom{n-r}{r-3} < \dots$$
$$< \binom{r}{\frac{r+1}{2}}\binom{n-r}{r-\frac{r+1}{2}} \tag{3.8}$$

Proof

The proof is accomplished by showing that the second term of the sequence is less than the third and by showing that the third term is less than the fourth but in a general manner.

$$\binom{r}{i}\binom{n-r}{i} < \binom{r}{r-i}\binom{n-r}{r-i}$$

Since

$$\binom{r}{r-i} = \binom{r}{i}$$

and

$$\binom{n-r}{r-i} > \binom{n-r}{i} \text{ when } i \geq \frac{1}{2}(n-r)$$

Also

$$\binom{r}{i}\binom{n-r}{i} > \binom{r}{r-(i-1)}\binom{n-r}{r-(i-1)}$$

To show this, note that

$$\binom{r}{r-(i-1)} = \binom{r}{i-1}$$

and

$$\binom{r}{i} > \binom{r}{i-1} \text{ for } i \leq \frac{1}{2}r$$

Also

$$\binom{n-r}{i} > \binom{n-r}{r-(i-1)} \text{ if } i < r-(i-1) \text{ and } i \geq \frac{1}{2}(n-r)$$

Combining these inequalities,

$$\binom{n-r}{i} > \binom{n-r}{r-(i-1)} \text{ if } \frac{1}{2}(n-r) \leq i < \frac{1}{2}(r+1)$$

and so

$$\binom{r}{i}\binom{n-r}{i} > \binom{r}{r-(i-1)}\binom{n-r}{r-(i-1)}$$

whenever $\frac{1}{2}(n-r) \leq i < \frac{1}{2}(r+1)$.

The proof is completed by showing that

$$\binom{r}{\frac{r+1}{2}}\binom{n-r}{r-\frac{r+1}{2}}$$

is the largest term. If the terms of expression 3.7 were a sequence of r terms, the median position of this sequence of subdegrees is given by $\frac{1}{2}(r+1)$ if r is an odd number. By Figure 3.1 the coefficient at the median position is the largest one.

□

Theorem 3.3.4

If r is an even number, and if $\frac{1}{2}(n-r) \leq i < \frac{1}{2}(r+1),\ i \in \mathbb{N}$, the arrangements of expression 3.7 in an increasing order is given by

$$\begin{pmatrix} n-r \\ r \end{pmatrix} < \begin{pmatrix} r \\ 1 \end{pmatrix}\begin{pmatrix} n-r \\ 1 \end{pmatrix} < \begin{pmatrix} r \\ r-1 \end{pmatrix}\begin{pmatrix} n-r \\ r-1 \end{pmatrix} < \begin{pmatrix} r \\ 2 \end{pmatrix}\begin{pmatrix} n-r \\ 2 \end{pmatrix} <$$
$$\begin{pmatrix} r \\ r-2 \end{pmatrix}\begin{pmatrix} n-r \\ r-2 \end{pmatrix} < \begin{pmatrix} r \\ 3 \end{pmatrix}\begin{pmatrix} n-r \\ 3 \end{pmatrix} < \begin{pmatrix} r \\ r-3 \end{pmatrix}\begin{pmatrix} n-r \\ r-3 \end{pmatrix} < \ldots$$
$$< \begin{pmatrix} r \\ \frac{r+2}{2} \end{pmatrix}\begin{pmatrix} n-r \\ r-\frac{r+2}{2} \end{pmatrix} \tag{3.9}$$

The proof for this Theorem is similar to that for Theorem 3.3.3.

We may rewrite the terms of expression 3.8 as

$$\begin{pmatrix} n-r \\ r \end{pmatrix} < \begin{pmatrix} r \\ r-1 \end{pmatrix}\begin{pmatrix} n-r \\ n-(r+1) \end{pmatrix} < \begin{pmatrix} r \\ r-1 \end{pmatrix}\begin{pmatrix} n-r \\ r-1 \end{pmatrix} <$$
$$\begin{pmatrix} r \\ r-2 \end{pmatrix}\begin{pmatrix} n-r \\ n-(r+2) \end{pmatrix} < \begin{pmatrix} r \\ r-2 \end{pmatrix}\begin{pmatrix} n-r \\ r-2 \end{pmatrix} <$$
$$\begin{pmatrix} r \\ r-3 \end{pmatrix}\begin{pmatrix} n-r \\ n-(r+3) \end{pmatrix} < \begin{pmatrix} r \\ r-3 \end{pmatrix}\begin{pmatrix} n-r \\ r-3 \end{pmatrix} < \ldots$$
$$< \begin{pmatrix} r \\ r-\frac{r+1}{2} \end{pmatrix}\begin{pmatrix} n-r \\ r-\frac{r+1}{2} \end{pmatrix} \tag{3.10}$$

while those of expression 3.9 may be rewritten as

$$\binom{n-r}{r} < \binom{r}{r-1}\binom{n-r}{n-(r+1)} < \binom{r}{r-1}\binom{n-r}{r-1} <$$
$$\binom{r}{r-2}\binom{n-r}{n-(r+2)} < \binom{r}{r-2}\binom{n-r}{r-2} <$$
$$\binom{r}{r-3}\binom{n-r}{n-(r+3)} < \binom{r}{r-3}\binom{n-r}{r-3} < ...$$
$$< \binom{r}{r-\frac{r+2}{2}}\binom{n-r}{r-\frac{r+2}{2}} \qquad (3.11)$$

Example 3.3.5

Let $G = S_{19}$ acting on $X^{(9)}$. By expression 3.10. The arrangements on the subdegrees of S_{19} in increasing order is given by

$$\binom{10}{9} < \binom{9}{8}\binom{10}{9} < \binom{9}{8}\binom{10}{8} < \binom{9}{7}\binom{10}{8} < \binom{9}{7}\binom{10}{7}$$
$$< \binom{9}{6}\binom{10}{7} < \binom{9}{6}\binom{10}{6} < \binom{9}{5}\binom{10}{5}$$

that is $10 < 90 < 405 < 1620 < 4320 < 10080 < 17640 < 26460 < 31752$

Example 3.3.6

Let $G = S_{17}$ acting on $X^{(8)}$. By expression 3.11. The arrangements on the subdegrees of S_{17} in increasing order is given by

$$\binom{9}{8} < \binom{8}{7}\binom{9}{8} < \binom{8}{7}\binom{9}{7} < \binom{8}{6}\binom{9}{7} < \binom{8}{6}\binom{9}{6}$$
$$< \binom{8}{5}\binom{9}{6} < \binom{8}{5}\binom{9}{5} < \binom{8}{4}\binom{9}{5}$$

That is $9 < 72 < 288 < 1008 < 2352 < 4704 < 7056 < 8820$

Remark 3.3.7

(a) Theorem 3.3.3 and Theorem 3.3.4 work when $2r \leq n \leq 2r+2$. Also when $n \geq 2r+3$, expressions 3.8 and 3.9 are preserved except that

$$\binom{r}{1}\binom{n-r}{1} < \binom{n-r}{r}$$

and

$$\binom{r}{r-i}\binom{n-r}{r-i} < \binom{r}{i}\binom{n-r}{i}$$

(b) When $n = 2r$, then the strict inequalities in Theorem 3.3.3 and 3.3.4 are replaced with the less than or equal to inequalities. In this case all the terms apart from the smallest and the largest one can be grouped into equal pairs in the sequence. Also, the terms contain the squares of a row of Pascal's triangle. This is because since $n = 2r$ implies $n-r = r$, so that the products $\binom{r}{i}\binom{n-r}{i}$ and $\binom{r}{r-i}\binom{n-r}{r-i}$

can be written as $\begin{pmatrix} r \\ i \end{pmatrix} \begin{pmatrix} r \\ i \end{pmatrix}$ and $\begin{pmatrix} r \\ r-i \end{pmatrix} \begin{pmatrix} r \\ r-i \end{pmatrix}$ respectively. The fact that $\begin{pmatrix} r \\ i \end{pmatrix} = \begin{pmatrix} r \\ r-i \end{pmatrix}$ explains why we have equal pairs in this case.

Table 3.2 serves to clarify Remark 3.3.7. In this table we analyse the arrangements of subdegrees of S_n as $n-2r$ increases from 0 to 2 and when $n-2r \geq 3$.

	S_{20} on $X^{(10)}$	S_{17} on $X^{(8)}$	S_{18} on $X^{(8)}$	S_{17} on $X^{(7)}$	S_{18} on $X^{(7)}$
${}_{n-r}C_r$	1	9	45	120	330
${}_rC_1 \times_{n-r} C_{r-1}$	100	288	960	147	3234
${}_rC_2 \times_{n-r} C_{r-2}$	2025	2352	5880	5292	9702
${}_rC_3 \times_{n-r} C_{r-3}$	14400	7056	14112	7350	11550
${}_rC_4 \times_{n-r} C_{r-4}$	44100	8820	14700	4200	5775
${}_rC_5 \times_{n-r} C_{r-5}$	63504	4704	6720	945	1155
${}_rC_6 \times_{n-r} C_{r-6}$	44100	1008	1260	70	77
${}_rC_7 \times_{n-r} C_{r-7}$	14400	72	80		
${}_rC_8 \times_{n-r} C_{r-8}$	2025				
${}_rC_9 \times_{n-r} C_{r-9}$	100				

Table 3.2: Some Subdegrees of S_n

3.4 Primitivity of S_n

The primitivity S_n acting on $X^{(r)}$ is determined in this section. This is given by the following result.

Theorem 3.4.1

If $2 \leq r < \frac{1}{2}n$, then the action of S_n on $X^{(r)}$ is primitive.

Proof (By contradiction)

From Theorem 1.1.4, suppose G is imprimitive with subdegrees given in Theorems 3.3.3 and 3.3.4, then

$$\binom{n-r}{r}\binom{r}{i}\binom{n-r}{i} \geq \binom{r}{r-i}\binom{n-r}{r-i} \tag{3.12}$$

This is because

$$\binom{r}{i} = \binom{r}{r-i}$$

and whenever

$$\binom{n-r}{i} < \binom{n-r}{r-i},$$

$$\binom{n-r}{r}\binom{n-r}{i} \geq \binom{n-r}{r-i}$$

Also,

$$\binom{n-r}{r}\binom{r}{r-i}\binom{n-r}{r-i} \geq \binom{r}{i+1}\binom{n-r}{i+1} \tag{3.13}$$

This is because

$$\binom{r}{r-i} \geq \binom{r}{i+1} \text{ and } \binom{n-r}{r-i} \geq \binom{n-r}{i+1}$$

for some choice of r and i and whenever

$$\binom{r}{r-i} \leq \binom{r}{i+1} \text{ and or } \binom{n-r}{r-i} \leq \binom{n-r}{i+1}$$

then 3.13 is true.

$$\binom{n-r}{r}$$

is the 'n_1' of Theorem 3.3.3 and Theorem 3.3.4 while

$$\binom{r}{i}\binom{n-r}{i}, \binom{r}{r-i}\binom{n-r}{r-i}$$

and

$$\binom{r}{r-i}\binom{n-r}{r-i}, \binom{r}{i+1}\binom{n-r}{i+1}$$

are consecutive terms in Theorems 3.3.3 and 3.3.4. By Theorem 1.1.4, inequality 3.12 and inequality 3.13 leads to a contradiction, and so, S_n acts primitively on $X^{(r)}$. $\square$

CHAPTER 4

SUBORBITAL GRAPHS OF S_n

4.1 Introduction

In this chapter the suborbital graphs of S_n acting on $X^{(r)}$ will be constructed. To each suborbit of S_n say Δ_i, there will be a corresponding suborbital graph Γ_i. The properties of these graphs shall then be analysed.

4.2 Suborbital Graphs of S_n Acting on $X^{(r)}$ and Their Properties

The construction of the suborbital graphs corresponding to the suborbits of S_n given in Theorem 2.5.3 is given as follows:

The suborbital O_1 corresponding to the suborbit Δ_1 is
$O_1 = \{[g\{1, 2, 3, ..., r\}, g\{1, r + 1, r + 2, ..., 2r - 1\}]\}$, where $g \in S_n$. Therefore, in Γ_1, the suborbital graph corresponding to O_1, there is an edge from vertex A to B if and only if $|A \cap B| = 1$.
The suborbital O_2 corresponding to the suborbit Δ_2 is
$O_2 = \{[g\{1, 2, 3, ..., r\}, g\{1, 2, r + 1, ..., 2r - 2\}]\}$, $g \in S_n$. Therefore, in Γ_2, the suborbital graph corresponding to O_2, there is an edge from vertex A to B if and only if $|A \cap B| = 2$.
The suborbital O_3 corresponding to the suborbit Δ_3 is
$O_3 = \{[g\{1, 2, 3, ..., r\}, g\{1, 2, 3, r + 1, ..., 2r - 3\}]\}$, $g \in S_n$. Therefore, in Γ_3, the suborbital graph corresponding to O_3, there is an edge from vertex A to B if and

only if $|A \cap B| = 3$.

....

The suborbital O_{r-1} corresponding to the suborbit Δ_{r-1} is

$O_{r-1} = \{[g\{1, 2, 3, ..., r\}, g\{1, 2, 3, ..., r-1, r+1\}]\}$, $g \in S_n$. Therefore, in Γ_{r-1}, the suborbital graph corresponding to O_{r-1}, there is an edge from vertex A to B if and only if $|A \cap B| = r - 1$.

The suborbital O_r corresponding to the suborbit Δ_r is

$O_r = \{[g\{1, 2, 3, ..., r\}, g\{r+1, r+2, ..., 2r\}]\}$, $g \in S_n$. Therefore, in Γ_r, the suborbital graph corresponding to O_r, there is an edge from vertex A to B if and only if $|A \cap B| = 0$

Theorem 4.2.1

(a) $\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_r$ are undirected.

(b) If $n \geq 3r$, $\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_r$ have girth 3.

(c) $\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_r$ are connected if $n > \frac{1}{2}r$.

Proof

(a) Using Theorem 2.6.1, $\Delta_i, i = 1, 2, 3, ..., r$ are self-paired, implying that $\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_r$ are undirected.

(b) Let $X = \{1, 2, 3, ..., n\}$ and suppose that $n \geq 3r$. Then there exists three unordered r-element subsets of X, say $A, B,$ and C such that,

$|A \cap B| = |A \cap C| = |B \cap C| = 1$; $|A \cap B| = |A \cap C| = |B \cap C| = 2$; ...

$|A \cap B| = |A \cap C| = |B \cap C| = r - 1$; $|A \cap B| = |A \cap C| = |B \cap C| = 0$.

Thus in each case $A, B,$ and C are adjacent vertices in $\Gamma_1, \Gamma_2, ..., \Gamma_r$ respectively. Therefore if $n \geq 3r$, $\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_r$ have girth 3.

(c) By Theorem 3.4.1, G acts primitively on $X^{(r)}$. So, by Theorem 1.1.5, $\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_r$ are connected. □

Remark 4.2.2

Theorem 4.2.1 (b) means that if $n < 3r$, then some $\Gamma_i, i = 1, 2, ..., r$ may be of girth 3 but not all of them. This theorem gives a sufficient condition for all of Γ_i to have girth 3.

Example 4.2.3

Let $G = S_5$ acting on $X^{(2)}$. We shall base our discussions on Δ_1, and Δ_2. The suborbital O_1 corresponding to the suborbit Δ_1 is
$O_1 = \{[g\{1,2\}, g\{1,3\}] | g \in G\}$. The suborbital graph Γ_1 corresponding to the suborbital O_1 has 2-element subsets A and B from X adjacent if and only if $|A \cap B| = 1$.
Secondly, the suborbital O_2 corresponding to the suborbit Δ_2 is
$O_2 = \{[g\{1,2\}, g\{3,4\}] | g \in G\}$. The suborbital graph Γ_2 corresponding to O_2 has 2-element subsets A and B from X adjacent if and only if $|A \cap B| = 0$.

Using Theorem 4.2.1, Γ_1 and Γ_2 are undirected. We construct Γ_1 as in Figure 4.1.

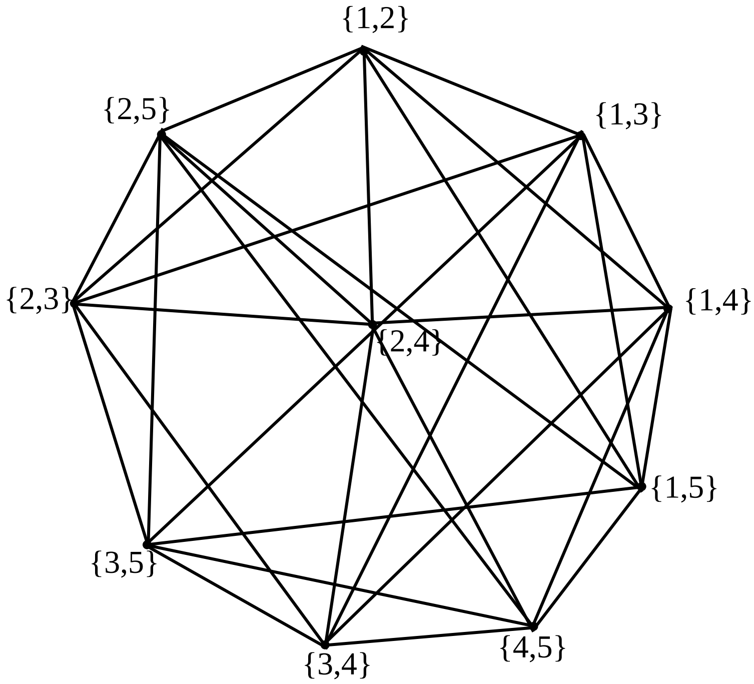

Figure 4.1: Γ_1, the suborbital graph of S_5 acting on $X^{(2)}$corresponding to Δ_1

From Figure 4.1, Γ_1 is connected, regular of degree 6 and has girth 3.

We also construct Γ_2 as in Figure 4.2

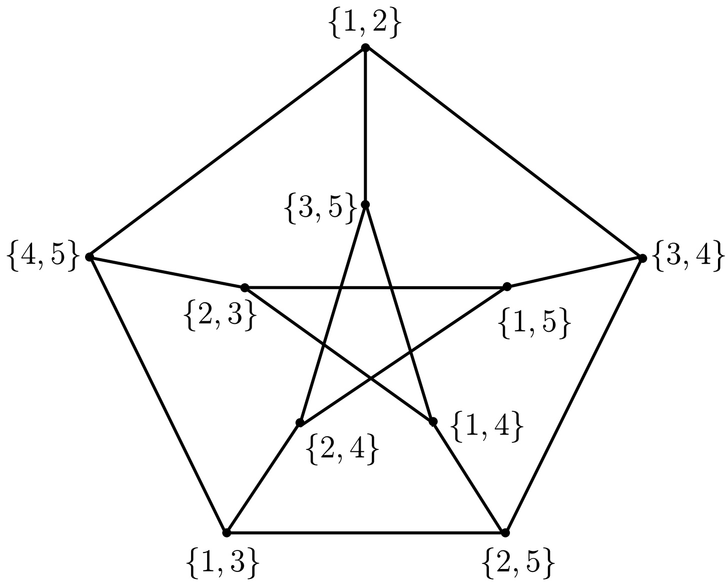

Figure 4.2: Γ_2, the suborbital graph of S_5 acting on $X^{(2)}$corresponding to Δ_2

From Figure 4.2, Γ_2 is regular of degree 3. It is connected and has girth 5.

Example 4.2.4

Let $G = S_6$ acting on $X^{(3)}$. The suborbital O_1 corresponding to the suborbit Δ_1 is $O_1 = \{[g\{1,2,3\}, g\{1,4,5\}] | g \in G\}$. The suborbital graph Γ_1 corresponding to the suborbital O_1 has 3-element subsets A and B from X adjacent if and only if $|A \cap B| = 1$.

Secondly, the suborbital O_2 corresponding to the suborbit Δ_2 is
$O_2 = \{[g\{1,2,3\}, g\{1,2,4\}] | g \in G\}$. The suborbital graph Γ_2 corresponding to O_2 has 3-element subsets A and B from X adjacent if and only if $|A \cap B| = 2$.

Lastly, the suborbital O_3 corresponding to the suborbit Δ_3 is
$O_3 = \{[g\{1,2,3\}, g\{4,5,6\}] | g \in G\}$. The suborbital graph Γ_3 corresponding to O_3 has 3-element subsets A and B from X adjacent if and only if $|A \cap B| = 0$.

By Theorem 1.1.5, Γ_1 and Γ_2 are connected. Also, by Theorem 4.2.1, Γ_1, Γ_2 and Γ_3 are undirected. Γ_1 and Γ_2 are of girth 3 while Γ_3 has no cycles. Γ_1 is regular of degree $|\Delta_1| = 9$, Γ_2 is also regular of degree 9 and Γ_3 is regular of degree 1 .

Example 4.2.5

Let $G = S_8$ acting on $X^{(4)}$. The suborbital O_1 corresponding to the suborbit Δ_1 is $O_1 = \{[g\{1,2,3,4\}, g\{1,5,6,7\}]|g \in G\}$, the suborbital graph Γ_1 corresponding to the suborbital O_1 has 4-element subsets A and B from X adjacent if and only if $|A \cap B| = 1$.

Secondly, the suborbital O_2 corresponding to the suborbit Δ_2 is $O_2 = \{[g\{1,2,3,4\}, g\{1,2,5,6\}]|g \in G\}$. The suborbital graph Γ_2 corresponding to O_2 has 4-element subsets A and B from X adjacent if and only if $|A \cap B| = 2$.

Thirdly,, the suborbital O_3 corresponding to the suborbit Δ_3 is $O_3 = \{[g\{1,2,3,4\}, g\{1,2,3,5\}]|g \in G\}$. The suborbital graph Γ_3 corresponding to O_3 has 4-element subsets A and B from X adjacent if and only if $|A \cap B| = 3$.

Lastly, the suborbital O_4 corresponding to the suborbit Δ_4 is $O_4 = \{[g\{1,2,3,4\}, g\{5,6,7,8\}]|g \in G\}$. The suborbital graph Γ_4 corresponding to O_4 has 4-element subsets A and B from X adjacent if and only if $|A \cap B| = 0$.

By Theorem 1.1.5, Γ_1, Γ_2, and Γ_3 are connected. Also, by Theorem 4.2.1, $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 are undirected.

Γ_1, Γ_2 and Γ_3 are of girth 3 since for $X = \{1,2,3,4,5,6,7,8\}$, there exists three 4-elements subsets of X say A, B, and C such that $|A \cap B| = 1$ or 2 or 3, $|A \cap C| = 1$ or 2 or 3 and $|B \cap C| = 1$ or 2 or 3. This implies that A, B, and C are vertices of Γ_1, Γ_2 and Γ_3 with cycles of length 3.

Γ_4 has no cycles and so has girth 0. $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 are regular of degree 16, 36, 16 and 1 respectively.

Example 4.2.6

Let $G = S_{15}$ acting on $X^{(5)}$. The suborbital O_1 corresponding to the suborbit Δ_1 is $O_1 = \{[g\{1,2,3,4,5\}, g\{1,6,7,8,9\}]|g \in G\}$, the suborbital graph Γ_1 corresponding to the suborbital O_1 has 5-element subsets A and B from

$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ adjacent if and only if $|A \cap B| = 1$.

Secondly, the suborbital O_2 corresponding to the suborbit Δ_2 is

$O_2 = \{[g\{1, 2, 3, 4, 5\}, g\{1, 2, 6, 7, 8\}]|g \in G\}$. The suborbital graph Γ_2 corresponding to O_2 has 5-element subsets A and B from X adjacent if and only if $|A \cap B| = 2$.

Thirdly, the suborbital O_3 corresponding to the suborbit Δ_3 is

$O_3 = \{[g\{1, 2, 3, 4, 5\}, g\{1, 2, 3, 6, 7\}]|g \in G\}$. The suborbital graph Γ_3 corresponding to O_3 has 5-element subsets A and B from X adjacent if and only if $|A \cap B| = 3$.

Also, the suborbital O_4 corresponding to the suborbit Δ_4 is

$O_4 = \{[g\{1, 2, 3, 4, 5\}, g\{1, 2, 3, 4, 6\}]|g \in G\}$. The suborbital graph Γ_4 corresponding to O_4 has 5-element subsets A and B from X adjacent if and only if $|A \cap B| = 4$.

Lastly, the suborbital O_5 corresponding to the suborbit Δ_5 is

$O_5 = \{[g\{1, 2, 3, 4, 5\}, g\{6, 7, 8, 9, 10\}]|g \in G\}$. The suborbital graph Γ_5 corresponding to O_5 has 5-element subsets A and B from X adjacent if and only if $|A \cap B| = 0$.

By Theorems 1.1.5 and 3.4.1, $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ and Γ_5 are connected. Also, by Theorem 4.2.1, $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ and Γ_5 are undirected and have girth 3. $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ and Γ_5 are regular of degree 1050, 1200, 450, 50, and 252 respectively.

CHAPTER 5

CONCLUSIONS AND AREAS FOR FURTHER RESEARCH

5.1 Introduction

Some conclusions of this study and suggestions of areas for further research are given in this chapter. A few applications of this study are also given.

5.2 Conclusions

In this study, some properties of the action of S_n on unordered r-element subsets of X, were discussed.

In chapter two it was shown that the order of the stabilizer of $\{1, 2, ..., r\}$ is $(n-r)!r!$. It was proved that S_n acts transitively on $X^{(r)}$. In addition, a proposition that gives a general formula for calculating the number of permutations of a given cycle type fixing an element of $X^{(r)}$ was given. A general formula that calculates $|\text{Fix}(g)|$ in $X^{(r)}$ was derived. We also showed that the rank of S_n acting on $X^{(r)}$ is $r + 1$ if $n \geq 2r$, and proved that all the suborbits of S_n acting on $X^{(r)}$ are self paired.

In chapter three it was proved that the subdegrees of S_n acting on $X^{(r)}$ are

$$1, r\binom{n-r}{r-1}, \binom{r}{2}\binom{n-r}{r-2}, \binom{r}{3}\binom{n-r}{r-3}, \ldots,$$
$$\binom{r}{r-1}\binom{n-r}{1}, \binom{n-r}{r}$$

In addition, the criteria for arranging the subdegrees in ascending order and the proof of S_n acts primitively on $X^{(r)}$ were given.

In chapter four the suborbital graphs of S_n acting on $X^{(r)}$ were constructed. It was shown that the suborbital graphs are undirected, and that $\Gamma_1, \Gamma_2, ..., \Gamma_r$ are of girth three if $n \geq 3r$. In addition, it was proved that $\Gamma_1, \Gamma_2, ..., \Gamma_r$ are connected if $n > 2r$.

5.3 Suggestions of Areas for Further Research

The following are some topics, which we hope will be tackled in future:

1. The proof of Proposition 2.4.4.

2. Proof of Proposition 2.2.6.

5.4 Applications

The following are some applications of this study.

5.4.1 Group Actions and Suborbital Graphs

Some properties of the suborbital graphs of a group G may be deduced by studying the properties of the group actions. For example one can tell whether the suborbital graphs are directed or undirected by determining whether the suborbits of G are self paired or not.

On the other hand we can deduce some properties of a group action of G by studying the suborbital graphs of G. For example one can tell whether G acts primitively or imprimitively by determining whether the suborbital graphs of G are connected or not.

5.4.2 Probability and Statistics

Consider the space X of the unordered pairs $\{i, j\}$ from X of cardinality $\begin{pmatrix} n \\ 2 \end{pmatrix}$. The symmetric group S_n acts on these pairs by $\pi\{i, j\} = \{\pi(i), \pi(j)\}$. The permutation representation generated by this action can be described as an $\begin{pmatrix} n \\ 2 \end{pmatrix}$ dimensional vector space spanned by basis vectors $e_{\{i,j\}}$. This space splits into three irreducibles: A one-dimensional trivial representation is spanned by $\bar{v} = \sum e_{\{i,j\}}$, an $n-1$ dimensional space is spanned by $v_i = \sum_j e_{\{i,j\}} - c\bar{v}, 1 \leq i \leq n$, with c chosen so v_i is orthogonal to $\bar{v}$. The complement of these two spaces is also an irreducible representation. We define the permutation representation associated to the action of S_n on tabloids as a vector space with basis $e_{\{t\}}$. It is denoted M^λ. Let G be a finite group acting on a set X. Extend the action to the product space X^k coordinatewise. The number of fixed points of the element $s \in G$ is $F(s) = |\{x : sx = x\}|$. For any positive integer k:

$$\frac{1}{|G|} \sum_{s} F(s)^k = |\text{orbits of } G \text{ acting on } X^k| \tag{5.1}$$

Here a few consequences of equation 5.1 according to Diaconis (1988):

1. With k = 1, this is the Cauchy-Frobenius lemma.

2. When G acts on itself we get back the decomposition of the regular representation.

3. There is a connection with probability problems. If G is considered as a probability space under the uniform distribution U, then $F(s)$ is a "random variable" corresponding to "pick an element of G at random and count how many fixed points it has." When $G = S_n$ and $X = \{1, 2, ..., n\}$, $F(s)$ is the number of fixed points of s. We know that this has an approximate Poisson distribution with mean 1. 5.1 gives a "formula" for all the moments of $F(s)$.

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APPENDICES

Appendix A

A Scilab program that calculates the number of permutations of S_n, $n \leq 10$, that fixes $X^{(r)}$, $3 \leq r \leq 5$, and having the same cycle type

A.1 Introduction

In this program, the inputs are n, r, c1, c2, c3, c4, c5, c6, c7, c8, c9, c10, where c1, c2, ..., c10 is the cycle type of a permutation $g \in S_n$. The program is developed using expressions 2.2, 2.3 and 2.4. The program is invoked by per1 = fix(n, r, c1, c2, c3, c4, c5, c6, c7, c8, c9, c10) where per1 is the output while the others are inputs. The program is typed in the Scilab Editor. We generate tables 2.1, 2.2, 2.3, and 2.4 by use of this program.

A.2 Scilab Program

function [per] = fix(n, r, c1, c2, c3, c4, c5, c6, c7, c8, c9, c10)

A1 = [factorial(c2) factorial(c3) factorial(c4) factorial(c5) factorial(c6) factorial(c7) factorial(c8) factorial(c9) factorial(c10);$2^{(c2)}3^{(c3)}4^{(c4)}5^{(c5)}6^{(c6)}7^{(c7)}8^{(c8)}9^{(c9)}10^{(c10)}$];

A2 = [factorial(c3) factorial(c4) factorial(c5) factorial(c6) factorial(c7) factorial(c8) factorial(c9) factorial(c10);$3^{(c3)}4^{(c4)}5^{(c5)}6^{(c6)}7^{(c7)}8^{(c8)}9^{(c9)}10^{(c10)}$];

A3 = [factorial(c4) factorial(c5) factorial(c6) factorial(c7) factorial(c8) factorial(c9) factorial(c10);$4^(c4)5^{(c5)}6^{(c6)}7^{(c7)}8^{(c8)}9^{(c9)}10^{(c10)}$];

A4 = [factorial(c5) factorial(c6) factorial(c7) factorial(c8) factorial(c9) factorial(c10);

$5^{(c5)}6^{(c6)}7^{(c7)}8^{(c8)}9^{(c9)}10^{(c10)}$];

A5 = [factorial(c6) factorial(c7) factorial(c8) factorial(c9) factorial(c10);

$6^{(c6)}7^{(c7)}8^{(c8)}9^{(c9)}10^{(c10)}$];

if (c1 - 3)<0 then a1 = 0; else a1 = factorial(n-3)/(factorial(c1-3)*prod(A1)); end

if (c1 - 1)<0 then a2 = 0; elseif (c2 - 1)<0 then a2 = 0

else a2 = 3*factorial(n-3)/(factorial(c1-1)*factorial(c2-1)*$2^{(c2-1)}$*prod(A2)); end

if (c3 - 1)<0 then a3 = 0; else

a3 = 2*factorial(n-3)/(factorial(c1)*factorial(c2)*factorial(c3-1)*$2^{(c2)} * 3^{(c3-1)}$*prod(A3));

end

if (c1 - 4)<0 then b1 = 0;

else b1 = factorial(n-4)/(factorial(c1-4)*prod(A1)); end

if (c1 - 2)<0 then b2 = 0; elseif (c2 - 1)<0 then b2 = 0

else b2 = 6*factorial(n-4)/(factorial(c1-2)*factorial(c2-1)*$2^{(c2-1)}$*prod(A2)); end

if (c3 - 1)<0 then b3 = 0; elseif (c1 - 1)<0 then b3 = 0 else

b3 = 8*factorial(n-4)/(factorial(c1-1)*factorial(c2)*factorial(c3-1)*$2^{(c2)} * 3^{(c3-1)}$*prod(A3));

end

if (c2 - 2)<0 then b4 = 0;

else b4 = 3*factorial(n-4)/(factorial(c1)*factorial(c2-2)*$2^{(c2-2)}$*prod(A2)); end

if (c4 - 1)<0 then b5 = 0;

else b5 = 6*factorial(n-4)/(factorial(c1)*factorial(c2)*factorial(c3)*factorial(c4-

1)*$2^{(c2)} * 3^{(c3)} * 4^{(c4-1)}$*prod(A4)); end

if (c1 - 5)<0 then d1 = 0;

else d1 = factorial(n-5)/(factorial(c1-5)*prod(A1)); end

if (c1 - 3)<0 then d2 = 0; elseif (c2 - 1)<0 then d2 = 0 else

d2 = 10*factorial(n-5)/(factorial(c1-3)*factorial(c2-1)*$2^{(c2-1)}$*prod(A2)); end

if (c3 - 1)<0 then d3 = 0; elseif (c1 - 2)<0 then d3 = 0 else

d3 = 20*factorial(n-5)/(factorial(c1-2)*factorial(c2)*factorial(c3-1)*$2^{(c2)} * 3^{(c3-1)}$*prod(A3));

end

if (c2 - 2)<0 then d4 = 0; elseif (c1 - 1)<0 then d4 = 0 else

d4 = 15*factorial(n-5)/(factorial(c1-1)*factorial(c2-2)*$2^{(c2-2)}$*prod(A2)); end

if (c4 - 1)<0 then d5 = 0; elseif (c1 - 1)<0 then d5 = 0 else

d5 = 30*factorial(n-5)/(factorial(c1-1)*factorial(c2)*factorial(c3)*factorial(c4-1)*$2^{(c2)} *$

$3^{(c3)} * 4^{(c4-1)}$*prod(A4)); end

if (c3 - 1)<0 then d6 = 0; elseif (c2 - 1)<0 then d6 = 0 else

d6 = 20*factorial(n-5)/(factorial(c1)*factorial(c2-1)*factorial(c3-1)*$2^{(c2-1)} * 3^{(c3-1)}$*prod(A3));

end

if (c5 - 1)<0 then d7 = 0;

else d7 = 24*factorial(n-5)/(factorial(c1)*factorial(c2)*factorial(c3)*factorial(c4)

factorial(c5-1)*$2^{(c2)} * 3^{(c3)} * 4^{(c4)} * 5^{(c5-1)}$*prod(A5)); end

if n<2*r then per = error; elseif n>10 then per = error

elseif r == 3 then per = a1+a2+a3 elseif r == 4 then per = b1+b2+b3+b4+b5

elseif r == 5 then per = d1+d2+d3+d4+d5+d6+d7 else per = 0; end endfunction

Appendix B

A Scilab program that calculates $|\mathbf{Fix}(g)|$ in $X^{(r)}$, $3 \leq r \leq 5$

B.1 Introduction

In this program, the inputs are r, c1, c2, c3, c4, c5, where c1, c2, ..., c5 is the cycle type of a permutation $g \in S_n$. The program is developed using expressions 2.7, 2.8 and 2.9. The program is invoked by f1 = fix(r, c1, c2, c3, c4, c5) where f1 is the output while the others are inputs. We generate tables 2.3 and 2.4 using this program.

B.2 Scilab Program

```
function [f] = fix(r, c1, c2, c3, c4, c5) if(c1 - 3)<0 then a = 0;
else a = factorial(c1)/(factorial(c1 - 3)*factorial(3)); end
b = c1*c2; c = c3;
    if (c1 - 4)<0 then d = 0;
else d = factorial(c1)/(factorial(c1 - 4)*factorial(4)); end
if (c1 - 2)<0 then e = 0;
else e = c2*factorial(c1)/(factorial(c1 - 2)*factorial(2));end
if (c2 - 2)<0 then g = 0;
else g = factorial(c2)/(factorial(c2 - 2)*factorial(2));end
h = c1*c3; i=c4;
    if (c1 - 5)<0 then j = 0;
else j = factorial(c1)/(factorial(c1 - 5)*factorial(5));
```

```
end if (c1 - 3)<0 then k = 0;
else k = c2*factorial(c1)/(factorial(c1 - 3)*factorial(3)); end
if (c1 - 2)<0 then l = 0;
else l = c3*factorial(c1)/(factorial(c1 - 2)*factorial(2)); end
if (c2 - 2)<0 then m = 0;
else m = c1*factorial(c2)/(factorial(c2 - 2)*factorial(2));
end n = c5; o = c1*c4;
p = c2*c3;
if r == 3 then f = a+b+c; elseif r == 4 then f = d+e+g+h+i elseif r == 5
then f = j+k+l+m+n+o+p else f = 0; end
endfunction
```

Appendix C

A Scilab program that calculates subdegrees of S_n for $n \leq 21$

C.1 Introduction

In this program, the inputs are n and r. The program is developed using expressions 3.1. The program is invoked by subds1=comb(n,r) where subds1 is the output while the others are inputs.

C.2 Scilab Program

```
function [subds] = comb(n,r)
a=1; if (r-1)<0 then b = 0; elseif (n-2*r+1)<0 then b = 0; else
b = factorial(r)/(factorial(r-1)*factorial(1))*(factorial(n-r)/(factorial(n-2*r+1)*factorial(r-1))); end
if (r-2)<=0 then c = 0; elseif (n-2*r+2)<0 then c = 0; else
c = factorial(r)/(factorial(r-2)*factorial(2))*(factorial(n-r)/(factorial(n-2*r+2)*factorial(r-2))); end
if (r-3)<=0 then d = 0; elseif (n-2*r+3)<0 then d = 0; else
d = factorial(r)/(factorial(r-3)*factorial(3))*(factorial(n-r)/(factorial(n-2*r+3)*factorial(r-3))); end
if (r-4)<=0 then e = 0; elseif (n-2*r+4)<0 then e = 0; else
e = factorial(r)/(factorial(r-4)*factorial(4))*(factorial(n-r)/(factorial(n-2*r+4)*factorial(r-4))); end
if (n-2*r)<0 then f=0; else
```

```
f=factorial(n-r)/(factorial(n-2*r)*factorial(r)); end
if (r-5)<=0 then g = 0; elseif (n-2*r+5)<0 then g = 0; else
g = factorial(r)/(factorial(r-5)*factorial(5))*(factorial(n-r)/(factorial(n-2*r+5)*factorial(r-5))); end
if (r-6)<=0 then h = 0; elseif (n-2*r+6)<0 then h = 0; else
h = factorial(r)/(factorial(r-6)*factorial(6))*(factorial(n-r)/(factorial(n-2*r+6)*factorial(r-6))); end
if (r-7)<= 0 then i = 0; elseif (n-2*r+7)<0 then i = 0; else
i = factorial(r)/(factorial(r-7)*factorial(7))*(factorial(n-r)/(factorial(n-2*r+7)*factorial(r-7))); end
if (r-8)<= 0 then j = 0; elseif (n-2*r+8)<0 then i = 0; else
j = factorial(r)/(factorial(r-8)*factorial(8))*(factorial(n-r)/(factorial(n-2*r+8)*factorial(r-8))); end
if (r-9)<=0 then k = 0; elseif (n-2*r+9)<0 then k = 0; else
k = factorial(r)/(factorial(r-9)*factorial(9))*(factorial(n-r)/(factorial(n-2*r+9)*factorial(r-9))); end
subds={a,f,b,c,d,e,g,h,i,j,k};
    endfunction
```