EUROPEAN OPTION PRICING UNDER DOUBLE WISHART STOCHASTIC VOLATILITY MODEL

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(Financial Option)

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Declaration

I, Raphael NARYONGO, declare that this research Thesis is my own original work and has not been submitted anywhere for any degree award in any other university.

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Dedication

I dedicate this work to my wife Mrs Sarah Naryongo, and my children Alpha Naryongo, Courtney Nandele Naryongo and Philiper Titini Naryongo. To my father Mr Naryongo Raphael and my mother Mrs Maria Titini Naryongo.

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Contents

Publications

1. European Option Pricing under Double Wishart Stochastic Processes (July 2021, Journal of mathematics, Hindawi)

2. The Log-Asset Dynamic with Euler-Maruyama Scheme under Wishart Processes (November 2021, International journal of mathematics and mathematical sciences, Hindawi)

Acronyms

ODEs : Ordinary differential equations

PDEs : Partial differential equations

SDEs : Stochastic differential equations

BSm : Black-Scholes model

$M_{n,m}(\mathbb{R})$:The space of $n \times m$ real matrices

$GL_n(\mathbb{R})$: The space of $n \times n$ invertible real matrices

S_n: The space of $n \times n$ symmetric real matrices

S_n^+, S_n^- : The space of $n \times n$ real symmetric positive, negative definite matrices

For $X \in M_n(\mathbb{R})$, $Tr(X)$ is a trace of matrix X

For A^T : Transpose of matrix A

For $\varSigma_t \in S_n^+(\mathbb{R})$, $\varSigma_t$ is a Wishart process

For processes $B_t, W_t, X_t, \bar{X}_t, \xi_t, \eta_t$ are Brownian motions

For a, a non-negative symmetric matrix,$\sqrt{a}$ is the unique non negative symmetric matrix such as $\sqrt{a}\sqrt{a} = a$.

For $\hat{S}^N$ Euler scheme approximate depend on N grid

β_1 and β_2 are parameters for existence and uniqueness of the solution

p and q are characteristic orders in perturbation method

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Abstract

This study considered Wishart affine diffusion processes, which are stochastic processes defined as matrix-valued square root processes or as matrix generalization of a squared Bessel process. The aim of the study was to develop a double Wishart stochastic volatility model to price European option: A multifactor Heston model whose volatility components follow Wishart affine processes for a single risky asset, with two dependence matrices describing the correlations between the asset dynamic and the Wishart processes, making it more flexible enough to price options or describe the market prices for short or long maturities. We constructed the double Wishart stochastic volatility model, through the generalization of Heston model into a multifactor nature of implied volatilities, together with the associated properties. The partial differential equation describing the behavior of prices associated to the dynamics of stock price under double Wishart volatility model was derived and solved through the application of Fourier techniques, combined with perturbation methods to obtain European call option pricing formula to confirm the applicability of the model in financial derivatives. Then the log asset return price dynamic under double Wishart model was derived using Ito lemma and its integrals are solved using corrected Euler-Maruyama discretization technique in order to obtain the numerical solution for the log-asset return in order to illustrate the behavior of the log asset returns. The numerical examples show that the call price predictions under double Wishart model exhibits similar pricing behavior with respect to the market price in short and long maturities due to the flexibility in the model. Additionally, the numerical illustrations on the log-asset price return shows the effect of model parameters more so the correlation matrices on the log-asset price return behavior under double Wishart volatility model in trading, which is of importance for investors to analyze the stock prices over time.

Chapter 1

Introduction

1.1 Background

The Wishart process was introduced in Bru (1991), and is a stochastic process which is defined as a positive semi definite matrix-valued generalization of the square root process. The square root process itself is a generalized square Bessel process. La Bua & Marazzina (2019), defines a Wishart process as a matrix valued continuous time stochastic process with a marginal Wishart distribution, (that is a generalization to multiple dimensions of chi-squared distribution). The Wishart affine process found its application in finance by Gourieroux & Sufana (2004).

Due to the limitation of Black & Scholes (1973), of assuming that volatility is constant and so not incorporating the observable phenomenon that implied volatility of derivative products is strike-and-maturity-dependent. This caused the introduction of Heston (1993) stochastic volatility model, and it has been powerful and widely applied in financial markets, due to its flexibility, financial interpretation of parameters and analytical tractability property since it belongs to the class of affine processes (see Filipovic & Mayerhofer (2009)). The affine property allows the model to form a closed form solution of the characteristic function of the log-price, in order to obtain European call option price by Fourier transform inversion.

However, despite the Heston model popularity, the existing literature has documented the limitation of the model, Da Fonseca et al. (2008); Christoffersen et al. (2009); Ahdida & Alfonsi

(2013); Kang et al. (2017); La Bua & Marazzina (2019), Gouriéroux (2006) point out that, the major deficiency of the model is that, does not generate the realistic term structure of the volatility smiles; Heston model provides too flat implied volatility surface to attain reality. Yet implied volatility curve generally has steep slope and it is convex in short maturity and tends to be linear for long maturity. Thus the Heston model is not flexible enough to describe the market prices in short or long maturities. More effort has been extended to solving this problem by generalizing the Heston model using two approaches; by adding jump in the stock dynamic or volatility and investigating the multifactor nature of the implied volatility.

It is well understanding that, the multifactor approach is capable of handling the pricing problem of derivative products and volatility skew. This usher in the application of the stochastic matrix defined process, that is the Wishart multidimensional stochastic volatility model. The Wishart process introduced in Bru (1991) as a matrix generalized square process posses complex parameter specification, but fully preserves tractability and analytical features, since it belongs to the class of affine processes. The Wishart model considers the volatility of the asset as trace of the Wishart process which allows to take into account a stochastic correlation between the underlying asset and the volatility process and gives nice but complex model. A large existing literature recommends the application of matrix-defined stochastic processes in derivative pricing or options pricing, the Wishart stochastic volatility model, is among the best flexible models which can match the term structure of implied volatility well, Da Fonseca et al. (2008) and Da Fonseca et al. (2005). The term structure of the realized volatilities in this model is explained by positive semi-definite matrix valued stochastic process.

This study aimed at developing a double Wishart stochastic volatility model (i.e can also be called Bi-Wishart stochastic volatility model) for pricing the options whose volatility components follow Wishart affine processes. This double Wishart diffusion model is obtained through generalizing Heston model into multifactor nature of implied volatilities with two dependence matrices for a single risky asset; that is the asset dynamics depends on two Wishart volatility processes (double Wishart volatility processes). Following Kang et al. (2017) and Benabid et al. (2008), the matrix specification of the Wishart model makes it flexible enough to describe the market prices.

Therefore in order to handle the European call options pricing problem, we consider the transform methods of Duffie et al. (2000) and the Fast Fourier transform (FFT) methods of Carr & Madan (1999), together with the perturbation techniques in Benabid et al. (2008) to obtain the analytical pricing formula. Then the corrected Euler-Maruyama discretization scheme is employed to obtain the numerical solution for log-asset price dynamic under double Wishart stochastic volatility model.

Although there are some aspects of the Wishart stochastic volatility model which have been studied like in Benabid et al. (2008), Da Fonseca et al. (2011), Da Fonseca et al. (2008), Gauthier & Possamaï (2010), Kang et al. (2017), La Bua & Marazzina (2019), however, there is no study which has been considered on pricing European options under double Wishart affine processes (i.e Bi-Wishart affine processes) through the generalization of Heston model; for a single risky asset pricing model with Fourier transform methods together with perturbation techniques. That is why the current study is carried out on European option pricing under double Wishart stochastic volatility model.

1.2 Problem statement

Due to the fact that volatility levels affect all sectors of the economy, more so option trading. The limitation of the famous standard Black-Scholes model in option pricing for assuming that volatility is constant which is not realistic in the financial market, this caused the introduction of Heston volatility model which has been widely applied in option pricing, but the literature shows that the model has its deficiency, it does not describe the market prices well due to lack of flexibility in the correlation factors and so can not capture the stylized facts of the market. This problem of option pricing can be solved through generalizing Heston model, although there are some aspects of Wishart affine processes which have been studied in Da Fonseca et al. (2008) and Kang et al. (2017). However, no attention has been put on pricing European options under double Wishart diffusion processes for a single risky asset pricing model, through generalization of Heston model into a multifactor form of implied volatility with two dependence matrices. This enables enough flexibility to provide fair option pricing predictions or describe the market prices well for short or long maturities. That is why it is imperative to consider the current study on European option pricing under double Wishart stochastic volatility model.

1.3 Objectives of the study

1.3.1 General objective

The general objective of the study was to develop double Wishart stochastic volatility model in option pricing.

1.3.2 Specific objectives

The specific objectives of this study are:

(i) To construct double Wishart stochastic volatility model.

(ii) To determine the call price by solving partial differential equation associated to the dynamic of the stock price under double Wishart affine processes.

(iii) To discretize the dynamic of stock price under double Wishart affine processes.

(iv) To compare the Bi-Wishart stochastic volatility model price behavior with market price.

1.4 Significance of the study

This study provides the required theoretical frame work to practitioners on how to derive the European option pricing formula under double Wishart stochastic volatility model; This study attempts to improve the existing methods to allow investors invest with minimum risks in financial markets. The study will also contribute to the literature on how to construct the model, derive the partial differential equation and to develop a discretization scheme for double Wishart stochastic volatility model. This will be of great importance to investors, policy makers, researchers, practitioners and data scientists as regard to the application of double Wishart model in financial markets.

1.5 Scope of the study

This study covers the multifactor Heston volatility model, with two correlation matrices describing the correlations between the Brownian matrices of the stock process and those of the Wishart processes. Ito calculus, complex integrals, Fourier techniques together with perturbation methods are employed in order to price the European options, also the Euler numerical scheme is taken into account.

1.6 Limitation of the study

This study does not analyze the stock distribution function, Greeks and also do not capture parameter estimations but rather focuses on developing a double Wishart stochastic volatility model in option pricing, in order to derive new European option pricing formula under double Wishart stochastic volatility model.

Chapter 2

Literature Review

This chapter presents the rational of carrying out a research on Wishart stochastic volatility model by different authors both theoretically and empirically, and identifying the gap to be alleviated by this study.

The double Heston model suggested by Christoffersen et al. (2009) introduced a second factor for the variance, which is driven by its own stochastic differential equation. The reason was to offer more flexibility in modeling volatility surface. In their study on "The shape and term structure of the index option smirk: Why multi-factor stochastic volatility models work so well". Observed that the state-of-the-art stochastic volatility models generate a volatility smirk, where this gives explanation of prices being high for out of money index puts compared to Black-Scholes model, and these models widely explains how volatility smirk moves up and down in response to change in risk. However the data reflected that the slope and level of smirk fluctuated largely independently. But in single factor model stochastic models they capture the smile and can not explain such largely independent fluctuations in its level and slope over time, hence these movements can only be modeled by two factor models, because the model generates stochastic correlation between volatility and stock returns because of two independent correlations with market returns. The empirical results shows that the model improves Heston model by 15% in-sample and 11% out of sample.

Da Fonseca et al. (2008), studied a Multi-factor volatility Heston model. They considered a model for a single risky asset whose volatility follows a matrix Wishart affine process, which was introduced by Gourieroux and Sufana (2004) in finance. As in the study of Duffie and Kan (1996) that the affine models, the pricing problem can be solved via Fast Fourier transforms of Carr and Madan (1999). The numerical results shows that the specification provides a separate fit of the long term and short term implied volatility surface, that is additional factor allows better fit of the smile skew effect at both short and long maturities, and it is possible to identify a specific factor accounting for a stochastic leverage effect as compared to previous stochastic diffusive models, as in a well known analyzed fact of FX option markets analyzed in Carr and Wu (2004). They concluded that multi-factor volatility extension of the Heston model is flexible enough to take into consideration of correlations with underlying asset returns and preserves analytical tractability feature which is useful in calibration procedure.

According to La Bua & Marazzina (2019), on the calibration and advanced simulation schemes for the Wishart stochastic volatility models, where they used calibration and Mont Carlo simulation of the Wishart stochastic volatility model. Regardless of the analytical tractability property of the considered model and being affine type, implementing Wishart based stochastic volatility models poses non-trivial challenges from the numerical part. The task was to provide a fast and accurate calibration procedure for efficient numerical schemes for Mont Carlo simulation. The matrix structure of Wishart based model gives a degree of flexibility in explaining the evolution of asset volatility. They concluded by showing a leverage on a complete analysis of distribution properties of Wishart process and possible solution to make the model suitable for market application.

Benabid et al. (2008) studied Wishart stochastic volatility: Asymptotic smile and numerical framework. The emphasis was on stochastic volatility model on asset pricing which was originally studied in Da Fonseca (2005,2008), the Wishart model takes the volatility of the asset as a trace of a Wishart process. Which is not the case with classical multi-factor Heston model because the model allows to add the degree of freedom with regard to stochastic correlation. They observed that the model‘s flexibility gives a better fit of the market data compared to the Heston model and maintains the clear interpretation of its parameters and preserves an efficient tractability. Having known the problem of data fitting for short and long maturities, Wishart model can solve the problem efficiently, since it preserves the advantage of affine models gives a semi-closed formula for price of some options like call and put options and better understanding of volatility effect through the model parameters. Using singular perturbation technique, closed forms are obtained concerning asymptotic smiles for short and medium maturities. This later can be used in practice as a calibration tool to demonstrate the effect of the parameter on the smile that can benefit businessmen. Generalizing the model in the world of interest rates where a great number of factors are required. One expects that type of model, where volatility is a function of a Wishart process, manifests specific market-existing source of risk and gives good smile flexibility.

Kang et al. (2017), Kang & Kang (2013) carried out a study on Exact simulation of Wishart multidimensional stochastic volatility model, the aim was to propose an exact simulation method of the Wishart multidimensional stochastic volatility model of a single asset model with a multidimensional Wishart variance process, the method was based on the analysis of conditional characteristic function of log price given volatility level. They obtained an explicit expression for the conditional characteristic function for Heston model and numerical experiment for the performance and accuracy of the method. The method showed better performance than Euler discretization method. The exact simulation of asset price and volatility factor of Wishart model is inspired by Fourier inversion method of Broadie and Kaya (2006) and based on the analysis of the conditional Laplace transformation of the asset price given a volatility.

Gauthier & Possamaï (2010), investigated efficient simulation of the double Heston model. They worked on the simulation and discretization issues of the double Heston model originally introduced in Christoffersen et al (2009), observed that not only the model is analytically tractable but also the additional volatility factor cause flexibility to capture the volatility term structure and this leads the model to be calibrated easily. The aim was to extend the original Heston model to new two-factor form model. They worked on the issue of European options pricing using the characteristic function technique and generalized some efficient discretization created for Mont Carlo pricing with the square root process of Alfonsi and Zhu schemes. Zhu scheme performs good compared to Euler scheme but Alfonsi scheme does better than them and is quite faster making it a better scheme to apply in Heston model or double Heston model. This because the Zhu scheme does not improve its convergence properties and also time consuming.

Filipovic & Mayerhofer (2009), looked at Affine diffusion processes; Theory and applications. The hard work on general and canonical space was performed on the affine diffusion processes. The focus was on theoretical and applied aspects of this class of Markov processes and also to derive admissibility conditions, existence and uniqueness through stochastic invariance of canonical state space, where affine transform formula and exponential moments are established, later applied in pricing of bonds by applying CIR, Vasicek and Heston models.

Grasselli & Tebaldi (2008), studied Structural Models of Solvable Affine Concepts. Here the pricing contingent claims in the Affine term structure models can be reduced to the solution of a set of Riccati ordinary differential equation as in Duffie, Pan and Singleton (2000) and also in Duffie, Filipovic and Schachermayer (2001). The discussion as on solvability of these equations, and admissibility is a necessary and sufficient condition in order to obtain the general solution as an analytical series expansion. Let factors follow continuous paths, the ode admit a fundamental system of solutions if and only if all positive factor are independent.Lastly all consistent polynomial term structure models relating to fundamental system of solutions were solved.

Da Fonseca et al. (2013), studied a flexible matrix Libor model with smiles. The objective was to explore a flexible approach for the valuation of interest rate derivative based on Affine process. By the help of the method suggested in Keller-Ressel et al (2009) of changing the choice of state space, semi-closed type solutions for the pricing of caps and floors was provided. In this multi-factor setting, the methodology can be adapted to this state space in a straightforward manner and can be efficiently priced by expansion . It can help identify effects of correlation that can not be addressed by a single-factor framework. Because of off-diagonal elements new possibilities emerge in regulating the form of the implied volatility surface due to the numerical work of the Wishart Libor model. The numerical exercise thus shows the versatility of the Wishart Libor model in defining the movements of the volatility surface predicted by it.

Gouriéroux (2006), the Wishart process for stochastic risk continuous time. In the study, Risks are defined and measured by volatility co-volatility matrices, while Wishart processes are models for dynamic multivariate risk analysis and describes the evolution of stochastic volatility co-volatility matrices, restricted to being symmetrically positive definite. The auto regressive Wishart process is the multivariate extended version of CIR process. The CIR process allows for a closed form solution to a variety of financial problems such as term structure of T-bonds and corporate bonds, derivative pricing in a multivariate stochastic volatility model, and credit risk model. However Wishart dynamics are very flexible and taken for less structural multivariate ARCH models to be serious competitors. The study shows that the Wishart process is suitable in various financial problems for modeling multivariate risk and provides highly closed form derivative prices and is sufficiently versatile enough to compete with multivariate ARCH models.

Exact and Higher-order discretization schemes for Wishart processes and their affine extensions, was studied in Ahdida & Alfonsi (2013) where the focus was on the simulation of Wishart processes and affine diffusion on positive semi-definite matrices and also the work dealt with the splitting of the infinitesimal generator in order apply the composition method. The Wishart processes initially introduced by Bru (1991), additionally Cuchiero et al. (2011) introduced a general framework for affine processes and includes possible jumps. The study sampled exactly every Wishart distribution and suggested a third-order scheme for Wishart system, second order scheme for general affine diffusion. Confirmation of convergence rates was supported with numerical tests and analysis of the time complexity of each method. This made them to recommend for the use of exact scheme simulation to compute expectations that depend on one time. For the case of path wise expectations the study recommend to use discretization schemes that is second order scheme. Further work was suggested on extending the current schemes to affine diffusion on positive semi definite matrices that include jumps.

In the study of moment non-explosions for Wishart-based stochastic volatility model, (Jose Da Fonseca,2006), showed a result on moment non-explosions for a stock following a Wishart multidimensional stochastic volatility dynamics, when the parameter values are conditional to certain constraints. The study reformulated the stock process in terms of volatility path together with standard results on matrix Lyapunov and matrix Riccati equations, a non-explosion results of the moment of order greater than one can be obtained; The study also emphasized a conditional sub filtration generated by volatility pat to rewrite the moment-generating function of Wishart dynamic and its integral for which explicit conclusions can be obtained, which might be useful for exotic option pricing.

The Autoregressive Conditional Heteroscedasticity (ARCH) model was introduced by Engle (1982) to model volatility by relating conditional variance of the disturbance term to the linear combination of the squared disturbances in the recent past. Later Bellerslev (1986) generalized the ARCH model through modeling the conditional variance to depend on its lagged values as well as squared lagged values of disturbances; Several variants of GARCH model have been developed to model volatility. The ARCH family models have been used to capture the volatility and stock returns volatility behavior, by practitioners in the financial markets.

Philipov and Glickman (2006), proposed a high dimensional factor multivariate stochastic volatility model in which factor covariance matrices are driven by Wishart random processes. The frame work allows for unrestricted specification of inter temporal sensitivities, which can handle the persistence in volatilities, kurtosis in returns, and correlation breakdowns and contagion effects in volatilities. This factor structure allows addressing high dimensional setups used in portfolio analysis and risk management together with modeling conditional means and conditional variances. The general model allows for the conditional volatility of an asset to depend not only on its past volatility but also on past covariances with other assets, incorporating the observed contagion among asset returns into their covariance structure. They concluded by showing that it is possible to implement factor multidimensional stochastic volatility models with higher dimensionality, making them feasible for practical applications in risk management, asset allocation and portfolio optimization.

This study considers continuous diffusion models where volatility in the stock price dynamic is driven by Wishart diffusion stochastic processes. The existing literature gives the evidence that, Heston stochastic volatility model and double Heston volatility model considers constant correlations making them not flexible enough to describe the market prices in short or long maturity. This current work is an extension of the Heston model to multifactor Heston model considering two dependence matrices describing the correlation between Brownian matrices of the stock process and those of Wishart affine dynamics, this makes it flexible enough to capture the stylized facts of the market: Fourier transform methods combined with perturbation methods will be the sophisticated techniques used in order to obtained European call option pricing formula. This provides a clear gap that there is no research study which has been conducted on pricing European option under double Wishart stochastic volatility processes for single risky asset pricing model.

Chapter 3

Methodology

This chapter discusses the methodology that is used throughout the entire thesis, it includes all the technical proofs needed in developing the model and achieving the analytical pricing formula which is the major objective of this study. Section 3.1 presents the definitions and properties, Wishart process, connection between Wishart process and OU process, double Wishart model construction, condition for existence and uniqueness of the solution, change of probability measure, derivation of log-price dynamic and correlation structure. Section 3.2 provides the derivation of the partial differential equation, derivation of Ito infinitesimal operator, the Laplace and Fourier transform methods, Riccati differential equations, perturbation techniques and pricing formula. Section 3.3 gives the numerical methods for stock price dynamics.

3.1 Wishart Processes and Double Wishart Volatility Model

This section provides some of the concepts, definitions and properties of Wishart processes in order to construct double Wishart model.

3.1.1 Definitions and properties

Definition 3.1.1 *Martingale. Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a fixed probability space equipped with an increasing sequence of sub-*σ*-fields* $(\mathcal{F}_t, t \in \mathbb{N})$ *or* $(\mathcal{F}_t)_{t\geq 0}$.

A stochastic process $(X_t)_{t\geq 0}$ is said to be $(\mathcal{F}_t)$-adapted, if for each t, X_t is $\mathcal{F}_t$-measurable.

Martingale. An $(\mathcal{F}_t)$-adapted stochastic sequence $(X_t)_{t\geq 0}$ is said to be an $(\mathcal{F}_t)$- martingale if for each $t \geq 0$, X_t is integrable such that

$$\mathbb{E}\left[X_{t+1}|\mathcal{F}_t\right] = X_t \quad a.s \tag{3.1}$$

Stopping time. A random variable $\tau : (\Omega, \mathcal{F}) \longrightarrow [t, +\infty]$ is called a stopping time or optimal time, if for each $s \geq t$,

$$[\tau \leq s] := [\omega \in \Omega : \quad \tau(\omega) \leq s] \in \mathcal{F}_s$$

.

Definition 3.1.2 *Standard Brownian motion. A continuous time stochastic process $W := (W)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a standard Brownian motion (SBM) or a standard Wiener process if the following properties hold;*

i) $W_0 = 0$ $\mathbb{P}$ *a.s*

ii) The process W_t has independent increment.

iii) Each increment $W_t - W_s, 0 \leq s < t$, is normally distributed with zero mean and variance $t - s, W_t - W_s \sim N(0, t - s)$.

iv) All sample paths $W(., \omega) : [0, \infty) \longrightarrow \mathbb{R}, \omega \in \Omega$, are continuous $\mathbb{P}$ -a.s.

Definition 3.1.3 *Let X be an $m \times n$ continuous matrix stochastic process and let Y be $n \times p$ matrix stochastic process for which the following differentials holds*

i) $d(AX)_t = AdX_t, \quad A \in \mathbb{R}^{p\times m}$

ii) $d(XA)_t = dX_tA, \quad A \in \mathbb{R}^{n\times p}$

iii) $d(XY)_t = X_tdY_t + dX_tY_t + dX_tdY_t$

We can consider the convention of two real-valued processes X and Y, $d[X,Y]_t = dX_t dY_t$, taking note for any two matrix-valued processes X and Y, $dX_t dY_t$ as a component-wise, where the product is understood to be the matrix multiplication

Proof.

i) $d(AX)_t = AdX_t$

$$\begin{aligned} d(AX)_t &= d(\sum_{k=1}^{m} a_{ik} X_t^{kj}) \\ &= \sum_{k=1}^{m} a_{ik} dX_t^{kj} \\ &= [AdX_t]_{ij} \end{aligned}$$

ii) $d(XA)_t = dX_t A$

$$\begin{aligned} d(XA)_t &= d(\sum_{k=1}^{m} X_t^{kj} a_{ik}) \\ &= \sum_{k=1}^{m} dX_t^{kj} a_{ik} \\ &= [dX_t A]_{ij} \end{aligned}$$

iii) $[d(XY)_t]_{ij} = d(\sum_{k=1}^{n} X_t^{ik} Y_{kj})$

$$\begin{aligned} [d(XY)_t]_{ij} &= d(\sum_{k=1}^{n} X_t^{ik} Y_{kj}) \\ &= \sum_{k=1}^{n} (X_t^{ik} dY_t^{kj} + dX_t^{ik} Y_t^{kj} + d[X^{ik}, Y^{kj}]_t) \\ &= \sum_{k=1}^{n} X_t^{ik} dY_t^{kj} + \sum_{k=1}^{n} dX_t^{ik} Y_t^{kj} + \sum_{k=1}^{n} dX_t^{ik}, dY_t^{kj} \\ &= [dX_t Y_t]_{ij} + [X_t dY_t]_{ij} + [dX_t dY_t]_{ij} \end{aligned}$$

Counter example. Let W be $m \times n$ Brownian matrix, then

$$d(W^TW)_t = W_t^T dW_t + dW_t^T W_t + dW_t^T dW_t$$

Since

$$\begin{aligned}
[dW_t^T dW_t]_{ij} &= \sum_{k=1}^{n} dW_t^{ki} dW_t^{kj} \\
&= \sum_{k=1}^{n} d[W^{ki}, W^{kj}]_t \\
&= \begin{cases} ndt \quad if \quad i = j \\ 0 \quad if \quad i \neq j \end{cases}
\end{aligned}$$

Hence $d(W^TW)_t = W_t^T dW_t + dW_t^T W_t + n\mathbb{I}dt$, where $\mathbb{I}$ is the $n \times n$ identity matrix. □

Definition 3.1.4 *Covariation of two processes. If X_t and Y_t are two continuous time process on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the (quadratic) variation of X_t and Y_t is defined as the following limiting process in probability when it exists*

$$[X, Y]_t := \lim_{\delta t_n \to 0} \sum_{k=0}^{n-1} (X_{t_{k+1}} - X_{t_k})(Y_{t_{k+1}} - Y_{t_k})$$

for every time $t \in [0, T]$.

Definition 3.1.5 *A process $X = (X_t)$ is called semimartingale, if X can be decomposed as follows*

$$X = M + A \tag{3.2}$$

where M is a local martingale, A is an adapted process with finite variation

Let $\mathcal{D}$ be a class of processes. The localized class of $\mathcal{D}$, denoted by $\mathcal{D}_{loc}$ is defined as follows: a process $X = (X_t)$ belongs to $\mathcal{D}_{loc}$ iff $X_0 \in \mathcal{F}$ and there exists a sequence T_n of stopping times with $T_n \uparrow \infty$ such that for each n the stopped process $X^{T_n} - X_0 \in \mathcal{D}$.

3.1.2 Wishart process

Definition. Let $W_t, t \geq 0$ be a $n \times n$ matrix-valued Brownian motion under the probability measure $\mathbb{Q}$. The matrix-valued process Σ is said to be a Wishart process if the following diffusion equation is satisfied

$$d\Sigma_t = (\beta QQ^T + M\Sigma_t + \Sigma_t M^T)dt + \sqrt{\Sigma_t} dW_t Q + Q^T dW_t^T \sqrt{\Sigma_t} \tag{3.3}$$

where $Q \in GL_n(\mathbb{R})$, M is a $n \times n$ non positive matrix, $\Sigma_0 \in S_n^+$ and β a real parameter, such that $\beta > (n-1)$. The condition $\beta > (n-1)$ is introduced to ensure existence and uniqueness of the solution $\Sigma_t \in S_n^+$ of equation (3.3) and eigen values of the solution are non negative for all $t \geq 0$ a.s $\Sigma_t \in S_n^+$. Like in Benabid et al. (2008), the probability measure $\mathbb{Q}$ corresponds to a risk-neutral measure.

Then for us to be able to construct the double Wishart stochastic volatility model, the following properties and assumptions should also be taken into consideration.

Infinitesimal generator. Let Σ_t be process, then the infinitesimal generator $\mathcal{A}$ of Σ_t defined on $C^2(\mathbb{R}^{n \times n}, \mathbb{R})$, is given as

$$\mathcal{A} = Tr[(M\Sigma + \Sigma M^T + \beta Q^T Q)D + 2\Sigma D Q^T Q D] \tag{3.4}$$

where D is $n \times n$ matrix differential operator defined by $D_{ij} = \frac{\partial}{\partial x_{ij}}$ and $C^2(\mathbb{R}^{n \times n}, \mathbb{R})$ denotes the real-valued functions defined on $\mathbb{R}^{n \times n}$.

The trace dynamics of Wishart process. For Brownian motion W, the trace of Σ_t follow stochastic differential equation

$$d(Tr(\Sigma_t)) = 2\sqrt{Tr(Q^T Q \Sigma_t)} dW_t + (2Tr(M\Sigma_t) + \beta Tr(Q^T Q))dt \tag{3.5}$$

Lemma 3.1.1 ***Semimartingale property***. *Let $(\mathcal{F}_t)_{t \geq 0}$ represents the filtration generated by $W_t, t \geq 0$. We consider continuous $(\mathcal{F}_t)$- adapted processes $(a_t)_{t \geq 0}, (b_t)_{t \geq 0}, (c_t)_{t \geq 0}$ respectively* *valued in $M_n(\mathbb{R}), M_n(\mathbb{R})$ and $S_n(\mathbb{R})$, and a process $(M_t)_{t\geq 0}$ that admits the following semi-martingale decomposition*

$$dM_t = c_t dt + b_t dW_t a_t + a_t^T dW_t^T b_t^T \tag{3.6}$$

For $i, j, m, n \in 1, ..., n$, then the quadratic covariation of $(M_t)_{i,j}$ and $(M_t)_{m,n}$ given as

$$\begin{aligned}\langle d(M_t)_{i,j}, d(M_t)_{m,n}\rangle = [&(b_t b_t^T)_{i,m}(a_t^T a_t)_{j,n} + (b_t b_t^T)_{i,n}(a_t^T a_t)_{j,m} \\ &+ (b_t b_t^T)_{j,m}(a_t^T a_t)_{i,n} + (b_t b_t^T)_{j,n}(a_t^T a_t)_{i,m}]dt\end{aligned}$$

Clearly we see that the quadratic covariation (3.6) depends on a_t and b_t only through the matrices a_t^T and b_t^T

Proof.

$$\begin{aligned}dM_t &= c_t dt + b_t dW_t a_t + a_t^T dW_t^T b_t^T \\ &= c_t dt + \sum_{k,l=1}^{n}((b_t)_{i,k}(a_t)_{l,j} + (b_t)_{j,k}(a_t)_{l,i})(dW_t)_{k,l}\end{aligned}$$

The quadratic covariation is calculated as follows

$$\begin{aligned}\langle d(M_t)_{i,j}, d(M_t)_{m,n}\rangle &= \sum_{k,l=1}^{n}((b_t)_{i,k}(a_t)_{l,j} + (b_t)_{j,k}(a_t)_{l,i})((b_t)_{m,k}(a_t)_{l,n} + (b_t)_{n,k}(a_t)_{l,im})dt \\ &= \sum_{k,l=1}^{n}(b_t)_{i,k}(b_t)_{m,k}(a_t)_{l,n}(a_t)_{l,j} + \sum_{k,l=1}^{n}(b_t)_{i,k}(b_t)_{n,k}(a_t)_{l,m}(a_t)_{l,j} \\ &+ \sum_{k,l=1}^{n}(b_t)_{j,k}(b_t)_{m,k}(a_t)_{l,n}(a_t)_{l,i} + \sum_{k,l=1}^{n}(b_t)_{j,k}(b_t)_{n,k}(a_t)_{l,m}(a_t)_{l,i} \\ &= (b_t b_t^T)_{i,m}(a_t^T a_t^T)_{n,j} + (b_t b_t^T)_{i,n}(a_t^T a_t^T)_{m,j} \\ &+ (b_t b_t^T)_{j,m}(a_t^T a_t^T)_{n,i} + (b_t b_t^T)_{j,n}(a_t^T a_t^T)_{m,i}\end{aligned}$$

$\square$

Particularly M_t can be written in form of trace, as follows

$$\begin{aligned} dTr(M_t) &= Tr(c_t)dt + 2Tr(a_t^T b_t dW_t), \\ d\langle Tr(M_t)\rangle &= \sum_{i,j=1}^{n} d\langle (M_t)_{i,i}, (M_t)_{j,j}\rangle = 4\sum_{i,j=1}^{n} (b_t b_t^T)_{i,j}(a_t^T a_t)_{i,j} \\ &= 4Tr(b_t b_t^T a_t^T a_t). \end{aligned}$$

Lemma 3.1.2 ***Characteristic function of Wishart process***. *Given a real symmetric matrix D, the conditional characteristic function of the Wishart processes Σ_t is given by,*

$$\begin{aligned} \tilde{\phi}_{\Sigma_t}^{D} &= \mathbb{E}\left[exp\{iTr[D\Sigma_t]\}\right] \\ &= exp\{Tr[A(\tau)\Sigma_t] + C(\tau)\} \end{aligned}$$

where the deterministic complex-valued function $A(\tau) \in M_n(\mathbb{C})$, $C(\tau) \in \mathbb{C}$.

3.1.3 Link between Wishart processes and Ornstein Uhlembeck processes

Following Benabid et al. (2008), a natural way to generate a Wishart process is to replace the Gaussian vector X_i in the definition of a Wishart distribution with Ornstein Uhlembeck processes $X_{i,t}$, with this representation it implies that β is an integer. Considering β independent n-dimensional Ornstein Uhlembeck processes. $dX_{k,t} = MX_{k,t}dt + Q^T dW_{k,t}$ and $\Sigma_t := \sum_{k=1}^{\beta} X_{k,t}X_{k,t}^T$. Where $d\Sigma(t) = \Sigma(t+dt) - \Sigma(t)$ then, the process Σ_t is a Wishart process with the following stochastic differential equation

$$d\Sigma_t = (\beta QQ^T + M\Sigma_t + \Sigma_t M^T)dt + \sqrt{\Sigma_t}dW_t Q + Q^T dW_t^T \sqrt{\Sigma_t}$$

With W_t a matrix valued Brownian motion, obtained by

$$\sqrt{\Sigma_t}dW_t = \sum_{k=1}^{\beta} X_{k,t}dW_{k,t}^T$$

Proof. For $dX_t = MX_t dt + QdW_t$ and $\Sigma_t := X_t X_t^T$

$$
\begin{aligned}
d\Sigma(t) &= \Sigma(t+dt) - \Sigma(t) \\
&= X(t+dt)X^T(t+dt) - X_t X_t^T \\
&= (X(t) + dX(t))(dX^T(t) + X^T(t)) - X_t X_t^T \\
&= X(t)dX^T(t) + dX(t)dX^T(t) + dX(t)X^T(t) \\
&= X(t)[M^T X_t^T dt + Q^T dW_t^T] + [(MX_t dt + QdW_t)(M^T X_t^T dt + Q^T dW_t^T)] \\
&+ (MX_t dt + QdW_t)X^T(t) \\
&= \Sigma_t M^T dt + Q^T dW_t^T X(t) + \beta QQ^T dt + M\Sigma_t dt + X_t^T dW_t Q \\
&= (\beta QQ^T + M\Sigma_t + \Sigma_t M^T)dt + \sqrt{\Sigma_t} dW_t Q + Q^T dW_t^T \sqrt{\Sigma_t}
\end{aligned}
$$

□

3.1.4 Mathematical illustration of the model

Let consider a stochastic differential equation (SDE) in Björk (2009), driven by the standard Brownian motion Z_t defined by a system

$$
dS_t = \mu(t, S_t)dt + \gamma(t, S_t)dZ_t, \quad S_0 = s \in \mathbb{R}, 0 \le t \le T
$$

to be satisfied by an unknown process S_t, where $\mu(t, S_t)$ and $\gamma(t, S_t)$ are drift and diffusion coefficients respectively.

Such that, the same SDE can be written in matrix form

$$
\begin{aligned}
dS_t &= \mu(t, S_t)dt + \gamma(t, S_t)dZ_t \\
&= \mu(t, S_t)dt + \gamma(t, S_t)dZ_t + 0d\bar{Z}_t \\
&= \mu(t, S_t)dt + Tr \begin{bmatrix} \gamma(t, S_t)dZ_t & 0 \\ 0 & 0d\bar{Z}_t \end{bmatrix}
\end{aligned}
$$

If

$$\begin{cases} \mu(t,S_t) & = \mu S_t \\ \gamma(t,S_t) & = \gamma S_t \end{cases} \tag{3.7}$$

that is

$$\frac{dS_t}{S_t} = \mu dt + Tr\begin{bmatrix} \gamma dZ_t & 0 \\ 0 & 0d\bar{Z}_t \end{bmatrix}$$

Considering two volatility components, gives

$$dS_t = \mu(t,S_t)dt + Tr\begin{bmatrix} \gamma(t,S_t) & 0 \\ 0 & \bar{\gamma}(t,S_t) \end{bmatrix} d\begin{bmatrix} Z_t & 0 \\ 0 & \bar{Z}_t \end{bmatrix}$$

let $\Sigma(t,S_t) \in S_n^+$ a positive semi-definite symmetric matrix and $Z_t \in M_n(\mathbb{R})$, such that

$$\Sigma(t,S_t) = \begin{bmatrix} \gamma(t,S_t) & 0 \\ 0 & \bar{\gamma}(t,S_t) \end{bmatrix}^T \begin{bmatrix} \gamma(t,S_t) & 0 \\ 0 & \bar{\gamma}(t,S_t) \end{bmatrix}$$

and

$$Z_t^{\Sigma} = \begin{bmatrix} Z_t & 0 \\ 0 & \bar{Z}_t \end{bmatrix}$$

Therefore

$$\begin{cases} dS_t = \mu(t,S_t)dt + Tr[\sqrt{\Sigma}(t,S_t)dZ_t^{\Sigma}] \\ S_0 = s \end{cases} \tag{3.8}$$

taking

$$\begin{cases} \Sigma(t,S_t) = S_t^2\Sigma_t \\ \mu(t,S_t) = \mu S_t \end{cases} \tag{3.9}$$

then

$$\begin{cases} dS_t = \mu S_t dt + S_t Tr[\sqrt{\Sigma_t}dZ_t^{\Sigma}] \\ S_0 = s \end{cases} \tag{3.10}$$

This can be written as

$$\frac{dS_t}{S_t} = \mu dt + Tr[\sqrt{\Sigma_t} dZ_t^{\Sigma}]$$

Considering $A = 0$ null matrix, then it is the same as the above diffusion process

$$\begin{aligned}
\frac{dS_t}{S_t} &= \mu dt + Tr[\sqrt{\Sigma_t} dZ_t^{\Sigma} + Ad\bar{Z}_t^{\bar{\Sigma}}] \\
&= \mu dt + Tr[\sqrt{\Sigma_t} dZ_t^{\Sigma} + (0)d\bar{Z}_t^{\bar{\Sigma}}] \\
&= \mu dt + Tr\left[\begin{pmatrix} \sqrt{\Sigma_t} dZ_t^{\Sigma} & (0) \\ (0) & (0)d\bar{Z}_t^{\bar{\Sigma}} \end{pmatrix}\right]
\end{aligned}$$

For non-zero matrix A, such that $A \neq 0$

$$\frac{dS_t}{S_t} = \mu dt + Tr\left[\begin{pmatrix} \sqrt{\Sigma_t} dZ_t^{\Sigma} & (0) \\ (0) & A_t d\bar{Z}_t^{\bar{\Sigma}} \end{pmatrix}\right]$$

Assumptions on A_t

let $\Sigma_t = A_t^T A_t$, for $A \in S_n^+(\mathbb{R})$

Therefore S_t is stock under Wishart process, as follows

$$\frac{dS_t}{S_t} = \mu dt + Tr[\sqrt{\Sigma_t} dZ_t^{\Sigma} + \sqrt{\bar{\Sigma}_t} d\bar{Z}_t^{\bar{\Sigma}}]$$

Then generalizing the idea with many Wishart volatility processes

$$\frac{dS_t}{S_t} = \mu dt + Tr \begin{pmatrix} A_t^1 dZ_t^{\Sigma_1} & \cdots & (0) \\ \vdots & \ddots & \vdots \\ (0) & \cdots & A_t^n dZ_t^{\Sigma_n} \end{pmatrix}$$

where $A_t^i, \quad i = 1, ..., n$ verify $\quad \Sigma_t^i = A_t^{iT} A_t^i$

The study focuses on two Wishart diffusion components, for $i = 1, 2$ that is;

$$\frac{dS_t}{S_t} = \mu dt + Tr \begin{pmatrix} A_t^1 dZ_t^{\Sigma_1} & (0) \\ (0) & A_t^2 dZ_t^{\Sigma_2} \end{pmatrix}$$

Which can be written as

$$\frac{dS_t}{S_t} = \mu dt + Tr[A_t^1 dZ_t^{\Sigma_1} + A_t^2 dZ_t^{\Sigma_2}]$$

From the assumption, the asset price dynamic disturbed by double Wishart processes (i.e double Wishart volatility model) can be written as follows

$$\frac{dS_t}{S_t} = \mu dt + Tr[\sqrt{\Sigma_t^1} dZ_t^{\Sigma_1} + \sqrt{\Sigma_t^2} dZ_t^{\Sigma_2}] \tag{3.11}$$

where the volatility follow Wishart process

$$d\Sigma_t^i = (\beta_i Q_i Q_i^T + M_i \Sigma_t^i + \Sigma_t^i M_i^T)dt + \sqrt{\Sigma_t^i} dW_t Q_i + Q_i^T dW_{it}^{iT} \sqrt{\Sigma_{ti}}, \quad \Sigma_0^i = \Sigma^i, i = 1, 2. \tag{3.12}$$

where the volatility components are trace of the Wishart diffusion processes, which follow multi-dimensional CIR processes.

3.1.5 The uniqueness and existence of the solution of Wishart process

The Wishart processes are affine processes defined on $S_n^+(\mathbb{R})$. Following Bru (1991), La Bua & Marazzina (2019) and Alfonsi (2015), the results on weak and strong solution of the Wishart differential equation (3.3) can be showed.

Lemma 3.1.3 *Let X_t be generic affine process with continuous trajectories defined in $S_n^+(\mathbb{R})$, following stochastic differential equation*

$$X_t = X_0 + \int_0^t (\bar{\alpha} + D(X_s))\, ds + \int_0^t \left(\sqrt{X_s} dW_s Q + Q^T (dW_s)^T \sqrt{X_s} \right) \tag{3.13}$$

where $X_0, \bar{\alpha} \in S_n^+(\mathbb{R})$, $Q \in M_n(\mathbb{R})$ and $D : S_n^+(\mathbb{R}) \longrightarrow S_n^+(\mathbb{R})$ is a linear transformation. The process admits a unique weak solution in $S_n^+(\mathbb{R})$ if

(i) $\bar{\alpha} - (n-1)QQ^T \in S_n^+(\mathbb{R})$

(ii) For all $X_1, X_2 \in S_n^+(\mathbb{R})$ such that $Tr[(X_1)X_2] = 0 \Longrightarrow Tr[D(X_1)X_2] \geq 0$, where $Tr[.]$ is a trace of a square matrix. So if X_0 contain in a set of real positive definite matrices $S_n^+(\mathbb{R})$, with condition (i) is replaced by a stronger requirement

(iii) $\bar{\alpha} - (n+1)QQ^T \in S_n^+(\mathbb{R})$. *Then there exist a unique strong solution for equation (3.13) in $S_n^+(\mathbb{R})$. By observation $\bar{\alpha} = \Omega\Omega^T$ and $D[X_0] = MX_0 + X_0M^T$ and when we assume restrictive parametrization for the deterministic part of the drift, $\Omega\Omega^T = \beta QQ^T$. The conditions (i) and (iii) are satisfied as long as $\beta \geq n-1$ and $\beta \geq n+1$ respectively, with direct comparison Wishart SDE (3.3) from (3.13) can be obtained. The real positive parameter β also plays a role in Feller's condition in the univariate case.*

3.1.6 Change of probability measure

From mathematics perspective, it is important to discuss the change of the probability measure to allow a change of the drift in the dynamics processes . In the financial application, more so in the practical aspects, Wishart process has to be simulated in its general form with $\beta \geq n+1$, such that $\beta \in \mathbb{R}$. The function allows to write $K = \beta + 2\lambda$ with $\beta = K \geq n+1$ and λ real number such that $0 \leq \lambda \leq \frac{1}{2}$.

The objective of this idea, is to find a change of probability measure in order to change the generalized Wishart diffusion into the simple one, where β is an integer. Therefore the new probability measure $\mathbb{P}$, following Benabid et al. (2008) and Shreve (2004), can be expressed as follows.

Theorem 3.1.1 *Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be probability space equipped with filtration $\mathcal{F}_t$ satisfying usual conditions.*

Let $q = \beta + \lambda - n - 1$. If $h_T(\mathbb{Q}, \mathbb{P}) = \frac{d\mathbb{Q}}{d\mathbb{P}}$ defines the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$, then

$$h_T(\mathbb{Q}, \mathbb{P}) = \left(\frac{det(\Sigma_T)}{det(\Sigma_0)}\right)^{\frac{\lambda}{2}} exp(-\lambda T Tr(M)) exp\left[\frac{-\lambda}{2} q \int_0^T Tr(\Sigma_s^{-1} Q^T Q) ds\right] \tag{3.14}$$

Proof. The probability measure $\mathbb{P}$ can be specified through an exponential martingale (see Benabid et al. (2008)),

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = exp\{\lambda \int_0^T Tr[\sqrt{\Sigma_s^{-1}} dW_s Q] - \frac{\lambda^2}{2} \int_0^T Tr(\Sigma_s^{-1} Q^T Q) ds\} \tag{3.15}$$

The new process is defined as follows

$$W_t = \tilde{W}_t + \lambda \int_0^t \sqrt{\Sigma_s^{-1}} Q^T ds \tag{3.16}$$

It is simple to check by Girsanov theorem that W is a matrix-valued Brownian motion under the probability measure $\mathbb{P}$. Consequently, the dynamics of Wishart process under the probability measure $\mathbb{P}$ can be written as follows

$$d\Sigma_t = (\beta Q Q^T + M\Sigma_t + \Sigma_t M^T)dt + \sqrt{\Sigma_t} dW_t Q + Q^T dW_t^T \sqrt{\Sigma_t} \tag{3.17}$$

where

$$\begin{aligned} W_t &= \tilde{W}_t + \lambda \int_0^t \sqrt{\Sigma_s^{-1}} Q^T ds \\ dW_t &= d\tilde{W}_t + \lambda \sqrt{\Sigma_t^{-1}} Q^T dt \end{aligned} \tag{3.18}$$

Replacing W_t in the Wishart dynamic process

$$\begin{aligned} d\Sigma_t = {} & (\beta Q Q^T + M\Sigma_t + \Sigma_t M^T)dt + \sqrt{\Sigma_t}(d\tilde{W}_t + \lambda \sqrt{\Sigma_t^{-1}} Q^T dt)Q + Q^T(d\tilde{W}_t \\ & + \lambda \sqrt{\Sigma_t^{-1}} Q^T dt)^T \sqrt{\Sigma_t} \end{aligned} \tag{3.19}$$

$$d\Sigma_t = (\beta Q Q^T + M\Sigma_t + \Sigma_t M^T)dt + \sqrt{\Sigma_t} d\tilde{W}_t Q + Q^T d\tilde{W}_t^T \sqrt{\Sigma_t} + \lambda Q^T Q dt + \lambda Q^T Q dt \tag{3.20}$$

Then

$$d\Sigma_t = \left((\beta+2\lambda)QQ^T + M\Sigma_t + \Sigma_t M^T\right)dt + \sqrt{\Sigma_t}d\tilde{W}_t Q + Q^T d\tilde{W}_t^T \sqrt{\Sigma_t} \qquad (3.21)$$

The Radon-Nikodym derivative can be handled using the determinant dynamics.

$$\begin{aligned}
\int_0^T d\log(det\Sigma_s) &= \int_0^T [(\beta-n-1)Tr(\Sigma_s^{-1}Q^TQ) + 2Tr(M)]ds + 2\int_0^T Tr[\sqrt{\Sigma_s^{-1}}dW_sQ] \\
\log\left(\frac{det\Sigma_T}{det\Sigma_0}\right) &= 2TTr(M) + \int_0^T (\beta-n-1)Tr(\Sigma_s^{-1}Q^TQ)ds + 2\int_0^T Tr[\sqrt{\Sigma_s^{-1}}dW_sQ] \\
\log\left(\frac{det\Sigma_T}{det\Sigma_0}\right)^{\frac{\lambda}{2}} &= \lambda TTr(M) + \frac{\lambda}{2}\int_0^T (\beta-n-1)Tr(\Sigma_s^{-1}Q^TQ)ds \\
&\quad + \lambda\int_0^T Tr[\sqrt{\Sigma_s^{-1}}dW_sQ] \\
\left(\frac{det\Sigma_T}{det\Sigma_0}\right)^{-\frac{\lambda}{2}} &= exp\{-\lambda TTr(M) - \frac{\lambda}{2}\int_0^T (\beta-n-1)Tr(\Sigma_s^{-1}Q^TQ)ds \\
&\quad - \lambda\int_0^T Tr[\sqrt{\Sigma_s^{-1}}dW_sQ]\} \\
e^{\lambda\int_0^T Tr[\sqrt{\Sigma_s^{-1}}dW_sQ]} &= \left(\frac{det\Sigma_T}{det\Sigma_0}\right)^{\frac{\lambda}{2}} exp\{-\lambda TTr(M)\}exp\{-\frac{\lambda}{2}\int_0^T (\beta-n-1)Tr(\Sigma_s^{-1}Q^TQ)ds\}
\end{aligned}$$

From the equation (3.15) the new measure with $\tilde{W}$ is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = exp\{\lambda\int_0^T Tr[\Sigma_s^{-1}dW_sQ]\}exp\{-\frac{\lambda^2}{2}\int_0^T Tr(\Sigma_s^{-1}Q^TQ)ds\} \qquad (3.22)$$

$$\begin{aligned}
\frac{d\mathbb{Q}}{d\mathbb{P}} &= \left(\frac{det\Sigma_T}{det\Sigma_0}\right)^{\frac{\lambda}{2}} exp\{-\lambda TTr(M)\}exp\{-\frac{\lambda}{2}\int_0^T (\beta+2\lambda-n-1)Tr(\Sigma_s^{-1}Q^TQ)ds\} \\
&\quad .exp\{-\frac{\lambda^2}{2}\int_0^T Tr(\Sigma_s^{-1}Q^TQ)ds\} \\
&= \left(\frac{det\Sigma_T}{det\Sigma_0}\right)^{\frac{\lambda}{2}} exp\{-\lambda TTr(M)\}exp\{[-\frac{\lambda}{2}(\beta+2\lambda-n-1) - \frac{\lambda^2}{2}]\int_0^T Tr(\Sigma_s^{-1}Q^TQ)ds\} \\
&= \left(\frac{det\Sigma_T}{det\Sigma_0}\right)^{\frac{\lambda}{2}} exp\{-\lambda TTr(M)\}exp\{-\frac{\lambda}{2}(\beta+\lambda-n-1)\int_0^T Tr(\Sigma_s^{-1}Q^TQ)ds\}
\end{aligned}$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \left(\frac{det\Sigma_T}{det\Sigma_0}\right)^{\frac{\lambda}{2}} exp\{-\lambda TTr(M)\}exp\{-\frac{\lambda}{2}(\beta+\lambda-n-1)\int_0^T Tr(\Sigma_s^{-1}Q^TQ)ds\} \qquad (3.23)$$

The change of the probability measure is obtained. $\square$

3.1.7 Wishart stochastic volatility model

In Da Fonseca et al. (2008) and Benabid et al. (2008), under the risk neutral probability measure, the risky asset price dynamic and its quadratic variation follows,

$$\begin{cases} \dfrac{dS(t)}{S(t)} &= rdt + Tr[\sqrt{\Sigma_t} dZ(t)], \quad S_0 = s \\ d\Sigma_t &= (\beta Q Q^T + M\Sigma_t + \Sigma_t M^T)dt + \sqrt{\Sigma_t} dW_t Q + Q^T dW_t^T \sqrt{\Sigma_t}, \quad \Sigma_0 = \Sigma \end{cases} \tag{3.24}$$

where r denotes risk free interest rate, Tr is trace operator, $Z \in M_n$ is a matrix Brownian motion under the risk-neutral measure, and Σ_t belongs to the set of symmetric $n \times n$ positive-definite matrices (as well as its square root $\sqrt{\Sigma_t}$). Observe that volatility of the risky asset is the trace of the matrix Σ_t, with $\Omega, M, Q \in M_n$, Ω invertible, and $W_t \in M_n$ is a matrix Brownian motion.

The dynamic of the Wishart process in Bru (1991), denotes a matrix analogue of the square root mean-reverting process. To ensure strict positivity and the typical mean reverting feature of the volatility, matrix, M is taken to be negative semi-definite, while Ω satisfies

$$\Omega\Omega^T = \beta Q^T Q$$

with the real parameter $\beta > n - 1$, for condition for uniqueness and existence of the solution of the dynamic of the Wishart processes.

3.1.8 The double Wishart stochastic volatility model

This section introduces a novel model, the double Wishart stochastic volatility model in the stock market. The model takes two underlying volatility components defined as the trace of Wishart processes. However, the diagonal components of the Wishart matrices will be the factors guiding the dynamics of volatilities.

In an arbitrage-free financial market and under the risk neutral probability measure, the risky asset price and Wishart process dynamics follows;

$$\begin{cases} \dfrac{dS_t}{S_t} &= rdt + Tr[\sqrt{\Sigma_t}dZ_t + \sqrt{\bar{\Sigma}_t}d\bar{Z}_t], \quad S_0 = s \\ d\Sigma_t &= (\beta QQ^T + M\Sigma_t + \Sigma_t M^T)dt + \sqrt{\Sigma_t}dW_t Q + Q^T dW_t^T \sqrt{\Sigma_t}, \quad \Sigma_0 = \Sigma \\ d\bar{\Sigma}_t &= (\bar{\beta}\bar{Q}\bar{Q}^T + \bar{M}\bar{\Sigma}_t + \bar{\Sigma}_t\bar{M}^T)dt + \sqrt{\bar{\Sigma}_t}d\bar{W}_t\bar{Q} + \bar{Q}^T d\bar{W}_t^T\sqrt{\bar{\Sigma}_t}, \quad \bar{\Sigma}_0 = \bar{\Sigma} \end{cases} \tag{3.25}$$

where $\beta, \bar{\beta}$ are real parameters such that $\beta, \bar{\beta} > n-1$, $Q, \bar{Q}, M, \bar{M} \in M_n$, Q is invertible matrix and $W_t, \bar{W}_t \in M_n$ are matrices Brownian motions, also $Z_t, \bar{Z}_t \in M_n$.

Lemma 3.1.4 *The correlations between the Brownian matrices of the stock price dynamics and the Brownian matrices of the Wishart processes in equation (3.25) are given by*

$$\rho_t = \frac{Tr(R^T Q\Sigma_t)}{\sqrt{Tr(\Sigma_t)}\sqrt{Tr(Q^T Q\Sigma_t)}}$$
$$\bar{\rho}_t = \frac{Tr(\bar{R}^T\bar{Q}\bar{\Sigma}_t)}{\sqrt{Tr(\bar{\Sigma}_t)}\sqrt{Tr(\bar{Q}^T\bar{Q}\bar{\Sigma}_t)}}$$

Proof. Deriving the correlations as follows

$$\begin{aligned} \frac{dS_t}{S_t} &= rdt + Tr[\sqrt{\Sigma_t}dZ_t + \sqrt{\bar{\Sigma}_t}d\bar{Z}_t] \\ &= rdt + \sqrt{Tr(\Sigma_t)}\frac{Tr(\sqrt{\Sigma_t}dZ_t)}{\sqrt{Tr(\Sigma_t)}} + \sqrt{Tr(\bar{\Sigma}_t)}\frac{Tr(\sqrt{\bar{\Sigma}_t}d\bar{Z}_t)}{\sqrt{Tr(\bar{\Sigma}_t)}} \\ &= rdt + \sqrt{Tr(\Sigma_t)}dX_t + \sqrt{Tr(\bar{\Sigma}_t)}d\bar{X}_t \end{aligned}$$

where X_t and $\bar{X}_t$ are standard Brownian motions and also taking the trace of the Wishart volatility dynamics (3.25), we have

$$dTr(\Sigma_t) = ((\beta Tr(Q^T Q)) + 2Tr(M\Sigma_t))dt + 2Tr(QdW_t\sqrt{\Sigma_t})$$

$$dTr(\bar{\Sigma}_t) = ((\bar{\beta}Tr(\bar{Q}^T\bar{Q})) + 2Tr(\bar{M}\bar{\Sigma}_t))dt + 2Tr(\bar{Q}d\bar{W}_t\sqrt{\bar{\Sigma}_t})$$

These processes can still be written in the form

$$dTr(\Sigma_t) = ((\beta Tr(Q^TQ)) + 2Tr(M\Sigma_t))dt + 2\sqrt{Tr(Q^TQ\Sigma_t)}\frac{Tr(QdW_t\sqrt{\Sigma_t})}{\sqrt{Tr(Q^TQ\Sigma_t)}}$$
$$dTr(\bar{\Sigma}_t) = ((\bar{\beta} Tr(\bar{Q}^T\bar{Q})) + 2Tr(\bar{M}\bar{\Sigma}_t))dt + 2\sqrt{Tr(\bar{Q}^T\bar{Q}\bar{\Sigma}_t)}\frac{Tr(\bar{Q}d\bar{W}_t\sqrt{\bar{\Sigma}_t})}{\sqrt{Tr(\bar{Q}^T\bar{Q}\bar{\Sigma}_t)}}$$

where ξ_t and η_t are Brownian motions, then

$$dTr(\Sigma_t) = ((\beta Tr(Q^TQ)) + 2Tr(M\Sigma_t))dt + 2\sqrt{Tr(Q^TQ\Sigma_t)}d\xi_t$$
$$dTr(\bar{\Sigma}_t) = ((\bar{\beta} Tr(\bar{Q}^T\bar{Q})) + 2Tr(\bar{M}\bar{\Sigma}_t))dt + 2\sqrt{Tr(\bar{Q}^T\bar{Q}\bar{\Sigma}_t)}d\eta_t$$

Now determining the covariation of the stock price and Wishart processes as follows

$$\begin{aligned}
Cov_t(dX_t, d\xi_t) &= Cov_t\left(\frac{Tr(\sqrt{\Sigma_t}dZ_t)}{\sqrt{Tr(\Sigma_t)}}, \frac{Tr(QdW_t\sqrt{\Sigma_t})}{\sqrt{Tr(Q^TQ\Sigma_t)}}\right) \\
&= \mathbb{E}_t\left(\frac{Tr(\sqrt{\Sigma_t}dW_tR^T)}{\sqrt{Tr(\Sigma_t)}}\frac{Tr(QdW_t\sqrt{\Sigma_t})}{\sqrt{Tr(Q^TQ\Sigma_t)}}\right) \\
&= \frac{\sum_{ij} Cov_t(e_i^T\sqrt{\Sigma_t}dW_tR^Te_i, e_j^TQdW_t\sqrt{\Sigma_t}e_j)}{\sqrt{Tr(\Sigma_t)}\sqrt{Tr(Q^TQ\Sigma_t)}} \\
&= \frac{\sum_{ij} \mathbb{E}_t(e_i^T\sqrt{\Sigma_t}dW_tR^Te_ie_j^TQ^TdW_t^T\sqrt{\Sigma_t}e_j)}{\sqrt{Tr(\Sigma_t)}\sqrt{Tr(Q^TQ\Sigma_t)}} \\
&= \frac{\sum_{ij} e_i\sqrt{\Sigma_t}Tr(R^Te_ie_j^TQ^T)\sqrt{\Sigma_t}e_jdt}{\sqrt{Tr(\Sigma_t)}\sqrt{Tr(Q^TQ\Sigma_t)}} \\
&= \frac{\sum_{ij} Tr(QRe_ie_j^T)e_i^T\Sigma_te_jdt}{\sqrt{Tr(\Sigma_t)}\sqrt{Tr(Q^TQ\Sigma_t)}} = \frac{\sum_{ij} e_j^TQRe_ie_i^T\Sigma_te_jdt}{\sqrt{Tr(\Sigma_t)}\sqrt{Tr(Q^TQ\Sigma_t)}} \\
&= \frac{\sum_{ij} e_j^TQR\Sigma_te_jdt}{\sqrt{Tr(\Sigma_t)}\sqrt{Tr(Q^TQ\Sigma_t)}} \\
Cov_t(dX_t, d\xi_t) &= \frac{Tr(R^TQ\Sigma_t)}{\sqrt{Tr(\Sigma_t)}\sqrt{Tr(Q^TQ\Sigma_t)}}
\end{aligned}$$

Similarly for the second stochastic differential equation of the Wishart processes, the determination of covariation follow the same procedures, that is

$$Cov_t(dX_t, d\eta_t) = \frac{Tr(\bar{R}^T\bar{Q}\bar{\Sigma}_t)}{\sqrt{Tr(\bar{\Sigma}_t)}\sqrt{Tr(\bar{Q}^T\bar{Q}\bar{\Sigma}_t)}}$$

□

Corollary 3.1.1 *Let us verify that the following processes $X_t, \bar{X}_t, \xi_t, \eta_t$ are Brownian motions (in lemma 3.3.1).*

Proof.

$$
\begin{aligned}
Cov_t(dX_t, dX_t) &= \mathbb{E}_t[(dX_t)(dX_t)^T] \\
&= \mathbb{E}_t\left[\frac{Tr(\sqrt{\Sigma_t}dZ_t)}{\sqrt{Tr(\Sigma_t)}}\frac{Tr(\sqrt{\Sigma_t}dZ_t)}{\sqrt{Tr(\Sigma_t)}}\right] \\
&= \frac{\sum_{i,j} Cov_t(e_i^T\sqrt{\Sigma_t}dZ_t e_i, e_j^T\sqrt{\Sigma_t}dZ_t e_j)}{\sqrt{Tr(\Sigma_t)}\sqrt{Tr(\Sigma_t)}} \\
&= \frac{\sum_{i,j} e_i^T\sqrt{\Sigma_t}Tr(e_i e_j^T)\sqrt{\Sigma_t}e_j dt}{Tr(\Sigma_t)} \\
&= \frac{\sum_{i,j} Tr(e_i e_j^T)e_i^T\Sigma_t e_j dt}{Tr(\Sigma_t)} = \frac{\sum_{i,j} e_j^T e_i e_i^T\Sigma_t e_j dt}{Tr(\Sigma_t)} \\
&= \frac{\sum_j e_j^T\Sigma_t e_j dt}{Tr(\Sigma_t)} = \frac{Tr(\Sigma_t)dt}{Tr(\Sigma_t)} \\
&= dt
\end{aligned}
$$

Similarly for Brownian motion $\bar{X}_t$, its proof can be obtained through the same procedures as above. □

Then showing that ξ_t is a Brownian motion as follows

Proof.

$$
\begin{aligned}
Cov_t(d\xi_t, d\xi_t) &= \mathbb{E}_t\left[\frac{Tr(QdW_t\sqrt{\Sigma_t})}{\sqrt{Tr(Q^TQ\Sigma_t)}}\frac{Tr(QdW_t\sqrt{\Sigma_t})}{\sqrt{Tr(Q^TQ\Sigma_t)}}\right] \\
&= \frac{\sum_{i,j} Cov_t(e_i^TQdW_t\sqrt{\Sigma_t}e_i, e_j^TQdW_t\sqrt{\Sigma_t}e_j)}{\sqrt{Tr(Q^TQ\Sigma_t)}\sqrt{Tr(Q^TQ\Sigma_t)}} \\
&= \frac{\sum_{i,j} e_i^T\sqrt{\Sigma_t}Tr(Q^Te_i, e_j^TQ)\sqrt{\Sigma_t}e_jdt}{\sqrt{Tr(Q^TQ\Sigma_t)}\sqrt{Tr(Q^TQ\Sigma_t)}} \\
&= \frac{\sum_{i,j} Tr(Q^TQe_ie_j^T)e_i^T\Sigma_te_jdt}{Tr(Q^TQ\Sigma_t)} = \frac{\sum_{i,j} e_j^TQ^TQe_ie_i^T\Sigma_te_jdt}{Tr(Q^TQ\Sigma_t)} \\
&= \frac{\sum_{i,j} e_j^TQ^TQ\Sigma_te_jdt}{Tr(Q^TQ\Sigma_t)} = \frac{Tr(Q^TQ\Sigma_t)dt}{Tr(Q^TQ\Sigma_t)} \\
&= dt
\end{aligned}
$$

Similarly the same procedures can be followed to show that η_t is a Brownian motion. $\square$

3.1.9 The correlation structure

The Brownian motion matrices $W_{k,t}, Z_{k,t}$, are correlated in a way that all Brownian motions belonging to the column i of $Z_{k,t}$ and that of column j of $W_{k,t}$ have the same correlation, say $R_{k,ij}$. This gives a constant correlated matrix $R_k \in M_n$ which completely describes the correlation structure, such that $Z_{k,t}$ can be written as

$$Z_{k,t} := W_{k,t}R_k^T + B_{k,t}\sqrt{\mathbb{I} - R_kR_k^T}, \quad for \quad k = 1, 2.$$

where $\mathbb{I}$ denotes the identity matrix, T is the transpose, and $B_{k,t}$ is an independent Brownian motion matrix from W_t. Observing that, in order to preserve the affine property, the correlation structure should be introduced in the model.

Proposition 3.1.1 *The correlation structure $Z_{k,t} := W_{k,t}R_k^T + B_{k,t}\sqrt{\mathbb{I} - R_kR_k^T}$ is a matrix Brownian motion.*

Proof. *Worthy noting that $Z_{k,t}$ is a matrix Brownian motion iff for $\alpha_k, \beta_k \in \mathbb{R}^n$*

$$
\begin{aligned}
Cov_t(dZ_{k,t}\alpha_k, dZ_{k,t}\beta_k) &= \mathbb{E}_t[(dZ_{k,t}\alpha_k)(dZ_{k,t}\beta_k)^T] = \alpha_k^T\beta_k\mathbb{I}dt \\
Cov_t(dZ_{k,t}\alpha_k, dZ_{k,t}\beta_k) &= \mathbb{E}_t[(dW_{k,t}R^T\alpha_k + dB_{k,t}\sqrt{\mathbb{I} - R_kR_k^T\alpha_k}) \\
&\quad .(dW_{k,t}R_k^T\beta_k + dB_{k,t}\sqrt{\mathbb{I} - R_kR_k^T\beta_k})] \\
&= Cov_t(dW_{k,t}R_k^T\alpha_k, dW_{k,t}R_k^T\beta_k) \\
&\quad + Cov_t(dB_{k,t}\sqrt{\mathbb{I} - R_kR_k^T\alpha_k}, dB_{k,t}\sqrt{\mathbb{I} - R_kR_k^T\beta_k}) \\
&= \alpha_k^T R_kR_K^T\beta_k\mathbb{I}dt + \alpha_k^T(\mathbb{I} - R_kR_k^T)\beta_k\mathbb{I}dt \\
&= \alpha_k^T\beta_k\mathbb{I}dt
\end{aligned}
$$

for correlation structures; $Z_{1,t} := W_{1,t}R_1^T + B_{1,t}\sqrt{\mathbb{I} - R_1R_1^T}$, *the* $Cov_t(dZ_{1,t}\alpha_1, dZ_{1,t}\beta_1)$ $= \alpha_1^T\beta_1\mathbb{I}dt$

and $Z_{2,t} := W_{2,t}R_2^T + B_{2,t}\sqrt{\mathbb{I} - R_2R_2^T}$, *the* $Cov_t(dZ_{2,t}\alpha_2, dZ_{2,t}\beta_2) = \alpha_2^T\beta_2\mathbb{I}dt$ □

3.1.10 The dynamic of log-price for the double Wishart model

The matrices R and $\bar{R}$ describes the correlations between the Brownian of the asset price and those of the Wishart processes.

Lemma 3.1.5 *The log-price dynamic $Y_t = \log(S_t)$ under double Wishart model is given as*

$$dY_t = (r - \frac{1}{2}Tr[\Sigma_t + \bar{\Sigma}_t])dt + Tr[\sqrt{\Sigma_t}dZ_t + \sqrt{\bar{\Sigma}_t}d\bar{Z}_t] \tag{3.26}$$

Proof. Let $Y_t = \log(S_t)$, the asset dynamic given as ,

$$\frac{dS(t)}{S(t)} = rdt + Tr[\sqrt{\Sigma_t}dZ_t + \sqrt{\bar{\Sigma}_t}d\bar{Z}_t] \tag{3.27}$$

By applying the Ito's formula on Y_t, see Björk (2009); Shreve (2004), we get

$$dY_t = d\log(S_t) = \frac{dS_t}{S_t} - \frac{1}{2}\frac{(dS_t)^2}{S_t^2} \tag{3.28}$$

Substituting the asset process (3.27) in the derivative expression (3.28) of Y_t

$$dY(t) = rdt + Tr[\sqrt{\Sigma_t}dZ_t + \sqrt{\bar{\Sigma}_t}d\bar{Z}_t] - \frac{1}{2}Tr[\Sigma_t + \bar{\Sigma}_t]dt \tag{3.29}$$

$$dY_t = (r - \frac{1}{2}Tr[\Sigma_t + \bar{\Sigma}_t])dt + Tr[\sqrt{\Sigma_t}dZ_t + \sqrt{\bar{\Sigma}_t}d\bar{Z}_t]$$

which can be written as

$$\begin{aligned} dY(t) = (r - \frac{1}{2}Tr[\Sigma_t + \bar{\Sigma}_t])dt + Tr[\sqrt{\Sigma_t}(dW_t R^T + dB_t\sqrt{\mathbb{I} - RR^T}) \\ + \sqrt{\bar{\Sigma}_t}(d\bar{W}_t\bar{R}^T + d\bar{B}_t\sqrt{\mathbb{I} - \bar{R}\bar{R}^T})], \quad Y_0 = y. \end{aligned} \tag{3.30}$$

□

3.2 The Solution of Partial Differential Equation under Double Wishart Model

In this section, the infinitesimal generator and the partial differential equation associated to double Wishart stochastic volatility model are derived. The Fourier techniques, combined with perturbation methods are applied in order to obtain European call option pricing formula.

3.2.1 Infinitesimal generator

The infinitesimal generator is obtained and also following Benabid et al. (2008), it is acceptable to write the log-price and its volatility dynamics as driven by two correlated Brownian motions, Z_s^Y, Z_s^{Σ}, and $\bar{Z}_s^Y, \bar{Z}_s^{\bar{\Sigma}}$; as follows for easy handling of the complexity of the computation of the problem

$$\begin{cases} dY_t = (r - \frac{1}{2}Tr[\Sigma_t + \bar{\Sigma}_t])dt + Tr[\sqrt{\Sigma_t}dZ_t^Y + \sqrt{\bar{\Sigma}_t}d\bar{Z}_t^Y] \\ dTr(\Sigma_t) = ((\beta Tr(Q^TQ)) + 2Tr(M\Sigma_t))dt + 2\sqrt{Tr(Q^TQ\Sigma_t)}(\rho_t dZ_t^Y + \sqrt{I-\rho_t^2}dZ_t^{\Sigma}) \\ dTr(\bar{\Sigma}_t) = ((\bar{\beta} Tr(\bar{Q}^T\bar{Q})) + 2Tr(\bar{M}\bar{\Sigma}_t))dt + 2\sqrt{Tr(\bar{Q}^T\bar{Q}\bar{\Sigma}_t)}(\bar{\rho}_t d\bar{Z}_t^Y + \sqrt{I-\bar{\rho}_t^2}d\bar{Z}_t^{\bar{\Sigma}}) \end{cases} \tag{3.31}$$

or

$$\begin{cases} dY_t = (r - \frac{1}{2}Tr[\Sigma_t + \bar{\Sigma}_t])dt + Tr[\sqrt{\Sigma_t}dZ_t + \sqrt{\bar{\Sigma}_t}d\bar{Z}_t] \\ dTr(\Sigma_t) = ((\beta Tr(Q^TQ)) + 2Tr(M\Sigma_t))dt + 2Tr(QdW_t\sqrt{\Sigma_t}) \\ dTr(\bar{\Sigma}_t) = ((\bar{\beta} Tr(\bar{Q}^T\bar{Q})) + 2Tr(\bar{M}\bar{\Sigma}_t))dt + 2Tr(\bar{Q}d\bar{W}_t\sqrt{\bar{\Sigma}_t}) \end{cases}$$

where

$$d < Z^Y, Z^{\Sigma} >_t = \rho_t = \frac{Tr(R^TQ\Sigma_t)}{\sqrt{Tr(\Sigma_t)}\sqrt{Tr(Q^TQ\Sigma_t)}}$$

$$d < \bar{Z}^Y, \bar{Z}^{\bar{\Sigma}} >_t = \bar{\rho}_t = \frac{Tr(\bar{R}^T\bar{Q}\bar{\Sigma}_t)}{\sqrt{Tr(\bar{\Sigma}_t)}\sqrt{Tr(\bar{Q}^T\bar{Q}\bar{\Sigma}_t)}}$$

Proposition 3.2.1 *The infinitesimal generator of double Wishart stochastic volatility model for couple of* $(Y_t, \Sigma_t, \bar{\Sigma}_t)$ *given as;*

$$\begin{aligned} \mathcal{L}_{Y,\Sigma,\bar{\Sigma}} = & \left(r - \frac{Tr[\Sigma + \bar{\Sigma}]}{2}\right)\frac{\partial}{\partial y} + \frac{Tr[\Sigma + \bar{\Sigma}]}{2}\frac{\partial^2}{\partial y^2} \\ & + (\beta Tr(Q^TQ) + 2Tr(M\Sigma))\frac{\partial}{\partial \Sigma} + 2Tr\left(\Sigma\frac{\partial}{\partial \Sigma}Q^TQ\frac{\partial}{\partial \Sigma}\right) \\ & + (\bar{\beta} Tr(\bar{Q}^T\bar{Q}) + 2Tr(\bar{M}\bar{\Sigma}))\frac{\partial}{\partial \bar{\Sigma}} + 2Tr\left(\bar{\Sigma}\frac{\partial}{\partial \bar{\Sigma}}\bar{Q}^T\bar{Q}\frac{\partial}{\partial \bar{\Sigma}}\right) \\ & + 2Tr\left(\Sigma RQ\frac{\partial}{\partial \Sigma}\right)\frac{\partial}{\partial y} + 2Tr\left(\bar{\Sigma}\bar{R}\bar{Q}\frac{\partial}{\partial \bar{\Sigma}}\right)\frac{\partial}{\partial y} \end{aligned} \tag{3.32}$$

Proof.

The infinitesimal generator has non-trivial term which arise from covariation $d < \Sigma_{\theta}^{ij}, Y >$ corresponding to the coefficients of the term

$$\frac{\partial^2}{\partial x_{\theta;ij}\partial y} = \frac{\partial}{\partial x_{\theta;ij}}\left(\frac{\partial}{\partial y}\right), \quad i,j = 1,...,n, \quad \theta = 1,2.$$

Let $V_{\theta;t} := \sqrt{\Sigma_{\theta;t}}$ be square root matrix such that

$$\Sigma_{\theta;t}^{ij} = \sum_{t=1}^{n} V_{\theta;t}^{il} V_{\theta;t}^{lj} = \sum_{t=1}^{n} V_{\theta;t}^{il} V_{\theta;t}^{jl}$$

Due to the fact that $V_{\theta;t}$ is symmetric, the covariation terms matching with $\frac{\partial^2}{\partial x_{\theta;ij}\partial y}$ coefficients can be determined

$$\begin{aligned}
d < \Sigma_{\theta}^{ij}, Y > &= \mathbb{E}_t \left[(\sum_{i,k=1}^{n} V_{\theta;t}^{il} dW_{lk}^{\theta} Q_{kj}^{\theta} + \sum_{l,k=1}^{n} V_{\theta;t}^{jl} dW_{lk}^{\theta} Q_{ki}^{\theta})(\sum_{l,k,h=1}^{n} V_{\theta;t}^{lk} dW_{kh}^{\theta} R_{lh}^{\theta}) \right] \\
&= \sum_{l,k,h=1}^{n} \left(V_{\theta;t}^{il} Q_{kj}^{\theta} + V_{\theta;t}^{jl} Q_{ki}^{\theta} \right) V_{\theta;t}^{hl} R_{hk}^{\theta} dt \\
&= \sum_{k,h=1}^{n} \left[(\sum_{l=1}^{n} V_{\theta;t}^{il} V_{\theta;t}^{hl}) Q_{kj}^{\theta} + (\sum_{l=1}^{n} V_{\theta;t}^{jl} V_{\theta;t}^{hl}) Q_{ki}^{\theta} \right] R_{hk}^{\theta} dt \\
&= \sum_{k,h=1}^{n} \left(\Sigma_{\theta;t}^{ih} Q_{kj}^{\theta} + \Sigma_{\theta;t}^{jh} Q_{ki}^{\theta} \right) R_{hk}^{\theta} dt
\end{aligned}$$

This gives the corresponding coefficients of the term since

$$2Tr(\Sigma_{\theta} R^{\theta} Q^{\theta} D_{\theta}) \frac{\partial}{\partial y} = 2 \sum_{i,j,k,h=1}^{n} D_{\theta}^{ij} \Sigma_{\theta}^{jh} R_{hk}^{\theta} Q_{ki}^{\theta} \frac{\partial}{\partial y}$$

The notation when $\theta = 1, \Sigma_1 = \Sigma, R^1 = R, Q^1 = Q, D_1 = \frac{\partial}{\partial x_{ij}}$ while for $\theta = 2, \Sigma_2 = \bar{\Sigma}, R^2 = \bar{R}, Q^2 = \bar{Q}, D_2 = \frac{\partial}{\partial \bar{x}_{ij}}$, since our D is symmetric. □

3.2.2 The Laplace transform of the asset returns

Following Duffie et al. (2000) and Da Fonseca et al. (2008), in order to solve the European options pricing problem, consider the Laplace transform of process (3.30). The Laplace transform of Wishart process is exponentially affine as in Bru (1991), hence the conditional moment generating function of the log-asset returns is the exponential of an affine combination of Y and Wishart components, so we provide for the deterministic functions $\lambda_1(t), \lambda_2(t) \in M_n$ and

$\delta(t), \varepsilon(t) \in \mathbb{R}$, as parameters of the Laplace transform

$$\begin{aligned}\psi_{\gamma,t}(\tau) &= \mathbb{E}\left(e^{\gamma Y_{t+\tau}}\right)\\ &= exp[Tr\lambda_1(\tau)\Sigma_t + Tr\lambda_2(\tau)\bar{\Sigma}_t + \delta(\tau)Y_t + \varepsilon(\tau)]\end{aligned} \tag{3.33}$$

where $\mathbb{E}_t$ denotes the conditional expected value with respect to the risk-neutral measure and $\gamma \in \mathbb{R}$.

By applying the Feynman-Kac argument, a set of ordinary differential equations for each of the functions mentioned above is obtained

$$\begin{cases} \frac{\partial \psi_{\gamma,t}}{\partial \tau} & = \mathcal{L}_{Y,\Sigma,\bar{\Sigma}} \psi_{\gamma,t} \\ \psi_{\gamma,t}(0) & = exp\{\gamma Y_t\} = e^{\gamma Y_t} \end{cases} \tag{3.34}$$

The setting of matrices for the Wishart dynamics implies a non standard definition of the infinitesimal generator for $(Y_t, \Sigma_t, \bar{\Sigma}_t)$. The infinitesimal generator for the double Wishart process, Σ_t and $\bar{\Sigma}_t$ can be written as

$$\begin{aligned}\mathcal{L}_{Y,\Sigma,\bar{\Sigma}} = & \left(r - \frac{Tr[\Sigma + \bar{\Sigma}]}{2}\right)\frac{\partial}{\partial y} + \frac{Tr[\Sigma + \bar{\Sigma}]}{2}\frac{\partial^2}{\partial y^2}\\ & + Tr[(\beta Q^T Q + 2M\Sigma)D]\\ & + Tr[(\bar{\beta}\bar{Q}^T\bar{Q} + 2\bar{M}\bar{\Sigma})\bar{D}] + 2Tr(\Sigma D Q^T Q D) + 2Tr(\bar{\Sigma}\bar{D}\bar{Q}^T\bar{Q}\bar{D})\\ & + 2\left[Tr(\Sigma R Q D) + Tr(\bar{\Sigma}\bar{R}\bar{Q}\bar{D})\right]\frac{\partial}{\partial y}\end{aligned} \tag{3.35}$$

where D is a matrix differential operator with elements
$D_{i,j} = (\frac{\partial}{\partial \Sigma^{i,j}})$ and $\bar{D}_{i,j} = (\frac{\partial}{\partial \bar{\Sigma}^{i,j}})$
From equation (3.34) and the Laplace transform (3.33), solve the problem by considering proposition (3.2.1), for double Wishart stochastic volatility model as follows

$$\psi_{\gamma,t}(\tau) = exp\{Tr[\lambda_1(\tau)\Sigma_t + \lambda_2(\tau)\bar{\Sigma}_t] + \delta(\tau)Y_t + \varepsilon(\tau)\} \tag{3.36}$$

$$
\begin{aligned}
\frac{\partial \psi_{\gamma,t}}{\partial \tau} &= \left(r - \frac{Tr[\Sigma+\bar{\Sigma}]}{2}\right)\frac{\partial \psi}{\partial y} + \frac{Tr[\Sigma+\bar{\Sigma}]}{2}\frac{\partial^2 \psi}{\partial y^2} \\
&+ (\beta Tr(Q^TQ) + 2Tr(M\Sigma))\frac{\partial \psi}{\partial \Sigma} + 2Tr\left(\Sigma\frac{\partial}{\partial \Sigma}Q^TQ\frac{\partial}{\partial \Sigma}\right)\psi \\
&+ (\bar{\beta} Tr(\bar{Q}^T\bar{Q}) + 2Tr(\bar{M}\bar{\Sigma}))\frac{\partial \psi}{\partial \bar{\Sigma}} + 2Tr\left(\bar{\Sigma}\frac{\partial}{\partial \bar{\Sigma}}\bar{Q}^T\bar{Q}\frac{\partial}{\partial \bar{\Sigma}}\right)\psi \\
&+ 2Tr\left(\Sigma RQ\frac{\partial}{\partial \Sigma}\right)\frac{\partial \psi}{\partial y} + 2Tr\left(\bar{\Sigma}\bar{R}\bar{Q}\frac{\partial}{\partial \bar{\Sigma}}\right)\frac{\partial \psi}{\partial y}
\end{aligned} \tag{3.37}
$$

Considering equation (3.34) to obtain the boundary conditions as follows

$$
\begin{cases}
\psi_{\gamma,t}(\tau) = e^{\{Tr[\lambda_1(\tau)\Sigma_t + \lambda_2(\tau)\bar{\Sigma}_t] + \delta(\tau)Y_t + \varepsilon(\tau)\}} \\
\lambda_1(0) = 0, \lambda_2(0) = 0, \delta(0) = \gamma, \varepsilon(0) = 0
\end{cases} \tag{3.38}
$$

$$
\begin{aligned}
\frac{\partial \psi_{\gamma,t}(\tau)}{\partial \tau} &= Tr\left(\frac{d\lambda_1(\tau)}{d\tau}\Sigma_t + \frac{d\lambda_2(\tau)}{\partial \tau}\bar{\Sigma}_t\right) + \frac{d}{d\tau}\delta(\tau)Y_t + \frac{d}{d\tau}\varepsilon(\tau)\psi_{\gamma,t}(\tau) \\
&= (r - \frac{Tr[\Sigma+\bar{\Sigma}]}{2})\delta(\tau)\psi_{\gamma,t}(\tau) + \frac{Tr(\Sigma+\bar{\Sigma})}{2}\delta^2(\tau)\psi_{\gamma,t}(\tau) \\
&+ [(\beta Tr(Q^TQ) + 2Tr(M\Sigma))\lambda_1(\tau) + (\bar{\beta}Tr(\bar{Q}^T\bar{Q}) + 2Tr(\bar{M}\bar{\Sigma}))\lambda_2(\tau)]\psi_{\gamma,t}(\tau) \\
&+ 2Tr[\Sigma\lambda_1(\tau)Q^TQ\lambda_1(\tau)]\psi_{\gamma,t}(\tau) + 2Tr[\bar{\Sigma}\lambda_2(\tau)\bar{Q}^T\bar{Q}\lambda_2(\tau)]\psi_{\gamma,t}(\tau) \\
&+ 2Tr[\Sigma RQ\delta(\tau)\lambda_1(\tau)]\psi_{\gamma,t}(\tau) + 2Tr[\bar{\Sigma}\bar{R}\bar{Q}\delta(\tau)\lambda_2(\tau)]\psi_{\gamma,t}(\tau).
\end{aligned} \tag{3.39}
$$

$$
\frac{\partial \psi_{\gamma,t}(\tau)}{\partial \tau} = [Tr(\acute{\lambda}_1(\tau)\Sigma_t + \acute{\lambda}_2(\tau)\bar{\Sigma}_t) + \acute{\delta}(\tau)Y_t + \acute{\varepsilon}(\tau)]\psi_{\gamma,t}(\tau) = \mathcal{L}_{Y,\Sigma,\bar{\Sigma}}\psi_{\gamma,t}(\tau) \tag{3.40}
$$

$$
-\frac{\partial \psi_{\gamma,t}(\tau)}{\partial \tau} + \mathcal{L}_{Y,\Sigma,\bar{\Sigma}}\psi_{\gamma,t}(\tau) = 0 \tag{3.41}
$$

By substituting in equation (3.41) provides the following expression,

$$
\begin{aligned}
&- [Tr(\acute{\lambda}_1(\tau)\Sigma_t + \acute{\lambda}_2(\tau)\bar{\Sigma}_t) + \acute{\delta}(\tau)Y_t + \acute{\varepsilon}(\tau)] + (r - \frac{Tr[\Sigma+\bar{\Sigma}]}{2})\delta(\tau) \\
&+ \frac{Tr(\Sigma+\bar{\Sigma})}{2}\delta^2(\tau) + Tr[(\beta Q^TQ + M\Sigma + \Sigma M^T)\lambda_1(\tau) + (\bar{\beta}\bar{Q}^T\bar{Q} + \bar{M}\bar{\Sigma} + \bar{\Sigma}\bar{M}^T)\lambda_2(\tau) \\
&+ 2\Sigma\lambda_1(\tau)Q^TQ\lambda_1(\tau) + 2\bar{\Sigma}\lambda_2(\tau)\bar{Q}^T\bar{Q}\lambda_2(\tau) + 2\Sigma RQ\delta(\tau)\lambda_1(\tau) + 2\bar{\Sigma}\bar{R}\bar{Q}\delta(\tau)\lambda_2(\tau)] = 0
\end{aligned} \tag{3.42}
$$

Identifying the coefficients in equation (3.42) as follows

$$\begin{aligned}&- Tr(\frac{d}{d\tau}\lambda_1(\tau)\Sigma) - \frac{1}{2}Tr[\Sigma]\delta(\tau) + \frac{1}{2}Tr[\Sigma]\delta^2(\tau)\\&+ Tr[(M\Sigma + \Sigma M^T)\lambda_1(\tau) + 2\Sigma\lambda_1(\tau)Q^TQ\lambda_1(\tau) + 2\Sigma RQ\delta(\tau)\lambda_1(\tau)] = 0\end{aligned} \tag{3.43}$$

$$\begin{aligned}&- Tr(\frac{d}{d\tau}\lambda_2(\tau)\bar{\Sigma}) - \frac{1}{2}Tr[\bar{\Sigma}]\delta(\tau) + \frac{1}{2}Tr[\bar{\Sigma}]\delta^2(\tau)\\&+ Tr[(\bar{M}\bar{\Sigma} + \bar{\Sigma}\bar{M}^T)\lambda_2(\tau) + 2\bar{\Sigma}\lambda_2(\tau)\bar{Q}^T\bar{Q}\lambda_2(\tau) + 2\bar{\Sigma}\bar{R}\bar{Q}\delta(\tau)\lambda_2(\tau)] = 0\end{aligned} \tag{3.44}$$

$$-\acute{\delta}(\tau)Y_t - \acute{\varepsilon}(\tau) + r\delta(\tau) + Tr[\beta Q^TQ]\lambda_1(\tau) + Tr[\bar{\beta}\bar{Q}^T\bar{Q}]\lambda_2(\tau) = 0 \tag{3.45}$$

Then arranging equations (3.43), (3.44) and (3.45), then matrix Riccati ordinary differential equations are obtained as below

$$\begin{cases}\frac{d}{d\tau}\lambda_1(\tau) & = M\lambda_1(\tau) + (M^T + 2\gamma RQ)\lambda_1(\tau) + 2\lambda_1(\tau)Q^TQ\lambda_1(\tau) + \frac{\gamma(\gamma-1)}{2}\mathbb{I}_n\\ \lambda_1(0) & = 0\end{cases} \tag{3.46}$$

$$\begin{cases}\frac{d}{d\tau}\lambda_2(\tau) & = \bar{M}\lambda_2(\tau) + (\bar{M}^T + 2\gamma\bar{R}\bar{Q})\lambda_2(\tau) + 2\lambda_2(\tau)\bar{Q}^T\bar{Q}\lambda_2(\tau) + \frac{\gamma(\gamma-1)}{2}\mathbb{I}_n\\ \lambda_2(0) & = 0\end{cases} \tag{3.47}$$

For the constant ε the matrix Riccati ordinary differential can be obtained as follows

$$-\acute{\delta}(\tau)Y_t - \acute{\varepsilon}(\tau) + r\delta(\tau) + Tr(\beta Q^TQ)\lambda_1(\tau) + Tr(\bar{\beta}\bar{Q}^T\bar{Q})\lambda_2(\tau) = 0$$

$$\delta(\tau) = C_0 = \gamma, \quad since \quad \acute{\delta(\tau)} = 0 \quad so \Rightarrow \delta(\tau) = C_0 = \gamma$$

$$\begin{cases}\frac{d\varepsilon(\tau)}{d\tau} = r\gamma + \beta Tr[(Q^TQ)\lambda_1(\tau)] + \bar{\beta}Tr[(\bar{Q}^T\bar{Q})\lambda_2(\tau)]\\ \varepsilon(0) = 0\end{cases} \tag{3.48}$$

Such that $\varepsilon(\tau)$ can be obtained by direct integration

$$\varepsilon(\tau) = \int_0^\tau r\gamma ds + \int_0^\tau Tr[\beta Q^TQ\lambda_1(s) + \bar{\beta}\bar{Q}^T\bar{Q}\lambda_2(s)]ds$$

Where $\lambda_1, \lambda_2 \in M_n(\mathbb{R})$ and $\delta(\tau), \varepsilon(\tau) \in \mathbb{R}$.

Matrix Riccati linearization; The matrix Riccati equations above are linearized to provide representation of the closed form solution, following the techniques of Benabid et al. (2008); Da Fonseca et al. (2008). So from Riccati equations (3.46) and (3.47),

$$\begin{cases} \frac{d}{d\tau}\lambda_1(\tau) & = M\lambda_1(\tau) + (M^T + 2\gamma RQ)\lambda_1(\tau) + 2\lambda_1(\tau)Q^T Q\lambda_1(\tau) + \frac{\gamma(\gamma-1)}{2}\mathbb{I}_n \\ \lambda_1(0) & = 0 \end{cases}$$

$$\begin{cases} \frac{d}{d\tau}\lambda_2(\tau) & = \bar{M}\lambda_2(\tau) + (\bar{M}^T + 2\gamma \bar{R}\bar{Q})\lambda_2(\tau) + 2\lambda_2(\tau)\bar{Q}^T \bar{Q}\lambda_2(\tau) + \frac{\gamma(\gamma-1)}{2}\mathbb{I}_n \\ \lambda_2(0) & = 0 \end{cases}$$

Let

$$\lambda_1(\tau) = H_1(\tau)^{-1}H_2(\tau)$$

where $H_1(\tau) \in GL_n(\mathbb{R})$ and $H_2(\tau) \in M_n(\mathbb{R})$ and therefore

$$\frac{d}{d\tau}[H_1(\tau)\lambda_1(\tau)] = \frac{dH_1(\tau)}{d\tau}\lambda_1(\tau) + H_1(\tau)\frac{d\lambda_1(\tau)}{d\tau}$$

$$H_1(\tau)\frac{d\lambda_1(\tau)}{d\tau} = \frac{d}{d\tau}[H_1(\tau)\lambda_1(\tau)] - \frac{dH_1(\tau)}{d\tau}\lambda_1(\tau) \tag{3.49}$$

$$\begin{aligned} &H_1(\tau)\lambda_1(\tau)M + H_1(\tau)(M^T + 2\gamma RQ)\lambda_1(\tau) + 2H_1(\tau)\lambda_1(\tau)Q^T Q\lambda_1(\tau) \\ &+ H_1(\tau)\frac{\gamma(\gamma-1)}{2} = H_1(\tau)\frac{d\lambda_1(\tau)}{d\tau} \end{aligned} \tag{3.50}$$

$$\begin{aligned} &H_2(\tau)M + H_1(\tau)(M^T + 2\gamma RQ)\lambda_1(\tau) + 2H_2(\tau)Q^T Q\lambda_1(\tau) \\ &+ H_1(\tau)\frac{\gamma(\gamma-1)}{2} = H_1(\tau)\frac{d\lambda_1(\tau)}{d\tau} \end{aligned} \tag{3.51}$$

Since $H_2(\tau) = H_1(\tau)\lambda_1(\tau)$, then

$$\frac{dH_2(\tau)}{d\tau} = \frac{dH_1(\tau)}{d\tau}\lambda_1(\tau) + H_1(\tau)\frac{d\lambda_1(\tau)}{d\tau}$$

This implies that

$$H_1(\tau)\frac{d\lambda_1(\tau)}{d\tau} = \frac{dH_2(\tau)}{d\tau} - \frac{dH_1(\tau)}{d\tau}\lambda_1(\tau) \tag{3.52}$$

Such that it provides an expression

$$\begin{aligned}\frac{dH_2(\tau)}{d\tau} - \frac{dH_1(\tau)}{d\tau}\lambda_1(\tau) &= H_2(\tau)M + H_1(\tau)(M^T + 2\gamma RQ)\lambda_1(\tau) \\ &+ 2H_2(\tau)Q^TQ\lambda_1(\tau) + H_1(\tau)\frac{\gamma(\gamma-1)}{2} \\ &= H_1(\tau)\frac{\gamma(\gamma-1)}{2} + H_2(\tau)M + [H_1(\tau)(M^T + 2\gamma RQ) \\ &+ 2H_2(\tau)Q^TQ]\lambda_1(\tau)\end{aligned} \tag{3.53}$$

Then

$$\frac{dH_2(\tau)}{d\tau} = H_1(\tau)\frac{\gamma(\gamma-1)}{2} + H_2(\tau)M \tag{3.54}$$

$$\frac{dH_1(\tau)}{d\tau} = -H_1(\tau)(M^T + 2\gamma RQ) - 2H_2(\tau)Q^TQ \tag{3.55}$$

$$\frac{d}{d\tau}(H_2(\tau) \quad H_1(\tau)) = \begin{pmatrix} H_2(\tau) & H_1(\tau)\end{pmatrix}\begin{pmatrix} M & -2Q^TQ \\ \frac{\gamma(\gamma-1)}{2}\mathbb{I}_n & -(M^T + 2\gamma RQ)\end{pmatrix} \tag{3.56}$$

The solution of differential equation (3.56) above is given as

$$\begin{pmatrix} H_2(\tau) & H_1(\tau)\end{pmatrix} = \begin{pmatrix} H_2(0) & H_1(0)\end{pmatrix} e^{Q\tau} \tag{3.57}$$

Proof.

Indeed by letting and differentiating

$$\begin{aligned} X(\tau) &= (H_2(\tau) \quad H_1(\tau)) \\ \frac{dX(s)}{ds} &= X(s)Q \\ \frac{dX(s)}{X(s)} &= Qds \\ \int_0^\tau \frac{dX(s)}{X(s)} &= \int_0^\tau Qds = \ln(\frac{X(\tau)}{X(0)}) \quad = Q\tau \\ \Rightarrow X(\tau) &= X(0)e^{Q\tau}\end{aligned}$$

$\square$

This can be writen in the form

$$(H_2(\tau) \quad H_1(\tau)) = (H_2(0) \quad H_1(0))e^{Q\tau} \tag{3.58}$$

Noting that, $\lambda_1(\tau) = H_1(\tau)^{-1}H_2(\tau)$ and from

$$\frac{dH_1(\tau)}{d\tau} = -H_1(\tau)(M^T + 2\gamma RQ) - 2H_2(\tau)Q^TQ \tag{3.59}$$

and solving the equation as follows

$$\frac{dH_1(\tau)}{d\tau} = -H_1(\tau)(M^T + 2\gamma RQ) \tag{3.60}$$

setting $G = (M^T + 2\gamma RQ)$

This gives

$$H_1(\tau) = Ke^{-G\tau}$$

Differentiating, gives

$$\frac{dH_1(\tau)}{d\tau} = \acute{K}e^{-G\tau} - Ke^{-G\tau}G \tag{3.61}$$

Now substituting in equation (3.59), provides

$$\acute{K}e^{-G\tau} - Ke^{-G\tau}G = -Ke^{-G\tau}G - 2H_2(\tau)Q^TQ \tag{3.62}$$

then

$$K = \int_0^\tau -2H_2(u)Q^TQe^{Gu}du$$

$$H_1(\tau) = (\int_0^\tau -2H_2(u)Q^TQe^{Gu}du)e^{-G\tau} = Ke^{-G\tau}$$

Such that $H_1(0) = K\mathbb{I}_n$ and $H_1^{-1}(0) = K^{-1}\mathbb{I}_n$ and for $K = 1, \quad H_1^{-1}(0) = \mathbb{I}_n$

and then

$$H_2(0) = H_1(0)\lambda_1(0) = \lambda_1(0)$$

$$(H_2(\tau) \quad H_1(\tau)) = (\lambda_1(0) \quad \mathbb{I}_n)e^{\tau Q}$$

Setting

$$e^{\tau Q} = \begin{pmatrix} \lambda_1^{11}(\tau) & \lambda_1^{12}(\tau) \\ \lambda_1^{21}(\tau) & \lambda_1^{22}(\tau) \end{pmatrix}$$

Such that

$$\begin{pmatrix} \lambda_1^{11}(\tau) & \lambda_1^{12}(\tau) \\ \lambda_1^{21}(\tau) & \lambda_1^{22}(\tau) \end{pmatrix} = exp\tau \begin{pmatrix} M & -2Q^TQ \\ \frac{\gamma(\gamma-1)}{2}\mathbb{I}_n & -(M^T + 2\gamma RQ) \end{pmatrix} \tag{3.63}$$

In conclusion

$$(H_2(0) \quad H_1(0)) = (\lambda_1(0) \quad \mathbb{I}_n) \begin{pmatrix} \lambda_1^{11}(\tau) & \lambda_1^{12}(\tau) \\ \lambda_1^{21}(\tau) & \lambda_1^{22}(\tau) \end{pmatrix} \tag{3.64}$$

Its solution is simply obtained through exponentiation

$$(H_2(\tau) \quad H_1(\tau)) = (H_2(0) \quad H_1(0))exp\tau \begin{pmatrix} M & -2Q^TQ \\ \frac{\gamma(\gamma-1)}{2}\mathbb{I}_n & -(M^T + 2\gamma RQ) \end{pmatrix}$$

$$= (\lambda_1(0) \quad \mathbb{I}_n)exp\tau \begin{pmatrix} M & -2Q^TQ \\ \frac{\gamma(\gamma-1)}{2}\mathbb{I}_n & -(M^T + 2\gamma RQ) \end{pmatrix}$$

since

$$e^{\tau Q} = \begin{pmatrix} \lambda_1^{11}(\tau) & \lambda_1^{12}(\tau) \\ \lambda_1^{21}(\tau) & \lambda_1^{22}(\tau) \end{pmatrix} = exp\tau \begin{pmatrix} M & -2Q^TQ \\ \frac{\gamma(\gamma-1)}{2}\mathbb{I}_n & -(M^T + 2\gamma RQ) \end{pmatrix}$$

$$(H_2(\tau) \quad H_1(\tau)) = (\lambda_1(0) \quad \mathbb{I}_n) \begin{pmatrix} \lambda_1^{11}(\tau) & \lambda_1^{12}(\tau) \\ \lambda_1^{21}(\tau) & \lambda_1^{22}(\tau) \end{pmatrix} \tag{3.65}$$

$$= [\lambda_1(0)\lambda_1^{11}(\tau) + \lambda_1^{21}(\tau) \qquad \lambda_1(0)\lambda_1^{12}(\tau) + \lambda_1^{22}(\tau)]$$

and since $\lambda_1(0) = 0$

$$
\begin{aligned}
(H_2(\tau) \quad H_1(\tau)) &= (\lambda_1^{21}(\tau) \quad \lambda_1^{22}(\tau)) \\
H_2(\tau) &= \lambda_1^{21}(\tau) \\
H_1(\tau) &= \lambda_1^{22}(\tau) \\
\lambda_1(\tau) &= H_1^{-1}(\tau)H_2(\tau) \\
\lambda_1(\tau) &= \lambda_1^{22}(\tau)^{-1}\lambda_1^{21}(\tau)
\end{aligned}
\tag{3.66}
$$

which represents the closed form solution of the matrix Riccati equation.

Now looking at the second Riccati equation. Setting;

$$\lambda_2(\tau) = I_1^{-1}(\tau)I_2(\tau) \tag{3.67}$$

$$\frac{dI_2(\tau)}{d\tau} = I_1(\tau)\frac{\gamma(\gamma-1)}{2} + I_2(\tau)\bar{M} \tag{3.68}$$

$$\frac{dI_1(\tau)}{d\tau} = -I_1(\tau)(\bar{M}^T + 2\gamma\bar{R}\bar{Q}) - 2I_2(\tau)\bar{Q}^T\bar{Q} \tag{3.69}$$

$$\frac{d}{d\tau}(I_2(\tau) \quad I_1(\tau)) = \begin{pmatrix} I_2(\tau) & I_1(\tau) \end{pmatrix} \begin{pmatrix} \bar{M} & -2\bar{Q}^T\bar{Q} \\ \frac{\gamma(\gamma-1)}{2}\mathbb{I}_n & -(\bar{M}^T + 2\gamma\bar{R}\bar{Q}) \end{pmatrix} \tag{3.70}$$

Recall;

$$Y(\tau) = Y(0)e^{\bar{Q}\tau}$$

Then,

$$(I_2(\tau) \quad I_1(\tau)) = (I_2(0) \quad I_1(0))e^{\bar{Q}\tau}$$

From equation (3.69)

$$I_1(\tau) = (\int_0^\tau -2I_2(u)\bar{Q}^T\bar{Q}e^{\bar{G}(u)}du)e^{-\bar{G}\tau} = \bar{K}e^{-\bar{G}\tau} \tag{3.71}$$

with conditions

$$I_1(0) = \bar{K}\mathbb{I} = \bar{K} = \mathbb{I}$$

such that

$$I_2(0) = I_1(0)\lambda_2(0) = \lambda_2(0)$$

$$\lambda_2(\tau) = I_1^{-1}(\tau)I_2(\tau)$$

$$\begin{pmatrix} I_2(\tau) & I_1(\tau) \end{pmatrix} = \begin{pmatrix} \lambda_2(0) & \mathbb{I}_n \end{pmatrix} \begin{pmatrix} \lambda_2^{11}(\tau) & \lambda_2^{12}(\tau) \\ \lambda_2^{21}(\tau) & \lambda_2^{22}(\tau) \end{pmatrix} \tag{3.72}$$

$$\begin{pmatrix} I_2(\tau) & I_1(\tau) \end{pmatrix} = \begin{pmatrix} \lambda_2(0)\lambda_2^{11}(\tau) + \lambda_2^{21}(\tau) & \lambda_2(0)\lambda_2^{12}(\tau) + \lambda_2^{22}(\tau) \end{pmatrix} \tag{3.73}$$

and since $\lambda_2(0) = 0$

$$\begin{pmatrix} I_2(\tau) & I_1(\tau) \end{pmatrix} = \begin{pmatrix} \lambda_2^{21}(\tau) & \lambda_2^{22}(\tau) \end{pmatrix}$$

Therefore we obtain

$$\begin{aligned} I_2(\tau) &= \lambda_2^{21}(\tau) \\ I_1(\tau) &= \lambda_2^{22}(\tau) \\ \lambda_2(\tau) &= I_1^{-1}(\tau)I_2(\tau) \\ \lambda_2(\tau) &= \lambda_2^{22}(\tau)^{-1}\lambda_2^{21}(\tau) \end{aligned} \tag{3.74}$$

Then proceed by computing the last Riccati equation ,

$$\begin{cases} \frac{d\varepsilon(\tau)}{d\tau} = r\gamma + \beta Tr[(Q^TQ)\lambda_1(\tau)] + \bar{\beta}Tr[(\bar{Q}^T\bar{Q})\lambda_2(\tau)] \\ \varepsilon(0) = 0 \end{cases} \tag{3.75}$$

Recalling equations

$$\begin{cases} \frac{dH_2(\tau)}{d\tau} = H_1(\tau)\frac{\gamma(\gamma-1)}{2} + H_2(\tau)M \\ \frac{dH_1(\tau)}{d\tau} = -H_1(\tau)(M^T + 2\gamma RQ) - 2H_2(\tau)Q^TQ \end{cases} \tag{3.76}$$

and

$$\begin{cases} \frac{dI_2(\tau)}{d\tau} = I_1(\tau)\frac{\gamma(\gamma-1)}{2} + I_2(\tau)\bar{M} \\ \frac{dI_1(\tau)}{d\tau} = -I_1(\tau)(\bar{M}^T + 2\gamma\bar{R}\bar{Q}) - 2I_2(\tau)\bar{Q}^T\bar{Q} \end{cases} \tag{3.77}$$

$$\frac{d\varepsilon(\tau)}{d\tau} = r\gamma + \beta Tr[(Q^TQ)H_1^{-1}(\tau)H_2(\tau)] + \bar{\beta}Tr[(\bar{Q}^T\bar{Q})I_1^{-1}(\tau)I_2(\tau)] \tag{3.78}$$

From equation (3.76)

$$H_2(\tau) = -\frac{1}{2}[\frac{dH_1(\tau)}{d\tau} + H_1(\tau)(M^T + 2\gamma RQ)](Q^TQ)^{-1} \tag{3.79}$$

and from equation (3.77)

$$I_2(\tau) = -\frac{1}{2}[\frac{dI_1(\tau)}{d\tau} + I_1(\tau)(\bar{M}^T + 2\gamma\bar{R}\bar{Q})](\bar{Q}^T\bar{Q})^{-1} \tag{3.80}$$

Such that

$$\lambda_1(\tau) = H_1^{-1}(\tau)H_2(\tau) = -\frac{1}{2}[H_1^{-1}(\tau)\frac{dH_1(\tau)}{d\tau} + (M^T + 2\gamma RQ)](Q^TQ)^{-1} \tag{3.81}$$

$$\lambda_2(\tau) = I_1^{-1}(\tau)I_2(\tau) = -\frac{1}{2}[I_1^{-1}(\tau)\frac{dI_1(\tau)}{d\tau} + (\bar{M}^T + 2\gamma\bar{R}\bar{Q})](\bar{Q}^T\bar{Q})^{-1} \tag{3.82}$$

Substituting equations (3.81) and (3.82) in equation (3.78), as follows

$$\begin{cases} \frac{d\varepsilon(\tau)}{d\tau} & = -\frac{\beta}{2}Tr[H_1^{-1}(\tau)\frac{dH_1(\tau)}{d\tau} + (M^T + 2\gamma RQ)] - \frac{\bar{\beta}}{2}Tr[I_1^{-1}(\tau)\frac{dI_1(\tau)}{d\tau} \\ & +(\bar{M}^T + 2\gamma\bar{R}\bar{Q})] + r\gamma \\ \varepsilon(0) & = 0 \end{cases} \tag{3.83}$$

$$d\varepsilon(\tau) = -\frac{\beta}{2}Tr[\frac{dH_1(\tau)}{H_1(\tau)} + (M^T + 2\gamma RQ)d\tau] - \frac{\bar{\beta}}{2}Tr[\frac{dI_1(\tau)}{I_1(\tau)} + (\bar{M}^T + 2\gamma\bar{R}\bar{Q})d\tau] + r\gamma d\tau \tag{3.84}$$

Now by integrating directly, provides

$$\varepsilon(\tau) = -\frac{\beta}{2}Tr[\log H_1(\tau) + (M^T + 2\gamma RQ)\tau] - \frac{\bar{\beta}}{2}Tr[\log I_1(\tau) + (\bar{M}^T + 2\gamma\bar{R}\bar{Q})\tau] + r\gamma\tau \tag{3.85}$$

This can be simplified as

$$\begin{aligned} \varepsilon(\tau) =& Tr\left[\log(\frac{1}{\sqrt{(H_1(\tau))^{\beta}(I_1(\tau))^{\bar{\beta}}}}) - \frac{1}{2}(\beta M^T + \bar{\beta}\bar{M}^T + 2\gamma(\beta RQ + \bar{\beta}\bar{R}\bar{Q}))\tau\right] \\ & + r\gamma\tau \end{aligned} \tag{3.86}$$

3.2.3 The characteristic function and Fast Fourier transform method

This section considers Fast Fourier Transform (FFT) method in Carr & Madan (1999), to price European call option with $\alpha > 0$, at time t, strike $k = \log(K)$ and time to maturity T, given as

$$C_t(T,K) = e^{-r(T-t)}\mathbb{E}[(X_T - K)^+ \mid \mathcal{F}_t] \tag{3.87}$$

Since $Y_T = \log X_T$, then $X_t = exp(Y_t)$. This provides;

$$C_t(T,K) = C_t(T,k) = e^{-r(T-t)}\mathbb{E}[(exp(Y_t) - exp(k))^+|\mathcal{F}_t] \tag{3.88}$$

The modified Price C_t^α and $\alpha = 1.1$ as considered in Carr & Madan (1999) as a good empirical value in the Heston model. This modified call price is introduced in order to obtain a square integrable function so that we can apply the inverse Fourier transform,

$$C_t^\alpha(T,k) = exp(\alpha k)C_t(T,K) \tag{3.89}$$

The Fourier transform of the modified price and also applying a Fubini integration theorem

$$\psi_t^\alpha(T,\theta) = \int_{-\infty}^{+\infty} C_t^\alpha(T,k)e^{i\theta k}dk \tag{3.90}$$

$$\begin{aligned}
&= \int_{-\infty}^{+\infty} e^{\alpha k}C_t(T,K)e^{i\theta k}dk \\
&= \int_{-\infty}^{+\infty} e^{\alpha k}e^{-r(T-t)}\mathbb{E}[(exp(Y_t) - exp(k))^+|\mathcal{F}_t]e^{i\theta k}dk \\
&= e^{-r(T-t)}\int_{-\infty}^{+\infty} exp[(\alpha + i\theta)k]\mathbb{E}[(exp(Y_t) - exp(k))^+|\mathcal{F}_t]dk \\
&= e^{-r(T-t)}\mathbb{E}\left[\int_{-\infty}^{Y_T}(e^{(\alpha+i\theta)k+Y_T} - e^{(\alpha+i\theta)k+k})dk\right]
\end{aligned}$$

$$\psi_t^\alpha(T,\theta) = e^{-r(T-t)}\mathbb{E}\left[\frac{e^{(\alpha+i\theta)k+Y_T}}{\alpha+i\theta} - \frac{e^{(\alpha+i\theta+1)k}}{\alpha+i\theta+1}\right]_{A=-\infty}^{Y}$$

$$\psi_t^\alpha(T,\theta) = e^{-r(T-t)}\mathbb{E}\left[\left(\frac{e^{(\alpha+i\theta)Y_T+Y_T}}{\alpha+i\theta} - \frac{e^{(\alpha+i\theta+1)Y_T}}{\alpha+i\theta+1}\right) - \lim_{A\to-\infty}\left(\frac{e^{(\alpha+i\theta)A+Y_T}}{\alpha+i\theta} - \frac{e^{(\alpha+i\theta+1)A}}{\alpha+i\theta+1}\right)\right]$$

as $A \to -\infty$, therefore this goes to zero

$$\lim_{A \to -\infty} \left(\frac{e^{(\alpha+i\theta)A+Y_T}}{\alpha+i\theta} - \frac{e^{(\alpha+i\theta+1)A}}{\alpha+i\theta+1} \right) \approx 0$$

$$\begin{aligned}
\psi_t^{\alpha}(T,\theta) &= e^{-r(T-t)}\mathbb{E}\left[\frac{e^{(\alpha+i\theta+1)Y_T}}{(\alpha+i\theta)(\alpha+i\theta+1)}(\alpha+i\theta+1-\alpha-i\theta)\right] \\
\psi_t^{\alpha}(T,\theta) &= e^{-r(T-t)}\mathbb{E}\left[\frac{e^{(\alpha+i\theta+1)Y_T}}{(\alpha+i\theta)(\alpha+i\theta+1)}\right] \\
&= \frac{e^{-r(T-t)}}{(\alpha+i\theta)(\alpha+i\theta+1)}\mathbb{E}\left[exp[i\theta+i(\frac{\alpha+1}{i})]Y_T\right] \\
&= \frac{e^{-r(T-t)}}{(\alpha+i\theta)(\alpha+i\theta+1)}\mathbb{E}\left[exp[i(\theta+(\frac{\alpha+1}{i}))]Y_T\right] \\
&= \frac{e^{-r(T-t)}}{(\alpha+i\theta)(\alpha+i\theta+1)}\mathbb{E}\left[exp[i(\theta-(\alpha+1)i)]Y_T\right] \\
&= \frac{e^{-r(T-t)}}{(\alpha+i\theta)(\alpha+i\theta+1)}\mathbb{E}\left[exp[i(\theta-(\alpha+1)i)]Y_T|\mathcal{F}_t\right]
\end{aligned}$$

$$\psi_t^{\alpha}(T,\theta) = \frac{e^{-r(T-t)}\phi_t(T,\theta-(1+\alpha)i)}{(\alpha+i\theta)(\alpha+i\theta+1)} \tag{3.91}$$

Now observing that the call price can be obtained by inversion of the Fourier transform using the fact that the function $\psi_t^{\alpha}(T,\theta)$ has an odd imaginary part and an even real part,

$$\begin{aligned}
C_t^{\alpha}(T,k) &= exp(\alpha k)C_t(T,K) \\
\int_{-\infty}^{+\infty} C_t^{\alpha}(T,k)e^{i\theta k}dk &= \frac{e^{-r(T-t)}\phi_t(T,\theta-(1+\alpha)i)}{(\alpha+i\theta)(\alpha+i\theta+1)} \\
C_t^{\alpha}(T,k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi_t^{\alpha}(T,\theta)e^{-i\theta k}d\theta
\end{aligned} \tag{3.92}$$

Such that

$$C_t(T,K) = \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty}\psi_t^{\alpha}(T,\theta)e^{-i\theta k}d\theta$$

$$C_t(T,K) = \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-r(T-t)}\tilde{\phi}_t(T,\theta-(1+\alpha)i)}{(\alpha+i\theta)(\alpha+i\theta+1)}e^{-i\theta k}d\theta$$

This is a Fourier transform

$$C_t(T,K) = \frac{e^{-\alpha k}}{2\pi} Re\left(\int_{-\infty}^{\infty} \frac{e^{-r(T-t)}\tilde{\phi}_t(T, \theta-(1+\alpha)i)}{(\alpha+i\theta)(\alpha+i\theta+1)} e^{-i\theta k} d\theta\right) \tag{3.93}$$

In order to price the European call option, it is sufficient to compute the characteristic function $\tilde{\phi}_{\gamma,0}(t,T)$, this computation will involve the Wishart processes.

Proposition 3.2.2 *Given a real symmetric matrix D, the conditional characteristic function of the Wishart processes Σ_t and $\bar{\Sigma}_t$ is given by,*

$$\begin{aligned}\tilde{\phi}^{D_1 D_2}_{\Sigma_t,\bar{\Sigma}_t} &= \mathbb{E}\left[exp\{iTr[D_1\Sigma_t + D_2\bar{\Sigma}_t]\}\right] \\ &= exp\{Tr[A_1(\tau)\Sigma_t + A_2(\tau)\bar{\Sigma}_t] + C(\tau)\}\end{aligned}$$

where the deterministic complex-valued function $A_1(\tau), A_2(\tau) \in M_n(\mathbb{C})$, $C(\tau) \in \mathbb{C}$

Now obtaining the ordinary differential equations for $A_1(\tau), A_2(\tau)$ and $C(\tau)$

$$\begin{cases} \frac{dA_1(\tau)}{d\tau} = A_1(\tau)M + M^T A_1(\tau) + 2A_1(\tau)Q^T Q A_1(\tau) \\ A_1(0) = iD_1 \end{cases} \tag{3.94}$$

$$\begin{cases} \frac{dA_2(\tau)}{d\tau} = A_2(\tau)\bar{M} + \bar{M}^T A_2(\tau) + 2A_2(\tau)\bar{Q}^T \bar{Q} A_2(\tau) \\ A_2(0) = iD_2 \end{cases} \tag{3.95}$$

$$\begin{cases} \frac{dC(\tau)}{d\tau} = \beta Tr[Q^T Q A_1(\tau)] + \bar{\beta} Tr[\bar{Q}^T \bar{Q} A_2(\tau)] \\ C(0) = 0 \end{cases} \tag{3.96}$$

$$C(\tau) = \int_0^\tau Tr[\beta Q^T Q A_1(\tau) du + \bar{\beta}\bar{Q}^T \bar{Q} A_2(\tau) du] \tag{3.97}$$

$$C(\tau) = Tr\left[\int_0^\tau \beta Q^T Q A_1(u) du + \int_0^\tau \bar{\beta}\bar{Q}^T \bar{Q} A_2(u) du\right] \tag{3.98}$$

Now

$$A_1(\tau) = h_1^{-1}(\tau) h_2(\tau)$$

$$\frac{d}{d\tau}(h_2(\tau) \quad h_1(\tau)) = \begin{pmatrix} h_2(\tau) & h_1(\tau) \end{pmatrix} \begin{pmatrix} M & -2Q^TQ \\ 0 & -M^T \end{pmatrix} \tag{3.99}$$

$$A_2(\tau) = i_1^{-1}(\tau)i_2(\tau)$$

$$\frac{d}{d\tau}(i_2(\tau) \quad i_1(\tau)) = \begin{pmatrix} i_2(\tau) & i_1(\tau) \end{pmatrix} \begin{pmatrix} \bar{M} & -2\bar{Q}^T\bar{Q} \\ 0 & -\bar{M}^T \end{pmatrix} \tag{3.100}$$

Provides

$$(h_2(\tau) \quad h_1(\tau)) = (iD_1 \quad \mathbb{I}_n) \begin{pmatrix} A_1^{11}(\tau) & A_1^{12}(\tau) \\ A_1^{21}(\tau) & A_1^{22}(\tau) \end{pmatrix} \tag{3.101}$$

$$(i_2(\tau) \quad i_1(\tau)) = (iD_2 \quad \mathbb{I}_n) \begin{pmatrix} A_2^{11}(\tau) & A_2^{12}(\tau) \\ A_2^{21}(\tau) & A_2^{22}(\tau) \end{pmatrix} \tag{3.102}$$

$$(h_2(\tau) \quad h_1(\tau)) = (iD_1A_1^{11}(\tau) + A_1^{21}(\tau) \quad iD_1A_1^{12}(\tau) + A_1^{22}(\tau)) \tag{3.103}$$

$$(i_2(\tau) \quad i_1(\tau)) = (iD_2A_2^{11}(\tau) + A_2^{21}(\tau) \qquad iD_2A_2^{12}(\tau) + A_2^{22}(\tau)) \tag{3.104}$$

Then

$$A_1(\tau) = h_1^{-1}(\tau)h_2(\tau) \tag{3.105}$$

$$h_1(\tau) = iD_1A_1^{12}(\tau) + A_1^{22}(\tau)$$

$$h_2(\tau) = iD_1A_1^{11}(\tau) + A_1^{21}(\tau)$$

$$A_1(\tau) = (iD_1A_1^{12}(\tau) + A_1^{22}(\tau))^{-1}(iD_1A_1^{11}(\tau) + A_1^{21}(\tau)) \tag{3.106}$$

And for

$$A_2(\tau) = i_1^{-1}(\tau)i_2(\tau) \tag{3.107}$$

$$i_1(\tau) = iD_2A_2^{12}(\tau) + A_2^{22}(\tau)$$

$$i_2(\tau) = iD_2A_2^{11}(\tau) + A_2^{21}(\tau)$$

$$A_2(\tau) = (iD_2A_2^{12}(\tau) + A_2^{22}(\tau))^{-1}(iD_2A_2^{11}(\tau) + A_2^{21}(\tau)) \tag{3.108}$$

Then generalizing as follows,

$$A_j(\tau) = (iD_jA_j^{12}(\tau) + A_j^{22}(\tau))^{-1}(iD_jA_j^{11}(\tau) + A_j^{21}(\tau)), \quad forj = 1, 2. \tag{3.109}$$

By letting $\tilde{\phi}(t) = \tilde{\phi}_{\gamma,0}(t, T)$ be the characteristic function of the log-return, we also observe that the price of the option depend on volatility, we consider log-asset price return Y_t

$$\begin{aligned} Y_{t,T} &= \ln\left(\frac{X_T}{X_t}\right) \\ &= \ln(X_T) - \ln(X_t) \\ &= Y_T - Y_t \end{aligned}$$

Then

$$\begin{aligned} \tilde{\phi}_{\gamma,0}(t, T) &= \mathbb{E}[exp\{i\gamma Y_{t,T}\}] \\ &= \mathbb{E}[\mathbb{E}exp\{i\gamma(Y_T - Y_t)\}] \\ &= \mathbb{E}[\mathbb{E}exp\{i\gamma Y_T - i\gamma Y_t\}] \\ &= \mathbb{E}[\mathbb{E}_t exp\{i\gamma Y_T\}exp\{-i\gamma Y_t\}] \\ &= \mathbb{E}[(exp(-i\gamma Y_t))\mathbb{E}_t exp\{i\gamma Y_T\}] \\ &= \mathbb{E}[exp(-i\gamma Y_t)exp\{Tr\left[\lambda_1(T-t)\Sigma_t + \lambda_2(T-t)\bar{\Sigma}_t\right] + i\gamma Y_t + \varepsilon(T-t)\}\big] \\ &= exp\{\varepsilon(T-t)\}\mathbb{E}\left[exp\{Tr[\lambda_1(T-t)\Sigma_t + \lambda_2(T-t)\bar{\Sigma}_t]\}\right] \\ &= exp\{\varepsilon(T-t)\}exp\{Tr[A_1(\tau)\Sigma_0 + A_2(\tau)\bar{\Sigma}_0] + C(\tau)\} \\ &= exp\{\varepsilon(T-t)\}exp\{Tr[A_1(t)\Sigma_0 + A_2(t)\bar{\Sigma}_0] + C(t)\} \\ &= exp\{Tr[A_1(t)\Sigma_0 + A_2(t)\bar{\Sigma}_0] + C(t) + \varepsilon(T-t)\} \end{aligned} \tag{3.110}$$

where

$$A_j(\tau) = \lambda_j(T-t), \quad j = 1, 2.$$

with

$$A_j(t) = (iD_jA_j^{12}(t) + A_j^{22}(t))^{-1}(iD_jA_j^{11}(t) + A_j^{21}(t)), \quad forj = 1, 2.$$

Then the European call price, can be written in the form as follows

$$C_t(T,K) = \frac{e^{-\alpha k}}{\sqrt{2\pi}} Re\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{-r(T-t)}e^{\{Tr[A_1(t)\Sigma_0+A_2(t)\bar{\Sigma}_0]+C(t)+\varepsilon(T-t)\}}}{(\alpha+i\theta)(\alpha+i\theta+1)} e^{-i\theta k} d\theta\right) \quad (3.111)$$

The call price can still be presented in the form

$$C_t(T,K) = \frac{e^{-\alpha k}}{2\pi} Re\left(\int_{-\infty}^{\infty} e^{-r(T-t)}\tilde{\phi}_{\Sigma,\bar{\Sigma}}(t)\phi(\theta)e^{-i\theta k} d\theta\right) \quad (3.112)$$

where

$$\tilde{\phi}_{\Sigma,\bar{\Sigma}}(t) = e^{\{Tr[A_1(t)\Sigma_0+A_2(t)\bar{\Sigma}_0]+C(t)+\varepsilon(T-t)\}}$$

$$\phi(\theta) = \frac{1}{(\alpha+i\theta)(\alpha+i\theta+1)}$$

$$C(\tau) = Tr\left[\int_0^{\tau} \beta Q^T Q A_1(u)du + \int_0^{\tau} \bar{\beta}\bar{Q}^T\bar{Q}A_2(u)du\right]$$

and

$$\varepsilon(\tau) = Tr\left[\log(\frac{1}{\sqrt{(H_1(\tau))^{\beta}(I_1(\tau))^{\bar{\beta}}}}) - \frac{1}{2}(\beta M^T + \bar{\beta}\bar{M}^T) + r\gamma\right]$$

3.2.4 Perturbation techniques for the Riccati differential equations

This section deals with implementation of perturbation techniques since the system of the Riccati differential equations does not allow analytical closed form solution. This is due to non commutativity of matrix multiplication, in order to approximate the call option price.

The method procedures retains the affine properties and allows higher orders comparable to standard perturbation schemes more so the singular perturbation technique on partial differential equations that is complex after the first order Benabid et al. (2008); Fouque et al. (2003). Considering the differential Riccati equations of double Wishart stochastic volatility model and taking dimension $n = 2$. The two characteristics orders in perturbation p and q are considered.

The solution of $A(\tau)$ is given in the form

$$A(\tau) = \sum_{i,j} p^{\frac{i}{2}} q^{\frac{j}{2}} A^{i,j}(\tau)$$

Then substituting this development in the perturbed differential Riccati equation and by comparing the coefficients and identifying the terms in p and q gives the expected approximation.

Recalling the ordinary Riccati differential equations which are functions to be obtained in the price function.

$$A_1^{'}(\tau) = A_1(\tau)M + M^T A_1(\tau) + 2A_1(\tau)Q^T Q A_1(\tau), \quad A_1(0) = iD_1 \tag{3.113}$$

$$A_2^{'}(\tau) = A_2(\tau)\bar{M} + \bar{M}^T A_2(\tau) + 2A_2(\tau)\bar{Q}^T \bar{Q} A_2(\tau), \quad A_2(0) = iD_2 \tag{3.114}$$

$$C^{'}(\tau) = \beta Tr[Q^T Q A_1(\tau)] + \bar{\beta} Tr[\bar{Q}^T \bar{Q} A_2(\tau)], \quad C(0) = 0 \tag{3.115}$$

$$\varepsilon^{'}(\tau) = r\gamma + \beta Tr[Q^T Q \lambda_1(\tau)] + \bar{\beta} Tr[\bar{Q}^T \bar{Q} \lambda_2(\tau)], \quad \varepsilon(0) = 0 \tag{3.116}$$

$$\begin{cases} \lambda_1^{'}(\tau) = M\lambda_1(\tau) + (M^T + 2\gamma RQ)\lambda_1(\tau) + 2\lambda_1(\tau)Q^T Q \lambda_1(\tau) + \frac{\gamma(\gamma-1)}{2}\mathbb{I}_2 \\ \lambda_1(0) = 0 \end{cases} \tag{3.117}$$

$$\begin{cases} \lambda_2^{'}(\tau) = \bar{M}\lambda_2(\tau) + (\bar{M}^T + 2\gamma \bar{R}\bar{Q})\lambda_2(\tau) + 2\lambda_2(\tau)\bar{Q}^T \bar{Q} \lambda_2(\tau) + \frac{\gamma(\gamma-1)}{2}\mathbb{I}_2 \\ \lambda_2(0) = 0 \end{cases} \tag{3.118}$$

Let $p = M_1$ and $q = M_2$ are small while ν_i quantities remain constant. The Wishart variance does not fluctuate much around initial value. The approximation at order one $(\sqrt{p}, \sqrt{q})$ and order two (p, q) with the following notations.

$$M = \begin{pmatrix} -p & 0 \\ 0 & -q \end{pmatrix} = -p \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - q \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = -pM_1 - qM_2$$

and

$$\bar{M} = \begin{pmatrix} -\bar{p} & 0 \\ 0 & -\bar{q} \end{pmatrix} = -\bar{p} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \bar{q} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = -\bar{p}M_1 - \bar{q}M_2$$

and also noting that Q is the volatility of the volatility

$$Q = \sqrt{p}\nu_1 M_1 + \sqrt{q}\nu_2 M_2, \quad Q^2 = p\nu_1^2 M_1 + q\nu_2^2 M_2$$

Then Q can be presented in the form

$$Q = \sqrt{p}Q_1 + \sqrt{q}Q_2$$

where

$$\begin{cases} Q_1 = \nu_1 M_1, & Q_1^2 = \nu_1^2 M_1 \\ Q_2 = \nu_2 M_2, & Q_2^2 = \nu_2^2 M_2 \end{cases} \tag{3.119}$$

Rewriting the Riccati equations in the form

$$A_1'(\tau) = A_1(\tau)(-pM_1 - qM_2) + (-pM_1 - qM_2)^T A_1(\tau) + 2A_1(\tau)(p\nu_1^2 M_1 + q\nu_2^2 M_2)A_1(\tau)$$

$$\begin{aligned} A_1'(\tau) &= p[-A_1(\tau)M_1 - M_1A_1(\tau) + 2A_1(\tau)Q_1^2A_1(\tau)] \\ &\quad + q[-A_1(\tau)M_2 - M_2A_1(\tau) + 2A_1(\tau)Q_2^2A_1(\tau)] \end{aligned} \tag{3.120}$$

and for

$$\begin{aligned} \lambda_1'(\tau) &= \frac{\gamma(\gamma-1)}{2}\mathbb{I}_2 + \lambda_1(\tau)(-pM_1 - qM_2) \\ &\quad + (-pM_1 - qM_2 + 2\gamma R(\sqrt{p}\nu_1 M_1 + \sqrt{q}\nu_2 M_2))\lambda_1(\tau) \\ &\quad + 2\lambda_1(\tau)(p\nu_1^2 M_1 + q\nu_2^2 M_2)\lambda_1(\tau) \end{aligned}$$

$$\begin{aligned} \lambda_1'(\tau) = \frac{\gamma(\gamma-1)}{2}\mathbb{I}_2 + p(-\lambda_1(\tau)M_1 - M_1\lambda_1(\tau) + 2\lambda_1(\tau)Q_1^2\lambda_1(\tau)) + 2\sqrt{p}\gamma RQ_1\lambda_1(\tau) \\ + 2\sqrt{q}\gamma RQ_2\lambda_1(\tau) + q(-\lambda_1(\tau)M_2 - M_2\lambda_1(\tau) + 2\lambda_1(\tau)Q_2^2\lambda_1(\tau)) \end{aligned} \tag{3.121}$$

$$\lambda_2'(\tau) = \frac{\gamma(\gamma-1)}{2}\mathbb{I}_2 + \bar{p}(-\lambda_2(\tau)M_1 - M_1\lambda_2(\tau) + 2\lambda_2(\tau)\bar{Q}_1^2\lambda_2(\tau)) + 2\sqrt{\bar{p}}\gamma\bar{R}\bar{Q}_1\lambda_2(\tau)$$
$$+ 2\sqrt{\bar{q}}\gamma\bar{R}\bar{Q}_2\lambda_2(\tau) + \bar{q}(-\lambda_2(\tau)M_2 - M_2\lambda_2(\tau) + 2\lambda_2(\tau)\bar{Q}_2^2\lambda_2(\tau)) \tag{3.122}$$

Development of the Riccati functions. The development of the Riccati functions $A_1, A_2, \lambda_1, \lambda_2, C$ and ε, by perturbation of the Riccati equations at order two (order 2), as follows

$$A_k(\tau) = A_k^{0,0}(\tau) + \sqrt{p}A_k^{1,0}(\tau) + \sqrt{q}A_k^{0,1}(\tau) + pA_k^{2,0}(\tau) + qA_k^{0,2}(\tau)$$
$$+ \sqrt{pq}A_k^{1,1}(\tau) + o(max(p,q)), \quad k = 1, 2. \tag{3.123}$$

$$\lambda_k(\tau) = \lambda_k^{0,0}(\tau) + \sqrt{p}\lambda_k^{1,0}(\tau) + \sqrt{q}\lambda_k^{0,1}(\tau) + p\lambda_k^{2,0}(\tau) + q\lambda_k^{0,2}(\tau)$$
$$+ \sqrt{pq}\lambda_k^{1,1}(\tau) + o(max(p,q)), \quad k = 1, 2. \tag{3.124}$$

Let us determine $A_k^{0,0}, ..., A_k^{1,1}$

$$A_1'(\tau) = p[-A_1(\tau)M_1 - M_1A_1(\tau) + 2A_1(\tau)Q_1^2A_1(\tau)]$$
$$+ q[-A_1(\tau)M_2 - M_2A_1(\tau) + 2A_1(\tau)Q_2^2A_1(\tau)]$$

$$A_1'(\tau) = (A_1^{0,0}(\tau))' + \sqrt{p}(A_1^{1,0}(\tau))' + \sqrt{q}(A_1^{0,1}(\tau))' + p(A_1^{2,0}(\tau))' + q(A_1^{0,2}(\tau))' + \sqrt{pq}(A_1^{1,1}(\tau))'$$

By identifying the terms in respective orders, gives

$$(A_1^{0,0}(\tau))' = 0, \quad A_1^{0,0}(\tau) \in M_2(\mathbb{C})$$

Since $A_1(0) = iD_1$, where

$$D_1 = \begin{pmatrix} d_1 & 0 \\ 0 & d_1 \end{pmatrix}$$

Then, having

$$A_1^{0,0}(0) + \sqrt{p}A_1^{1,0}(0) + \sqrt{q}A_1^{0,1}(0) + pA_1^{2,0}(0) + qA_1^{0,2}(0) + \sqrt{pq}A_1^{1,1}(0) = iD_1 \tag{3.125}$$

then

$$A_1^{0,0}(0) = iD_1 = id_1\mathbb{I}_2$$

$$A_1^{1,0}(0) = A_1^{0,1}(0) = A_1^{2,0}(0) = A_1^{0,2}(0) = A_1^{1,1}(0) = (0)$$

where

$$\begin{aligned}
(A_1^{2,0}(\tau))' &= -A_1^{0,0}(\tau)M_1 - M_1A_1^{0,0}(\tau) + 2\nu_1^2A_1^{0,0}(\tau)M_1A_1^{0,0}(\tau) \\
&= -id_1M_1 - M_1id_1 + 2\nu_1^2id_1M_1id_1 \\
&= -2id_1M_1 - 2d_1^2\nu_1^2M_1 \\
&= -2d_1(i + d_1\nu_1^2)M_1
\end{aligned}$$

then by integrating, gives

$$A_1^{2,0}(\tau) = -2d_1(i + d_1\nu_1^2)\tau M_1$$

and for

$$\begin{aligned}
(A_1^{0,2}(\tau))' &= -A_1^{0,0}(\tau)M_2 - M_2A_1^{0,0}(\tau) + 2\nu_2^2A_1^{0,0}(\tau)M_2A_1^{0,0}(\tau) \\
&= -id_1M_2 - M_2id_1 + 2\nu_2^2id_1M_2id_1 \\
&= -2id_1M_2 - 2d_1^2\nu_2^2M_2 \\
&= -2d_1(i + d_1\nu_2^2)M_2
\end{aligned}$$

After integration, gives

$$A_1^{0,2}(\tau) = -2d_1(i + d_1\nu_2^2)\tau M_2$$

and

$$(A_1^{1,1}(\tau))' = 0, \quad \Longrightarrow A_1^{1,1}(\tau) = C$$

since $A_1^{1,1}(0) = 0$ for all $\tau \in \mathbb{R}_+$ the $C = 0$.

Computing for function $A_2(\tau)$, go through the same procedures

$$
\begin{aligned}
A_2'(\tau) &= \bar{p}[-A_2(\tau)M_1 - M_1A_2(\tau) + 2A_2(\tau)\bar{Q}_1^2A_2(\tau)] \\
&\quad + \bar{q}[-A_2(\tau)M_2 - M_2A_2(\tau) + 2A_2(\tau)\bar{Q}_2^2A_2(\tau)] \\
A_2'(\tau) &= (A_2^{0,0}(\tau))' + \sqrt{\bar{p}}(A_2^{1,0}(\tau))' + \sqrt{\bar{q}}(A_2^{0,1}(\tau))' + \bar{p}(A_2^{2,0}(\tau))' + \bar{q}(A_2^{0,2}(\tau))' + \sqrt{\bar{p}\bar{q}}(A_2^{1,1}(\tau))'
\end{aligned}
$$

then

$$(A_2^{0,0}(\tau))' = 0, \quad \Longrightarrow A_2^{0,0}(\tau) \in M_2(\mathbb{C})$$

Since $A_2(0) = iD_2$, where

$$D_2 = \begin{pmatrix} d_2 & 0 \\ 0 & d_2 \end{pmatrix}$$

Having

$$(A_2^{0,0}(0))' + \sqrt{\bar{p}}(A_2^{1,0}(0))' + \sqrt{\bar{q}}(A_2^{0,1}(0))' + \bar{p}(A_2^{2,0}(0))' + \bar{q}(A_2^{0,2}(0))' + \sqrt{\bar{p}\bar{q}}(A_2^{1,1}(0))' = iD_2 \tag{3.126}$$

$$A_2^{0,0}(0) = iD_2 = id_2\mathbb{I}_2$$

$$A_2^{1,0}(0) = A_2^{0,1}(0) = A_2^{2,0}(0) = A_2^{0,2}(0) = A_2^{1,1}(0) = (0)$$

Proceed by obtaining

$$
\begin{aligned}
(A_2^{2,0}(\tau))' &= -A_2^{0,0}(\tau)M_1 - M_1A_2^{0,0}(\tau) + 2\bar{\nu}_1^2A_2^{0,0}(\tau)M_1A_2^{0,0}(\tau) \\
&= -id_2M_1 - M_1id_2 + 2\bar{\nu}_1^2id_2M_1id_2 \\
&= -2id_2M_1 - 2d_2^2\bar{\nu}_1^2M_1 \\
&= -2d_2(i + d_2\bar{\nu}_1^2)M_1
\end{aligned}
$$

then integrating, provides

$$A_2^{2,0}(\tau) = -2d_2(i + d_2\bar{\nu}_1^2)\tau M_1$$

and for

$$(A_2^{0,2}(\tau))' = -A_2^{0,0}(\tau)M_2 - M_2A_2^{0,0}(\tau) + 2\bar{\nu}_2^2 A_2^{0,0}(\tau)M_2A_2^{0,0}(\tau)$$
$$= -id_2M_2 - M_2id_2 + 2\bar{\nu}_2^2 id_2M_2id_2$$
$$= -2id_2M_2 - 2d_2^2\bar{\nu}_2^2M_2$$
$$= -2d_2(i + d_2\bar{\nu}_2^2)M_2$$

Then after integrating, provides

$$A_2^{0,2}(\tau) = -2d_2(i + d_2\bar{\nu}_2^2)\tau M_2$$

and $(A_2^{1,1}(\tau))' = 0, \quad A_2^{1,1}(\tau) = 0$, since $A_2^{1,1}(0) = 0$ for all $\tau \in \mathbb{R}_+$, then $C = 0$ constant. Continue to obtain λ_1 and λ_2 for the following ordinary differential equations

$$\lambda_1'(\tau) = \frac{\gamma(\gamma-1)}{2}\mathbb{I}_2 + p(-\lambda_1(\tau)M_1 - M_1\lambda_1(\tau) + 2\lambda_1(\tau)Q_1^2\lambda_1(\tau)) + 2\sqrt{p}\gamma RQ_1\lambda_1(\tau)$$
$$+2\sqrt{q}\gamma RQ_2\lambda_1(\tau) + q(-\lambda_1(\tau)M_2 - M_2\lambda_1(\tau) + 2\lambda_1(\tau)Q_2^2\lambda_1(\tau)) \tag{3.127}$$

$$\lambda_1(\tau) = \lambda_1^{0,0}(\tau) + \sqrt{p}\lambda_1^{1,0}(\tau) + \sqrt{q}\lambda_1^{0,1}(\tau) + p\lambda_1^{2,0}(\tau) + q\lambda_1^{0,2}(\tau) + \sqrt{pq}\lambda_1^{1,1}(\tau) + o(max(p,q)) \tag{3.128}$$

For

$$(\lambda_1^{0,0}(\tau))' = \frac{\gamma(\gamma-1)}{2}\mathbb{I}_2 \quad then \Rightarrow \lambda_1^{0,0}(\tau) = \frac{\gamma(\gamma-1)}{2}\tau\mathbb{I}_2$$

$$(\lambda_1^{1,0}(\tau))' = 2\gamma RQ_1\lambda_1^{0,0}(\tau)$$
$$= 2\gamma\nu_1RM_1\lambda_1^{0,0}(\tau)$$
$$= \gamma^2(\gamma-1)\nu_1\tau RM_1$$

Such that

$$\lambda_1^{1,0}(\tau) = \frac{\gamma^2(\gamma-1)}{2}\nu_1\tau^2(RM_1)$$

Then for

$$\begin{aligned}(\lambda_1^{0,1}(\tau))' &= 2\gamma RQ_2\lambda_1^{0,0}(\tau)\\ &= 2\gamma\nu_1 RM_2\lambda_1^{0,0}(\tau)\\ &= \gamma^2(\gamma-1)\nu_2\tau RM_2\end{aligned}$$

$$\lambda_1^{0,1}(\tau) = \frac{\gamma^2(\gamma-1)}{2}\nu_2\tau^2(RM_2)$$

And

$$\begin{aligned}(\lambda_1^{2,0}(\tau))' &= -\lambda_1^{0,0}(\tau)M_1 - M_1\lambda_1^{0,0}(\tau) + 2\lambda_1^{0,0}(\tau)\nu_1^2 M_1\lambda_1^{0,0}(\tau) + 2\gamma RQ_1\lambda_1^{1,0}(\tau)\\ &= -\gamma(\gamma-1)\tau M_1 + \frac{\gamma^2(\gamma-1)^2}{2}\nu_1^2\tau^2 M_1 + \gamma^3(\gamma-1)\nu_1^2\tau^2(RM_1)^2\end{aligned}$$

taking the integral, gives

$$\lambda_1^{2,0}(\tau) = -\frac{\gamma(\gamma-1)}{2}\tau^2 M_1 + \frac{\gamma^2(\gamma-1)^2}{6}\nu_1^2\tau^3 M_1 + \frac{\gamma^3(\gamma-1)}{3}\nu_1^2\tau^3(RM_1)^2$$

and for

$$\begin{aligned}(\lambda_1^{0,2}(\tau))' &= -\lambda_1^{0,0}(\tau)M_2 - M_2\lambda_1^{0,0}(\tau) + 2\lambda_1^{0,0}(\tau)\nu_2^2 M_2\lambda_1^{0,0}(\tau) + 2\gamma RQ_2\lambda_1^{0,1}(\tau)\\ &= -\gamma(\gamma-1)\tau M_2 + \frac{\gamma^2(\gamma-1)^2}{2}\nu_2^2\tau^2 M_2 + \gamma^3(\gamma-1)\nu_2^2\tau^2(RM_2)^2\end{aligned}$$

taking the integral, gives

$$\lambda_1^{0,2}(\tau) = -\frac{\gamma(\gamma-1)}{2}\tau^2 M_2 + \frac{\gamma^2(\gamma-1)}{6}\nu_2^2\tau^3 M_2 + \frac{\gamma^3(\gamma-1)}{3}\nu_2^2\tau^3(RM_2)^2$$

$$\begin{aligned}(\lambda_1^{1,1}(\tau))' &= \gamma^3(\gamma-1)\nu_1\nu_2\tau^2(RM_2)(RM_1) + \gamma^3(\gamma-1)\nu_1\nu_2\tau^2(RM_1)(RM_2)\\ &= \gamma^3(\gamma-1)\nu_1\nu_2\tau^2[(RM_1)(RM_2) + (RM_2)(RM_1)]\end{aligned}$$

then

$$\lambda_1^{1,1}(\tau) = \frac{\gamma^3(\gamma-1)}{3}\nu_1\nu_2\tau^3[(RM_1)(RM_2) + (RM_2)(RM_1)]$$

Then for λ_2, the following approximation results are obtained

$$\lambda_2^{0,0}(\tau) = \frac{\gamma(\gamma-1)}{2}\tau\mathbb{I}_2$$
$$\lambda_2^{1,0}(\tau) = \frac{\gamma^2(\gamma-1)}{2}\bar{\nu}_1\tau^2(\bar{R}M_1)$$
$$\lambda_2^{0,1}(\tau) = \frac{\gamma^2(\gamma-1)}{2}\bar{\nu}_2\tau^2(\bar{R}M_2)$$
$$\lambda_2^{2,0}(\tau) = -\frac{\gamma(\gamma-1)}{2}\tau^2 M_1 + \frac{\gamma^2(\gamma-1)^2}{6}\bar{\nu}_1^2\tau^3 M_1 + \frac{\gamma^3(\gamma-1)}{3}\bar{\nu}_1^2\tau^3(\bar{R}M_1)^2$$
$$\lambda_2^{0,2}(\tau) = -\frac{\gamma(\gamma-1)}{2}\tau^2 M_2 + \frac{\gamma^2(\gamma-1)}{6}\bar{\nu}_2^2\tau^3 M_2 + \frac{\gamma^3(\gamma-1)}{3}\bar{\nu}_2^2\tau^3(\bar{R}M_2)^2$$
$$\lambda_2^{1,1}(\tau) = \frac{\gamma^3(\gamma-1)}{3}\bar{\nu}_1\bar{\nu}_2\tau^3[(\bar{R}M_1)(\bar{R}M_2) + (\bar{R}M_2)(\bar{R}M_1)]$$

Then computing the differential equation for $C(\tau)$

$$C^{'}(\tau) = \beta Tr[Q^T Q A_1(\tau)] + \bar{\beta} Tr[\bar{Q}^T\bar{Q}A_2(\tau)], \quad C(0) = 0 \tag{3.129}$$

By perturbing the equation

$$C(\tau) = C^{0,0}(\tau)+\sqrt{p}C^{1,0}(\tau)+\sqrt{q}C^{0,1}(\tau)+pC^{2,0}(\tau)+qC^{0,2}(\tau)+\sqrt{pq}C^{1,1}(\tau)+o(max(p,q)) \tag{3.130}$$

considering the differential

$$C^{'}(\tau) = \beta Tr[Q^2 A_1(\tau)] + \bar{\beta} Tr[\bar{Q}^2 A_2(\tau)]$$

$$C^{'}(\tau) = \beta Tr[p\nu_1^2 M_1 A_1(\tau) + q\nu_2^2 M_2 A_1(\tau)] + \bar{\beta} Tr[p\bar{\nu}_1^2 M_1 A_2(\tau) + q\bar{\nu}_2^2 M_2 A_2(\tau)]$$

$$(C^{0,0}(\tau))^{'} = 0, \quad C^{0,0}(\tau) = constant = 0$$

since $C(0) = 0$

$$(C^{1,0}(\tau))^{'} = 0, \quad C^{1,0}(\tau) = constant = 0$$

$$(C^{0,1}(\tau))^{'} = 0, \quad C^{0,1}(\tau) = constant = 0$$

$$
\begin{aligned}
\left(C^{2,0}(\tau)\right)' &= \beta Tr[\nu_1^2 M_1 A_1^{0,0}(\tau)] + \bar{\beta} Tr[\bar{\nu}_1^2 M_1 A_2^{0,0}(\tau)] \\
&= \beta Tr[\nu_1^2 M_1 i d_1 \mathbb{I}_2] + \bar{\beta} Tr[\bar{\nu}_1^2 M_1 i d_2 \mathbb{I}_2] \\
&= i\beta \nu_1^2 d_1 + i\bar{\beta}\bar{\nu}_1^2 d_2
\end{aligned}
$$

Integrating it yields

$$C^{2,0}(\tau) = i\beta\nu_1^2 d_1 \tau + i\bar{\beta}\bar{\nu}_1^2 d_2 \tau$$

$$
\begin{aligned}
\left(C^{0,2}(\tau)\right)' &= \beta Tr[\nu_2^2 M_2 A_1^{0,0}(\tau)] + \bar{\beta} Tr[\bar{\nu}_2^2 M_2 A_2^{0,0}(\tau)] \\
&= \beta Tr[\nu_2^2 M_2 i d_1 \mathbb{I}_2] + \bar{\beta} Tr[\bar{\nu}_2^2 M_2 i d_2 \mathbb{I}_2] \\
&= i\beta \nu_2^2 d_1 + i\bar{\beta}\bar{\nu}_2^2 d_2
\end{aligned}
$$

Integrating it yields

$$C^{2,0}(\tau) = i\beta\nu_2^2 d_1 \tau + i\bar{\beta}\bar{\nu}_2^2 d_2 \tau$$

and

$$C^{1,1}(\tau) = 0$$

For ε differential equation

$$
\begin{aligned}
\varepsilon'(\tau) &= r\gamma + \beta Tr[Q^T Q \lambda_1(\tau)] + \bar{\beta} Tr[\bar{Q}^T \bar{Q} \lambda_2(\tau)] \\
&= r\gamma + \beta Tr[Q^2 \lambda_1(\tau)] + \bar{\beta} Tr[\bar{Q}^2 \lambda_2(\tau)]
\end{aligned}
\tag{3.131}
$$

$$\varepsilon'(\tau) = r\gamma + \beta Tr[p\nu_1^2 M_1 \lambda_1(\tau) + q\nu_2^2 M_2 \lambda_1(\tau)] + \bar{\beta} Tr[p\bar{\nu}_1^2 M_1 \lambda_2(\tau) + q\bar{\nu}_2^2 M_2 \lambda_2(\tau)]$$

with

$$\varepsilon(\tau) = \varepsilon^{0,0}(\tau) + \sqrt{p}\varepsilon^{1,0}(\tau) + \sqrt{q}\varepsilon^{0,1}(\tau) + p\varepsilon^{2,0}(\tau) + q\varepsilon^{0,2}(\tau) + \sqrt{pq}\varepsilon^{1,1}(\tau) + o(max(p,q)) \tag{3.132}$$

$$\left(\varepsilon^{0,0}(\tau)\right)' = r\gamma, \quad \Rightarrow \varepsilon^{0,0}(\tau) = r\gamma\tau$$

and for

$$\varepsilon^{1,0}(\tau) = 0, \quad \varepsilon^{0,1}(\tau) = 0$$

$$
\begin{aligned}
(\varepsilon^{2,0}(\tau))' &= \beta Tr[\nu_1^2 M_1 \lambda_1^{0,0}(\tau)] + \bar{\beta} Tr[\bar{\nu}_1^2 M_1 \lambda_2^{0,0}(\tau)] \\
&= \beta Tr[\nu_1^2 M_1 \frac{\gamma(\gamma-1)}{2}\tau] + \bar{\beta} Tr[\bar{\nu}_1^2 M_1 \frac{\gamma(\gamma-1)}{2}\tau]
\end{aligned}
$$

take integral

$$\varepsilon^{2,0}(\tau) = \beta \nu_1^2 \frac{\gamma(\gamma-1)}{4}\tau^2 + \bar{\beta}\bar{\nu}_1^2 \frac{\gamma(\gamma-1)}{4}\tau^2$$

and

$$\varepsilon^{0,2}(\tau) = \beta \nu_2^2 \frac{\gamma(\gamma-1)}{4}\tau^2 + \bar{\beta}\bar{\nu}_2^2 \frac{\gamma(\gamma-1)}{4}\tau^2$$

then for

$$\varepsilon^{1,1}(\tau) = 0$$

Substituting back in the perturbation function, gives

$$A_1(\tau) = A_1^{0,0}(\tau) + \sqrt{p}A_1^{1,0}(\tau) + \sqrt{q}A_1^{0,1}(\tau) + pA_1^{2,0}(\tau) + qA_1^{0,2}(\tau) + \sqrt{pq}A_1^{1,1}(\tau) + o(max(p,q)) \tag{3.133}$$

$$A_1(\tau) = \begin{pmatrix} id_1 & 0 \\ 0 & id_1 \end{pmatrix} + \begin{pmatrix} -2pd_1(i+d_1\nu_1^2)\tau & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -2qd_1(i+d_1\nu_2^2)\tau \end{pmatrix} + o(max(p,q))$$

$$A_1(\tau) = \begin{pmatrix} -2pd_1^2\nu_1^2\tau + id_1(1-2p\tau) & 0 \\ 0 & -2qd_1^2\nu_2^2\tau + id_1(1-2q\tau) \end{pmatrix}$$

and

$$A_2(\tau) = \begin{pmatrix} -2pd_2^2\bar{\nu}_1^2\tau + id_2(1-2p\tau) & 0 \\ 0 & -2qd_2^2\bar{\nu}_2^2\tau + id_2(1-2q\tau) \end{pmatrix}$$

Let us take up the values of Σ_0 and $\bar{\Sigma}_0$

$$\Sigma_0 = \begin{pmatrix} u & v \\ v & w \end{pmatrix}, \quad \bar{\Sigma}_0 = \begin{pmatrix} \bar{u} & \bar{v} \\ \bar{v} & \bar{w} \end{pmatrix}$$

$$A_1(\tau)\Sigma_0 = \begin{pmatrix} -2pd_1^2\nu_1^2\tau + id_1(1-2p\tau) & 0 \\ 0 & -2qd_1^2\nu_2^2\tau + id_1(1-2q\tau) \end{pmatrix} \begin{pmatrix} u & v \\ v & w \end{pmatrix}$$

$$A_1(\tau)\Sigma_0 = \begin{pmatrix} -2upd_1^2\nu_1^2\tau + id_1(1-2p\tau)u & -2vpd_1^2\nu_1^2\tau + id_1(1-2p\tau)v \\ -2qvd_1^2\nu_2^2\tau + id_1(1-2q\tau)v & -2qwd_1^2\nu_2^2\tau + id_1(1-2q\tau)w \end{pmatrix}$$

By comparison we get for

$$A_2(\tau)\bar{\Sigma}_0 = \begin{pmatrix} -2\bar{u}pd_2^2\bar{\nu}_1^2\tau + id_2(1-2p\tau)\bar{u} & -2p\bar{v}d_2^2\bar{\nu}_1^2\tau + id_2(1-2p\tau)\bar{v} \\ -2q\bar{v}d_2^2\bar{\nu}_2^2\tau + id_2(1-2q\tau)\bar{v} & -2q\bar{w}d_2^2\bar{\nu}_2^2\tau + id_2(1-2q\tau)\bar{w} \end{pmatrix}$$

Continue by taking the trace

$$Tr(A_1(\tau)\Sigma_0) = -2d_1^2[up\nu_1^2 + wq\nu_2^2]\tau + i[(1-2p\tau)u + (1-2q\tau)w]d_1$$

and

$$Tr(A_2(\tau)\bar{\Sigma}_0) = -2d_2^2[\bar{u}p\bar{\nu}_1^2 + \bar{w}q\bar{\nu}_2^2]\tau + i[(1-2p\tau)\bar{u} + (1-2q\tau)\bar{w}]d_2$$

Obtaining the approximation of the function $C(\tau)$

$$\begin{aligned} C(\tau) &= pC^{2,0}(\tau) + qC^{0,2}(\tau) + o(max(p,q)) \\ &= p(i\beta\nu_1^2 d_1\tau + i\bar{\beta}\bar{\nu}_1^2 d_2\tau) + q(i\beta\nu_2^2 d_1\tau + i\bar{\beta}\bar{\nu}_2^2 d_2\tau) + o(max(p,q)) \\ &= i\beta d_1(p\nu_1^2 + q\nu_2^2)\tau + i\bar{\beta}d_2(p\bar{\nu}_1^2 + q\bar{\nu}_2^2)\tau + o(max(p,q)) \end{aligned} \tag{3.134}$$

for ε

$$\begin{aligned} \varepsilon(\tau) &= \varepsilon^{0,0}(\tau) + p\varepsilon^{2,0}(\tau) + q\varepsilon^{0,2}(\tau) + o(max(p,q)) \\ &= r\gamma\tau + \frac{\gamma(\gamma-1)}{4}\beta(p\nu_1^2 + q\nu_2^2)\tau^2 + \frac{\gamma(\gamma-1)}{4}\bar{\beta}(p\bar{\nu}_1^2 + q\bar{\nu}_2^2)\tau^2 + o(max(p,q)) \end{aligned} \tag{3.135}$$

Then substitute these functions back in the characteristic function $\tilde{\phi}_{\Sigma\bar{\Sigma}}(\tau)$ in equation

(3.110)

$$\begin{aligned}\tilde{\phi}_{\Sigma\bar{\Sigma}}(\tau) = exp\{ & r\gamma(T-t) \\ & + p[-2d_1^2 u\nu_1^2 t - 2d_2^2\bar{u}\bar{\nu}_1^2 t + \frac{\gamma(\gamma-1)}{4}\beta\nu_1^2(T-t)^2 + \frac{\gamma(\gamma-1)}{4}\bar{\beta}\bar{\nu}_1^2(T-t)^2] \\ & + q[-2d_1^2 w\nu_2^2 t - 2d_2^2\bar{w}\bar{\nu}_2^2 t + \frac{\gamma(\gamma-1)}{4}\beta\nu_2^2(T-t)^2 + \frac{\gamma(\gamma-1)}{4}\bar{\beta}\bar{\nu}_2^2(T-t)^2]\} \\ & \times exp\{i[(1-2pt)ud_1 + (1-2qt)wd_1 + (1-2pt)\bar{u}d_2 \\ & + (1-2qt)\bar{w}d_2 + \beta d_1(p\nu_1^2 + q\nu_2^2)t + \bar{\beta}d_2(p\bar{\nu}_1^2 + q\bar{\nu}_2^2)t] + o(max(p,q))\}\end{aligned} \tag{3.136}$$

where

$$\begin{aligned}\Delta_1(t) = \{ & r\gamma(T-t) \\ & + p[-2d_1^2 u\nu_1^2 t - 2d_2^2\bar{u}\bar{\nu}_1^2 t + \frac{\gamma(\gamma-1)}{4}\beta\nu_1^2(T-t)^2 + \frac{\gamma(\gamma-1)}{4}\bar{\beta}\bar{\nu}_1^2(T-t)^2] \\ & + q[-2d_1^2 w\nu_2^2 t - 2d_2^2\bar{w}\bar{\nu}_2^2 t + \frac{\gamma(\gamma-1)}{4}\beta\nu_2^2(T-t)^2 + \frac{\gamma(\gamma-1)}{4}\bar{\beta}\bar{\nu}_2^2(T-t)^2]\}\end{aligned}$$

and

$$\Delta_2(t) = \{(1-2pt)ud_1 + (1-2qt)wd_1 + (1-2pt)\bar{u}d_2$$

$$+(1-2qt)\bar{w}d_2 + \beta d_1(p\nu_1^2 + q\nu_2^2)t + \bar{\beta}d_2(p\bar{\nu}_1^2 + q\bar{\nu}_2^2)t + o(max(p,q))\}$$

This can still be written in form

$$\tilde{\phi}_{\Sigma\bar{\Sigma}}(\tau) = e^{\Delta_1(t)}e^{i\Delta_2(t)}$$

From the call option price equation (3.112)

$$\begin{aligned}C_t(T,K) &= \frac{1}{2\pi}Re\left(\int_{-\infty}^{\infty} e^{-r(T-t)}\tilde{\phi}_{\Sigma\bar{\Sigma}}(t)\phi(\theta)e^{-i\theta k}d\theta\right) \\ &= \frac{1}{2\pi}Re\left(e^{-r(T-t)}\tilde{\phi}_{\Sigma\bar{\Sigma}}(t)\int_{-\infty}^{\infty}\phi(\theta)e^{-i\theta k}d\theta\right)\end{aligned}$$

Let us compute the complex integral from the price function, that is

$$\int_{-\infty}^{\infty}\phi(\theta)e^{-i\theta k}d\theta = \int_{-\infty}^{\infty}\frac{e^{-i\theta k}}{(\alpha+i\theta)(\alpha+1+i\theta)}d\theta$$

So this integral

$$I(\theta) = \int_{-\infty}^{\infty} \frac{e^{-i\theta k}}{(\alpha + i\theta)(\alpha + 1 + i\theta)} d\theta$$
$$= \int_{-\infty}^{\infty} \frac{e^{-i\theta k}}{i(\frac{\alpha}{i} + \theta)i(\frac{\alpha+1}{i} + \theta)} d\theta$$
$$= \int_{-\infty}^{\infty} \frac{-e^{-i\theta k}}{(\theta - i\alpha)(\theta - i(\alpha + 1))} d\theta$$

since $\theta = i\alpha, \quad \theta = i(\alpha + 1)$

This can be written as

$$I(z) = \int_{-\infty}^{\infty} \frac{-e^{-izk}}{(z - i\alpha)(z - i(\alpha + 1))} dz$$

Considering the integral on $\gamma = [-R, R]UC_R$

$$I(z) = \int_{-R}^{R} \frac{-e^{-izk}}{(z - i\alpha)(z - i(\alpha + 1))} dz + \int_{C_R} \frac{-e^{-izk}}{(z - i\alpha)(z - i(\alpha + 1))} dz \tag{3.137}$$

Then $R > \alpha + 1$, then $i(\alpha + 1)$ and $i\alpha$ are inside γ

$$\int_{\gamma} f(z)dz = 2i\pi \sum_{z_k \in Int(\gamma)} Res(f, z_k), \quad for \quad z_1 = i\alpha, \quad z_2 = i(\alpha + 1)$$

$$\int_{\gamma} f(z)dz = 2i\pi Res(f, i\alpha) + 2i\pi Res(f, i(\alpha + 1))$$

By computing the residuals

$$Res(f, i\alpha) = \lim_{z \to i\alpha} \{(z - i\alpha)f(z)\}$$

and

$$Res(f, i(\alpha + 1) = \lim_{z \to i(\alpha+1)} \{z - i(\alpha + 1)f(z)\}$$

Then

$$\begin{aligned}
Res(f, i\alpha) &= \lim_{z \to i\alpha} \{(z - i\alpha)f(z)\} \\
&= \lim_{z \to i\alpha} \{\frac{-(z - i\alpha)e^{-ikz}}{(z - i\alpha)(z - i(\alpha + 1))}\} \\
&= \lim_{z \to i\alpha} \frac{-e^{-ikz}}{z - i(\alpha + 1)} = -ie^{\alpha k}
\end{aligned}$$

and for

$$\begin{aligned}
Res(f, i(\alpha + 1)) &= \lim_{z \to i(\alpha+1)} \{\frac{-(z - i(\alpha + 1))e^{-ikz}}{(z - i\alpha)(z - i(\alpha + 1))}\} \\
&= \lim_{z \to i(\alpha+1)} \{\frac{-e^{-ikz}}{z - i\alpha}\} \\
&= ie^{k(\alpha+1)}
\end{aligned}$$

This imply that

$$\begin{aligned}
\int_\gamma f(z)dz &= 2i\pi(-ie^{k\alpha}) + 2i\pi(ie^{(\alpha+1)k}) \\
&= 2\pi(e^{k\alpha} - e^{k(\alpha+1)})
\end{aligned}$$

$$\begin{aligned}
\int_{-R}^{R} \frac{-e^{-izk}}{(z - i\alpha)(z - i(\alpha + 1))} dz = 2\pi(e^{k\alpha} - e^{k(\alpha+1)}) \\
= \int_{-R}^{R} \frac{-e^{-izk}}{(z - i\alpha)(z - i(\alpha + 1))} dz \\
+ \int_{C_R} \frac{-e^{-izk}}{(z - i\alpha)(z - i(\alpha + 1))} dz
\end{aligned} \tag{3.138}$$

$$as \quad R \longrightarrow \infty$$

$$= \int_{-\infty}^{\infty} \frac{-e^{-izk}}{(z - i\alpha)(z - i(\alpha + 1))} dz + 0$$

Therefore

$$\int_{-\infty}^{\infty} \frac{-e^{-i\theta k}}{(\theta - i\alpha)(\theta - i(\alpha + 1))} d\theta = 2\pi(e^{k\alpha} - e^{k(\alpha+1)})$$

Hence similarly the complex integral

$$\int_{-\infty}^{\infty} \frac{e^{-ik\theta}}{(\alpha + i\theta)(\alpha + 1 + i\theta)} d\theta = 2\pi(e^{k\alpha} - e^{k(\alpha+1)})$$

From our call option price

$$
\begin{aligned}
C_t(T,K) &= \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\theta k}e^{-r(T-t)}}{(\alpha+i\theta)(\alpha+1+i\theta)}\tilde{\phi}_{\Sigma,\bar{\Sigma}}(t)d\theta \\
&= \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-r(T-t)}e^{\Delta_1(t)}e^{i\Delta_2(t)}e^{-i\theta k}}{(\alpha+i\theta)(\alpha+1+i\theta)}d\theta \\
&= \frac{e^{-\alpha k}e^{-r(T-t)}e^{\Delta_1(t)}e^{i\Delta_2(t)}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\theta k}}{(\alpha+i\theta)(\alpha+1+i\theta)}d\theta \\
&= \frac{e^{-r(T-t)}e^{\Delta_1(t)}}{2\pi}e^{-\alpha k}e^{i\Delta_2(t)}[2\pi(e^{\alpha k}-e^{k(\alpha+1)})] \\
&= e^{(-r(T-t)+\Delta_1(t))}e^{i\Delta_2(t)}(1-e^k) \\
&= exp\{-r(T-t)+\Delta_1(t)\}e^{i\Delta_2(t)}(1-e^k) \\
&= exp\{-r(T-t)+\Delta_1(t)\}exp\{i\Delta_2(t)\}(1-e^k) \\
&= e^{-r(T-t)+\Delta_1(t)}(1-e^k)Re(e^{i\Delta_2(t)}) \\
&= exp\{-r(T-t)+\Delta_1(t)\}(1-e^k)cos\Delta_2(t) \\
&= e^{(-r(T-t)+\Delta_1(t))}(1-e^k)cos\Delta_2(t) \\
&= e^{(-r(T-t)+\Delta_1(t))}(1-e^k)cos\Delta_2(t)
\end{aligned}
$$

Therefore, the approximated price of the call option finally obtained as follows

$$
C_t(T,K) \approx e^{(-r(T-t)+\Delta_1(t))}(1-e^k)cos\Delta_2(t). \tag{3.139}
$$

3.3 Discretization of the Log-asset price return Dynamic

This section basically deals with the discretization schemes of Bi-variate Wishart model, Heston model and double Heston model, using the Euler-Maruyama discretization technique.

3.3.1 Euler-Maruyama discretization method

This is one of the best approximation methods to handle sophisticated stochastic differential equations as in Ahdida & Alfonsi (2013), Fadugba et al. (2013), Dereich et al. (2012), Berkaoui et al. (2008), Alfonsi (2015), Gauthier & Possamaï (2009), Berkaoui et al. (2008) and Mao

(2015). It is a time discrete approximation of an Ito process.

Considering S_t as an Ito process on $t \in [t_0, T]$ satisfying the stochastic differential equation

$$S_t = S_0 + \int_0^t \mu(t,s)ds + \int_0^t \gamma(t,s)dW_s, \quad t \geq 0 \tag{3.140}$$

For discretization $t_0 < t_1 < t_2 ... < t_N = T$ or on a regular grid $t_i = \frac{iT}{N}$, given as follows

$$\hat{S}_{t_{i+1}} = \hat{S}_{t_i} + \mu(\hat{S}_{t_i})\frac{T}{N} + \gamma(\hat{S}_{t_i})(W_{t_{i+1}} - W_{t_i}), \quad 1 \leq i \leq N-1 \tag{3.141}$$

and also considering a continuous case we have

$$\hat{S}_{t_{i+1}} = \hat{S}_{t_i} + \mu(\hat{S}_{t_i})(t_{i+1} - t_i) + \gamma(\hat{S}_{t_i})(W_{t_{i+1}} - W_{t_i}), \quad t \in [t_i, t_{i+1})$$

One can observe that, the $\hat{S}$ the Euler scheme depend on N throughout the discretization grid and it is similarly better to denote it as $\hat{S}^N$.

3.3.2 Heston stochastic volatility model

The Heston (1993) model where the underlying asset pricing dynamic process under the risk-neutral measure follows

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + \sqrt{\Sigma_t} dZ_t \\ d\Sigma_t &= (\alpha - \beta\Sigma_t)dt + \sigma\sqrt{\Sigma_t}dW_t^{\Sigma} \end{aligned} \tag{3.142}$$

where Z_t and W_t are correlated Brownian motions, and B_t is independent Brownian motion while the correlation structure of the model is given by

$$Z_t = \rho W_t + B_t\sqrt{1-\rho^2}$$

where μ is interest rate, S stock price, Σ variance, σ volatility of volatility, β speed of mean reversion, α long term variance. The volatility in Heston model follow CIR process.

Log-price for Heston model. The log-price Y_t of the model is obtained by applying the

Ito lemma. Let $Y_t = \log(S_t) = f(t,s)$, by applying Ito formula

$$\begin{aligned}
dY_t &= f_t dt + f_s dS + \frac{1}{2} f_{ss}(dS)^2 \\
&= 0.dt + \frac{dS}{S} - \frac{1}{2}\frac{(dS)^2}{S^2} \\
&= \mu dt - \frac{1}{2}\Sigma dt + \sqrt{\Sigma_t} dZ_t \\
&= (\mu - \frac{1}{2}\Sigma)dt + \sqrt{\Sigma_t} dZ_t
\end{aligned}$$

We can now apply Euler-scheme on the CIR process in equation (3.142),

$$\hat{\Sigma}_{t^N_{i+1}} = \hat{\Sigma}_{t^N_i} + (\alpha - \beta \hat{\Sigma}_{t^N_i})\frac{T}{N} + \sigma \sqrt{\hat{\Sigma}^+_{t^N_i}} \sqrt{\frac{T}{N}} \xi_i, \quad \hat{\Sigma}_{t^N_0} = \Sigma_0 \tag{3.143}$$

where the random variable $\xi_i \sim \mathcal{N}(0,1)$. Now we discretize the log-price process as follows

$$\hat{Y}^y_{t_{i+1}} = \hat{Y}^y_{t_i} + (\mu - \frac{1}{2}\hat{\Sigma}_{t^N_i})\frac{T}{N} + \sqrt{\hat{\Sigma}^+_{t^N_i}}(Z_{t^N_{i+1}} - Z_{t^N_i}), \quad \hat{Y}_{t_0} = y \tag{3.144}$$

simplifying the Brownian motion term

$$Z_{t^N_{i+1}} - Z_{t^N_i} = \rho(W_{t^N_{i+1}} - W_{t^N_i}) + (B_{t^N_{i+1}} - B_{t^N_i})\sqrt{1-\rho^2} \tag{3.145}$$

hence substituting equation (3.145) into equation (3.144), we obtain

$$\hat{Y}^y_{t_{i+1}} = \hat{Y}^y_{t_i} + (\mu - \frac{1}{2}\hat{\Sigma}_{t^N_i})\frac{T}{N} + \sqrt{\hat{\Sigma}^+_{t^N_i}}\rho \Delta W_{t_i} + \sqrt{\hat{\Sigma}_{t^N_i}(1-\rho^2)}\Delta B_{t_i}$$

3.3.3 The discretization of Bi-Heston model

This section describes the double Heston model and obtain the solution of the SDEs using Euler-Maruyama discretization method. The double Heston model by Christoffersen et al. (2009), added second volatility factor $H_{2,t}$ driven by independent stochastic differential equation

$$
\begin{aligned}
\frac{dS_t}{S_t} &= \mu dt + \sqrt{H_{1,t}} dZ_{1,t} + \sqrt{H_{2,t}} dZ_{2,t} \\
dH_{1,t} &= (\alpha_1 - \beta_1 H_{1,t}) dt + \sigma_1 \sqrt{H_{1,t}} dW_{1,t} \\
dH_{2,t} &= (\alpha_2 - \beta_2 H_{2,t}) dt + \sigma_2 \sqrt{H_{2,t}} dW_{2,t}
\end{aligned}
\tag{3.146}
$$

and the Brownian motions $W_{1,t}, W_{2,t}$ are correlated to the Brownian motions of the asset price $H_{1,t}, H_{2,t}$ as follows

$$
\begin{aligned}
d < Z_{1,t}, W_{1,t} > &= \rho_1, \quad d < Z_{2,t}, W_{2,t} >= \rho_2 \\
d < W_{1,t}, W_{2,t} > &= 0, \quad d < Z_{1,t}, Z_{2,t} >= 0
\end{aligned}
$$

with the correlation structures, given independent Brownian motions $B_{1,t}$ and $B_{2,t}$

$$
\begin{aligned}
Z_{1,t} &= \rho_1 W_{1,t} + B_{1,t} \sqrt{1 - \rho_1^2} \\
Z_{2,t} &= \rho_2 W_{2,t} + B_{2,t} \sqrt{1 - \rho_2^2}
\end{aligned}
\tag{3.147}
$$

Derivation of the log-asset dynamic for the system, using Ito formula (or Ito lemma). let $V_t = \log(S_t) = f(t, s)$

$$
\begin{aligned}
dV_t &= f_t dt + f_s dS + \frac{1}{2} f_{ss} (dS)^2 \\
&= 0.dt + \frac{dS}{S} - \frac{1}{2} \frac{(dS)^2}{S^2} \\
&= \mu dt + \sqrt{H_{1,t}} dZ_{1,t} + \sqrt{H_{2,t}} dZ_{2,t} - \frac{1}{2}(H_{1,t} + H_{2,t}) dt \\
&= (\mu - \frac{1}{2}(H_{1,t} + H_{2,t})) dt + \sqrt{H_{1,t}} dZ_{1,t} + \sqrt{H_{2,t}} dZ_{2,t}, \quad V_0 = v
\end{aligned}
$$

Euler-Maruyama method for double Heston model, let us consider a time horizon T and the regular time grid $t_i^N = \frac{iT}{N}, \quad for \quad i = 0, ..., N.$

Indeed we recall that $dt = t_{i+1}^N - t_i^N = \frac{(i+1)T}{N} - \frac{iT}{N} = \frac{T}{N}$, we note that, it is important to first discretize the Brownian motions in the system of equations of the model as follows

$$
\begin{aligned}
dW_{1,t} &= W_{1,t_{i+1}^N} - W_{1,t_i^N}, \quad dB_{1,t} = B_{1,t_{i+1}^N} - B_{1,t_i^N} \\
dW_{2,t} &= W_{2,t_{i+1}^N} - W_{2,t_i^N}, \quad dB_{2,t} = B_{2,t_{i+1}^N} - B_{2,t_i^N}
\end{aligned}
$$

Therefore substituting in the correlation structures, gives

$$
\begin{aligned}
dZ_{1,t} &= \rho_1(W_{1,t_{i+1}^N} - W_{1,t_i^N}) + \sqrt{1-\rho^2}(B_{1,t_{i+1}^N} - B_{1,t_i^N}) \\
dZ_{2,t} &= \rho_2(W_{2,t_{i+1}^N} - W_{2,t_i^N}) + \sqrt{1-\rho^2}(B_{,t_{i+1}^N} - B_{2,t_i^N})
\end{aligned}
$$

hence this can still be simplified as

$$
dZ_{1,t} = \rho_1\sqrt{\frac{T}{N}}\xi_{1,i} + \sqrt{\frac{T}{N}(1-\rho^2)}\zeta_{1,i}
$$

similarly the second correlation structure as

$$
dZ_{2,t} = \rho_2\sqrt{\frac{T}{N}}\xi_{2,i} + \sqrt{\frac{T}{N}(1-\rho^2)}\zeta_{2,i}
$$

where $\xi_{k,i}$ and $\zeta_{k,i}$, for $k = 1, 2$ are Gaussian random variables with mean 0 and standard deviation 1, that is $\xi_{k,i}$,$\zeta_{k,i} \sim \mathcal{N}(0,1)$.

Then discretizing the CIR volatility processes in (3.146) and for computational purposes these stochastic differential equations can be written in the form

$$
\begin{aligned}
H_{1,t} &= H_{1,0} + \int_0^t (\alpha_1 - \beta_1 H_{1,s})ds + \sigma_1 \int_0^t \sqrt{H_{1,s}}dW_{1,s}, \quad H_{1,0} = h_1 \\
H_{2,t} &= H_{2,0} + \int_0^t (\alpha_2 - \beta_2 H_{2,s})ds + \sigma_2 \int_0^t \sqrt{H_{2,s}}dW_{2,s}, \quad H_{2,0} = h_2
\end{aligned}
$$

From Euler scheme, yields

$$
\begin{aligned}
\hat{H}^{h_1}_{1,t_{i+1}^N} &= \hat{H}_{1,t_0^N} + (\alpha_1 - \beta_1 \hat{H}_{1,t_i^N})\frac{T}{N} + \sigma_1\sqrt{\hat{H}^+_{1,t_i^N}}\sqrt{\frac{T}{N}}\xi_{1,i}, \quad \hat{H}_{1,t_0^N} = h_1 \\
\hat{H}^{h_2}_{2,t_{i+1}^N} &= \hat{H}_{2,t_0^N} + (\alpha_2 - \beta_2 \hat{H}_{2,t_i^N})\frac{T}{N} + \sigma_2\sqrt{\hat{H}^+_{2,t_i^N}}\sqrt{\frac{T}{N}}\xi_{2,i}, \quad \hat{H}_{2,t_0^N} = h_2
\end{aligned}
\tag{3.148}
$$

Then we apply the Euler scheme on the log-price dynamic for double Heston model.

$$\begin{aligned}
\hat{V}^v_{t^N_{i+1}} &= \hat{V}_{t^N_0} + (\mu - \frac{1}{2}(\hat{H}_{1,t^N_i} + \hat{H}_{2,t^N_i}))\frac{T}{N} + \sqrt{\hat{H}^+_{1,t^N_i}}(Z_{1,t^N_i} - Z_{1,t^N_i}) + \sqrt{\hat{H}^+_{2,t^N_i}}(Z_{2,t^N_i} - Z_{2,t^N_i}) \\
\hat{V}_{t^N_0} &= v
\end{aligned} \tag{3.149}$$

3.3.4 The corrected Euler-Maruyama discretization scheme for double Wishart affine processes

This section discuses the discretization techniques for Bi-variate Wishart model, particularly the discretization of log-asset price dynamics and Wishart processes using the corrected Euler-Maruyama scheme. We notice that the Euler scheme is not well defined, in fact the Gaussian increment may lead the scheme to negative values with some positive probability, and the square root is then no long defined. Therefore this calls for the need to apply the corrected Euler-Maruyama scheme to avoid negative values, see Alfonsi (2015) and Gauthier & Possamaï (2009).

The asset dynamics is disturbed by two volatility components driven by Wishart matrix process

$$\frac{dS_t}{S_t} = \mu dt + Tr[\sqrt{\Sigma}_{1,t} dZ^{\Sigma_1}_t + \sqrt{\Sigma}_{2,t} dZ^{\Sigma_2}_t], \quad S_0 = s \tag{3.150}$$

where the Wishart matrix processes are given as follows

$$\begin{aligned}
d\Sigma_{1,t} &= (\beta_1 Q_1 Q_1^T + M_1 \Sigma_{1,t} + \Sigma_{1,t} M_1^T)dt + \sqrt{\Sigma}_{1,t} dW^{\Sigma_1}_t Q_1 + Q_1^T (dW^{\Sigma_1}_t)^T \sqrt{\Sigma_{1,t}}, \\
&\quad \Sigma_{1,0} = \Sigma_1 \\
d\Sigma_{2,t} &= (\beta_2 Q_2 Q_2^T + M_2 \Sigma_{2,t} + \Sigma_{2,t} M_2^T)dt + \sqrt{\Sigma}_{2,t} dW^{\Sigma_2}_t Q_2 + Q_2^T (dW^{\Sigma_2}_t)^T \sqrt{\Sigma_{2,t}}, \\
&\quad \Sigma_{2,0} = \Sigma_2
\end{aligned}$$

These stochastic differential equations above can still be presented in the form

$$\begin{aligned}\Sigma_{1,t} &= \Sigma_{1,0} + \int_0^t (\Omega_1 + \eta_1(\Sigma, s))ds + \int_0^t \sqrt{\Sigma}_{1,s} dW_s^{\Sigma_1} Q_1 + \int_0^t Q_1^T (dW_s^{\Sigma_1})^T \sqrt{\Sigma_{1,s}} \\ \Sigma_{2,t} &= \Sigma_{2,0} + \int_0^t (\Omega_2 + \eta_2(\Sigma, s))ds + \int_0^t \sqrt{\Sigma}_{2,s} dW_s^{\Sigma_2} Q_2 + \int_0^t Q_2^T (dW_s^{\Sigma_2})^T \sqrt{\Sigma_{2,s}}\end{aligned} \tag{3.151}$$

where $\Omega_1 = \beta_1 Q_1 Q_1^T$ and $\Omega_2 = \beta_2 Q_2 Q_2^T$ such that $\Sigma_{1,0}, \Sigma_{2,0}, \Omega_1, \Omega_2 \in S_n^+(\mathbb{R})$ while $\eta_1, \eta_2 \in \mathcal{L}(S_n^+(\mathbb{R}))$ and $M_1, M_2 \in M_n(\mathbb{R})$

The log-price dynamic under Bi-variate Wishart processes is as follows

$$dY_t = (\mu - \frac{1}{2} Tr[\Sigma_{1,t} + \Sigma_{2,t}])dt + Tr[\sqrt{\Sigma}_{1,t} dZ_t^{\Sigma_1} + \sqrt{\Sigma}_{2,t} dZ_t^{\Sigma_2}] \tag{3.152}$$

Then discretizing the system of equations (3.151) and (3.152), by considering a time horizon T and regular time grid $t_i^N = \frac{iT}{N}$, for $i = 0, ..., N$.

$$\hat{\Sigma}_{1,t_{i+1}^N} = \hat{\Sigma}_{1,t_i^N} + (\Omega_1 + \eta_1(\hat{\Sigma}_1, t_i^N))\frac{T}{N} + \sqrt{\hat{\Sigma}^+_{1,t_i^N}} (W^{\Sigma_1}_{t_{i+1}^N} - W^{\Sigma_1}_{t_i^N}) Q_1 + Q_1^T (W^{\Sigma_1}_{t_{i+1}^N} - W^{\Sigma_1}_{t_i^N}) \sqrt{\hat{\Sigma}^+_{1,t_i^N}}$$

where $\hat{\Sigma}_{1,t_0^N} = \Sigma_{1,0} = \Sigma_1$ and for the second volatility process as follows

$$\hat{\Sigma}_{2,t_{i+1}^N} = \hat{\Sigma}_{2,t_i^N} + (\Omega_2 + \eta_2(\hat{\Sigma}_2, t_i^N))\frac{T}{N} + \sqrt{\hat{\Sigma}^+_{2,t_i^N}} (W^{\Sigma_2}_{t_{i+1}^N} - W^{\Sigma_2}_{t_i^N}) Q_2 + Q_2^T (W^{\Sigma_2}_{t_{i+1}^N} - W^{\Sigma_2}_{t_i^N}) \sqrt{\hat{\Sigma}^+_{2,t_i^N}}$$

with $\hat{\Sigma}_{2,t_0^N} = \Sigma_{2,0} = \Sigma_2$.

Now proceed to discretize the the log-price dynamics under double Wishart model as follows

$$\begin{aligned}\hat{Y}^y_{t_{i+1}^N} &= \hat{Y}^y_{t_i^N} \\ &+ \left(\mu - \frac{1}{2} Tr[\hat{\Sigma}_{1,t_i^N} + \hat{\Sigma}_{2,t_i^N}]\right)\frac{T}{N} + Tr\left[\sqrt{\hat{\Sigma}^+_{1,t_i^N}} (Z^{\Sigma_1}_{t_{i+1}^N} - Z^{\Sigma_1}_{t_i^N}) + \sqrt{\hat{\Sigma}^+_{2,t_i^N}} (Z^{\Sigma_2}_{t_{i+1}^N} - Z^{\Sigma_2}_{t_i^N})\right]\end{aligned} \tag{3.153}$$

Discretizing the Brownian motions in equation (3.153) by considering the correlation structures in proposition (3.1.1), that is $Z_t^{\Sigma_j} = W_t^{\Sigma_j} R_j^T + B_t^{\Sigma_j} \sqrt{\mathbb{I} - R_j R_j^T}, \quad for \quad j = 1, 2$

$$\begin{aligned} Z^{\Sigma_1}_{t^N_{i+1}} - Z^{\Sigma_1}_{t^N_i} &= W^{\Sigma_1}_{t^N_{i+1}} R_1^T + B^{\Sigma_1}_{t^N_{i+1}} \sqrt{\mathbb{I} - R_1 R_1^T} - \left(W^{\Sigma_1}_{t^N_i} R_1^T + B^{\Sigma_1}_{t^N_i} \sqrt{\mathbb{I} - R_1 R_1^T} \right) \\ &= \left(W^{\Sigma_1}_{t^N_{i+1}} - W^{\Sigma_1}_{t^N_i} \right) R_1^T + \left(B^{\Sigma_1}_{t^N_{i+1}} - B^{\Sigma_1}_{t^N_i} \right) \sqrt{\mathbb{I} - R_1 R_1^T} \end{aligned} \tag{3.154}$$

and for the second Brownian motion, proceed with the same discretizing procedures

$$\begin{aligned} Z^{\Sigma_2}_{t^N_{i+1}} - Z^{\Sigma_2}_{t^N_i} &= W^{\Sigma_2}_{t^N_{i+1}} R_2^T + B^{\Sigma_2}_{t^N_{i+1}} \sqrt{\mathbb{I} - R_2 R_2^T} - \left(W^{\Sigma_2}_{t^N_i} R_2^T + B^{\Sigma_2}_{t^N_i} \sqrt{\mathbb{I} - R_2 R_2^T} \right) \\ &= \left(W^{\Sigma_2}_{t^N_{i+1}} - W^{\Sigma_2}_{t^N_i} \right) R_2^T + \left(B^{\Sigma_2}_{t^N_{i+1}} - B^{\Sigma_2}_{t^N_i} \right) \sqrt{\mathbb{I} - R_2 R_2^T} \end{aligned} \tag{3.155}$$

These Brownian matrices Z^{Σ_1} and B^{Σ_1} follow normal distribution, that is

$$\begin{aligned} W^{\Sigma_1}_{t^N_{i+1}} - W^{\Sigma_1}_{t^N_i} &\sim \mathcal{N}(0, (t^N_{i+1} - t^N_i)\mathbb{I}_n) \\ W^{\Sigma_1}_{t^N_{i+1}} - W^{\Sigma_1}_{t^N_i} &\sim \sqrt{t^N_{i+1} - t^N_i}\mathcal{N}(0, \mathbb{I}) \end{aligned} \tag{3.156}$$

and

$$B^{\Sigma_1}_{t^N_{i+1}} - B^{\Sigma_1}_{t^N_i} \sim \sqrt{t^N_{i+1} - t^N_i}\mathcal{N}(0, \mathbb{I})$$

Hence simplifying the computation we substitute equations (3.156) into equation (3.154) we get

$$Z^{\Sigma_1}_{t^N_{i+1}} - Z^{\Sigma_1}_{t^N_i} \sim \left(R_1^T + \sqrt{\mathbb{I} - R_1 R_1^T} \right) \sqrt{t^N_{i+1} - t^N_i}\mathcal{N}(0, \mathbb{I}) \tag{3.157}$$

similarly for

$$Z^{\Sigma_2}_{t^N_{i+1}} - Z^{\Sigma_2}_{t^N_i} \sim \left(R_2^T + \sqrt{\mathbb{I} - R_2 R_2^T} \right) \sqrt{t^N_{i+1} - t^N_i}\mathcal{N}(0, \mathbb{I})$$

The equation (3.157) is reduced in the form

$$Z^{\Sigma_1}_{t^N_{i+1}} - Z^{\Sigma_1}_{t^N_i} \sim \left(R_1^T + \sqrt{\mathbb{I} - R_1 R_1^T} \right) \sqrt{\frac{T}{N}}\mathcal{N}(0, \mathbb{I})$$

as well as

$$Z^{\Sigma_2}_{t^N_{i+1}} - Z^{\Sigma_2}_{t^N_i} \sim \left(R_2^T + \sqrt{\mathbb{I} - R_2 R_2^T} \right) \sqrt{\frac{T}{N}}\mathcal{N}(0, \mathbb{I})$$

Let the log-price eturn dynamics $Y_t = \log(S_t)$ in equation (3.153) be $\hat{Y}_{t^N_{i+1}} = \log(\hat{S}_{t^N_{i+1}})$ and $\log(\hat{S}_{t^N_i})$

$$\log(\hat{S}_{t^N_{i+1}}) = \log(\hat{S}_{t^N_i}) + \left(\mu - \frac{1}{2}Tr[\hat{\Sigma}_{1,t^N_i} + \hat{\Sigma}_{2,t^N_i}]\right)\frac{T}{N} + Tr\left[\sqrt{\hat{\Sigma}^+_{1,t^N_i}}(Z^{\Sigma_1}_{t^N_{i+1}} - Z^{\Sigma_1}_{t^N_i}) + \sqrt{\hat{\Sigma}^+_{2,t^N_i}}(Z^{\Sigma_2}_{t^N_{i+1}} - Z^{\Sigma_2}_{t^N_i})\right]$$

This log-price expression can be represented in the form

$$\log(\hat{S}_{t^N_{i+1}}) = \log(\hat{S}_{t^N_i}) + \mu\frac{T}{N} - \frac{1}{2}\int_{t^N_i}^{t^N_{i+1}} Tr[\Sigma_1(s) + \Sigma_2(s)]ds + Tr\left[\int_{t^N_i}^{t^N_{i+1}} \sqrt{\Sigma^+_1(s)}dZ^{\Sigma_1}_s + \int_{t^N_i}^{t^N_{i+1}} \sqrt{\Sigma^+_2(s)}dZ^{\Sigma_2}_s\right] \tag{3.158}$$

The increment $\xi_{k,i_{1\leq k\leq\beta}}$, allows to simulate $Tr[\int_{t^N_i}^{t^N_{i+1}} \sqrt{\Sigma^+_1(s)}dW^{\Sigma_1}_s R^T_1]$ and for other terms the standard Euler scheme is selected, that is

$$\int_{t^N_i}^{t^N_{i+1}} \sqrt{\Sigma^+_1(s)}dW^{\Sigma_1}_s = \sum_{k=1}^{\beta}\int_{t^N_i}^{t^N_{i+1}} X_{k,s}dW^{\Sigma_1}_{k,s} = \sqrt{\Delta t}\sum_{k=1}^{\beta} X_{k,t}\xi^T_{k,i}$$

and

$$Tr\left[\int_{t^N_i}^{t^N_{i+1}} \sqrt{\Sigma^+_1(s)}dB^{\Sigma_1}_s\sqrt{\mathbb{I} - R_1R^T_1}\right] = \sqrt{\Delta t}\sqrt{Tr(\hat{\Sigma}^+_{1,t^N_i}(\mathbb{I} - R_1R^T_1))}\zeta_{1,i}$$

Observe that the integral term in the equation (3.158) need to be simplified as follows

$$\begin{aligned}\int_{t^N_i}^{t^N_{i+1}} \sqrt{\Sigma^+_1(s)}dZ^{\Sigma_1}_s &= \int_{t^N_i}^{t^N_{i+1}} \sqrt{\Sigma^+_1(s)}dW^{\Sigma_1}_s R^T_1 + \int_{t^N_i}^{t^N_{i+1}} \sqrt{\Sigma^+_1(s)}dB^{\Sigma_1}_s\sqrt{\mathbb{I} - R_1R^T_1} \\ &= \sqrt{t^N_{i+1} - t^N_i}\sum_{k=1}^{l}\hat{X}_1(k, t^N_i)\xi^T_{k,i}R^T_1 + \sqrt{t^N_{i+1} - t^N_i}\sqrt{\hat{\Sigma}^+_{1,t^N_i}(\mathbb{I} - R_1R^T_1)}\zeta_{1,i} \\ &= \sqrt{\frac{T}{N}}\sum_{k=1}^{l}\hat{X}_1(k, t^N_i)\xi^T_{k,i}R^T_1 + \sqrt{\frac{T}{N}}\sqrt{\hat{\Sigma}^+_{1,t^N_i}(\mathbb{I} - R_1R^T_1)}\zeta_{1,i}\end{aligned} \tag{3.159}$$

where the Gaussian vector $\zeta_{1,i} \sim \mathcal{N}(0, \mathbb{I})$ are the components of an independent Gaussian vector and being independent with the increment $\xi_{k,i}$ and the same applies for the second term

$$\int_{t_i^N}^{t_{i+1}^N} \sqrt{\Sigma_2^+(s)} dZ_s^{\Sigma_2} = \sqrt{\frac{T}{N}} \sum_{k=1}^{l} \hat{X}_2(k, t_i^N) \xi_{k,i}^T R_2^T + \sqrt{\frac{T}{N}} \sqrt{\hat{\Sigma}^+_{2,t_i^N} (\mathbb{I} - R_2 R_2^T)} \zeta_{2,i} \quad (3.160)$$

where $\zeta_{2,i} \sim \mathcal{N}(0, \mathbb{I})$ then can substitute equations (3.159) and (3.160) in equation (3.158) as follows

$$\begin{aligned} \log\left(\frac{\hat{S}_{t_{i+1}^N}}{\hat{S}_{t_i^N}}\right) &= \mu\frac{T}{N} - \frac{1}{2}\int_{t_i^N}^{t_{i+1}^N} Tr[\Sigma_1(s) + \Sigma_2(s)] ds \\ &+ \sqrt{\frac{T}{N}} Tr\left[\sum_{k=1}^{l} \hat{X}_1(k, t_i^N)\xi_{k,i}^T R_1^T + \sum_{k=1}^{l} \hat{X}_2(k, t_i^N)\xi_{k,i}^T R_2^T\right] \\ &+ \sqrt{\frac{T}{N}} \left[\sqrt{Tr(\hat{\Sigma}^+_{1,t_i^N}(\mathbb{I} - R_1 R_1^T))}\zeta_{1,i} + \sqrt{Tr(\hat{\Sigma}^+_{2,t_i^N}(\mathbb{I} - R_2 R_2^T))}\zeta_{2,i}\right] \end{aligned} \quad (3.161)$$

Finally the numerical solution for log price return under double Wishart diffusion processes is obtained as follows

$$\begin{aligned} \log\left(\frac{\hat{S}_{t_{i+1}^N}}{\hat{S}_{t_i^N}}\right) &= \left(\mu - \frac{1}{2} Tr[\hat{\Sigma}_{1,t_i^N} + \hat{\Sigma}_{2,t_i^N}]\right)\frac{T}{N} \\ &+ \sqrt{\frac{T}{N}} Tr\left[\sum_{k=1}^{l} \hat{X}_1(k, t_i^N)\xi_{k,i}^T R_1^T + \sum_{k=1}^{l} \hat{X}_2(k, t_i^N)\xi_{k,i}^T R_2^T\right] \\ &+ \sqrt{\frac{T}{N}} \left[\sqrt{Tr(\hat{\Sigma}^+_{1,t_i^N}(\mathbb{I} - R_1 R_1^T))}\zeta_{1,i} + \sqrt{Tr(\hat{\Sigma}^+_{2,t_i^N}(\mathbb{I} - R_2 R_2^T))}\zeta_{2,i}\right] \end{aligned}$$

Chapter 4

Numerical Illustrations

4.1 Numerical Illustration

This section deals with illustrative numerical examples to examine the implications of volatility in option pricing under double Wishart stochastic volatility model.

4.1.1 Comparison between the double Wishart model price and the market price

The volatility specification of double Wishart model remarkably makes it flexible, to produce price predictions which exhibit similar price behavior with the market. From Benabid et al. (2008) and market data drawn from QQQ (a fund by Invesco that tracks the performance of the stocks listed under the NASDAQ Index) options, April 2020 is quoted.

Taking the values for variance matrices Σ and $\bar{\Sigma}$ as

$$\Sigma_0 = \begin{pmatrix} 0.00008 & 0 \\ 0 & 0.000095 \end{pmatrix} \quad and \quad \bar{\Sigma}_0 = \begin{pmatrix} 0.00012 & 0 \\ 0 & 0.0001 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0.10 & 0 \\ 0 & 0.23 \end{pmatrix} \quad and \quad \bar{Q} = \begin{pmatrix} 0.20 & 0 \\ 0 & 0.25 \end{pmatrix}$$

While the strike price $K = 189$ and interest rate $r = 0.05$.

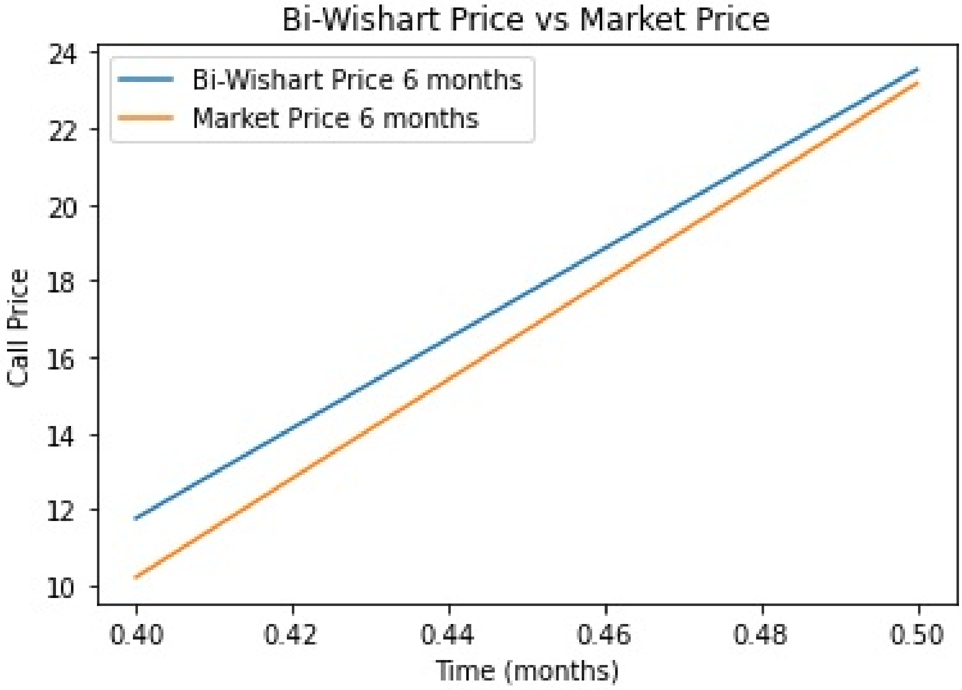

Figure 4.1: Short maturity $T = 6$ months, $d_1 = 0.45$, $d_2 = 0.479$ and $\gamma = 0.5$, the model call price predictions with respect to the market price behavior

Figure 4.1 illustrates the effect of model parameters on the call price under double Wishart model. Given the parameter values, observe that, the price predictions under double Wishart model exhibits the same behavior with respect to the market in short maturity time. This depends also on the choice of the model parameters, such as $\gamma = 0.5, \beta = 4$, $\bar{\beta} = 3$, $d_1 = 0.45, d_2 = 0.479$. This example proves that the double Wishart volatility model has greater flexibility.

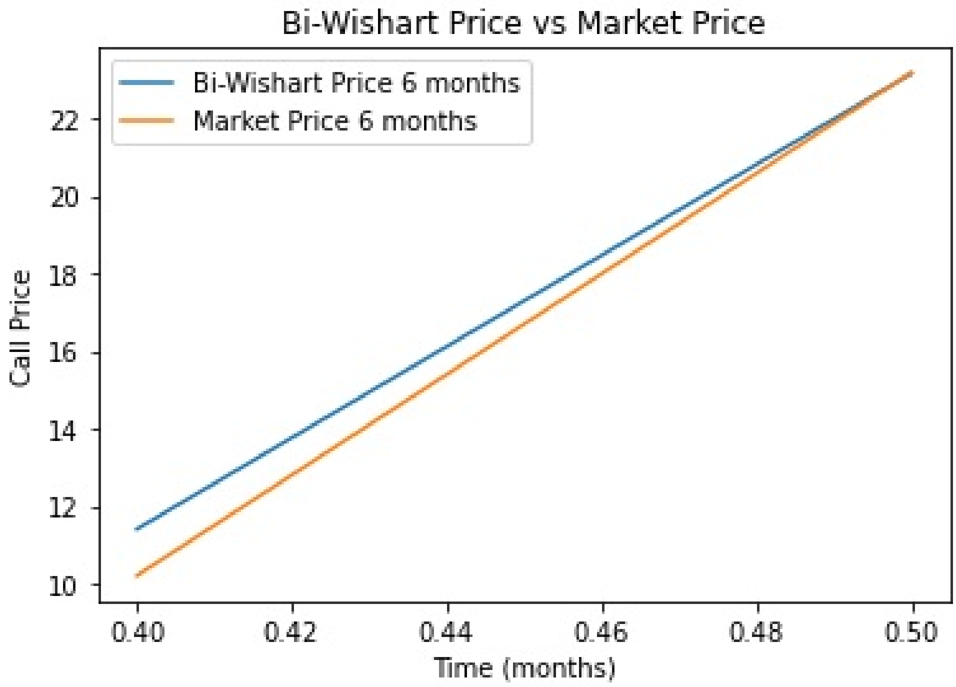

Figure 4.2: Short maturity $T = 6$ months, with change in $d_2 = 0.478$, $d_1 = 0.45$ and $\gamma = 0.5, \beta = 4, \bar{\beta} = 3$ the model call price predictions versus market price behavior

Figure 4.2, for a given change in d_2 from 0.479 to 0.478 the model provides optimal pricing almost similar to the market for holding above 0.48 on 6 months maturities. This is due to model parameter specification impact on the call European option price under double Wishart model.

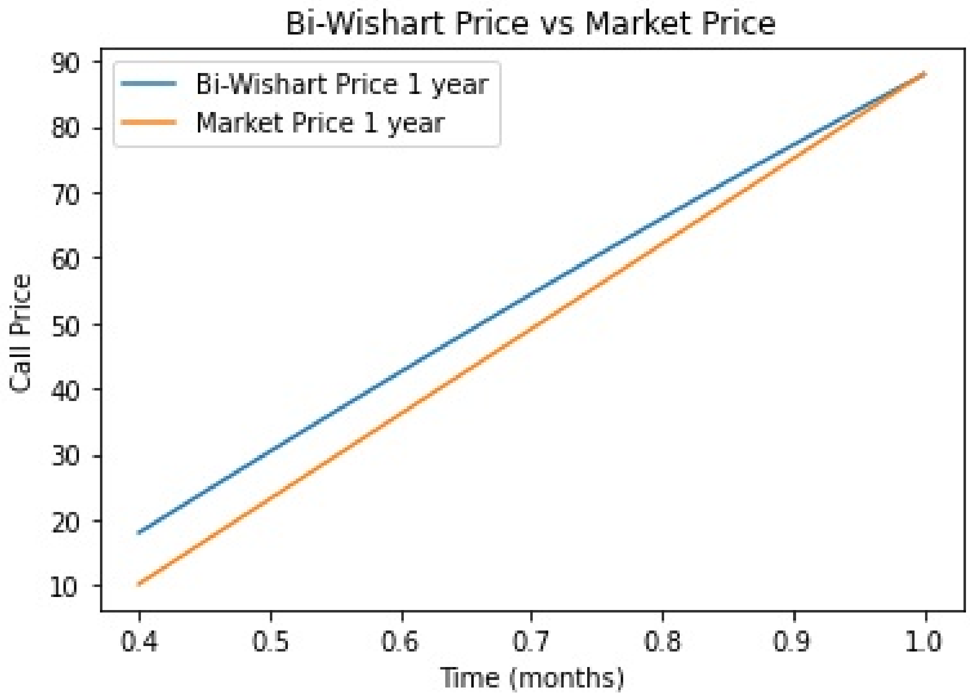

Figure 4.3: Long maturity $T = 1$ year, when $d_1 = 0.45$, $d_2 = 0.4968$ and $\gamma = 1$, model price predictions with respect to the market price behavior

Figure 4.3 demonstrates the effect of parameters on the European call option price under double Wishart model. For long maturity time of 1 year the predictive power of the model is optimal as from 0.9 months of holding to expiry of the contract.The model shows price predictions exhibiting the same market price behavior, in long maturity for $\gamma = 1, \beta = 4, \bar{\beta} = 3$, $d_1 = 0.45$ and $d_2 = 0.4968$. It clearly shows that the model parameters influence the prices greatly and helps long position holders not to experience arbitrage profits.

4.1.2 The effect of model parameters on log-asset return

This section deals with numerical examples to examine the effects of volatility and other model parameters on the log-asset price under double Wishart volatility model, for given number of paths N.

The volatility specification under double Wishart volatility model makes it mathematically flexible, to influence the log-asset price behavior. It is important for investors to study the stock behavior since options or derivative products derive their values from the underlying asset.

Taking the correlation values as;

$$R_1 = \begin{pmatrix} -0.25 & 0.1 \\ -0.35 & 0.3 \end{pmatrix} \qquad and \quad R_2 = \begin{pmatrix} -0.5 & -0.60 \\ 0.18 & -0.45 \end{pmatrix}$$

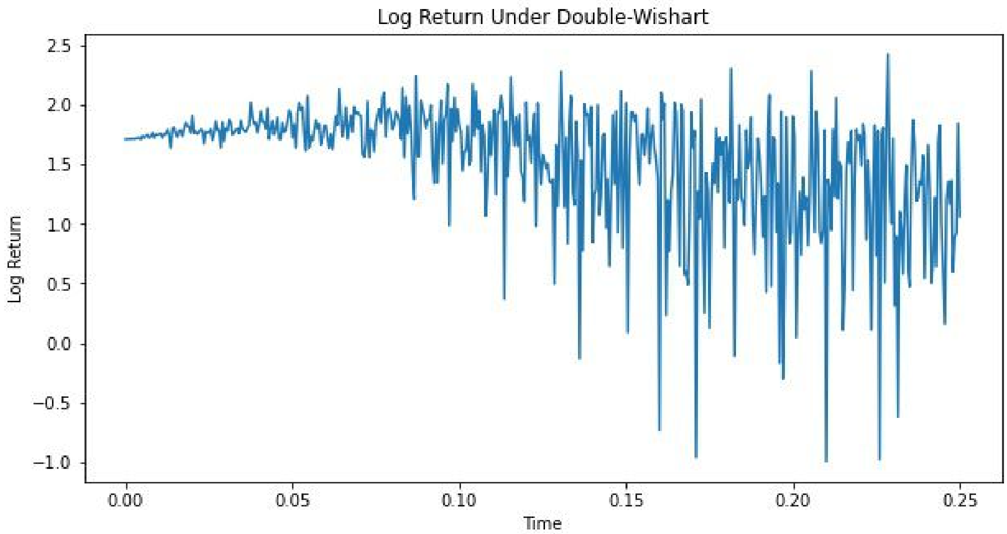

Figure 4.4: Short maturity $T = 3$ months, $N = 500$, $\beta_1 = 4$, $\beta_2 = 3$ $r = 0.05$, $K = 189$, log-asset price behaviors under double Wishart model

Figure 4.4 demonstrates the effect of implied volatilities and correlation matrices on the log-asset price under double Wishart model. Given the parameter values, we observe that, the log-asset returns under double Wishart model exhibits interesting behaviors. This depends also on the choice of the model parameters, such as $\beta_1 = 4$, $\beta_2 = 3$ and the number of paths $N = 500$. This example proves that the double Wishart volatility model has greater flexibility. The log-returns under the double-Wishart model are more volatile towards the end of the trading period. The asset prices increase from the higher deviations with the number of paths under trading from the values of volatility matrices. It leads to higher returns in 3-months asset trading period under the Double-Wishart process.

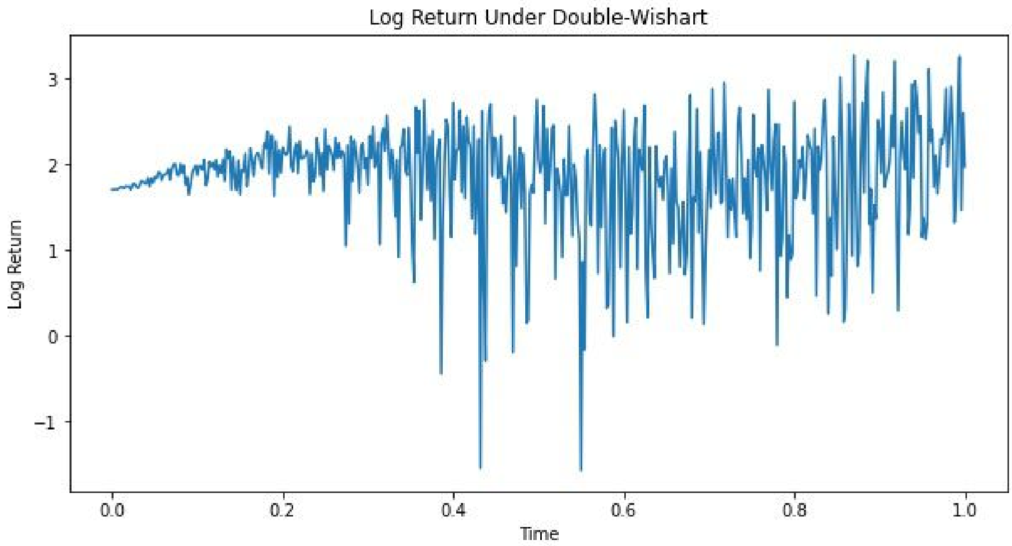

Figure 4.5: Long maturity $T = 1$ year, $N = 500$, $r = 0.05$, $K = 189$ with change in correlation matrices R_i, the log-price trend behavior

Figure 4.5 shows that a change in correlation matrices affects the the behavior of the asset returns, that is, when

$$R_1 = \begin{pmatrix} 0.0 & 0.85 \\ -0.05 & 0.75 \end{pmatrix} \qquad and \quad R_2 = \begin{pmatrix} 0.20 & 0.0 \\ 0.01 & -0.20 \end{pmatrix}$$

this provides different stock price behavior pattern due to flexibility in the model parameters. This can help investors to study and predict the behavior of stock price over time. The parameters $\beta_1 = 4, \beta_2 = 3$ are maintained. The stock price trend under double Wishart model when $N = 500$ at one year, the asset price increases with time from the implied volatility as it approaches annual trading period from the volatility matrices.

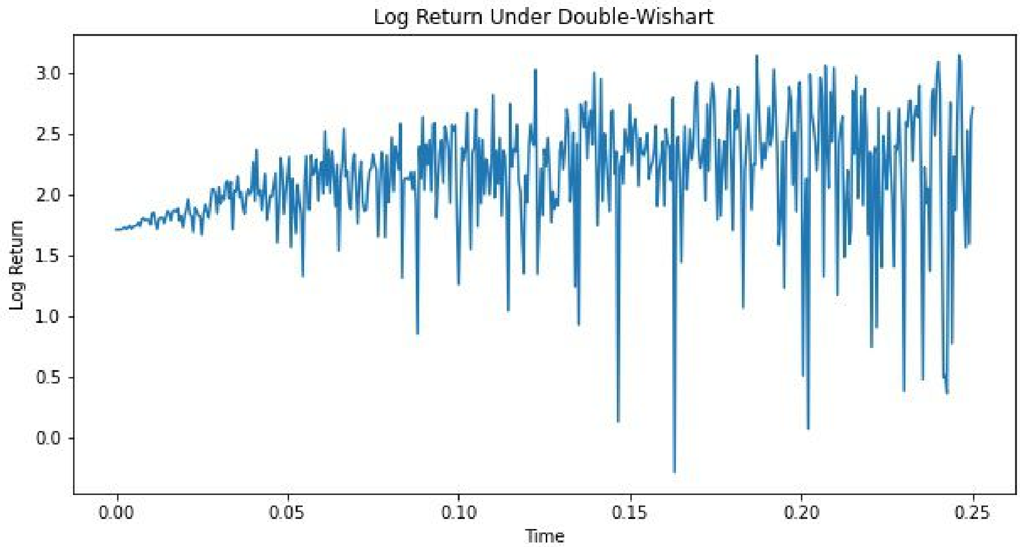

Figure 4.6: Short maturity $T = 3$ months, $r = 0.05$, $K = 189$, log price behavior under double Wishart model with different correlation matrices R_i

Figure 4.6 illustrates the effect of correlation matrices and $N = 500$ on the stock price trend under double Wishart model while considering parameters $\beta_1 = 4, \beta_2 = 3$, while taking the values of,

$$R_1 = \begin{pmatrix} 0.0 & 0.05 \\ 0.05 & 0.6 \end{pmatrix} \qquad and \quad R_2 = \begin{pmatrix} 0.8 & 0.5 \\ 0.45 & 0.65 \end{pmatrix}$$

The model shows stock price trend predictions. We note that the model parameters influence the log-asset returns greatly due to mathematical flexibility of the model. We can observe that there is higher asset returns at the end of the trading period.

Chapter 5

Conclusion and Recommendation

5.1 Conclusion

The Wishart affine diffusion process is a matrix-valued defined square root Bessel process. The main objective of the study was to develop the double Wishart stochastic volatility model in option pricing. This was divided into four specific objectives; that is, to construct the double Wishart stochastic volatility model, to determine the call option price by solving the partial differential equation associated to the dynamic of the stock price under double Wishart model, to discretize the dynamic of stock price under Bi-Wishart affine process and to compare the Bi-Wishart Stochastic Volatility model price behavior with market price.

The study constructed the double Wishart stochastic volatility model, by generalizing the Heston stochastic volatility model into a multifactor form of implied volatilities. The volatilities in the stock diffusion process are considered as positive symmetric matrices and Brownian matrices as real matrices. The stock price dynamic is disturbed by two volatility components which follow multidimensional CIR processes, that is the Wishart processes. In this new constructed volatility model, there are two dependence matrices describing the correlations between the Brownian matrices in the stock process and those in the Wishart affine diffusion processes. This multifactor Heston model is flexible enough to solves the pricing problem for financial derivatives in short or long maturities.

The study derived and solved the partial differential equation associated to the dynamic of the stock price in order to obtain the analytical European call option price formula. The Laplace and Fourier transform methods were used since the Wishart diffusion model preserves the property of analytical tractability, this allows the model to form a closed form solution of conditional characteristic function for the call option price expression; And the use of perturbation techniques confirms the applicability of the model in option pricing.

The study also derived and solved the log-price dynamic under double Wishart volatility model. The numerical solution of the log-price was obtained by applying the corrected Euler-Maruyama discretization scheme. The numerical solution of the log-asset price helps practitioners to analyze stock market price and invest with minimal risks in the financial derivatives since financial derivatives derive their values from stock.

The numerical illustrations shows that the double Wishart stochastic volatility model produces price predictions exhibiting similar behaviors with market prices. While on the log-asset returns, the numerical illustrations shows the effect of model parameters more so the correlation matrices on the log-asset price behavior under double Wishart volatility model in trading. This provides a theoretical framework for investors to study the market behavior over time.

5.2 Recommendation

The study recommends more work to be done on the analysis of the stock distribution function under double Wishart stochastic volatility model, using Malliavin calculus. This will enable to obtain the sensitivity of the market or the Greeks under the model.

5.3 Further research work

The study opens access to future work on exact simulation method, higher order discretization schemes for double Wishart volatility model and also to investigate the behavior of non-diagonal matrix components in the model and parameter estimations.

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Appendix

Infinitesimal generator

Given the n-dimensional, Björk (2009),

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t, S(t) \in \mathbb{R}^n, S(0) = M \in \mathbb{R}^n$$

the partial differential operator $\mathcal{A}$ is called infinitesimal operator of S, defined for any function h(s) with $h \in C^2(\mathbb{R}^n)$, by

$$\mathcal{A}h(t, s) = \sum_{t=1}^{n} \mu_i(t, S(t)) \frac{\partial h(t, S(t))}{\partial S_i}(t, s) + \frac{1}{2} \sum_{j=1}^{n} (\sigma\sigma^*)_{ij} \frac{\partial^2 h}{\partial S_i \partial S_j}$$

where $\mu(t, s)$ is a vector and $\sigma(t, s)$ is a matrix

The correlation structure

The Brownian matrices W_t, Z_t are correlated to give a constant correlated matrix $R \in M_n$, in Da Fonseca et al. (2008), which describes the correlation structure for Z_t

$$Z_t := W_t R^T + B_t \sqrt{\mathbb{I} - RR^T}$$

where $\mathbb{I}$ is identity matrix, T is transpose, and B_t is an independent Brownian motion matrix from W_t. The correlation structure is a Brownian motion

Proof. Z_t is matrix Brownian motion if and only if $a, b \in \mathbb{R}^n$

$$Cov_t(dZ_t a, dZ_t b) = \mathbb{E}_t[(dZ_t a)(dZ_t b)^T] = a^T b \mathbb{I} dt$$

Since

$$\begin{aligned}
Cov_t(dZ_t a, dZ_t b) &= \mathbb{E}_t[(dW_t R^T a + dB_t \sqrt{\mathbb{I} - RR^T a})(dW_t R^T b + dB_t \sqrt{\mathbb{I} - RR^T b})] \\
&= Cov_t(dW_t R^T a, dW_t R^T b) + Cov_t(dB_t \sqrt{\mathbb{I} - RR^T a}, dB_t \sqrt{\mathbb{I} - RR^T b}) \\
&= a^T RR^T b \mathbb{I} dt + a^T (\mathbb{I} - RR^T) b \mathbb{I} dt \\
&= a^T b \mathbb{I} dt
\end{aligned}$$

□

Matrix Girsanov Theorem

Let X be an $\mathbb{R}^{m \times n}$ valued process of the form

$$dX_t = A_t dt + dW_t, \quad 0 \leq t \leq T$$

and $X_0 = 0$ with W being $m \times n$ Brownian motion, A is adopted, $(t, \omega) \longrightarrow A_t(\omega)$ is $B([0, \infty)) \times \mathcal{F}$-measurable
$\mathbb{P}(\int_0^t \parallel a_s \parallel ds < \infty, \forall_t \geq 0) = 1$
and $\parallel . \parallel$ denotes the Hilbert-Schmidt norm for matrices and M process is defined by

$$M_t = exp\{-[\int_0^t \langle A_s, dW_s \rangle + \frac{1}{2} \int_0^t \parallel A_s \parallel^2 ds]\}$$

and a measure $\mathbb{Q}$ on $\mathcal{F}_T$ by $\mathbb{Q}(E) = \int_E M_T d\mathbb{P}, \quad E \in \mathcal{F}_T$
If M is a martingale with respect to the filtration, $(\mathcal{F}_{t \geq 0})$, then $\mathbb{Q}$ is a probability measure on $\mathcal{F}_T$ and X is an $m \times n$ matrix Brownian motion on $(\Omega, (\mathcal{F}_t)_{t \leq T}, \mathbb{Q})$.

Proof. Let Y be vec(X), a be vect(A), W be vec(W), then

$$dY_t = a_t dt + dW_t, \quad 0 \leq t \leq T, \quad Y_0 = 0$$

Y is an mn-dimensional Brownian motion under the probability measure $\mathbb{Q}$ defined on $\mathcal{F}_T$ by $\mathbb{Q}(E) = \int_E M_T d\mathbb{P}, \quad E \in \mathcal{F}_T$

where

$$M_t = exp\{-[\int_0^t a_s, dW_s + \frac{1}{2}\int_0^t \parallel a_s \parallel^2 ds]\}$$

If M is Martingale with filtration $(\mathcal{F}_t)_{t\geq 0}$, since $\parallel a \parallel^2 = Tr(A^T A)$ and $a_t dW_t = Tr(A^T dW_t)$

$$M_t = exp\{-[\int_0^t Tr(A_s^T dW_s) + \frac{1}{2}\int_0^t Tr(A_s^T A_s) ds]\}$$

Lastly note that, Vec(X) being mn-dimensional Brownian motion implies that X is an $m \times n$ matrix Brownian motion. $\square$