STUDY OF W_6-CURVATURE TENSOR ON LORENTZIAN PARA-SASAKIAN MANIFOLDS AND OTHER RELATED MANIFOLDS

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(Applied Mathematics)

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OF

AGRICULTURE AND TECHNOLOGY

2023

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A Thesis Submitted in Full Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Applied Mathematics of the Jomo Kenyatta University of Agriculture and Technology

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2023

DECLARATION

This thesis is my own work and has not been presented for a degree in any other university.

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DEDICATION

To my grandmother and my mother Mrs Elizabeth Mwangi and Mrs Purity Mwangi for your unconditional support. My Wife: Thanks for believing in me and allowing me to further my studies. My daughters Shi and Bri: Your smile every morning gave me hope and encouragement.

ACKNOWLEDGMENT

I must first thank my supervisors Prof. Moindi, Dr. Kiogora and Dr. Esekon and also my external reviewer Prof.G.Pokhariyal (India). It has been my pleasure to interact with them both in and outside the classroom. I must acknowledge the Department of Pure and Applied Mathematics and family members not forgetting my friends who helped me both emotionally and financially.

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LIST OF ABBREVIATIONS AND SYMBOLS

$\bigcap$	Intersection of sets
$\bigcup$	Union of sets
$\forall$	For all
$\in$	Belong to
$>$	greater than
$<$	less than
$\leq$	less or equal to
$\geq$	greater or equal to
$\mapsto$	Maps to
LP	Lorentzian-Para
W_6	Weyl's six
$R(X,Y)$	Ricci tensor
$W(X,Y,Z)$	Weyl's curvature tensor
$R(X,Y,Z)$	Reimannian curvature tensor
$V(X,Y,Z)$	Conformal curvature tensor
$C(X,Y,Z)$	Concircular curvature tensor
$L(X,Y,Z)$	Conharmonic curvature tensor

ABSTRACT

In this research we study W_6 curvature tensor on Lorentzian para Sasakian Manifold and other related Manifold where curvature tensor have been defined on the lines of Weyls projective curvature tensor.It has been shown that distribution (order in which the vectors in question are arranged before being acted upon by the tensor in question) of vector field over the metric potential and matter tensor plays an important role in shaping various physical and geometrical properties of a tensor and the formulation of gravitational waves,reduction of electromagnetic field to a purely electric field,vanishing of the contracted tensor in an Einstein space and cyclic property. The study deals with curvature tensor of Semi-Riemannian and generalized Sasakian space forms admitting Semi-symmetric metric connection. More specifically, we study the geometry of Semi-Rienmannian and generalized Sasakian space forms when they are W_6-flat,W_6-Symmetric,W_6-Semi Symmetric and W_6-Recurrent and compared to result of projectively Semi-Symmetric, Weyl's Semi-Symmetric and concircularly Semi-Symmetric on these spaces. Our main methodology will be use of definitions,manifold transformation and covariant differentiation. This study will add applicable knowledge in mathematics,physics and chemistry in the analysis of curvature tensor to generate equations which describe the nature of forces existing in black holes,spinning planets,Electrons and Protons in atoms.

CHAPTER ONE

INTRODUCTION

Riemannian geometry was first put forward in generality by Bernhard Riemann in the nineteenth century. It deals with a broad range of geometries whose metric properties vary from point to point, including the standard types of Non-Euclidean geometry.

Any smooth manifold admits a Riemannian metric, which often helps to solve problems of differential topology. It also serves as an entry level for the more complicated structure of pseudo-Riemannian manifolds, which (in four dimensions) are the main objects of the theory of general relativity. Other generalizations of Riemannian geometry include Finsler geometry. There exists a close analogy of differential geometry with the mathematical structure of defects in regular crystals.

1.1 Background Information

A topological space is said to be Hausdorff if for each pair of its distinct points,there exists neighborhoods with empty interaction. A locally Euclidean space is a topological space such that each point has a neighborhood homeomorphic to an open subset of Euclidean space. A manifold of dimension n is defined as Hausdorff, locally Euclidean space of dimension n. A topological manifold is said to be differentiable [or smooth], if differentiable structure is defined on it. The manifolds are classified on basis of their structures .Given a curve ‘r’, on smooth manifold , its tangent vector [or simply a vector] is defined as the derivative of differentiable function ‘f’,in direction of ‘r’ at origin. A vector field on smooth manifold ‘m’ is an assignment at tangent vector at each point of M. A geodesic is a curve , such that its vector field is parallel along the given curve A. A Riemannian metric is a positive definite bilinear, which is symmetrical by nature. The Riemannian metric tensor is useful in definition of metric property on differentiable manifolds

, such as angles between vectors ,curvature tensor ,Riemannian curvature tensor, Ricci tensor and geodesic.
A manifold M is said to be Riemannian manifold if a Riemannian metric tensor is defined on its tangent vector space. A pseudo-Riemannian manifold is a pair [M,g] when M is a smooth manifold and g a metric tensor that is not positive definite. A real manifold is differential manifold whose tangent vector space is real vector space. By introducing the complex structure in the manifold we obtain complex manifolds. Due to different structure we can introduce various manifolds. Manifolds are classified as even or odd dimensional according to the dimension of their respective tangent vector space. An odd dimensional manifolds is said to be sasakian if the sasakian structure is defined on it. An odd dimensional manifold is said to be para-sasakian If para-sasakian structure is given on it.
A lorentzian para sasakian manifolds is an atmost –contact structure is given.
If vector space and its dual have same geodesic[are in geodesic correspondence] then the expression of Weyl's curvature is obtained.
New curvature tensor have been obtained by Porkhariyal and Mishra [1970] and pokhariyal [1982a]on basis of Weyl's curvature tensor having different combination of vector field associated to Ricci tensor and metric tensor of Riemannian manifold. Ricci flows are partial differential equation whose variable is a metric tensor of Riemannian manifold. Einstein manifolds are fixed points of Ricci solutions are also used in quasi-einstein manifolds. Ricci solutions on antisymmetric and semisymetric para kenmonsu with respect to W_6 and on semisymmetric and antisymmetric lorentzian para sasakian with respect to W_6 have been studied. Pokhariyal[2] defined some curvature tensors and obtained their physical and geometrical properties of which Matsumoto and Mihai 1998[3] defined Lorentizian Para Sasakian Manifold.
In this thesis properties of W_6-curvature tensor will be studied in LP-Sasaskian manifold and some theorem proved.Since Pokhariyan[1982] defined W_6-curvature tensor,we will break this tensor into symmetric and skew symmetric parts in two ways and various relationship and obtained in part one. Part two we will look at W_6-curvature tensor in LP sasakian maniford and obtain various equation and prove them then part three n-dimentional LP- Sasakian maniford symmetric and

skew symmetric tensor field will be studied, obtain equations and proof them. We shall investigate LP-sasakian maniford in which [1] C=O Where C is weyl conformal curvature tensor. Then study LP-SM in which $C = O$ Where$C$ is weyl conformal curvature tensor.Then study LP-sasakian manifold in which $\bar{C} = 0$ where $\bar{C}$ is a quasi conformal curvature tensor. In both cases its shown that an LP-Sasakian maniford is isometric with unit sphere $S^n(1)$.Conformally flat L-P Sasakian maniford will be studieed and finally consider Weyl- semi symmetric LP Sasakian maniford.

1.2 Definitions and explanations

1.2.1 Differentiable manifold

The basic idea that leads to differentiable manifold is to try to select a family or sub collection of neighbourhood so that the change of cordinates is always given by differentiable functions. As to definitions of differentiable manifold we first look at n-dimensional real space R^n as product space of R. where R is set of real numbers. R^n is obtained taking n-copies of R

Example

R^n = R× R×R×R.........×R;

R n-times where n is any integer greater than zero in R^n each element can be represented by $n - tuples$ so that for every $X \epsilon R^n$

$x = (x_1, x_2, ..., x_n)$, where $x_i \epsilon$R

and i=1,2,3.....n

let us take two arbitrary points in R^n then for every such pair we can define metric on R^n by

d(x,y)=$(\sum_{i=1}^{n}(x_i - y_i)^2)^{\frac{1}{2}}$

then R^n becomes metric space with metric topology as defined above for future discussion R^n is been considered as topological space with M being open subset. The definition of topological manifold M of dimensional n is a Hausdorff space with countable basis of open sets and with further property that each point has a neighborhood homomorphic to open subset of R^n.

Definition

Let v_n be non empty para compact Hausdorff space.Then v_n is said to be n-dimensional topological manifold if every point $x\epsilon$ v_n has open neighborhood U in v_n which is homomorphic to an open subspace of the n-dimensional euclidean space R^n

1.2.2 Differentiable structure

Concept of differentiable structure is studied in this section which form basis of differentiable manifold. First we look at element of differentiable structure namely chart and atlases.

1. **Charts**

   For chart X we mean imbeddingΦ: U $\rightarrow \mathbb{R}^n$ of open subspace U of X into $\mathbb{R}^n$ such that Φ(U) in open subspace of $\mathbb{R}^n$. where U domain of chart.

   Let P_i: $\mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $i = 1, 2, ...n$ denote natural projection defined as

   $P_i(t_1, t_2, ...t_n) = t_i$ for all $t_1, t_2, t_3.....t_n\epsilon\mathbb{R}^n$. Then for every chart Φ:U $\rightarrow$ $\mathbb{R}^n$ on X. The Φ=p_i Φ: U $\rightarrow \mathbb{R}^n$ is known as i^{th} coordinate function in U with respect to the chart Φ and for every xϵU the real number $t_1 = \Phi_i(x)$ is the i^{th} cordinate of point x with respect to chart Φ.

   The chart Φ: U $\rightarrow \mathbb{R}^n$ is called local coordinate system u eor every xϵ U then real numbers $(t_1, t_2......t_n)$=$(\Phi_1......., \Phi_n) = \Phi(x)\epsilon\mathbb{R}^n$ are said to be coordinates of point x with respect to Φ.

Let f: W $\rightarrow \mathbb{R}^n$ denotes function of non-empty space W of $\mathbb{R}^n$ .Then f is said to be

(a) of class c^k,k=1,2,3.......... if and only if f has partial continous derivative of all order r$\leq$k

(b) of all class c^0 if and only if its continous

(c) of class c^∞or smooth if its of class c^k for every for every positive interger.

(d) of class c^w if it is analytic function.

A function f:w$\rightarrow$ $\mathbb{R}^n$ for an open subspace w of $\mathbb{R}^n$ into $\mathbb{R}^n$ is said to be of class c^k if and only if for every i=1,2,3..........n, the composed function $f_i = p_i : W \rightarrow \mathbb{R}^n$ of class C^k

2. **Atlas**

It is a collection of charts $(U_\alpha, \varphi_\alpha)$, $\alpha\epsilon I$ of X satisfying following condition

(a) The domain of the chart in α cover the n-manifold

(b) For any two chart Φ:U $\rightarrow \mathbb{R}^n$ and φ:V $\rightarrow \mathbb{R}^n$ in α with $U \bigcap V \neq \emptyset$ the function

f(Φ, φ):U $\cap$ V $\rightarrow \mathbb{R}^n$

defined by f(Φ, φ)(t)=$\varphi[\Phi^-1(t)]$, for every point t$\epsilon\Phi(U \cap V)$ is of class c^k.

Function f$(\Phi, \varphi)$is know as connecting function of 2 charts Φ and φ.for every x$\epsilon U \cap V$ we have
f$(\Phi, \varphi)[\Phi(x)]$ = $\varphi(x)$, hence f(Φ, φ) is ussually called the transformation for change of local cordinate system from Φ to φ.Thus, we have enough concept to define differentiable structure.

Definition 1.2.2.1

Let c^k be set of all atlases on X of class c^k .if K $\neq$0, this set maybe empty.The relation of X defined by α β if and only if αUβ is an atlas in c^k(x) for any two atlases $\alpha, \beta \epsilon c^k$(x). This is an equivalence relation in c^k(x)into disjoint equivalence classes. Each of equivalence classes is called is called differentiable structure of class c^k in the given n-manifold X. Two atlases α and β are known to be compatible if their union is an atlas.

Definition 1.2.2.2

A differentiable or c^∞ (or smooth) structure on topological manifold M is a family $U = (U_\alpha, \varphi_\alpha)$ of coordinates neighborhood such that the following are satisfied

1. The u_α cover U
2. for any α, β the neighborhood $((U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are c^∞ compatible
3. Any coordinate neighborhood $(U_\alpha, \varphi_\alpha)$ compatible with every conditions $(U_\beta, \varphi_\beta)\epsilon$U is itself U.

1.2.3 Differentiable n-manifold

An n-manifold M together with given differentiable structure A of class c^k on M is called differentiable n manifold .

1.2.4 Diffeomorphism

Let M and N be differentiable n-manifold of class c^k. Let also h≤k. If the function f: M → N is homeomorphism and both f and f^-1(its inverse) are function of class c^h then f is called diffeomorphism.

1.2.5 Tangent vector and tangent spaces

Definition 1.2.5.1

Let p be an element of v_n and let c^∞(p) be set of real valued function that are c^∞ on some neighborhood U of p. A vector$X$ at p is said to be a tangent vector at p if it satisfy the following properties

1. $X \epsilon v_n$ fϵC(p) then $X f \epsilon C^\infty(p)$

2. $X(f+g) = Xf + Xg$:f,g $\epsilon C^{\infty}(p)$
3. $X(fg) = fXg + gXf$
4. $X(af) = aXf$ aϵR

The system consist of

1. The set T_p containing all tangent vectors at P
2. The binary operation + satisfying $(X+Y)f = Xf + Yf$
3. An operation scalar multiplication fXϵ T_p and (aX)f=aXf where aϵ R is a vector space called **Tangent space** to V_n at P. t_p approximates to V_n at P and is n dimensional.

Definition 1.2.5.2

Let m_n be n- dimensional c^{∞} manifold. if $p\epsilon$ M_n and X be c^{∞} real valued function of some neighborhood of P and satisfies

X$(a_1 f_1 + a_2 f_2) = a_1(Xf_1) + a_2(Xf_2)$
and
$X(f_1 f_2) = (Xf_1)f_2 + f_1(Xf_2)$

where $a_1 a_2$ ϵR and $f_1 f_2$ ϵc^{∞} are real valued function at P. Then X is called **tangent vector at point P**.
The set of all tangent vector at point P with operation of addition (+) and multiplication (.) given

(X+Y)f_1=$Xf_1 + Yf_1$
and
$(f_1 X)f_2 = f_1(Xf_2)$

is a vector space and is called tangent space to M_n at P and is denoted by $T_(p)M$ or $t_(p)$.

1.2.6 Vector field

A vector field **X** on set B is a mapping that assigns A to each p in B a vector $\mathbf{X}_p$ in the tangent space T_p. A vector field **X** on B is c^∞ if

1. B is open

2. function Xf at P is c^∞ on $A\bigcap B$, f is being a c^∞ real valued function on A in v_n

1.3 Tensors

1.3.1 Tensor Algebra

In this section tensors are defined as elements of a vector space. The classical notation in definition is used but on most of the work index free notation is used. Let v' be an n-dimensional space and let e_i and $\bar{e_i}$ be two basis of v' then each vector of set $[\bar{e_i}]$ is a linear combination of elements of the set $[e_i]$ $i = 1, 2, 3, 4........, n$ and vice versa.
Let us take

1. $\bar{e^i} = p_i^j e_j$: where $p_i^j, q_i^j \epsilon$ F

2. $e_i = q_i^j \bar{e_j}$: where f is scalar field

Putting 2 in 1 above we shall get
$\bar{e_i} = p_i^k q_k^j \bar{e_j}$,
Since $[\bar{e_j}]$ is linear independent we have
$p_i^j q_j^k \bar{e_j} = \delta_i^k$.
Consequently
(p)(q)= I_n

i.e (p) and (q) are inverse to each other for any vector **X** ϵ V_n. We have

X=$\bar{\mathbf{X}}^k$ $\bar{e_k}$ = $\mathbf{X}^i e_j$,where $\bar{X_k}$ and $\mathbf{X}^i$ are component of **X** with respect to $\bar{e_1}$

and e_i.

From 1 and 2 we have

1. $\bar{X}^k = q_i^k \bar{X}^i$
2. $\bar{X}^k = p_i^k \bar{X}^i$

which are equations of laws of transformations of vector **X**.The vector **X** or any vector in v_n is called contravariant vector of order 1 or tensor type (1,0).

1. **Dual spaces**

   Consider V' consisting of

   (a) a set of all linear scalar function on V' where V is a vector space.

   (b) a binary operation $'' +''$ satisfying $(A+B)(x) = A(x) + B(x)$

   $A, B \epsilon V$;$X \epsilon V'$ then V' is a vector space called dual of V'.

   A bi-linear scalar function T over V×W is a mapping T:V×W→F. i.e T(X,A) are Xϵ V and AϵW is scalar such that

   $$T(aX + bY, cA + dB) = acT(X, A) + adT(X, B) + bcT(X, A) + bdT(Y, B)$$

   where $A, B \epsilon W$; $X, Y \epsilon V$ and $a, b, c, d \epsilon F$.

   Consider a system denoted by $V' \times V'$ or v^2 consisting of

   (a) a set v^{*2} of all bilinear scalar function of $V' \times V'$.

   (b) A binary operation say $'' +''$ satisfying

   $(T+S)(A, B) = T(A, B) + S(A, B)$; $T, S \epsilon V^{*2}$; $A, B \epsilon v_1$

   (c) an operation of scalar multiplication satisfying

   $(rT)(A, B) = rT(A, B)$; $r \epsilon F$; $A, B \epsilon v_1$ then v^2 is vector space called the tensor product of V' with itself.

2. **Tensors** A linear scalar function or form of V' is a linear mapping such that $A(X)$, $X \epsilon V'$ is a scalar and $A(fX + gY) = fA(X) + gA(Y)$;$f, g \epsilon F$ and $X, Y \epsilon V'$.

3. **Higher order tensors**

   We can define mixed tensor as $A^{t_1 t_2 \dots t_s}_{q^1 q^2 \dots\dots q^p}$

   This tensor is then called mixed tensor of contravariant order s and covariant order p. If by interchanging two indices the sign of tensors remain same then we say tensor is symmetric in those indices.

   If sign changes then it is skew-symmetric with respect to two indices. The properties on symmetry and skew-symmetric are independent of the cordinate system. A significant result from transformation laws of tensors is that "if components of a tensor are zero in one coordinate system , then they are zero in any coordinate system".It is this property of tensor that is useful in physical application and when a tensor is defined at all points of a curve in space v_n then we say it consists of a tensor field.

1.3.2 Fundamental operation of tensors

1. **Outer product** The outer product of two tensors is equal to a tensor whose rank is sum of rank of given tensor and it also involves multiplication of components of the tensor.

2. **Contraction** If we set one covariant index of tensor equal to one contravariant index then the resulting tensor will be of rank two less than original tensor. This process is contraction.

3. **Inner multiplication** The outer multiplication of two tensors followed by contraction will result to a tensor known as inner product of given tensor.

4. Addition and subtraction of tensors of same rank and type result in tensor of same rank and type.

   NB: Two operations are defined only for tensor of same rank and type.

   For us to verify whether functions would form components of tensor, we can use transformation laws of which they can be cumbersome so instead we can use the quotient law which is more convenient.

5. **Quotient law** If an inner product of any quantity X with arbitrary tensor is also a tensor then X is also a tensor.

A tensor Q of type $(r, 0)$ is said to be symmetric in h^{th} and k^{th} places if

$$S_{h,k}(Q) = Q$$

and skew symmetric if

$$S_{h,k}(Q) = -Q$$

where $1 \leq h < k \leq r$ and $S_{h,k}$ is a linear mapping which interchanges vector at h^{th} and k^{th} places

Note that it is also applies to a tensor of type (0,1).

1.3.3 Connexion

A connexion ∇ is type preserving mapping assigned to each pair of c^{∞} field (X,Y), a c^{∞} vector fields $\nabla_x P$ such that if **X,Y,Z** are c^{∞} vector field and f is a c^{∞} function then

1. $\nabla_X f = Xf$
2. $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$
3. $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$
4. $\nabla_{fX} Z = f\nabla_X Z$
   and also
   $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z.$

1.3.4 Affine connexion

An Affine Connexion ∇ on manifold m is map T(M)×T(M)........ > T(M),(X,Y).........> $\Delta_x Y$ such that for all $X_i, Y_i \epsilon$T(M),i=1,2 we have

1. $\nabla_{X_1+X_2}(Y) = \nabla_{X_1}Y + \nabla_{X_2}Y$

2. $\nabla_X(Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$

3. $\nabla_X(fY) = (xf)Y + f\nabla_X Y$

4. $\nabla_{fX}(Y) = f\nabla_X Y$

where f is a c^∞ real valued function on M.

Definition 1

A c^∞ vector field X is said to be parallel along a smooth curve $\gamma : t....... > \gamma(t)$ on M (with respect to Δ) iff

$$\Delta_T X = 0$$

along γ where T=d(γ(t))/dt. So if

$$\Delta_T Y = 0$$

everywhere along γ then X is parallel along γ.

Definition 2

A Riemannian structure on M is covariant tensor field of order 2(degree) called Riemannian metric with the following properties

1. g(X,Y)=g(Y,X) for X,Y ϵT(m)

2. $g_X : T_p(m) \times T_p(m)............ > \mathbb{R}^n$: $p\epsilon M$

   where g_X is a non-degenerate bilinear form on $T_p(m) \times T_p(m)$ i.e an inner product on $T_p(M)g_X(Y)$

3. $g_X(Y, X) = 0$; for all $p\epsilon T_p(m)$ if and only if $y = 0$.

4. $g(X, Y) \geq 0$ for all $T(m) : g(X, X) = 0$ which implies $Y = 0$.

Definition 3

A connection Δ is compatible with Riemannian metric g if a parallel transformation along any smooth curve γ in m preserves the inner product. i.e whenever x(t) and y(t) are parallel along γ then <x(t),y(t)> is independent to t.

1.3.5 Lie algebra

Let M be the set of all infinity vector field. The brackets [ ] is defined by mapping

$$[\,]:M \times M \to M$$

Such that for X,Y in M and

$$[X,Y]f = XYf - YXf$$

where f is smooth function for X,Y,Z in M we have

1. $[X,Y] = -[Y,X]$
   skew commutative(symmetric)
2. $[X+Y,Z] = [X,Z] + [Y,Z]$
3. $[fX,gY] = fg[X,Y] + f(XgY) - g(Yf)X$
4. $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$
   The last equation is known as **Jacobi identity**

Example

Let $M_n(R)$ denote the algebra of n×n matrices over R with X,Y denoting the usual matrix product of X and Y. Then

$$[X,Y] = XY - YX$$

the "commuter" of X and Y defines a lie algebra structure on M_n(R) as easily verified. If f is c^∞ on any open set UcM then so is $(XY - YZ)f$ and therefore Z is a c^∞ vector field on M as said.

We may define a product on T(m) using the fact ; namely ,define the product of X and Y by $[X, Y] = XY - YX$

Let us consider the following theorem;

Theorem 1.2.4.1

T(M) with the product $[X, Y]$ is a lie algebra.

Proof

If $\alpha, \beta \epsilon$R and X_1, X_2, Y are c^∞ vector field then it is straight forward to verify that

$$[\alpha X_1 + \beta X_2, Y]f = \alpha[X_1, Y]f + \beta[X_2, Y]f.$$

Thus $[X, Y]$ is linear in the first variable. Since the skew commutative $[X, Y] = -[Y, X]$ is clear from definition. We see linearity in the first variable implies linearity in the second variable. Therefore $[X, Y]$ is bilinear and skew commutative. There remains Jacobi identity which follows if we evaluate

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]$$

and apply to c^∞ function f. We obtain

$$[X, [Y, Z]]f = X(([Y, Z])f) - [Y, Z](Xf)$$
$$= X(Y(Zf)) - X(Z(Yf)) - Y(Z(Xf)) + Z(Y(Xf))$$

Permutting cyclically and adding establishes the identity.

1.3.6 Lie bracket and covariant Derivatives

Let **X,Y,Z** be c^∞ vector field on m_n. Then lie brackets is the mapping

$$[\,] : M_n \times M_n .... > M_n$$

Such that

$$[XY]f = X(Yf) - Y(Xf)$$

f being c^∞ function.

This satisfies the following properties;

1. $[X, Y](f_1 + f_2) = [X, Y]f_1 + [X, Y]f_2$

2. $[X, Y](f_1.f_2) = f_1[X, Y]f_2 + f_2[X, Y]f_1$

3. $[X, Y] + [Y, X] = 0$

4. $[X + Y, Z] = [X, Z] + [Y, Z]$

5. $[f_1X, f_2Y] = f_1f_2[X, Y] + f_1(Xf_2)Y - f_2(Yf_1)X$

and further it satisfies Jacobi identity i.e

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

The covariant derivative Δ is a mapping $\Delta : T^r_s .... > T^r_{s+1}$ such that $\Delta_p(a_1, ...a_r, X_1..........X_{s+1})$=$(\Delta_{s+1}p)(a_1, ..., a_r, X_1, ..., X_s)$

where p ϵT^r_s: $a_, a_2, ...a_r \epsilon T_{(p)}$ and $X_1, X_2.........X_s \epsilon T^*_{(p)}$

1.3.7 Lie bracket and Exterior Derivatives

let X be c^∞ vector field on an open set A. Lie derivative via X is a type preserving mapping

$$L_X : T^r_s \longrightarrow T^r_s$$

such that

1. $L_X f = Xf$, where f is c^∞

2. $L_X a = 0$, aϵ R
   $L_X Y = [X, Y], Y \epsilon T_{(p)}$

$$(L_X A)(Y) = X(A(Y)) - A(X, Y)$$

where $A \epsilon T^*_p$
and
$(L_X p)(A_1.......A_r, X_1...........X_s) = X(p(A_1, A_2......, X_s), ...p(A_1, ..., [X, X_s])$

where pϵT^r_s. Let v_p be c^∞. p forms an open set A. Then the mapping

$$d : V_p \longleftrightarrow V_{p+1}$$

given by

$$(df)(X) = Xf,$$

where X$\epsilon T(p)$ and f is c^∞ function on A thus from above it is clear now we can define the following as

(dA)$(X_1, ...X_{p+1}) = X_1(A(X_2, ..., X_{p+1}))+X_2(A(X_1, X_3, ...X_{p+1}))+$ ....

$$+X_{p+1}(A(X_1, X_2......X_p)) - A([X_1, X_2]X_3........X_{p+1})$$
$$-A([X_1, X_3], X_2, X_4.............X_{p+1}) - A([X_2, X_3], X_1, X_4..........Xp + 1)...........$$

for all arbitrary c^∞ fields $X\epsilon V$ and $A\epsilon V_p$ is called **exterior derivative**

1.3.8 Torsion tensor of a connexion

The torsion tensor of a connexion ∇ is defined as a vector valued bilinear function T which assigns to each pair of c^∞ vector **X** and **Y** with domain A, a c^∞ vector field $T(X, Y)$ with domain A and is given by
$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$.

A connexion is said to be symmetric if torsion tensor vanishes(Torsion=0) and a connexion ∇ is said to be Riemannian if

1. $T(X, Y) = 0$
   and

2. $\nabla_X g = 0$

1.3.9 Curvature Tensor

Consider a connexion D. Then the operator K_{XY} defined by

$$K_{XY} = [D_X, D_Y] - D_{[X,Y]}$$

is called the curvature operator.
Then curvature K of the connexion D

$$K(X, Y, Z) = K_{XY}Z$$

which can be written as

$$K(X, Y, Z) = [D_X, D_Y]Z - D_{X,Y}Z$$
$$= D_X D_Y Z - D_Y D_X Z - D_{X,Y}Z$$

where k is vector valued function. The curvature tensor K satisfies two identities

1. $K(X, Y, Z) + K(Y, Z, X) + K(Z, X, Y) = 0$

   and

2. $(D_X K)(Y, Z, W) + (D_Y K)(Z, X, W) + (D_z K)(X, Y, W) = 0$

   which are called Bianch first and second identities respectively.

   Proof

   Let D be a symmetric connexion then

   $K(X, Y, Z) + K(Y, Z, X) + K(Z, X, Y)$

   $=D_X D_Y Z - D_Y D_X Z - D_[X, Y]Z + D_Y D_Z X - D_Z D_Y X - D_[Z, Y]X + D_Z D_X Y - D_X D_Z Y$

   $- D_[X, Z]Y$

   $=D_X[Y, Z] - D_{[Y,Z]}X + D_Y[X, Z] - D_{[Z,X]}Y + D_Z[X, Y] - D_{[X,Y]}Z$

   $=[[X, Y], Z]+[[Y, X], Z]+[-[Z, X], Y]=0$ by Jacobi identities.Thus we have

   $K(X, Y, Z) + K(Y, Z, X) + K(Z, X, Y) = 0.$

   Similarly we also get

   $(D_X K)(Y, Z, W) + (D_Y K)(Z, X, W) + (D_Z K)(X, Y, W) = 0$

   Let us put $K'(X, Y, Z, W) = g(K(X, Y, Z), W)$.

   It can be noted that K' satisfy the following conditions

   (a) is skew symmetric in the first two slot as well as in the last two slot,

(b) satisfy first and the second banachi identity,

(c) symetric in two pair of slot,
i.e (XY) and (Z,W).

1.3.10 Difference tensor of two connexion

Consider a smooth manifold M and let D and $\bar{D}$ be two connexion on M for two field **X** and **Y** on M . We define difference tensor as

B(X,Y)=$\bar{D}_X$Y-D_XY

Linearlity of B slot is trivial result from properties of connexion. Let us consider slot 2 and Let f be c^∞ on domain X and Y. Then

$$B(X, fY) = (Xf)Y + fD_XY - (Xf)Y - f\bar{D}_XY = fB(X, Y).$$

If we decompose $B(X, Y)$ into symmetric and skew symmetric pieces we have

$$B(X, Y) = S(X, Y) + A(X, Y)$$

where

$$S(X, Y) = \tfrac{1}{2}[B(X, Y) + B(Y, X)]$$

is symmetric part
and

$$A(X, Y) = \tfrac{1}{2}[B(X, Y) - B(Y, X)]$$

is skew symmetric part.
Then we can express A in terms of torsion tensors T and $\bar{T}$ of connexion D and $\bar{D}$ respectively as for

$$\begin{aligned} 2A(X, Y) = B(X, Y) - B(Y, X)\bar{D}_XY - D_XY - \bar{D}_YX - D_YX \\ = \bar{T}(X, Y) - T(X, Y) + [X, Y] - [X, Y] \\ = \bar{T}(X, Y) - T(X, Y). \end{aligned}$$

Let the two connexion D and $\bar{D}$ be related in V_n by

$$\bar{D}_X Y = D_X Y + A(X)Y + A(Y)X.$$

Where A is a 1-form and **X** and **Y** are vector fields in V_n. Then D and $\bar{D}$ are said to be projectively related.

1.3.11 Ricci Tensor

The tensor defined by $Ric(Y, Z) = (C', K)(Y, Z)$ is called tensor of type (0,2), where C' denote contraction. Its symmetric tensor

$$Ric(X, Y) = Ric(Y, X),$$

the Ricci tensor of type (1,1) is defined by

$$g(R(X), Y) = Ric(X, Y),$$

the scalar curvature r is defined by

$$C_1' R = def \text{ r}$$

1.3.12 The weyl projective curvature tensor

This is defined by

$$W(X, Y, Z) = K(X, Y, Z) + \tfrac{1}{n+1}[L(X, Y) - L(Y, X)]Z + \tfrac{n}{n^2-1}[L(X, Y)Y - L(Y, Z)Y] + \tfrac{1}{n^2-1}[L(Z, X)Y - L(Z, Y)X]$$

Where

$$L(X, Y, Z, T) = R(X, Y, Z, T) - \tfrac{1}{n-2}[g(Y, Z)Ric(X, T) - g(X, Z)Ric(Y, T) + g(X, T)Ric(Y, Z) - g(Y, T)Ric(X, Z)]$$

is conharmonic curvature tensor.

It can be shown that symmetric connexion which are projectively related have the same curvature tensor.

The weyl's projective curvature tensor w satisfies the following properties:-

1. $W(X, Y, Z) = -W(Y, X, Z)$
2. $(trW)(X, Y) = (C_3' W)(X, Y) = 0$
3. $W(X, Y, Z) + W(Y, Z, X) + W(Z, X, Y) = 0$

1.4 Statement of the Problem

The aim of this study is to study Geometric and Physical properties of W_6–curvature tensors on lorentzian para sasakian manifold on semi-Riemannian generalized sasakian space forms endowed with semi symmetric metric connection in general theory of Relativity.

1.5 Objectives

1.5.1 General Objective

Study geometric and physical properties of W_6 curvature tensor on Lorentzian Para-Sasakian manifold and other related manifolds with emphasis on producing new geometric results having physical meaning.

1.5.2 Specific Objectives

The specific objectives of the study are:-

(a) To determine the geometric properties of W_6 curvature tensor on Sasakian space i.e flatness, cyclic,symmetric and semi-symmetric properties.

(b) To analyze the geometric properties of W_6 curvature tensor on Lorentzian-Para sasakian manifold i.e flatness, cyclic,symmetric and semi-symmetric properties and relation with other tensors.

(c) To determine physical properties of W_6 curvature tensor on Lorentzian-Para sasakian manifold i.e flatness,irrotational and conservative and also obtain results on Einstein Lorentzian para-Sasakian Manifold.

(d) To determine the relativistic significance of W_6-curvature tensor on LP-Sasakian Space.

1.6 Justifications

This study will add to the existing applicable knowledge in mathematics, physics and chemistry in the analysis of curvature tensors to generate equations which describe the nature of forces existing in: Black holes, that is regions of space time from which gravity prevents anything ,including light,from escaping. Spinning planets and their shapes as they transverse their orbits in space. Electrons and protons in an atom and the shapes of atomic orbitals.Bermuda triangle i.e region in the western part of the north Atlantic ocean where a number of air crafts and ships are said to have disappeared under mysterious circumstances.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

A set of new curvature tensors was defined on the line of Weyl tensor by Pokhariyal and Mishra (1970), and Pokhariyal (1982); to study Relativistic significance of curvature tensors. The Weyl's projective curvature tensor was defined on the basis of geodesic correspondence due to a particular type of distribution of vector fields and metric potential contained in it.

These new tensors were not necessary due to its in variance in two spaces V_n and $\bar{v}_n$, but showed that the "distribution" (order in which the vectors in question are arranged before being acted upon by the tensor in question), of vector field over the metric potentials and matter tensors plays an important role in shaping the various physical and geometrical properties of a tensor, viz the formulation of gravitational waves, reduction of electromagnetic field to a purely electric field, vanishing of the contracted tensor in an Einstein space and the cyclic property. The relativistic significance of Weyl's projective curvature tensor has also been explored by Singh et al., (1965).

The concept of curvature is very common in Differential Geometry. In this work we try to show its evolution along history, as well as some of its applications. This survey is limited both in number of topics dealt with and the extent with which they are treated. Some of them, like minimal submanifolds,Kahler manifolds or Morse Theory are completely omitted. Though in an implicit way, the curvature is already present in the Fifth Euclid's Postulate.

However it does not emerge explicitly in Mathematics until the appearance of the theory of curves and surfaces in the euclidean space. Taking basically the work of Gauss's as a starting point, Riemann defines the curvature tensor in an abstract and rigorous way.

The introduction of multilinear algebra in the second half of the 19^{th} century allowed a better analytic formulation and its further development. It is worth stressing its fundamental role in the development of the Theory of Relativity. Besides, the curvature is present, not only in Riemannian manifolds, but also in many other geometric structures, like homogeneous and symmetric spaces, the theory of connections, characteristic classes, etc. Having in mind that the physical world cannot be explained in a linear way, the curvature also arises in the theories of Mathematical Physics. Likewise, it seems interesting to note its presence in applied sciences, like Estereology.

The world we live in, and the mathematical models describing the geometrical and physical objects, cannot be properly explained with only linear constructions. In order to obtain an adequate description of Nature, it is necessary to introduce models in which the relations between parameters go beyond the linear ones. That is why the concept of curvature appears in a natural way.

According to Osserman, the notion of curvature is one of the main concepts of differential geometry; it could be argued that it is indeed the central one, by distinguishing the geometrical core of the subject from those aspects that are analytic, algebraic or topological. According to Berger, the curvature is the most important invariant of Riemann's Geometry, and the most natural one. In Gromov writes: "the curvature tensor of a Riemann manifold is a little monster of multilinear algebra whose complete geometrical meaning remains obscure".

Thus, for Riemannian manifolds without additional structures, the curvature is a complicated magnitude. Its properties in the simplest manifolds were the first to be studied. Later, the situation in a more general manifold could be compared to that in the simplest ones. The latter are often called "model spaces". The curvature also plays a fundamental role in Physics and other experimental sciences.

For example, the force required to move an object at a constant speed is, according to Newton's laws, a constant multiple of the curvature of its trajectory; and the movement of a body in a gravitational field is determined, according to Einstein, by the curvature of the space-time.

2.2 On Manifolds

The relativistic significance of weyl's projective curvature tensor has been explored by Singh,Radhakrishoa and Sharan and many other authors.Differential geometry builds on the following disciplines as its prerequisites: the analytic geometry of Descartes and Calculus (Leibniz 1646-1716, Newton (1645-1727)
The first isolated results on curves and surfaces date from the eighteenth century. Gauss (1777-1785) transformed the theory of surfaces into its modern systematic mound. A foundation of intrinsic geometry independent of embedding was given by Riemann (1826- 1866). Riemann also dropped the restriction to 3 dimensions. Around the 20th century the tensor calculus was developed as a powerful tool for differential geometry by Ricci and Levi Civita together with the general relativity of Einstein (1879-1955) .This signaled the development of other geometric structures in differentiable manifolds.
Calculus of variatons is closely linked to differential geometry. In 1918, Finsler wrote his dissertation in which this connection was used to construct a new metric differential geometry that has since developed considerably.
The chief aim of tensor calculus has been the investigation of relations which remain valid when we change from one coordinate system to another. This makes tensor calculus desirable as a mathematical tool for developing physical laws. Tensors also allow complex expressions to be represented in a compact way and thus simplify the mechanics of development of theory.
Mishra and Pokhariyal (1970) studied various geometric and physical properties of the curvature tensors. They defined a new tensor W2 based on the Weyl projective curvature tensor and investigated its relativistic significance. Based on the same Weyl Projective curvature tensor, Pokhariyal (1971-1982) has defined other tensors W3, W4, W5, W6, W7, W8 and W9. Some of the physical and

geometric properties of these and other tensors in different manifolds have been studied by Pokhariyal,Muindi,Njori and Others. The results obtained in these manifolds are reviewed in the following sections.

2.2.1 On Sasakian Manifolds

Mishra (1970) studied some properties of the Riemann curvature tensors as well as the Weyl projective curvature tensor and the conharmonic curvature tensors in Sasakian manifolds. He showed that a concircular symmetric Sasakian manifold is a manifold of constant curvature and that the concircular and Riemann curvature tensors do not vanish in a Sasakian manifold.

Pokhariyal (1971) studied the properties of the Bochner curvature tensor in the Kahler Manifold, in particular the relationship between conharmonic recurrence, Bochner recurrence and Ricci recurrence in the Kahler manifold.

B. Sinha and J. P. Sinha (1975) studied the properties of a Sasakian manifold with constant F-holomorphic sectional curvature in connection with Ricci tensor and a parallel field of null planes.

Sinha and Sharma (1979) have studied the structure induced on the hypersurface of a Sasakian manifold and subsequently its infinites variations in various modes and remarked that the discussions could be used to study infinitesimal deformations of the universe (with unified field structure) as hypersurface of five dimensional Sasakian manifold.

Matsumoto (1980) investigated curvature preserving transformations of P-Sasakian manifolds. He showed that each curvature preserving infinitesimal transformation is necessarily an infinitesimal automorphism.

Khan (2006) studied Einstein Projective Sasakian Manifold. He showed that a projectively flat Sasakian manifold is an Einstein Manifold and is a manifold of constant curvature. He also showed that if an Einstein Sasakian Manifold is projectively flat, then it is locally Isometric with the unit sphere Sn(l).

Tripathi and Dwivedi (2008) studied the structure of some classes of K-contact Manifolds. They showed that a (2m + 1) dimensional Sasakian Manifold is quasi projectively flat if and only if it is locally isometric to the unit sphere S'2n+1(l).

De, Jun and Gazi (2008) studied Sasakian Manifolds with quasi-conformal curvature tensor. It is proved that a Quasi-conformally flat Sasakian manifold is an n-Einstein Manifold and is necessarily locally isomorhpic to the unit sphere. They also showed that a compact orientable quasi-conformally flat Sasakian manifold cannot admit a non-isometric conformal transformation. They also proved that an n—dimensional Sasakian Manifold (n ¿ 3) is quasi conformally flat if and only if it is Quasi conformally semi-symmetric.

2.2.2 On LP— Sasakian Manifolds

Matsumoto and Mihai (1988) studied some properties of a transformation in a LP-Sasakian manifold and came up with some new results.
Ki and Kim (1990) Studied Sasakian manifolds whose C-Bodmer curvature tensor vanishes. They showed that such a manifold has constant scalar curvature and at most three constant Ricci curvatures provided that the square of the length of the Ricci tensor is constant.
Gebarowski (1991) has studied conformal collineations in a LP-Sasakian manifold and showed that any conformal collineation of an LP- Sasakian manifold is necessarily a conformal motion.
Pokhariyal (1996) studied the symmetric and skew symmetric properties of the W_1 tensor in LP-Sasakian manifolds and showed W_1 symmetric LP-Sasakian manifold is not W1 flat. These tensors have been used to explain some Physical and geometric behaviors of the four dimensional space time, Kahler, Sasakian and other complex manifolds.
Tarafdar and Bhatacharya (2000) studied LP-Sasakian manifolds with conformally flat and quasi conformally flat curvature tensor and showed in both cases that manifold is isometric to the unit sphere Sn(l).
Ozgur (2003) considered u-conformally flat, u-conharmonically flat and u-projectively flat LP—Sasakian manifolds. He showed that a u-conformally flat LP—Sasakian manifold is an n-Einstein manifold and further, that a u-conharmonically flat LP—Sasakian manifold is an n-Einstein manifold with zero scalar curvature and that a u-projectively flat LP—Sasakian manifold is an Einstein manifold with

constant scalar curvature.

Murathan et al (2006) studied certain classifications of the LP Sasakian Manifold which satisfy the conditions P.C = 0,P.Z — Z.P = 0, and P.Z + Z.P = 0, where P is the v-Weyl projective curvature tensor,Z is the concircular curvature tensor and C is the Weyl conformal curvature tensor.

Venkatesh and Bagewadi (2008) studied concircular u-recurrent LP—Sasakian manifold and showed that such a manifold is an Einstein manifold. They also showed that a u-recurrent LP—Sasakian manifold having nonzero constant sectional curvature is locally concircular u-symmetric.

2.2.3 On W_6-Curvature Tensor

Weyl introduced the notion of weyls tensor which he defined as a measure of the curvature of space time or more generally a pseudo-Riemannian manifold.

Kimetto (2015) studied W_6 –curvature tensor in K-contact Riemannian manifold and proved W6-flat,semi symmetric, symmetric and W_6 semi symmetric K-contact Riemannian Manifold is a W_6 –flat manifold.

Since W_6 has been defined by various author i.e Pokhariyal and Kimetto studied W_6 on K-contact manifold,there is still a gap on studying W_6 curvature tensor on the other manifold including lorentzian para sasakian manifold and Kenmutso manifold which will now led to our study.

CHAPTER THREE

RIEMANNIAN AND COMPLEX MANIFOLD

The methodology used is by use of covariant differentiation, manifold transformation and making use of the definition of W_6-curvature tensor,the physical and geometric properties shall be studied. The symmetry as well as cyclic properties of the tensor shall studied and the result obtained shall be combined with the result of the other tensors.

3.1 Riemannian manifold

3.1.1 Riemannian manifold

Let T be a tangent space at point P of differentiable manifold V_n. Let us single out in V_n a real valued bilinear symmetric and positive definite function g on the ordered pair of tangent vectors at each point P on V_n. Then V_n is called Riemannian manifold and g is called the metric tensor of V_n.

We thus have two vector **X,Y** of T at P. Such that

1. $g(X, Y)\epsilon$R,

2. $g(X, Y) = g(Y, X)$; g is symmetric,

3. $g(aX + bY, Z) = ag(X, Z) + bg(Y, Z)$,

4. $g(X, Y) >0$,

5. if X,Y are c^∞ fields with domain A then g(X,Y) at P is a c^∞ function on A. Let (G(X)(Y))=g(X,Y)then G is non singular and Let G^{-1} be the inverse map. Then $G^{-1}OG = GOG^{-1} = I_n$.

The angle θ between two vectors is defined by

$$||X||||Y||cos\theta = g(X, Y)$$

where

$$||X|| = g(X, X).$$

Thus two vectors **X** and **Y** are perpendicular if g(X,Y)=0

A connexion D is said to be Riemannian if it satisfies

1. D is symmetric

   $D_X Y - D_Y X = [X, X]$

2. g is covariant constant with respect to D which gives

   $$D_X g = 0$$

   and

   $$g(D_X Y, Z) + g(Y, D_X Z) = X(g(Y, Z))$$

   An affine connexion D is said to be metric if $D_X g = 0$.

   The Riemannian manifold is said to be Einsteinian manifold if

$Ric(X, Y) = \frac{r}{n} g(X, Y)$.

A Riemannian manifold is said to be flat if

$K(X, Y, Z) = 0$.

The torsion tensor Tor is vector valued linear dunction and is defined by

$$Tor(X, Y) = D_X Y - D_Y X - [X, Y]$$

If torsion vanishes then connexion is said to be torsion free or symmetric.

3.1.2 Riemannian curvature tensor

The curvature tensor with respect to the Riemannian connexion is called the Riemannian curvature tensor.

Let K be Riemannian curvature tensor given by

$$K(X, Y, Z) = (D_X D_Y - D_Y D_X - D_{X,Y})Z$$

3.1.3 Riemannian connexion

Let **X** and **W** be vectors as P in R_n and $\bar{D}$ be connexion. Let Y and Z be c^∞ field about P and let f be a c^∞real valued function about P. Then we have

1. $\bar{D}_X(Y + Z) = \bar{D}_X Y + \bar{D}_Y Z$
2. $\bar{D}_{X+W}(Y) = \bar{D}_X Y + \bar{D}_W Y$
3. $\bar{D}_{f(p)X} Y = f \bar{D}_X Y$
4. $\bar{D}_X(fY) = (Xf)Y_p + f_{(p)} \bar{D}_X Y$ **(3.1.3.1)**

Using $\bar{D}$ we can define parallel vector field along a curve and geodesics. Let r be a c^∞ curve on R_n with tangent T and let Y be an R_n vector field that is parallel along r if $\bar{D}_r Y$=0 along r.
The curve γ is geodesic if $\bar{D}_r T$=0 i.e if its tangent T is parallel along γ. Thus generalization of a definition of covariant differentiation or connexion on c^∞ manifold is clear i.e we merely need the existence of operator D which satisfies all four condition of above properties (3.1.3.1) listed for $\bar{D}$ and assigns to c^∞ vectors field **X** and **Y** with domain A,a c^∞ field $D_X Y$ on A.

NB: Connexion can be more than one on a given manifold.

Let us denote dot or inner product of X and Y tangent to R_n by
$< X, Y >= \sum_{i=1}^{n} X_i Y_i$.
If X,Y and Z are c^∞ field then $< X, Y >$ is also c^∞ field and if A is the domain of X,Y and X Y are c^∞ fields then one easily checks that

$$\bar{D}_Y Z - \bar{D}_Z Y = [Y, Z] \text{ on A}$$

(3.1.3.2)

and

$$X_p < Y, Z >=< \bar{D}_X Y, Z > P+ < Y, \bar{D}_X Z > P$$

(3.1.3.3)

for every X at p in A. From above we can generalize and fix some terms.
A Riemannian manifold is a c^∞ manifold M on which one has singled out a c^∞ real valued , bilinear , symmetric and positive define function$<, >$ on ordered pair of tangent vector at each point. Thus if X,Y and Z are in M_P then X,Y are real numbers and $<, >$ satisfies the following properties

1. $< X, Y >=< Y, X >$ symmetric

2. $< X + Y, Z >=< X, Z > + < Y, Z >$ bilinear
   $< aX, Y >= a < X, Y >$ for all aϵr

3. $< X, X >$ >0 for all X$\neq$ 0

4. If X and Y are c^{∞} fields with domain A then $< X, Y > p =< X_p, Y_p >$ is ac^{∞} function on A when (3) is placed by (3*) (non singular). $< X, Y >= 0$ for all X implies Y=0 then M is semi-Riemannian (or pseudo Riemannian) manifold. In either case the function is inner product, metric tensor, the Riemannian metric or infinite semi metric on M not the topological metric function.
   If D is c^{∞} connexion in semi-Riemannian manifold M then D is Riemannian connexion if it satisfies (3.1.3.2) and (3.1.3.3)

3.1.4 Properties of Riemannian curvature tensor

The Riemannian curvature tensors is linear over the ring of smooth function are coefficient on the right hand side and satisfy the following properties

1. $K(X, Y, Z) = -K(Y, X, Z)$

2. if f is smooth function then $K(fX, Y, Z) = -fK(Y, X, Z)$ where D is Riemannian connexion.
   Let us define

$$'K(X, Y, Z, W) = g(K(X, Y, Z), W)$$

   Then 'K is skew symmetric in the first two slots and the last two slots. The Riemannian curvature tensor K satisfies Binanchi's first identity and Bianchi's second identity.

Curvature Tensors

In a Riemannian manifold the weyl projective tensor reduces to

$$W(X, Y, Z) = K(X, Y, Z) + \tfrac{1}{n-1}[Ric(X, Z)Y - Ric(Y, Z)X]$$

Conformal curvature tensor

The tensor V defined by

$$V(X, Y, Z) = K(X, Y, Z) + \tfrac{1}{n-2}[Ric(Y, Z)X - Ric(X, Z)Y - g(X, Z)RY + g(Y, Z)RX] + \tfrac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y]$$

is same for manifolds in conformal correspondence.

This tensor is called the conformal curvature tensor.

A manifold whose conformal curvature tensor vanishes at every point is said to be conformaly flat.A conformal curvature V satisfies Bianchi's first identity

$$V(X,Y,Z) + V(Y,Z,X) + V(Z,X,Y) = 0$$

Concircular curvature tensor

The concircular curvature tensor is defined by

$$C(X,Y,Z) = K(X,Y,Z) - \tfrac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]$$

Conharmonic curvature tensor

The conharmonic curvature tensor is defined by

$$L(X,Y,Z) =$$
$$K(X,Y,Z) - \tfrac{1}{n-2}[Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)RX - g(X,Z)RY]$$

Riemannian curvature

Let **X** and **Y** be unit tangent vector at a point P of Riemannian manifold V_n. These vectors determine a pensil of direction at P if the unit vectors along that direction are U. Then

$U = fX + gY$

and

$f^2 + g^2 = 1$,where $f, g\epsilon F$

The geodesic of V_n whose unit tangent vector are U, generate a two dimensional sub manifold of the tangent manifold T at P.

The gaussian curvature $K(X,Y)$ at P of this two dimensional sub manifold was defined by Riemannian as sectional curvature at P of V_n in direction of X and Y. Thus

$K = -K(X,Y,X,Y)/||X||^2||Y||^2[1 - cos^2\theta]$;

where θ is the angle between X and Y.

A necessary and sufficient condition on V_n to be locally flat in the neighbourhood U of a point P is that Riemannian curvature of V_n at P vanishes.

If the Riemannian curvature K of V_n at P of the direction X and Y then

$K(X, Y, Z) = K[g(Y, Z)X - g(X, Z)Y]$. **(3.1.4.1)**

Contracting we get

1. $Ric = K(n-1)g$

2. $R = [K(n-1)]n$

contracting (2) we get

$R = Kn(n-1)$ **(3.1.4.2)**

Hence a Riemannian manifold of constant curvature is an Einstein manifold.

Schur's theorem

If a Riemannian curvature K of V_n at every point of neighborhood U of V_n is independent of the direction choosen then K is constant throughout the neighborhood U provided $n > 2$.

Proof

Putting (3.1.4.1) and (3.1.4.2) together we get $W = 0$

Conversely, if $W = 0$

$$K(X, Y, Z) = \tfrac{1}{n-1}[g(Y, Z)RX - g(X, Z)RY]$$

Contracting equation we get

$$Ric(Y, Z) = \tfrac{r}{n} g(Y, Z)$$

which is sometimes expressed as $RX = \frac{r}{n}X$ and putting the two equation into the first one we get

$$K(X, Y, Z) = \tfrac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y]$$

which shows that a manifold is constant Riemaniian curvature. Hence a necessary and sufficient condition for the manifold V_n to be of constant Riemannian curvature is not the weyl projective curvature tensor to vanish identically throughout V_n.

Similarly the conformal curvature tensor vanishes from manifold with constant Riemannian curvature.

3.1.5 Difference tensor of two connexions

Consider a smooth manifold M and let D and $\bar{D}$ be two connexion on M for two fields X and Y on M. We define difference tensor by

$$B(X,Y) = \bar{D}_X\text{Y-}D_XY$$

Linearlity of B slot is a trivial result from properties of connexion and let us consider slot 2.

Let f be c^{∞} on domains X and Y. Then

$$B(X,fY) = (Xf)Y + fD_XY - (Xf)Y - f\bar{D}_XY = fB(X,Y).$$

If we decomposed $B(X,Y)$ into symmetric and skew symmetric pieces we have;

$$B(X,Y) = S(X,Y) + A(X,Y)$$

where

$$S(X,Y) = \tfrac{1}{2}[B(X,Y) + B(Y,X)] \text{ (symmetric part).}$$

and

$$A(X,Y) = \tfrac{1}{2}[B(X,Y) - B(Y,X)]\text{(skew symmetric part)}$$

Then we can express A in terms of torsion tensors T and $\bar{T}$ of connexion D and $\bar{D}$ respectively as for

$$2A(X,Y) = B(X,Y) - B(Y,X) = \bar{D}_XY - D_XY - \bar{D}_YX - D_YX =$$
$$\bar{T}(X,Y) - T(X,Y) + [X,Y] - [X,Y] = \bar{T}(X,Y) - T(X,Y)$$

Theorem 3.1.5.1

The following statements are equivalent;

1. The connexion D and $\bar{D}$ have the same geodesic,
2. $B(X,X) = 0$ for all vector **X**,
3. $S = 0$,

4. $B = A$.

   Proof omitted

Theorem 3.1.5.2

The connexion D and $\bar{D}$ are equal if they have the same geodesic and the same torsion tensors.

Proof

That the first part implies the second is trival. Conversely ,if the geodesic are the same then $S = 0$ and if the torsion tensors are equal then $A = 0$; hence $B = 0$ and $D = \bar{D}$

3.1.6 Riemannian curvature tensor

The curvature tensor of connexion D is a linear transformation valued tensor R that assigns to each pair of vector **X** and **Y** at linear transformation R(X,Y) of M_n into itself. We define $R(X,Y)Z$ by imbedding X,Y and Z in c^∞ field about M and setting

$$R(X,Y)Z = (D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z)_m$$

(3.1.6.1)

Hence we notice that $R(X,Y) = -R(Y,X)$ and if f is c^∞ then

$$R(fX,Y)Z = fD_X D_Y Z - (Yf)D_X Z - fD_Y D_X Z + (Yf)D_X Z - fD_{X,Y} Z = fR(X,Y)Z$$

(3.1.6.2)

Also

$$R(X,Y)(fZ) =$$
$$D_X(Yf)X + fD_Y Z - D_Y((Xf)Z - fD_X Z) - ([X,Y]f)Z - fD_{[X,Y]} Z$$
$$= (XY)(fZ) + (Yf)D_X Z + (Xf)D_Y Z + fD_X D_Y Z - (YX)(fZ) - (Xf)D_Y Z - (Yf)D_X Z - fD_Y D_X Z - ([X,Y]f)Z - fD_{[X,Y]} Z$$
$$= fR(X,Y)Z$$

(3.1.6.3)

The linearlity of R(X,Y)Z with respect to addition (in each slot) is trivial to check. The curvature of symmetric linear connexion on M satisfies Bianchi identities

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

(3.1.6.4)

for all vector **X,Y,Z** in M for which the left hand side is defined to prove this,we recall that for symmetric connexion

$$D_A B - D_B A = [a, b]$$

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = D_X[Y,Z] + D_Y[Z,X] + D_Z[X,Y] -$$
$$D_{[Y,Z]}X - D_{[Z,X]}Y - D_{[X,Y]}Z = [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

by Jacobi identity.

if we define

$$Z < X, Y > = < D_Z X, Y > + < X, D_Z Y >$$

(3.1.6.5)

for all vector **X,Y,Z** with common domain, then using above definition we can define a 4 rank covariant tensor called Riemann-Christoffel curvature tensor as

$$K(X,Y,Z,W) = < X, R(Z,W)Y >$$

(3.1.6.6)

for all X,Y,Z and W is same domain.

Thus from the above definition the following result arises

1. $K(X,Y,Z,W) = -K(Y,X,Z,W)$,
2. $K(X,Y,Z,W) = -K(X,Y,W,Z)$,
3. $K(X,Y,Z,W) = K(Z,W,X,Y)$ **(3.1.6.7)**

Theorem 3.1.6.1

Let M be differential i.e Riemannian n-maniford. Then there is unique torsion free connexion D such that D on M satisfies

1. D is symmetric
2. $D_X g = 0$ for all X ϵT(M).
3. Parallel translation preserves inner products. This connexion is called the Riemannian or Levi-civita connexion.

Proof

Uniqueness from proposition (3.1.6.4) gives

$$Xg(Y,Z) - g(D_X Y, Z) - g(Y, D_X Z) = 0$$

and since D is torsion free yields

1. $Xg(Y,Z) = g(D_X Y, Z) = g([X,Y],Z) + g(Y, D_X Z)$
2. Cyclically permuting X,Y and Z we get
   $Yg(Z,X) = g(D_X Y, X) + g([Y,Z],X) + g(Z, D_Y X)$,
3. $Zg(X,Y) = g(D_X Z, Y) + g([Z,X],Y) + g(X, D_Z Y)$.

Substituting (1) from (2)+(3) we get

$$2g(D_Z Y, X) = -X < Y, Z > +Y < Z, X > +Z < X, Y > - < [Z,X], Y > - < [Y,Z], X > + < [X,Y], Z >.$$

The right hand of this last expression does not involve D, so we have a formula for $g(D_Z Y)$ on X. As $<,>$ is non singular i.e
the map $T(M)$........$T^*(m)$ induced by g being an isomorphism and X is arbitrary, $D_Z Y$ is uniquely determined so D is unique.

If we define $D_Z Y$ by using the expression $2g$ above then D is a connexion and we find condition (i) and (ii) of the theorem satisfied.

3.2 Complex Manifold

3.2.1 Complex Manifold

An even dimensional differentiable manifold V_n; $n = 2m$ which can be endowed by a system of complex coordinate neighborhood (U, α) in such a way that in the intersection $U \bigcap U'$ of two complex coordinate patches (U, α), (U', α'), α' are complex analytic function of α is called a complex manifold.

3.2.2 Almost complex manifold

If on an even dimensional differentiable manifold V_n;n=2m of differentiability class C^{r+1} there exist a vector valued real linear function f of differentiability class C^r satisfying

1. $f^2 + I_n = 0$
   which implies
2. $\bar{\bar{X}} + X = 0$ where $\bar{X} = fX$

Then V_n is said to be an almost complex manifold and f is said to be an almost complex structure V_n. We shall apply the following notation

1. The operation of pre-multiplying a vector by f will be known as barring the vector.
2. We shall denote T(V_n) the set of c^∞ vector field of V_n.
3. In this and what follows the equation containing $X, Y, Z...$ hold for arbitrary vectors fields $X, Y, Z...\epsilon T(V_n)$ unless explicitly stated otherwise.

CHAPTER FOUR

THE WELYS CURVATURE TENSOR

In differential geometry,Hermann Weyl introduced Weyl curvature tensor which is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. Just like the Riemann curvature tensor, Weyl expresses the tidal force that a body feels when moving along a geodesic.

The Weyl tensor differs from the Riemann curvature tensor in that it does not convey information on how the volume of the body changes, but rather only how the shape of the body is distorted by the tidal force.

The Ricci curvature, or trace component of the Riemann tensor contains precisely the information about how volumes change in the presence of tidal forces, so the Weyl tensor is the traceless component of the Riemann tensor. This tensor has the same symmetries as the Riemann tensor, but satisfies the extra condition that it is trace-free: metric contraction on any pair of indices yields zero. It is obtained from the Riemann tensor by subtracting a tensor that is a linear expression in the Ricci tensor.

In general relativity, the Weyl curvature is the only part of the curvature that exists in free space—a solution of the vacuum Einstein equation—and it governs the propagation of gravitational waves through regions of space devoid of matter. More generally, the Weyl curvature is the only component of curvature for Ricci-flat manifolds and always governs the characteristics of the field equations of an Einstein manifold.

In dimensions 2 and 3 the Weyl curvature tensor vanishes identically. In dimensions ≥ 4, the Weyl curvature is generally nonzero. If the Weyl tensor vanishes in dimension ≥ 4, then the metric is locally conformally flat: there exists a local coordinate system in which the metric tensor is proportional to a constant tensor. This fact was a key component of Nordström's theory of gravitation, which was a precursor of general relativity.

Weyl's projective curvature tensoris given by

$W_6(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)Ric(Y,U) - g(X,U)Ric(Y,Z).$

The other tensors have been defined by (Pokhariyal and mishra)(1970,1982) are given by

$W_1(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(X,U)Ric(Y,Z) - g(Y,U)Ric(X,Z)],$

$W_2(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)Ric(Y,U) - g(Y,Z)Ric(X,U)],$

$W_3(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(Y,Z)Ric(X,U) - g(Y,U)Ric(X,Z)],$

$W_4(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)Ric(Y,U) - g(X,Y)Ric(Z,U)],$

$W_5(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)Ric(Y,U) - g(Y,U)Ric(X,Z)],$

The W_6 which is studied in this thesis is given by

$W_6(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)Y - XRic(Y,Z)]$

$W_6(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z)]$

which is from the following definition

Definition

In a (2n+1) dimensional Riemannian manifold M the τ-curvature tensor is given by Tripathi and Gupta(2011)

$T(X,Y)Z = a_0R(X,Y)Z + a_1S(Y,Z)X + a_2S(X,Z)Y + a_3S(X,Y)Z + a_4g(Y,Z)QX + a_5g(X,Z)QY + a_6g(X,Y)QZ + a_7r(g(Y,Z)X - g(X,Z)Y)$ (4.1)

where R is curvature tensor, S is ricci tensor, Q is Ricci operator and r is scalar curvature.

The Weyls curvature tensor is W_6 curvature tensor if in the equation (4.1)

$$a_0 = 1, a_1 = -a_6 = \frac{-1}{2n}, a_2 = a_3 = a_5 = a_5 = a_7 = 0$$

Thus

$$W_6(X,Y)Z = R(X,Y)Z - \frac{1}{2n}Ric(Y,Z)X + \frac{1}{2n}g(X,Y)QZ,$$

$$W_6(X,Y)Z = R(X,Y)Z + \frac{1}{2n}[g(X,Y)QZ - XRic(Y,Z)],$$

$$g(W_6(X,Y,Z),U) = g(R(X,Y,Z),U) + \frac{1}{2n}[g(X,Y)g(QZ,U) - g(X,U)Ric(Y,Z)],$$

$$'W_6(X,Y,Z,U) =' R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z)],$$

$$'W_6(X,Y,Z,U) =' R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z)].$$

CHAPTER FIVE

SASAKIAN AND LP-SASAKIAN MANIFOLD

5.1 Sasakian Manifold

5.1.1 Introduction

Let (M, F, T, A, g) be $(2n+1)$-dimensional almost contact metric manifold consisting of a (1,1) tensor field F, a covariant (C^{∞}) vector field T, a C^{∞} 1 form A and a Riemannian metric g which satisfies

$A(T) = -1,$ **(5.1.1.1)**

$\bar{\bar{X}} = X + A(X)T$ where $\bar{X} = f(X)$. Then $A(\bar{X}) = 0$ and $\bar{T} = 0,$ **(5.1.1.2)**

$g(\bar{X}, \bar{Y}) = g(X, Y) + A(X)A(Y),$ **(5.1.1.3)**

$g(X, T) = A(X)$ and $g(x, \bar{Y}) = -g(\bar{X}, Y),$

where X and Y are arbitrary vector fields on M, **(5.1.1.4)**

$dA(X, Y) = g(X, \bar{Y})$ and almost contact metric manifold is K-contact metric manifold if **(5.1.1.5)**

$\Delta_x T = -\bar{X}$ where Δ is levi-civita connection. **(5.1.1.6)**

An almost contact metric manifold is K-contact metric manifold in a sasakian manifold if

$(\Delta_x F)Y = g(X, Y)T - A(Y)X.$ **(5.1.1.7)**

Δ denote operator covariant differentiation with respect to the Riemannian metric g.

A sasakian manifold is a K-Contact but the converse is only true if dimension is 3.

A contact metric manifold is sasakian if and only if

$R(X, Y)T = A(Y)X - A(X)Y.$ **(5.1.1.8)**

In sasakian Manifold (M, F, T, A, g) we easily get

$R(T, X)Y = g(X, Y)T - A(Y)X.$ **(5.1.1.9)**

Generally in $(2n+1)$-dimensional sasakian manifold with structure (F,T,A,g) we have

$rank(F) = n - 1.$ **(5.1.1.10)**

$$R'(X,Y,Z,T) = g(R(X,Y),Z,U) = g(g(Y,Z)X - g(X,Z)Y,U)$$
$$= g(Y,Z)g(X,U) - g(X,Z)g(Y,U) = g(Y,Z)A(X) - g(X,Z)A(Y),$$

(5.1.1.11)

where R is Riemannian Curvature tensor.

$Ric(X,T) = (n-1)A(X).$ **(5.1.1.12)**

$S(X,Y) = g(QX,Y) = (n-1)g(X,Y) = Ric(X,Y),$

where Q is Ricci operator and Ric(X,Y) denote Ricci tensor. **(5.1.1.13)**

5.1.2 W_6-Curvature Tensor in Sasakian Manifold

Pokhariyah(1982) have defined a tensor

$W_6'(X,Y,Z,U) = R'(X,Y,Z,U) + \frac{1}{n-1}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z)]$

OR

$W_6(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)Y - S(Y,Z)X].$ **(5.1.2.1)**

Definition 5.1.2.1

A Sasakian manifold M is said to be flat if the Riemannian curvature tensor vanishes identically i.e $R(X,Y)Z = 0$.

Definition 5.1.2.2

A Sasakian manifold M is said to be W_6-flat if W_6 curvature tensor vanishes identically i.e $W_6(X,Y)Z = 0$

Theorem 5.1.2.3

A W_6-flat Sasakian manifold is a flat manifold.

Proof

If the sasakian space is flat then $W_6 = 0$ in

$$W_6'(X,Y,Z,U) = R'(X,Y,Z,U) + \frac{1}{n-1}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z)]$$

OR

$$W_6(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)Y - S(Y,Z)X]$$

Therefore, if LP-Sasakian manifold M is W_6-flat then

$$0 = R'(X,Y,Z,U) + \frac{1}{n-1}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z)]$$

and

$$R'(X,Y,Z,U) = \frac{1}{n-1}[g(X,U)Ric(Y,Z) - g(X,Y)Ric(Z,U)]$$

From (5.1.1.13) where $Ric(X,Y) = (n-1)g(X,Y)$

we replace in our equation and we get

$$R'(X,Y,Z,U) = \frac{1}{n-1}[g(X,U)(n-1)g(Y,Z) - g(X,Y)(n-1)g(Z,U)]$$
$$R'(X,Y,Z,U) = \frac{n-1}{n-1}[g(X,U)g(Y,Z) - g(X,Y)g(Z,U)]$$
$$R'(X,Y,Z,U) = [g(X,U)g(Y,Z) - g(X,Y)g(Z,U)] =$$
$$[g(Y,Z)A(X) - g(X,Y)A(Z)]$$

But in Sasakian manifold we have

$$R'(X,Y,Z,U) = [g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]$$

Thus, for this to hold, we must have

$$R'(X,Y,Z,U) = 0$$

since

$$[g(X,U)g(Y,Z) - g(X,Y)g(Z,U)] \neq [g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]$$

Hence the theorem proved.

5.1.3 A W_6-Semi-Symmetric Sasakian Manifold

De and Guha (1998) gave definition of Semi-Symmetric as $R(X,Y)R(Z,U)V = 0$.

Definition 5.1.3.1

A Sasakian manifold M is said to be W_6 flat if $R(X,Y)W_6(Z,U)V = 0$.

Theorem 5.1.3.2

A W_6-Semi-Symmetric Sasakian Manifold is a W_6-flat manifold.

Proof

If the sasakian space is W_6-Semi-Symmetric then $R(X,Y)W_6(Z,U)V = 0$ in

$$R(X,Y)W_6(Z,U)V = g(Y,W_6(Z,U)V)X - g(X,W_6(Z,U)V)Y$$
$$0 = g(Y,W_6(Z,U)V)X - g(X,W_6(Z,U)V)Y$$
$$0 = W_6'(Y,Z,U,V)X - W_6'(X,Z,U,V)Y$$
$$0 = g(W_6(Y,Z,U,V)X,T) - g(W_6(X,Z,U,V)Y,T)$$
$$0 = W_6'(Y,Z,U,V)A(X) - W_6'(X,Z,U,V)A(Y)$$

But since $A(X) \neq 0$ and $A(Y) \neq 0$ then it follows that $W_6'(Y,Z,U,V) = 0$ and $W_6'(X,Z,U,V) = 0$ hence the theorem is proved.

5.1.4 A W_6-Symmetric Sasakian Manifold

Definition 5.1.4.1

A Sasakian manifold M is said to be W_6-Symmetric if

$$\Delta_u W_6(X,Y)Z = W_6'(U,X,Y)Z = 0$$

Theorem 5.1.4.2

A W_6-Symmetric Sasakian Manifold is a W_6-flat manifold.

Proof

If the LP-Sasakian space is a W_6- symmetric then it follows that

$\Delta_u W_6(X,Y)Z = R(X,Y)W_6(Z,U)V - W_6(R(X,Y)Z,U)V - W_6(Z,R(X,Y)U)V - W_6(Z,U)R(X,Y)V = 0.$ **(5.1.4.1)**

Computing each of the above four term and subject them to same conditions we have ;

$$R(X,Y)W_6(Z,U)V = g(Y,W_6(Z,U)V)X - g(X,W_6(Z,U)V)Y$$
$$= W_6'(Y,Z,U,V)X - W_6'(X,Z,U,V)Y$$
$$g(R(X,Y),W_6(Z,U)V,T) = g(W_6'(Y,Z,U,V)X,T) - g(W_6'(X,Z,U,V)Y,T)$$
$$= W_6'(Y,Z,U,V)A(X) - W_6'(X,Z,U,V)A(Y)$$

(5.1.4.2)

Recall

$$W_6(X,Y)Z = R(X,Y)Z + \tfrac{1}{n-1}[g(X,Z)Y - S(Y,Z)X]$$

Now

$$W_6(R(X,Y)Z,U)V =$$
$$R(R(X,Y)Z,U)V + \frac{1}{n-1}[g(R(X,Y)Z,V)U - S(U,V)R(X,Y)Z]$$

But

$$S(U,V) = (n-1)g(U,V) \text{ and } R'(X,Y,Z,U) = g(R(X,Y)Z,U)$$

SO

$$W_6(R(X,Y)Z,U)V =$$
$$R(R(X,Y)Z,U)V + \frac{1}{n-1}[R'(X,Y,Z,V)U - (n-1)g(U,V)R(X,Y)Z]$$
$$= g(U,V)R(X,Y)Z - g(R(X,Y)Z,V)U + \frac{1}{n-1}R'(X,Y,Z,V)U - g(U,V)R(X,Y)Z$$
$$= \frac{1}{n-1}R'(X,Y,Z,V)U - g(R(X,Y)Z,V)U$$
$$= \frac{1}{n-1}R'(X,Y,Z,V)U - R'(X,Y,Z,V)U$$

(5.1.4.3)

Again

$$W_6(Z,R(X,Y)U)V =$$
$$R(Z,R(X,Y)U)V + \frac{1}{n-1}[g(Z,V)R(X,Y)U - S(R(X,Y)U,V)Z]$$
$$= g(R(X,Y)U,V)Z - g(Z,V)R(X,Y)U + \frac{1}{n-1}[g(Z,V)R(X,Y)U - (n-1)g(R(X,Y)U,V)Z]$$
$$= \frac{1}{n-1}[g(Z,V)R(X,Y)U - g(Z,V)R(X,Y)U]$$

(5.1.4.4)

Also

$$W_6(Z,U)R(X,Y)V =$$
$$R(Z,U)R(X,Y)V + \frac{1}{n-1}[g(Z,R(X,Y)V)U - S(U,R(X,Y)V)Z]$$
$$=$$
$$g(U,R(X,Y)V)Z - g(Z,R(X,Y)V)U + \frac{1}{n-1}g(Z,R(X,Y)V)U - g(U,R(X,Y)V)Z]$$
$$= \frac{1}{n-1}g(Z,R(X,Y)V)U - g(Z,R(X,Y)V)U$$

(5.1.4.5)

Next in (5.1.4.1) we substitute (5.1.4.2), (5.1.4.3), (5.1.4.4) and (5.1.4.5) and we have

$$\Delta_u W_6(X,Y)Z =$$
$$W_6'(Y,Z,U,V)A(X) - W_6'(X,Z,U,V)A(Y) - \frac{1}{n-1}R'(X,Y,Z,V)U -$$
$$R'(X,Y,Z,V)U + \frac{1}{n-1}g(Z,V)R(X,Y)U - g(Z,V)R(X,Y)U +$$
$$\frac{1}{n-1}g(Z,V)R(X,Y)U - g(Z,V)R(X,Y)U$$
$$= W_6'(Y,Z,U,V)A(X) - W_6'(X,Z,U,V)A(Y) - \frac{2-n}{n-1}[R'(X,Y,Z,V)U +$$
$$g(Z,V)R(X,Y)U + g(Z,U)R(X,Y)V] = 0$$

But since

$$\Delta_x W_6'(Y,Z,U,V) = 0 \text{ and } g(Z,U) \neq g(Z,V) \neq 0$$

it implies that $R'(X,Y,Z,V) = 0$.
Thus follows the theorem.

5.2 Lp-sasakian manifold

5.2.1 Introduction

Matsumoto and Mihai (1988) have introduced the notion of Lorentzian para sasakian and studied certain transformation. Later Sasaki [16] introduced certain structures which are closely related to almost contact and later studied almost contact manifold.
An n-dimensional differentiable manifold M is said to be lorentzian para sasakian manifold if it admits a (1,1) tensor field F,covariant (C^∞) vector field T, C^∞ 1 form A and lorentzian metric g which satisfies

$A(T) = -1,$ **(5.2.1.1)**

$\bar{\bar{X}} = X + A(X)T$, where $\bar{X} = F(X)$. **(5.2.1.2)**

$g(\bar{X},\bar{Y}) = g(X,Y) + A(X)A(Y)$. **(5.2.1.3)**

$g(X,T) = A(X)$. **(5.2.1.4)**

$(\Delta_x F)(Y) = [g(X,Y) + A(X)A(Y)]T + [X + A(X)T]A(Y)$ where $\Delta_x T = \bar{X}$. **(5.2.1.5)**

Δ denote operator covariant differentiation with respect to the lorentzian metric g

In LP-Sasakian manifold M with structure (F,T,A,g) then

$\bar{T} = \Phi$, $A(\bar{X}) = \Phi$ **(5.2.1.6)**

$rank(F) = n - 1$ **(5.2.1.7)**

Further more if we put

$'F(X, Y) = g(\bar{X}, Y)$ **(5.2.1.8)**

then sensor field $'F(X, Y)$ is symmetric in X and Y.

In an n-dimensional LP-Sasakian manifold with structure (F, T, A, g) we have

$'R(X, Y, Z, T) = g(Y, Z)A(X) - g(X, Z)A(Y).$ **(5.2.1.9)**

$Ric(X, T) = (n - 1)A(X).$ **(5.2.1.10)**

$'R(X, Y, \bar{Z}, \bar{U}) =' R(X, Y, Z, U) + 2A(Z)[g(X, u)A(Y) - g(Y, U)A(X)] + 2A(U)[A(Y)g(X, Z) - A(X)g(Y, Z)] + 'F(Y, U)'F(X, Z) - 'F(X, U)'F(Y, Z) + g(Y, Z)g(X, U) - g(X, Z)g(Y, U),$

where $R(X, Y, Z)$ denote curvature and $Ric(X, Y)$ denote Ricci tensor. **(5.2.1.11)**

5.2.2 W_6-Curvature Tensor in LP-Sasakian Manifold

Pokhariyah (1982) have defined a tensor

$$W_6'(X, Y, Z, U) = R'(X, Y, Z, U) + \tfrac{1}{n-1}[g(X, Y)Ric(Z, U) - g(X, U)Ric(Y, Z)]$$

OR

$$W_6(X, Y)Z = R(X, Y)Z + \tfrac{1}{n-1}[g(X, Z)Y - S(Y, Z)X]$$

(5.2.2)

Definition 5.2.2.1

A LP-Sasakian manifold M is said to be flat if the Riemannian curvature tensor vanishes identically i.e $R(X, Y)Z = 0$.

Definition 5.2.2.2

A LP-Sasakian manifold M is said to be W_6-flat if W_6 curvature tensor vanishes identically i.e $W_6(X, Y)Z = 0$

Theorem 5.2.2.3

A W_6-flat LP-Sasakian manifold is a flat manifold.

Proof

If our hypothesis is true then $W_6 = 0$ in

$$W_6'(X, Y, Z, U) = R'(X, Y, Z, U) + \tfrac{1}{n-1}[g(X, Y)Ric(Z, U) - g(X, U)Ric(Y, Z)]$$

OR

$$W_6(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)Y - S(Y,Z)X]$$

Therefore, if LP-Sasakian manifold M is W_6-flat then

$$0 = R'(X,Y,Z,U) + \frac{1}{n-1}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z)]$$
$$R'(X,Y,Z,U) = \frac{1}{n-1}[g(X,U)Ric(Y,Z) - g(X,Y)Ric(Z,U)]$$

From (5.2.1.10) $Ric(X,Y) = (n-1)g(X,Y)$

$$R'(X,Y,Z,U) = \frac{1}{n-1}[g(X,U)(n-1)g(Y,Z) - g(X,Y)(n-1)g(Z,U)]$$
$$R'(X,Y,Z,U) = \frac{n-1}{n-1}[g(X,U)g(Y,Z) - g(X,Y)g(Z,U)]$$
$$R'(X,Y,Z,U) = [g(X,U)g(Y,Z) - g(X,Y)g(Z,U)] =$$
$$[g(Y,Z)A(X) - g(X,Y)A(Z)]$$

But in LP-Sasakian manifold we have $R'(X,Y,Z,U) = [g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]$

Thus, for this to hold, we must have $R'(X,Y,Z,U) = 0$

since $[g(X,U)g(Y,Z) - g(X,Y)g(Z,U)] \neq [g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]$

hence the theorem proved.

5.2.3 W_6-Semi-Symmetric LP-Sasakian Manifold

De and Guha (1992) gave definition of Semi-Symmetric as $R(X,Y)R(Z,U)V = 0$.

Definition 5.2.3.1

A LP-Sasakian manifold M is said to be W_6flat if $R(X,Y)W_6(Z,U)V = 0$.

Theorem 5.2.3.2

W_6-Semi-Symmetric LP-Sasakian Manifold is a W_6-flat manifold.

Proof

If our hypothesis is true then $R(X,Y)W_6(Z,U)V = 0$ in

$$R(X,Y)W_6(Z,U)V = g(Y,W_6(Z,U)V)X - g(X,W_6(Z,U)V)Y$$
$$0 = g(Y,W_6(Z,U)V)X - g(X,W_6(Z,U)V)Y$$
$$0 = W_6'(Y,Z,U,V)X - W_6'(X,Z,U,V)Y$$
$$0 = g(W_6(Y,Z,U,V)X,T) - g(W_6(X,Z,U,V)Y,T)$$
$$0 = W_6'(Y,Z,U,V)A(X) - W_6'(X,Z,U,V)A(Y)$$

But since $A(X) \neq 0$ and $A(Y) \neq 0$ then it follows that $W'_6(Y,Z,U,V) = 0$ and $W'_6(X,Z,U,V) = 0$ hence the theorem proved.

5.2.4 W_6-Symmetric LP-Sasakian Manifold

Definition 5.2.4.1

A LP-Sasakian manifold M is said to be W_6-Symmetric if

$\Delta_u W_6(X,Y)Z = W'_6(U,X,Y)Z = 0.$

Theorem 5.2.4.2

A W_6-Symmetric LP-Sasakian Manifold is a W_6-flat manifold.

Proof

If the LP-Sasakian space is a W_6- symmetric then it follows

$\Delta_u W_6(X,Y)Z = R(X,Y)W_6(Z,U)V - W_6(R(X,Y)Z,U)V - W_6(Z,R(X,Y)U)V - W_6(Z,U)R(X,Y)V = 0.$ **(5.2.4.1)**

Computing each of above four term and subject them to same conditions we have ;

$$\begin{aligned} R(X,Y)W_6(Z,U)V &= g(Y,W_6(Z,U)V)X - g(X,W_6(Z,U)V)Y \\ &= W'_6(Y,Z,U,V)X - W'_6(X,Z,U,V)Y \\ g(R(X,Y),W_6(Z,U)V,T) &= g(W'_6(Y,Z,U,V)X,T) - g(W'_6(X,Z,U,V)Y,T) \\ &= W'_6(Y,Z,U,V)A(X) - W'_6(X,Z,U,V)A(Y) \end{aligned}$$

(5.2.4.2)

Again if

$$W_6(X,Y)Z = R(X,Y)Z + \tfrac{1}{n-1}[g(X,Z)Y - Ric(Y,Z)X] \; W_6(R(X,Y)Z,U)V = R(R(X,Y)Z,U)V + \tfrac{1}{n-1}[g(R(X,Y)Z,V)U - Ric(U,V)R(X,Y)Z]$$

But

$$S(U,V) = (n-1)g(U,V) \text{ and } R'(X,Y,Z,U) = g(R(X,Y)Z,U)$$

So

$$W_6(R(X,Y)Z,U)V =$$
$$R(R(X,Y)Z,U)V + \frac{1}{n-1}[R'(X,Y,Z,V)U - (n-1)g(U,V)R(X,Y)Z]$$
$$= g(U,V)R(X,Y)Z - g(R(X,Y)Z,V)U + \frac{1}{n-1}R'(X,Y,Z,V)U - g(U,V)R(X,Y)Z$$
$$= \frac{1}{n-1}R'(X,Y,Z,V)U - g(R(X,Y)Z,V)U$$
$$= \frac{1}{n-1}R'(X,Y,Z,V)U - R'(X,Y,Z,V)U$$

(5.2.4.3)

Also

$$W_6(Z,R(X,Y)U)V =$$
$$R(Z,R(X,Y)U)V + \frac{1}{n-1}[g(Z,V)R(X,Y)U - S(R(X,Y)U,V)Z]$$
$$= g(R(X,Y)U,V)Z - g(Z,V)R(X,Y)U + \frac{1}{n-1}[g(Z,V)R(X,Y)U - (n-1)g(R(X,Y)U,V)Z]$$
$$= \frac{1}{n-1}[g(Z,V)R(X,Y)U - g(Z,V)R(X,Y)U]$$

(5.2.4.4)

Also

$$W_6(Z,U)R(X,Y)V =$$
$$R(Z,U)R(X,Y)V + \frac{1}{n-1}[g(Z,R(X,Y)V)U - S(U,R(X,Y)V)Z]$$
$$=$$
$$g(U,R(X,Y)V)Z - g(Z,R(X,Y)V)U + \frac{1}{n-1}g(Z,R(X,Y)V)U - g(U,R(X,Y)V)Z]$$
$$= \frac{1}{n-1}g(Z,R(X,Y)V)U - g(Z,R(X,Y)V)U$$

(5.2.4.5)

Next in (5.2.4.1) we put (5.2.4.2), (5.2.4.3), (5.2.4.4) and (5.2.4.4) and we have

$$\Delta_u W_6(X,Y)Z =$$
$$W_6'(Y,Z,U,V)A(X) - W_6'(X,Z,U,V)A(Y) - \frac{1}{n-1}R'(X,Y,Z,V)U -$$
$$R'(X,Y,Z,V)U + \frac{1}{n-1}g(Z,V)R(X,Y)U - g(Z,V)R(X,Y)U +$$
$$\frac{1}{n-1}g(Z,V)R(X,Y)U - g(Z,V)R(X,Y)U$$
$$= W_6'(Y,Z,U,V)A(X) - W_6'(X,Z,U,V)A(Y) - \frac{2-n}{n-1}[R'(X,Y,Z,V)U +$$
$$g(Z,V)R(X,Y)U + g(Z,U)R(X,Y)V] = 0$$

But since $\Delta_x W_6'(Y,Z,U,V) = 0$ and $g(Z,U) \neq g(Z,V) \neq 0$.

It implies that $R'(X,Y,Z,V) = 0$.

Thus follows the theorem.

5.2.5 Flat W_6-Curvature Tensor in a Lorentzian Para-Sasakian Manifolds

If the Lorentzian para-Sasakian manifold has flat W_6-curvature tensor, then

$g(W_6(X,Y)Z,U)) = 0.$ **(5.2.5.1)**

Substituting (5.2.5.1) in our W_6 equation (5.2.2) we have

$$R(X,Y)Z + \frac{1}{n-1}[g(X,Z)Y - S(Y,Z)X] = 0$$
$$g(R(X,Y)Z,F(W)) + \frac{1}{n-1}[g(X,Z)QY - S(Y,Z)QX] = 0$$

$g(R(X,Y)Z,F(W)) + \frac{1}{n-1}[g(X,Z)S(Y,F(W)) - S(Y,Z)S(X,F(W)] = 0$ **(5.2.5.2)**

Putting Y = Z = T in (5.2.5.2), we have

(5.2.5.3) $g(R(X,T)T,F(W)) + \frac{1}{n-1}[g(X,T)S(T,F(W)) - S(T,T)S(X,F(W)] = 0$

Using (5.2.1.10) and (5.2.1.9) in (5.2.4.3), we get

$-g(X,F(W)) + \frac{1}{n-1}[S(X,F(W)] = 0$ **(5.2.5.4)**

On simplification, we have

$(n-1)g(X,F(W)) = S(X,F(W))$ **(5.2.5.5)**

Replacing by W with F(W) we have

$(n-1)g(X,W) = S(X,W)$ **(5.2.5.6)**

On contracting the above relation, we obtain

$r = n(n-1)$ **(5.2.5.7)**

Thus we can state the following theorem:

Theorem 5.2.5.1 If a Lorentzian para-Sasakian manifold the W_6-curvature tensor is flat then it is an Einstein manifold and also a space of constant scalar curvature.

5.2.6 Irrotational W_6-Curvature Tensor in an LP-Sasakian Manifold

Definition 5.2.6.1

Let ∇ be a Riemannian connection. Then the rotation (curl) of W_6-curvature tensor in a Lorentzian para-Sasakian manifold M is defined as

$RotW_6 = (\nabla_U W_6)(X,Y)Z + (\nabla_X W_6)(U,Y)Z + (\nabla_Y W_6)(X,U)Z - (\nabla_Z W_6)(X,Y)U.$

. **(5.2.6.1)**

Using Bianchi's second identity for Riemannian connection ∇, (5.2.6.1) becomes

$RotW_6 = -(\nabla_Z W_6)(X,Y)U.$ **(5.2.6.2)**

If W_6 is Irrotational then $RotW_6 = 0$ and then

$(\nabla_Z W_6)(X,Y)U = 0,$ **(5.2.6.3)**

which will give

$\nabla_Z(W_6(X,Y)U) = W_6(\nabla_Z X,Y)U + W_6(X,\nabla_Z Y)U + W_6(X,Y)\nabla_Z U.$ **(5.2.6.4)**

Replacing U with T in (5.2.6.4) we will have

$\nabla_Z(W_6(X,Y)T) = W_6(\nabla_Z X,Y)T + W_6(X,\nabla_Z Y)T + W_6(X,Y)\nabla_Z T.$ **(5.2.6.5)**

If we substitute T=Z in (5.2.1)

$W_6(X,Y)T = R(X,Y)T + \frac{1}{n-1}[g(X,T)Y - S(Y,T)X].$ **(5.2.6.6)**

Using (5.2.1.2), (5.2.1.10) and (5.2.1.4) we have

$W_6(X,Y)T = k[A(X)Y].$ **(5.2.6.7)**

Where $k = \frac{2-n}{n-1}.$ **(5.2.6.8)**

Using (5.2.6.7) in (5.2.6.5),we obtain

$W_6(X,Y) = k[S(Y,Z)X - g(X,Z)Y].$ **(5.2.6.9)**

Also equation (5.2.1) and (5.2.6.9) gives

$S(Y,Z) = Ric(Y,Z) = (n-1)g(Y,Z).$ **(5.2.6.10)**

which gives

$r = n(n-1)$ **(5.2.6.11)**

In consequence of (5.2.1.1),(5.2.6.8),(5.2.6.9),(5.2.6.10) and (5.2.6.11)

$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y.$ **(5.2.6.12)**

hence we can state:

Theorem 5.2.6.1 If the W_6-curvature tensor In a Lorentzian para-Sasakian manifold is irrotational then the manifold is a space of constant curvature.

5.2.7 Conservative W_6-Curvature Tensor in an LP-Sasakian Manifold

Differentiating (5.2.1) with respect to U , we have

$(\nabla_U W_6)(X,Y)Z = (\nabla_U R)(X,Y)Z + \frac{1}{n-1}[g(X,Z)(\nabla_U Q)Y - S(Y,Z)(\nabla_U Q)X].$

, **(5.2.7.1)**

On contracting (5.2.7.1), we get

$(divW_6)(X,Y)Z = [(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)]+\frac{1}{2(n-1)}[g(X,Z)drY-S(Y,Z)drX].$

. **(5.2.7.2)**

If W_6-curvature tensor is conservative ($divW_6 = 0$), then (5.2.6.2) can be written as

$[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] = \frac{1}{2(n-1)}[S(Y,Z)drX - g(X,Z)drY]$ **(5.2.7.3)**

Putting X=T in (5.2.7.3), we have

$[(\nabla_T S)(Y,Z) - (\nabla_Y S)(T,Z)] = \frac{1}{2(n-1)}[S(Y,Z)drT - g(T,Z)drY]$ **(5.2.7.4)**

Since T is a killing vector, r remains invariant under it, that is,$L_T r = 0$, where L denotes the Lie derivative. But then the relation,

$(\nabla_T S)(Y,Z) = TS(Y,Z) - S(\nabla_T Y,Z) - S(Y,\nabla_T Z) = L_T S(Y,Z) - S(\nabla_T Y,Z) - S(Y,\nabla_Z T)$ **(5.2.7.5)**

yields

$[(\nabla_T S)(Y,Z) = 0.$ **(5.2.7.6)**

Now by substituting (5.2.7.6) in (5.2.7.4), we have

$-[(\nabla_Y S)(T,Z) - S(\nabla_Y T,Z) - S(T,\nabla_Y Z)] = \frac{1}{2(n-1)}[S(Y,Z)drT - g(T,Z)drY]$

. **(5.2.7.7)**

By using (5.2.1.4),(5.2.1.5), (5.2.1.10) and $dr(T) = 0$ in (5.2.7.7), we get

$-[(\nabla_Y[(n-1)A(Z)]-S(F(Y),Z)-(n-1)A(\nabla_Y Z)] = \frac{1}{2(n-1)}[-A(Z)drY].$ **(5.2.7.8)**

By Simplifying (5.2.7.8), we get

$[-(n-1)g(F(Y),Z) + S(F(Y),Z)] = -\frac{1}{2(n-1)}[A(Z)drY].$ **(5.2.7.9)**

Putting $F(Z) = Z$in (5.2.6.9), we obtain

$[-(n-1)g(F(Y),F(Z))+S(F(Y),F(Z))] = -\frac{1}{2(n-1)}[AF(Z)drY]$. **(5.2.7.10)**

But $A(F(Z)) = 0$ and $S(F(Y),F(Z)) = (n-1)g(F(Y),F(Z))$

it implies that

$S(Y,Z) = (n-1)g(Y,Z)$, **(5.2.7.11)**

$r = n(n-1)$. **(5.2.7.12)**

Thus we state:

Theorem 5.2.7.1 If W_6-curvature tensor in a Lorentzian para-Sasakian manifold is conservative then it is an Einstein manifold and also of constant scalar curvature.

5.2.8 Einstein Lorentzian Para-Sasakian Manifold satisfying $R(X,Y).W_6 = 0$

In consequence of $Qx = hx$ (5.2.1) becomes

$W_6(X,Y)Z = R(X,Y)Z + \frac{h}{n-1}[g(X,Z)Y - S(Y,Z)X]$. **(5.2.8.1)**

Using (5.2.1.10) and (5.2.8.1), we obtain

$A(W_6(X,Y)Z) = g(Y,Z)A(X) - g(X,Z)A(Y) + \frac{h}{n-1}[g(X,Z)Y - S(Y,Z)X]$ **(5.2.8.2)**

Replacing Z with T in (5.2.8.2), we have

$A(W_6(X,Y)T) = g(Y,T)A(X) - g(X,T)A(Y) + \frac{h}{n-1}[g(X,T)Y - S(Y,T)X]$

$A(W_6(X,Y)T) = \frac{h}{n-1}[A(X)Y - (n-1)A(Y)X]$.

$A(W_6(X,Y)T) = \frac{h(2-n)}{n-1}[A(X)A(Y)]$ **(5.2.8.3)**

$A(W_6(X,Y)T) = 0$.

Now

$(R(X,Y).W_6)(Z,U)V = R(X,Y)W_6(Z,U)V - W_6(R(X,Y)Z,U)V - W_6(Z,R(X,Y)U)V - W_6(Z,U)R(X,Y)V$ **(5.2.8.4)**

Using $R(X,Y).W_6 = 0$ in the above equation, we obtain

$R(X,Y)W_6(Z,U)V - W_6(R(X,Y)Z,U)V - W_6(Z,R(X,Y)U)V - W_6(Z,U)R(X,Y)V = 0$. **(5.2.8.5)**

By taking the inner product of the above relation with T, we get

$g(R(X,Y)W_6(Z,U)V,T) - g(W_6(R(X,Y)Z,U)V,T) - g(W_6(Z,R(X,Y)U)V -$

$W_6(Z,U)R(X,Y)V,T) = 0.$ **(5.2.8.6)**

Putting in X=T (5.2.8.6) and then using (5.1.1.11), we obtain

$-W_6(Z,U,V,Y) - A(Y)A(W_6(Z,U)V) + A(Z)A(W_6(Y,U)V) + A(U)A(W_6(Z,Y)V) + A(V)A(W_6(Z,U)Y) - g(Y,Z)A(W_6(T,U)V) - g(Y,U)A(W_6(Z,T)V) - g(Y,V)A(W_6(Z,U)T) = 0$ **(5.2.8.7)**

In consequence of (5.2.8.2)and simplification the above equation gives

$W_6(Z,U,V,Y) = g(Y,Z)g(U,V) - g(Y,U)g(Z,V) + \frac{h}{n-1}[g(Z,U)S(V,Y) - g(Z,Y)S(U,V)]$. **(5.2.8.8)**

and so

$W_6(Z,U)V = g(U,V)Z - g(Z,V)U + \frac{h}{n-1}[g(Z,V)U - S(U,V)Z].$ **(5.2.8.9)**

Thus in view of (5.2.8.1) and (5.2.8.9), we obtain

$R(Z,U)V = g(U,V)Z - g(Z,V)U.$ **(5.2.8.10)**

Thus we have the following:

Theorem 5.2.8.1 A Lorentzian para-Sasakian manifold satisfying $R(X,Y).W_6 = 0$ is a space of constant curvature.

5.2.9 Deriving Equations of W_6-Curvature tensor on Lp-sasakian manifold

In our equation

${}'W_6(X,Y,Z,U) = {}' R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z)],$

We break the equation into symmetric P and skew symmetric Q parts in X and Y to have new model equation and determine their geometric and physical properties on it .

We start with symmetric part P

$$'P(X,Y,Z,U) = \frac{1}{2}['W_6(X,Y,Z,U) +' W_6(Y,X,Z,U)]$$
$$= \frac{1}{2}['R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z)] +' R(Y,X,Z,U) + \frac{1}{n-1}[g(Y,X)Ric(Z,U) - g(Y,U)Ric(X,Z)]]$$
$$= \frac{1}{2}'R(X,Y,Z,U) + \frac{1}{2}'R(Y,X,Z,U) + \frac{1}{2(n-1)}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z) + g(Y,X)Ric(Z,U) - g(Y,U)Ric(X,Z)]$$
$$= \frac{1}{2(n-1)}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z) + g(Y,X)Ric(Z,U) - g(Y,U)Ric(X,Z)].$$

$'P(X,Y,Z,U) = \frac{1}{2(n-1)}[2g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z) - g(Y,U)Ric(X,Z)]$

. **(5.2.9.1)**

Now we take a look at skew-symmetric part Q

$$'Q(X,Y,Z,U) = \tfrac{1}{2}['W_6(X,Y,Z,U) -' W_6(Y,X,Z,U)]$$
$$= \tfrac{1}{2}['R(X,Y,Z,U) + \tfrac{1}{n-1}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z)] -'$$
$$R(Y,X,Z,U) - \tfrac{1}{n-1}[g(Y,X)Ric(Z,U) - g(Y,U)Ric(X,Z)]]$$
$$= \tfrac{1}{2}'R(X,Y,Z,U) - \tfrac{1}{2}'R(Y,X,Z,U) + \tfrac{1}{2(n-1)}[g(X,Y)Ric(Z,U) -$$
$$g(X,U)Ric(Y,Z) - g(Y,X)Ric(Z,U) + g(Y,U)Ric(X,Z)]$$
$$=' R(X,Y,Z,U) + \tfrac{1}{2(n-1)}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z)] -$$
$$g(Y,X)Ric(Z,U) + g(Y,U)Ric(X,Z)]$$

OR

$'Q(X,Y,Z,U) =' R(X,Y,Z,U) - \frac{1}{2(n-1)}[+g(X,U)Ric(Y,Z)] - g(Y,U)Ric(X,Z)]$

. **(5.2.9.2)**

1.LP-Sasakian manifold

In this section we study properties of W_6,P,q curvature tensors in LP-sasakian manifold.

Theorem 5.2.9.1

In an n-dimensional LP-Sasakian manifold we have

1. $'W_6(T,Y,Z,T) = -g(Y,Z) + \frac{1}{n-1}Ric(Y,Z)$
2. $W_6(X,Y,T) = YA(X)\frac{2-n}{n-1}$
3. $W_6(T,Y,T) = Y\frac{n-2}{n-1}$

Proof (5.2.9.1)i

Substituting U=T in (5.2.1.1) we get

$$'W_6(X,Y,Z,T) =' R(X,Y,Z,T) + \tfrac{1}{n-1}[g(X,Y)Ric(Z,T) - g(X,T)Ric(Y,Z)]$$

Equation (5.2.1.4) gives

$$'W_6(X,Y,Z,T) =' R(X,Y,Z,T) + \tfrac{1}{n-1}[g(X,Y)Ric(Z,T) - A(X))Ric(Y,Z)]$$

From (5.2.1.9) we get

$$'W_6(X,Y,Z,T) =$$
$$g(Y,Z)A(X) - g(X,Z)A(Y)) + \frac{1}{n-1}[g(X,Y)Ric(Z,T) - A(X))Ric(Y,Z)]$$

Using (5.2.1.10) we get

$$'W_6(X,Y,Z,T) =$$
$$g(Y,Z)A(X) - g(X,Z)A(Y)) + \frac{1}{n-1}[g(X,Y)(n-1)A(Z) - A(X))Ric(Y,Z)]$$
$$= g(Y,Z)A(X) - g(X,Z)A(Y)) + g(X,Y)A(Z) - A(X)\frac{1}{n-1}Ric(Y,Z) \quad (5.2.9.1)$$

Replacing X=T in(5.2.9.1) we have

$$'W_6(T,Y,Z,T) = g(Y,Z)A(T) - g(T,Z)A(Y)) + g(T,Y)A(Z) - A(T)\frac{1}{n-1}Ric(Y,Z)$$

Using equation (5.2.1.1),we get

$$'W_6(T,Y,Z,T) = -g(Y,Z) - g(T,Z)A(Y)) + g(T,Y)A(Z) + \frac{1}{n-1}Ric(Y,Z)$$

Again using (5.2.1.4),we shall have

$$'W_6(T,Y,Z,T) = -g(Y,Z) - A(Z)A(Y)) + A(Y)A(Z) + \frac{1}{n-1}Ric(Y,Z)$$
$$'W_6(T,Y,Z,T) = -g(Y,Z) + \frac{1}{n-1}Ric(Y,Z)$$

Hence proved

Proof (5.2.9.1)ii

Using $'W_6(X,Y,Z,U) = g(W_6(X,Y,Z),U)$ and equation (5.2.1.1) we get

$$W_6(X,Y,Z) = R(X,Y,Z) + \frac{1}{n-1}[g(X,Z)Y - XRic(Y,Z)]$$

Putting T=Z

$$W_6(X,Y,T) = R(X,Y,T) + \frac{1}{n-1}[g(X,T)Y - XRic(Y,T)]$$

Using $XA(Y) - YA(X)$ and (4.2.1.4)(5.2.1.10) we get

$$W_6(X,Y,T) = XA(Y) - YA(X) + \frac{1}{n-1}[A(X)Y - X(n-1)A(Y)]$$
$$W_6(X,Y,T) = XA(Y) - YA(X) + \frac{1}{n-1}A(X)Y - XA(Y)$$
$$W_6(X,Y,T) = -YA(X) + \frac{1}{n-1}A(X)Y)$$
$$W_6(X,Y,T) = YA(X)\frac{2-n}{n-1}$$

Hence proved

Proof (5.2.9.1)iii

Putting X=T in (5.2.9.1)ii we get

$$W_6(T,Y,T) = YA(T)\frac{2-n}{n-1}$$

Using (5.2.1.1) we get

$$W_6(T,Y,T) = -Y\frac{2-n}{n-1}$$
$$W_6(T,Y,T) = Y\frac{n-2}{n-1}$$

Hence proved

Theorem 5.2.9.2

In a n-dimensional LP-Sasakian manifold P tensor field satisfies

1. $'P(X,Y,Z,T) = g(X,Y)A(Z) - \frac{1}{2(n-1)}[A(X)Ric(Y,Z) + A(Y)Ric(X,Z)]$
2. $'P(T,Y,Z,U) = -\frac{1}{2}g(Y,U)A(Z) + \frac{1}{2(n-1)}[2A(Y)Ric(Z,U) - A(U)Ric(Y,Z)]$
3. $'P(T,Y,Z,T) = \frac{1}{2}[A(Y)A(Z) + \frac{1}{n-1}Ric(Y,Z)]$

Proof (5.2.9.2)i

Using (5.2.9.1) and putting T=U we have

$'P(X,Y,Z,T) = \frac{1}{2(n-1)}[g(X,Y)Ric(Z,T) - g(X,T)Ric(Y,Z)] + g(Y,X)Ric(Z,T) - g(Y,T)Ric(X,Z)]$

Using (5.2.1.4) and (5.2.1.10) we get

$$'P(X,Y,Z,T = \frac{1}{2(n-1)}[g(X,Y)(n-1)A(Z) - A(X)Ric(Y,Z)] + g(Y,X)(n-1)A(Z) - A(Y)Ric(X,Z)]$$
$$'P(X,Y,Z,T = \frac{1}{2}[g(Y,X)A(Z) + g(X,Y)A(Z)] - \frac{1}{2(n-1)}[A(X)Ric(Y,Z)] - A(Y)Ric(X,Z)]$$
$$'P(X,Y,Z,T = g(X,Y)A(Z) - \frac{1}{2(n-1)}[A(X)Ric(Y,Z) + A(Y)Ric(X,Z)]$$

Hence proved

Proof (5.2.9.2)ii

Using (5.2.9.1) and putting T=X we have

$'P(T,Y,Z,U) = \frac{1}{2(n-1)}[2g(T,Y)Ric(Z,U) - g(T,U)Ric(Y,Z) - g(Y,U)Ric(T,Z)]$

Using (5.2.1.4) and (5.2.1.10) we have

$$'P(T,Y,Z,U) = \frac{1}{2(n-1)}[2A(Y)Ric(Z,U) - A(U)Ric(Y,Z) - g(Y,U)(n-1)A(Z)]$$
$$'P(T,Y,Z,U) = -\frac{1}{2}g(Y,U)A(Z) + \frac{1}{2(n-1)}[2A(Y)Ric(Z,U) - A(U)Ric(Y,Z)]$$

Hence proved

Proof (5.2.9.2)iii

Putting X=T in (5.2.9.2)i we get

$$'P(T,Y,Z,T) = g(T,Y)A(Z) - \frac{1}{2(n-1)}[A(T)Ric(Y,Z) + A(Y)Ric(T,Z)]$$

Using (5.2.1.1),(5.2.1.4) and (5.2.1.10) we have

$$'P(T,Y,Z,T) = A(Y)A(Z) - \frac{1}{2(n-1)}[-Ric(Y,Z) - A(Y)(n-1)A(Z)]$$
$$'P(T,Y,Z,T) = A(Y)A(Z) - \frac{1}{2}A(Y)A(Z) + \frac{1}{2(n-1)}Ric(Y,Z)$$
$$'P(T,Y,Z,T) = \frac{1}{2}[A(Y)A(Z) + \frac{1}{n-1}Ric(Y,Z)].$$

Hence proved

Theorem 5.2.9.3

In an n-dimensional LP-Sasakian manifold Q tensor field satisfies

1. $'Q(X,Y,Z,T) = A(X)[g(Y,Z) - \frac{1}{2(n-1)}Ric(Y,Z)] - A(Y)[g(X,Z) - \frac{1}{2(n-1)}Ric(X,Z)]$

2. $'Q(T,Y,Z,U) = A(U)[g(Y,Z) - \frac{1}{2(n-1)}Ric(Y,Z)] - \frac{1}{2}g(Y,U)A(Z)$

3. $'Q(T,Y,Z,T) = -g(Y,Z) + \frac{1}{2}[\frac{1}{n-1}Ric(Y,Z) - A(Y)A(Z)]$

Proof (5.2.9.3)i

Using (5.2.9.2) and putting T=U we have

$$'Q(X,Y,Z,T) =' R(X,Y,Z,T) - \frac{1}{2(n-1)}[+g(X,T)Ric(Y,Z)] - g(Y,T)Ric(X,Z)]$$

Using (5.2.1.4),(5.2.1.9) we have

$$'Q(X,Y,Z,T) =$$
$$g(Y,Z)A(X) - g(X,Z)A(Y) - \frac{1}{2(n-1)}[+A(X)Ric(Y,Z)] - A(Y)Ric(X,Z)]$$

$$'Q(X,Y,Z,T) =$$
$$A(X)[g(Y,Z) - \frac{1}{2(n-1)}Ric(Y,Z)] - A(Y)[g(X,Z) - \frac{1}{2(n-1)}Ric(X,Z)]$$

Hence proved

Proof (5.2.9.3)ii

Using (5.2.9.2) and putting T=X we have

$$'Q(T,Y,Z,U) =' R(T,Y,Z,U) - \frac{1}{2(n-1)}[g(T,U)Ric(Y,Z)] - g(Y,U)Ric(T,Z)]$$

From equation (5.2.1.4),(5.2.1.9),(5.2.1.10) we have

$$'Q(T,Y,Z,U) =$$
$$g(Y,Z)A(U) - g(Y,U)A(Z) - \frac{1}{2(n-1)}[A(U)Ric(Y,Z)] - g(Y,U)(n-1)A(Z)]$$
$$'Q(T,Y,Z,U) = A(U)[g(Y,Z) - \frac{1}{2(n-1)}Ric(Y,Z)] - \frac{1}{2}g(Y,U)A(Z)$$

Hence proved

Proof (5.2.9.3)iii

Let X=T in (5.2.9.3)i

$$'Q(T,Y,Z,T) = A(T)[g(Y,Z) - \frac{1}{2(n-1)}Ric(Y,Z)] - A(Y)[g(T,Z) - \frac{1}{2(n-1)}Ric(T,Z)]$$

Using (5.2.1.1),(5.2.1.9),(5.2.1.10) we have

$$'Q(T,Y,Z,T) = -[g(Y,Z) - \frac{1}{2(n-1)}Ric(Y,Z)] - A(Y)[A(Z) - \frac{1}{2(n-1)}(n-1)A(Z)]$$
$$'Q(T,Y,Z,T) = -g(Y,Z) + \frac{1}{2(n-1)}Ric(Y,Z) - A(Y)[A(Z) + \frac{1}{2}A(Y)A(Z)$$
$$'Q(T,Y,Z,T) = -g(Y,Z) + \frac{1}{2}[\frac{1}{n-1}Ric(Y,Z) - A(Y)A(Z)]$$

Hence proved

5.2.10 The Relativistic Significance of W_6-Curvature Tensor in LP-Sasakian Space

In the n-dimensional space V_n the tensors

1.INTRODUCTION

$C(X,Y,Z) = R(X,Y,Z,T) - \frac{R}{n(n-1)}[g(X,T)g(Y,Z) - g(Y,T)g(X,Z)]$ **(5.2.10.1.1)**

$L(X,Y,Z,T) = R(X,Y,Z,T) - \frac{1}{n-2}[g(Y,Z)Ric(X,T) - g(X,Z)Ric(Y,T) + g(X,T)Ric(Y,Z) - g(Y,T)Ric(X,Z)]$ **(5.2.10.1.2)**

and

$V(X,Y,Z,T) = R(X,Y,Z,T) - \frac{1}{n-2}[g(X,T)Ric(Y,Z) - g(Y,T)Ric(X,Z) + g(Y,Z)Ric(X,T) - g(X,Z)Ric(Y,T)] + \frac{R}{(n-1)(n-2)}[g(X,T)g(Y,Z) - g(Y,T)g(X,T)]$ **(5.2.10.1.3)**

are called concircular curvature tensor, conharmonic curvature tensor and conformal curvature tensor respectively. These satisfy the symmetric and skew symmetric as well as the cyclic property possessed by curvature tensor $R(X,Y,Z,T)$.

Pokhariyah(1982) have define the projective curvature tensor as

$W_6'(X,Y,Z,T) = R'(X,Y,Z,T) + \frac{1}{n-1}[g(X,Y)Ric(Z,T) - g(X,T)Ric(Y,Z)]$

OR

$W_6(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)Y - S(Y,Z)X]$ **(5.2.10.1.4)**

We shall now define a tensor and obtain its properties

Definition(5.2.10.2)

We define the tensors

(5.2.10.2.1) $W_6'(X,Y,Z,T) = R'(X,Y,Z,T) + \frac{1}{n-1}[g(X,Y)Ric(Z,T) - g(X,T)Ric(Y,Z)]$ **(5.2.10.2.1)**

From equation (5.2.10.1.1) to (5.2.10.2.1), its clear that for an empty gravitational field characterised by $Ric(X,Y) = 0$, these five fourth rank tensors are identical.

In the space V_n,from (5.2.10.1.1),(5.2.10.1.2) and (5.2.10.1.3),we have

$V(X,Y,Z,T) = L(X,Y,Z,T) + \frac{n}{n-2}[R(X,Y,Z,T) - C(X,Y,Z,T)]$. **(5.2.10.2.2)**

which in V_4 reduces to

$V(X,Y,Z,T) = L(X,Y,Z,T) + 2R(X,Y,Z,T) - 2C(X,Y,Z,T)$. **(5.2.10.2.3)**

We notice that (5.2.10.2.1) is skew symmetric in Z,T and also satisfies

$W_6'(X,Y,Z,T) + W_6'(Y,Z,X,T) + W_6'(Z,X,Y,T) = 0$ **(5.2.10.2.4)**

Breaking $W_6'(X,Y,Z,T)$ into two parts, gives

$$'G(X,Y,Z,T) = \frac{1}{2}[W_6'(X,Y,Z,T) - W_6'(Y,X,Z,T)]$$

and

$$'H(X,Y,Z,T) = \frac{1}{2}[W_6'(X,Y,Z,T) + W_6'(Y,X,Z,T)]$$

which are respectively skew symmetric and symmetric in X,Y. From (5.2.10.2.1) it follows that

$$G(X,Y,Z,T) = \frac{1}{2}[W_6'(X,Y,Z,T) - W_6'(Y,X,Z,T)]$$
$$G(X,Y,Z,T) = \frac{1}{2}[R(X,Y,Z,T) + \frac{1}{n-1}[g(X,Y)Ric(Z,T) - g(X,T)Ric(Y,Z)] - R(Y,X,Z,T) - \frac{1}{n-1}[g(Y,X)Ric(Z,T) - g(Y,T)Ric(X,Z)]]$$
$$G(X,Y,Z,T) = \frac{1}{2}R(X,Y,Z,T) - \frac{1}{2}{}'R(Y,X,Z,T) + \frac{1}{2(n-1)}[g(X,Y)Ric(Z,T) - g(X,T)Ric(Y,Z) - g(Y,X)Ric(Z,T) + g(Y,T)Ric(X,Z)]$$
$$G(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{2(n-1)}[g(X,Y)Ric(Z,T) - g(X,T)Ric(Y,Z)] - g(Y,X)Ric(Z,T) + g(Y,T)Ric(X,Z)]$$

OR

$$G(X,Y,Z,T) = R(X,Y,Z,T) - \frac{1}{2(n-1)}[g(X,T)Ric(Y,Z)] - g(Y,T)Ric(X,Z)] \quad \textbf{(5.2.10.2.5)}$$

$$H(X,Y,Z,T) = \frac{1}{2}[W_6'(X,Y,Z,T) + W_6'(Y,X,Z,T)]$$
$$H(X,Y,Z,T = \frac{1}{2}[R(X,Y,Z,T) + \frac{1}{n-1}[g(X,Y)Ric(Z,T) - g(X,T)Ric(Y,Z)] + R(Y,X,Z,T) + \frac{1}{n-1}[g(Y,X)Ric(Z,T) - g(Y,T)Ric(X,Z)]]$$
$$H(X,Y,Z,T = \frac{1}{2}R(X,Y,Z,T) + \frac{1}{2}R(Y,X,Z,T) + \frac{1}{2(n-1)}[g(X,Y)Ric(Z,T) - g(X,T)Ric(Y,Z) + g(Y,X)Ric(Z,T) - g(Y,T)Ric(X,Z)]$$
$$H(X,Y,Z,T = \frac{1}{2(n-1)}[g(X,Y)Ric(Z,T) - g(X,T)Ric(Y,Z) + g(Y,X)Ric(Z,T) - g(Y,T)Ric(X,Z)].$$

or

$$H(X,Y,Z,T) = \frac{1}{2(n-1)}[2g(X,Y)Ric(Z,T) - g(X,T)Ric(Y,Z) - g(Y,T)Ric(X,Z)] \quad \textbf{(5.2.10.2.6)}$$

From (5.2.10.2.5) we see that $G(X,Y,Z,T)$ also possesses all the symmetric and skew symmetric properties of $R(X,Y,Z,T)$as well as the cyclic property.

$$G(X,Y,Z,T) + G(Y,Z,X,T) + G(Z,X,Y,T) = 0 \quad \textbf{(5.2.10.2.7)}$$

combining equations (5.2.10.1.3) and (5.2.10.2.5), simplifying we get

$$G(X,Y,Z,T) = \frac{1}{2(n-1)}[nR(X,Y,Z,T)+(n-2)V(X,Y,Z,)-\frac{R}{n-1}g(X,T)g(X,Z) - g(Y,T)g(X,Z$$

. **(5.2.10.2.8)**

which for electromagnetic field(or more generally in the space which vanishing scalar curvature) in v_4 becomes

$$3G(X,Y,Z,T) = 2R(X,Y,Z,T) + V(X,Y,Z,T) \quad \textbf{(5.2.10.2.9)}$$

Also from equation (5.2.10.1.2) and (5.2.10.2.5), for V_4,we have

$$3G(X,Y,Z,T) = 2R(X,Y,Z,T) + L(X,Y,Z,T) \quad \textbf{(5.2.10.2.10)}$$

Thus equation (5.2.10.2.9) is the consequence of (5.2.10.2.10) for a space of vanishing scalar curvature

We notice that $G(X,Y,Z,T)$ is identically equal to the skew symmetric part $P(X,Y,Z,T)$[5] of the projective curvature tensor unlike its symmetric part $Q(X,Y,Z,T)$ is different for $H(X,Y,Z,T)$.

On contracting W_{6hijk}, we obtain

$$W_{6ij} = (\frac{n}{n-1})(R_{ij} - \frac{R}{n}g_{ij}) \quad \textbf{(5.2.10.2.11)}$$

which vanishes in an Einstein space.

The scalar invariant

$$W_6 = g^{ij}W_{6ij} = 0 \quad \textbf{(5.2.10.2.12)}$$

identically. Now considering the scalar invariant of second degree in W_{6ij} Viz

$$(W_6)_{II}W_{6ij}W_6^{ij} = (\frac{n}{n-1})^2(R_6 - \frac{R^2}{n}) \quad \textbf{(5.2.10.2.13)}$$

where $R_6 = R_{ij}R^{ij}$.

From (5.2.10.2.11), we have

$$W_{6ij}R^{ij} = \frac{n}{n-1}(R_6 - \frac{R^2}{n}) \quad \textbf{(5.2.10.2.14)}$$

hence

$W_{6ij}W_2^{ij} = (\frac{n}{n-1})W_{6ij}R^{ij}$ **(5.2.10.2.15)**

From (5.2.10.2.5), we notice that contracted G_{ij} vanishes identically for Einstein space.This enables us to extend the Pirani formalism of gravitational waves to the Einstein space with the help of G_{hijk}.

For an Einstein space G_{hijk},W_{6hijk},W_{hijk} and V_{hijk} are identically equal.

We can show that the vanishing of the symmetric part H_{hijk} is necessary and sufficient condition for a space to be an Einstein space.

The vector

$Q_i = \frac{g_{ij}E^{jklm}R^p_k R_{pl;m}}{\sqrt{-g}R_{ab}R^{ab}}$ **(5.2.10.2.16)**

is called the comlexion field with no matter by $Misner$ and $Wheeler$ (1957) and its vanishing implies that the field is purely electrical.

A semi-colon stands for covariant differentiation.

It seems that we cannot get purely electrical field with the help of W_{6hijk}.

Rainich (1952) has shown that the necessary and sufficient conditions for the existence of the non-null electrovariance are

$R = 0$ **(5.2.10.2.17)**

$R^i_j R^j_k = (\frac{1}{4})\delta^i_k R_{ab}R^{ab}$ **(5.2.10.2.18)**

$Q_{i;j} = Q_{j;i}.$ **(5.2.10.2.19)**

In an electromagnetic field

$W_{6ij} = (\frac{4}{3})R_{ij}$ **(5.2.10.2.20)**

We can substitute W_{6ij} in place of R_{ij} in (5.2.10.2.16) and (5.2.10.2.18) such that the Rainich conditions so obtained are similar to those obtained with the help of W_{hijk}.

From the above discussion we conclude that except the vanishing of complexion vector and property of being identical in two spaces which are in geodesic correspondence, the tensor W_{6hijk} possess the properties almost similar to W_{hijk}.

Thus we can very well use W_{6hijk} in various physical and geometrical spheres in place of the Projective curvature tensor.

CHAPTER SIX

CONCLUSION AND RECOMMENDATION

6.1 Conclusion

Having determined the geometric properties on Sasakian space such as flatness, Semi-Symmetric and Symmetric it was clear that a W_6-flat Sasakian manifold is a flat manifold, W_6-Semi-Symmetric Sasakian manifold is a W_6-flat manifold and also W_6-Symmetric Sasaskian manifold is W_6-flat manifold.

On geometric properties on LP-Sasakian space it was also clear that a W_6-flat LP-Sasakian manifold is a flat manifold, W_6-Semi-Symmetric LP-Sasakian manifold is a W_6-flat manifold and also W_6-Symmetric LP-Sasaskian manifold is W_6-flat manifold and conclude that if LP-Sasakian manifold W_6-curvature tensor is flat then its an Einstein manifold and also a space of constant scalar curvature.

Having also determined the physical properties on LP-Sasakian manifold such as irrotational and conservative I found that if the W_6-curvature tensor in an LP-Sasakian manifold is irrotational then the manifold is a space of constant curvature and if it is conservative then is an Einstein manifold also of constant scalar curvature. On Einstein LP-Sasakian manifold satisfying $R(X,Y).W_6 = 0$ is a space of constant curvature.
On finding out the relativistic significance of W_6-curvature tensor on LP-Sasakian Space, we notice that $G(X,Y,Z,T)$ is identically equal to the skew symmetric part $P(X,Y,Z,T)$ of the projective curvature tensor unlike its symmetric part $Q(X,Y,Z,T)$ is different for $H(X,Y,Z,T)$ and on contracting W_{6hijk}, we obtain $W_{6ij} = (\frac{n}{n-1})(R_{ij} - \frac{R}{n}g_{ij})$ which vanishes in an Einstein space and contracted G_{ij} vanishes identically for Einstein space. This enables us to extend the Pirani formalism of gravitational waves to the Einstein space with the help of G_{hijk}. For an Einstein space G_{hijk}, W_{6hijk}, W_{hijk} and V_{hijk} are identically equal, this show that the vanishing of the symmetric part H_{hijk} is necessary and sufficient

condition for a space to be an Einstein space. The vector $Q_i = \frac{g_{ij}E^{jklm}R^p_k R_{pl;m}}{\sqrt{-g}R_{ab}R^{ab}}$ is called the comlexion field with no matter by Misner and Wheeler (1957) and its vanishing implies that the field is purely electrical. A semi-colon stands for covariant differentiation. It seems that we cannot get purely electrical field with the help of W_{6hijk}. In an electromagnetic field $W_{6ij} = (\frac{4}{3})R_{ij}$ We can substitute W_{6ij} in place of R_{ij} in (5.2.10.2.16) and (5.2.10.2.18) such that the Rainich conditions so obtained are similar to those obtained with the help of W_{hijk}. From the above discussion we were able to conclude that except the vanishing of complexion vector and property of being identical in two spaces which are in geodesic correspondence, the tensor W_{6hijk} possess the properties almost similar to W_{hijk}. Thus we can very well use W_{6hijk} in various physical and geometrical spheres in place of the Projective curvature tensor.

6.2 Recommendation

This study can be extended on other manifolds such as K-contact, A-Einstein, Komatsu and many more to investigate on geometric and physical properties and combine with theory of relativity to study space.

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APPENDICES

Publications

1. Kamami,W., Moindi, S.K., Kiogora,R., and Esekon, J. (2021). On some geometric properties of W_6-Curvature Tensor in LP-Sasakian manifold, *IJSTR*, 10(7) , 1-3.

2. Kamami,W., Moindi, S.K., Kiogora,R., and Esekon, J. (2021). On Lorentzian Para-Sasakian manifolds with W_6-Curvature Tensor satisfying certain conditions, *IJSTR*, 10(7) , 4-6.