# MODELING TRAFFIC FLOW ON MULTI-LANE 

ROAD: EFFECTS OF LANE-CHANGE MANOEUVRE DUE TO RAMPS AND WEAVING SECTIONS

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Modeling Traffic Flow on Multi-Lane Road: Effects of Lane-Change Manoeuvre Due to Ramps and Weaving Sections

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A Thesis Submitted in Partial Fulfillment of the Requirements of the Degree of Doctor of Philosophy in Applied Mathematics of the Jomo Kenyatta University of Agriculture and Technology

## DECLARATION

This thesis is my original work and has not been presented for a degree in any other University.
$\qquad$
Signature
Date

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This thesis has been submitted for examination with our approval as university supervisors.

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## DEDICATION

This work is dedicated to my mum late Leah Wanjiru, dear wife late Catherine Nduku and our beloved children Evalyne, Vanessa and Ryan.

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## LIST OF ABBREVIATIONS

| PDE | Partial Differential Equation |
| :--- | :--- |
| FVM | Finite Volume Method |
| FDM | Finite Difference Method |
| LHS | Left Hand Side |
| RHS | Right Hand Side |
| IVP | Initial Value Problem |
| BVP | Boundary Value Problem |
| 3-PTT | Three Phase Traffic Theory |
| SP | Synchronized Pattern |
| LSP | Localized Synchronized Pattern |
| WSP | Widening Synchronized pattern |
| MSP | Moving Synchronized Pattern |
| CFL | Courant - Friedrichs - Levy Condition |
| MATLAB | Matrix Laboratory |
| MFD | Macroscopic Fundamental Diagram |
| MLC | Mandatory Lane Change |
| ILC | Immediate Lane Change |

## NOMENCLATURE

| $\mathbf{F}$ | Free flow |
| :--- | :--- |
| $\mathbf{S}$ | Synchronized flow |
| $\mathbf{J}$ | Wide moving jam |
| $\mathbf{x}$ | Length of the section of the highway considered $[\mathrm{m}]$ |
| $\mathbf{t}$ | Duration of time considered [s] |
| $\boldsymbol{P}_{\boldsymbol{R}}$ | Probability of a vehicle changing lane to the right lane |
| $\boldsymbol{P}_{\boldsymbol{L}}$ | Probability of a vehicle changing lane to the left lane |
| $\boldsymbol{\rho}_{\boldsymbol{\alpha}}$ | Average traffic density on lane $\alpha[$ vehicles $/ \mathrm{m}]$ |
| $\boldsymbol{q}_{\boldsymbol{\alpha}}$ | Average traffic flow rate on lane $\alpha[$ vehicles $/ \mathrm{s}]$ |
| $\boldsymbol{u}_{\boldsymbol{\alpha}}$ | Average vehicle speed in lane $\alpha[\mathrm{m} / \mathrm{s}]$ |
| $\boldsymbol{u}_{\boldsymbol{\alpha}}^{+}$ | Speed of the vehicle ahead during interaction $[\mathrm{m} / \mathrm{s}]$ |
| $\boldsymbol{u}_{\boldsymbol{\alpha}}^{-}$ | Speed of the following vehicle during interaction $[\mathrm{m} / \mathrm{s}]$ |
| $\boldsymbol{\rho}_{\boldsymbol{\alpha}, \boldsymbol{f r e e}}$ | Traffic density for free flow [vehicles $/ \mathrm{m}]$ |
| $\boldsymbol{\rho}_{\boldsymbol{\alpha}, \boldsymbol{s y n}}$ | Traffic density for synchronized flow $[$ vehicles $/ \mathrm{m}]$ |
| $\boldsymbol{\rho}_{\boldsymbol{\alpha}, \boldsymbol{j a m}}$ | Traffic density for wide moving jam $[v e h i c l e s / \mathrm{m}]$ |
| $\boldsymbol{u}_{\boldsymbol{\alpha}, \boldsymbol{f r e e}}$ | Minimum vehicle speed in free flow $[\mathrm{m} / \mathrm{s}]$ |
| $\boldsymbol{u}_{\boldsymbol{\alpha}, \text { syn }}$ | Vehicle speed in synchronized flow $[\mathrm{m} / \mathrm{s}]$ |
| $\boldsymbol{u}_{\boldsymbol{\alpha}, \boldsymbol{j a m}}$ | Maximum vehicle speed in wide moving jam $[\mathrm{m} / \mathrm{s}]$ |
| $\boldsymbol{\delta}$ | Dirac delta |
| $\boldsymbol{G}_{\boldsymbol{\boldsymbol { R }}}^{+}$ | Gain term from the right lane |
| $\boldsymbol{G}_{\boldsymbol{L}}^{+}$ | Gain term from the left lane |
| $\boldsymbol{G}_{\boldsymbol{A}}^{+}$ | Gain term from acceleration |
| $\boldsymbol{G}_{\boldsymbol{B}}^{+}$ | Gain term from breaking interaction |
| $\boldsymbol{L}_{\boldsymbol{R}}^{+}$ | Loss term to the right lane |
| $\boldsymbol{\boldsymbol { L } _ { \boldsymbol { L } } ^ { + }}$ | Loss term to the left lane |
| $\boldsymbol{L}_{\boldsymbol{A}}^{+}$ | Loss term from acceleration |
| $\boldsymbol{L}_{\boldsymbol{B}}^{+}$ | Loss term from breaking interaction |


#### Abstract

Traffic breakdown is the main cause of vehicle traffic congestion in multi-lane roads due to highway bottlenecks such as lane-drops, on and off-ramps. In this study, the nature of traffic congestion and phase transition at the highway bottlenecks is explained. A multi-lane macroscopic traffic flow model of Aw-Rascle type is derived from the kinetic traffic flow model in which the lane-change terms are expressed explicitly. This is achieved by first developing the Kinetic traffic flow interaction operators and then applying the method of moments to get the corresponding macroscopic equations. For simulation of the traffic congestion, we consider a highway with three traffic lanes that have stationary bottlenecks. The macroscopic model equations for each lane are solved numerically using finite volume method (Godunov scheme), whereby the Euler's method is used for the source term. The results of the simulations of the congestion near and within the bottlenecks are presented in form of graphs and space-time plots. These results show that vehicle lane-change manoeuvre near the bottlenecks lead to traffic breakdown and congestion on the lane adjacent to the bottlenecks compared to the other express lanes in the highway. This prompts the following vehicles moving in the outermost lane to either slow down upon reaching the disturbance region or change to the other lanes before they reach the merging region. The model results provide important insight into improving the road network design and safety management strategies especially when new roads are constructed.


## CHAPTER ONE

## INTRODUCTION

### 1.1 Background Information

Vehicular traffic congestion is a condition on transport networks that occurs when a volume of traffic generates demand for space greater than the available road capacity and is one of the major problems experienced in most of the roads within urban areas. It exhibits a spatiotemporal traffic pattern which is a distribution of traffic flow variables in space and time. One way of solving this problem is to add more lanes on the existing roadways to increase the road capacity. Sometimes, this remedy is restricted by lack of spaces, resources, environments, politics and poor governance. Therefore, a proper understanding of empirical traffic congestion for effective traffic management, control and organization is necessary. Traffic flow theories and models which describe in a precise mathematical way the vehicle to vehicle and vehicle to infrastructure interactions are required to explain the real cause of traffic congestion.
One of the main causes of traffic congestion in road networks is traffic breakdown in an initially free flowing traffic, (Kerner, 2010). The traffic breakdown is described as the abrupt decrease in average vehicle speed in a free flow to a lower speed in congested traffic and usually occurs at highway bottlenecks such as lane-drops, road construction areas, accident areas, weaving section, on-ramps and off-ramps. This traffic breakdown is due to dynamic competition of the 'speed adaptation effect' which describes a tendency of traffic towards synchronized flow and the 'over-acceleration effect' describing a tendency of traffic towards free flow. Traffic congestion may also lead to various negative effects to motorists such as; delays in arrivals to various destinations, fuel wastage, wear and tear. However, traffic congestion has the advantage of encouraging travelers to re-time their trips early enough so that valuable road space is in full use for the most number of hours per day. Thus, there is a need to develop macroscopic traffic flow models which describe the traffic flow dynamics by averaging vehicles' density, velocity and flow rate.

### 1.2 The Three-Phase Traffic Flow Theory

The three-phase traffic flow theory is a qualitative flow introduced by Kerner (2010) based on common spatiotemporal features of measured congested traffic patterns. Traffic flow is a complex dynamics behavior of spatiotemporal traffic
pattern which occurs in space and time. This implies that traffic variables such as flow rate $(q)$, vehicles density $(\rho)$ and vehicle speeds are measured in real traffic flow in space and time. According to Kerner (2010), a traffic phase is a traffic state in space and time that possesses some unique empirical spatiotemporal features of traffic flow. Normally, traffic flow is considered to be either in free flow $(F)$ or in congested state. However, in the congested state there exist two different phases namely synchronized flow $(S)$ and wide moving jams $(J)$. Thus there are three traffic phases in the three-phase traffic theory namely: free flow $(F)$, synchronized flow $(S)$ and wide moving jam ( $J$ ) described henceforth.
The free traffic flow phase is normally observed when the vehicles' density in traffic stream is small and the vehicles are able to move freely with negligible vehicle to vehicle interactions. Therefore, the vehicles have an opportunity to move with their desired maximum speeds if not restricted by the traffic regulations or road conditions. When the vehicle density increases in free flow, an increase in flow rate follows and a limit for this phase is reached since the vehicle interactions can no longer be neglected, leading to a decrease in the vehicle speeds. At this limit point of free flow, the flow rate and density attain their maximum values at which the probability of a phase transition to a congested traffic phase is one.
A congested traffic state is the one in which the average speed of the vehicles is lower than the minimum average speed that is possible in free flow state. To distinguish between synchronized flow $(S)$ and wide moving jam $(J)$ in the congested state, we start by defining the later.
A wide moving jam is a localized congested traffic pattern with a low average vehicle speed and a high vehicle density spatially restricted by two jam fronts i.e. downstream and upstream jam fronts. Within the downstream jam front, vehicles accelerate from low speed states inside the jam to higher speeds in traffic flow downstream of the jam while within the upstream jam front, vehicles slow down to the speed within the jam. These two jam fronts move upstream which gives the wide moving jam a special feature of propagating through any other traffic states and through highway bottlenecks while maintaining the mean speed of the downstream jam front.
Synchronized flow $(S)$ is any congested traffic with no significant stoppage that does not exhibit the characteristic feature of wide moving jam phase. Unlike in the later phase, the downstream front of the synchronized flow phase does not maintain the mean velocity of the downstream front. In particular the downstream jam front is often fixed at bottlenecks such as lane-drops, on and
off ramps. Moreover within the downstream front of synchronized flow, vehicles accelerate from lower speeds in synchronized flow upstream of the front to higher speeds in free flow downstream of the front. Therefore if a congested traffic state is not associated with the wide moving jam phase then the remaining states are related to the synchronized flow phase.

### 1.3 Traffic Breakdown at the Bottlenecks

Traffic bottlenecks are disruptions of free traffic flow on a highway caused by either roadway design, traffic lights or accidents. There are two types of bottlenecks namely stationary and moving bottlenecks. Stationary bottlenecks are caused by lane drops which occur when a multi-lane road loses one or more lanes due to accidents or highway repairs. Consequently, the vehicular traffic are forced to merge on or diverge to the remaining lanes causing traffic breakdown around these areas. On the other hand moving bottlenecks result from slow moving vehicles such as heavy trucks and tractors which cause disruptions in free traffic flow. Thus, there is a permanent speed disturbance in free flow around these bottlenecks where the speed is lower and vehicles' density is greater than the other part of the highway. Traffic breakdown is a term associated with the nature of traffic congestion where ramps are regarded as a major source of traffic turbulence and congestion on highways. Generally traffic congestion occurs mostly in the vicinity of highway bottlenecks such as lane-drops, on and off ramps, accident areas, weaving sections and work zones.

### 1.4 Statement of the Problem

In the last decades and of late, traffic congestion has been a major problem in roads and is frequently experienced at the highway bottlenecks. Traffic congestion in real traffic network is caused by traffic breakdown at the highway bottlenecks where vehicles interact by changing their current lanes to their target lanes where possible. However, the earlier developed traffic flow models did not account for the discontinuous character of over-acceleration and failed to explain the coexistence of the three traffic phases as observed in real measured traffic data. Since the experimental approach is too expensive compared to numerical models which are cheaper and flexible, then the study of such macroscopic traffic flow models which exhibit the coexistence of free flow, synchronized flow and wide moving jam is necessary. This calls for development of macroscopic traffic flow model that adequately describes the vehicles interaction due to lane-change manoeuvres, acceleration and deceleration which are likely to happen at highway bottlenecks.

### 1.5 Objectives of the Study

### 1.5.1 General Objective

To determine the effects of lane-change manoeuvre due to ramps and weaving sections in a multi-lane road using macroscopic traffic flow models.

### 1.5.2 Specific Objectives

1. To derive macroscopic lane-change terms from the kinetic model equations.
2. To determine numerically the macroscopic lane-change terms near ramps and weaving sections in multi-lane roads.
3. To determine the effects of vehicles' lane-change manoeuvre due to an onramp in a 3-lanes highway.
4. To determine the effects of vehicles' lane-change manoeuvre due to an offramp in a 3-lanes highway.
5. To determine the effects of vehicles' lane-change manoeuvre due to a weaving section in a 3-lanes highway.

### 1.6 Justification of the Study

Vehicular traffic congestion on highways and within urban areas is a major problem experienced worldwide in roads networks. It is one of the drawback on peoples' quality way of life causing delays, accidents and environmental pollutions. Thus, a proper understanding of traffic congestion in the roads is necessary for effective traffic flow management and control. The major cause of traffic congestion is traffic breakdown which normally occurs at highway bottlenecks, such as roadworks, lane-drops, weaving areas, on and off-ramps. Research on macroscopic traffic flow models of multi-lane freeways have proved to be more suitable for applications in predicting traffic state and optimizing traffic control. This is because of their ability to describe the dynamics of traffic flow at bottlenecks without requiring as many parameters and variables as is the case in the microscopic models. The derived macroscopic traffic flow model is able to reproduce the real traffic conditions at the bottlenecks and therefore innovative traffic control measures can be developed to mitigate the traffic congestion in the highway.

Literature review of various researchers on traffic flow models and theories is devoted to the next chapter.

## CHAPTER TWO

## LITERATURE REVIEW

### 2.1 Introduction

In this chapter, literature review of various researchers contribution towards development of the traffic flow models is presented. Generally, traffic flow models are classified as: macroscopic, kinetic and microscopic. The macroscopic models describe the traffic flow dynamics by averaging the traffic characteristics such as density, velocity and flow rate while microscopic models describe the traffic flow in terms of individual vehicles interaction. For the Kinetic models, they characterize the traffic flow in less aggregate manner than macroscopic models and describe the traffic flow by use of probability distribution functions. The main advantage of using macroscopic traffic flow models over the kinetic models is that they are simple and have low computational complex.

### 2.2 Literature Review

According to Kerner (2010), vehicular traffic flow is a complex dynamic process associated with the spatiotemporal behavior of many particles systems. This is mainly due to nonlinear interactions between travel decision behavior, routing of vehicles in traffic network and traffic congestion occurrence within the network. The word spatiotemporal suggests that empirical traffic congestion occurs in real space and time. Normally, traffic flow is considered to be either in free flow $F$ or congested state but the later exists in two different phases known as synchronized flow $S$ and wide moving jam $J$. Investigations on traffic flow models from several researchers have revealed that traffic congestion in traffic network results from traffic breakdown in an initially traffic free flow state. Thus traffic congestion can be viewed as a form of vehicular queuing system categorized into stop-and-go waves and standing queues.

Whitham (1974) started the macroscopic modeling of vehicular traffic flow by considering the equation of continuity for traffic density $\rho$ and closed the equation by an equilibrium assumption on the mean velocity $u$. Later Payne (1979) introduced an additional momentum equation for the mean velocity to the Whitham model in analogy to fluid dynamics. The Payne-Whitman models predicted that if in front of a driver traveling at a certain speed and the vehicles' density is increasing but the vehicles ahead are faster, then the driver will slow down. This showed that the models produced unrealistic traffic
behavior especially when the density changes abruptly. However, a common observation is that a reasonable driver will obviously accelerate when the traffic in front is moving at speed higher than he is. This inconsistency was pointed out by Daganzo (1995) and was resolved by Aw \& Rascle (2000) who developed a new heuristic macroscopic traffic flow model from kinetic equations describing all situation correctly. In the recent past, the Payne-Whitman model has been improved by Khan (2016) using the fact that driver anticipation is based on the velocity of the forward traffic and therefore the traffic behavior depends on the velocity during transition. The new model results provided a more realistic traffic flow pattern than the Pane-Whitman model. Klar \& Wegener (2000) derived macroscopic traffic flow equations from the underlying kinetic models by considering a highway with $N$ lanes involving the vehicle interactions when changing lanes to either left or right. They obtained a general framework for the macroscopic traffic flow equations which exhibit the desired features like stop and go behavior.
Ahmed (1999) found that mandatory lane-change (MLC) exhibits different behavior compared to the immediate lane-changing models (ILC) of Hoogendoorn \& Bovy (1998) who included driver behavior. That is, vehicle lane change can be classified as either mandatory or discretionary according to driving incentives. The mandatory lane-change is mostly performed at bottlenecks where vehicles are forced to change to a fixed target lane and lead to traffic breakdown due to the increased traffic demand on this road sections, Ngoduy (2006a). On the other hand discretionary lane-change is conducted when the drivers perceive that driving conditions in the target lane are better. However, both lane-change sometime can be performed one after the other, especially when an aggressive driver decide to overtake a heavy vehicle in front first before performing mandatory lane-change to the off-ramp. Ahmed (1999) used a random utility theory in lane changing behavior modeling and defined a lane changing choice as a sequence of decision to make lane change, choice of the lane and then look for an acceptable space gap to decide the way forward. Otherwise, for a vehicle to attempt any lane change under the condition of traffic congestion, it requires the co-operation from at least one of the following vehicle in the target lane.
Another researcher Ngoduy (2006b) developed the gas-kinetic equations for interrupted traffic stream at weaving sections and used the method of moments to derive the corresponding macroscopic model. He modeled the lane-change manoeuvre terms using the renewal theory to calculate the lane-changing probabilities which depend on traffic density, speed, speed variance and vehicles
composition on the target lane. Steenberghen et al. (2012) noted that these models could not accurately show acceleration and deceleration traffic flow characters such as stop and start traffic, capacity drop and instantaneous changes. Kondyli \& Elefteriadou (2012) found that, at off-ramps the exiting vehicles must change lanes to the outside lane of the highway to access the shoulder lane. This increased traffic density on the outside lane and possible deceleration of exiting traffic is likely to force the express motorway vehicles in the outer lane to change lanes to the inner lanes to avoid exiting traffic. Interaction of vehicles while making these lanes change causes traffic congestion upstream of the off-ramps and adversely affect the available road capacity of expressways at these sections. Thus, even those vehicles on expressway experience long delays due to this congestion. Studies on traffic management by Wei (2001) revealed that lane-changing manoeuvres contribute to traffic flow disturbance on multi-lane freeways especially near the off-ramps. Amin \& Banks (2005) showed that, just upstream of an off-ramp, the fraction of flow on the outside lane is expected to be higher compared to an outside lane on a normal continuous stretch of motor way.
Kimathi (2012) derived macroscopic 3-phase traffic models of Aw-Rascle type from the kinetic traffic flow models based on integro-differential equations by specifying the relaxation term differently. He obtained three kinds of traffic flow models namely macroscopic speed adaptation, switching curve and modified switching curve traffic flow models. When the simulated results of the three models were compared, the macroscopic Speed Adaptation and modified Switching Curve model gave a better prediction of 3-phase traffic flow theory principle than the Switching Curve model. Marczak et al. (2014) analyzed detailed microscopic trajectory data collected on a weaving section in France and was able to show that lane changing behavior depends strongly on the prevailing traffic conditions. That is, when the traffic is heavy the lane changes occur at the beginning of the weaving section independently of their direction but when the traffic conditions are conducive, the lane changing positions are more distributed along the weaving area.

Tiaprasert et al. (2017) did simulations on urban roads using cell transmission model (CTM) where the results showed that CTM could accurately predict the occurrence time and spreading range of traffic congestion. Zhong et al. (2016) and Xu \& Wang (2016) in their studies on traffic flow in highway concluded that traffic congestion on expressway usually occurs in merging and weaving areas. Further more unreasonable lane changing behavior, forced lane change and abrupt deceleration behavior are the main reasons for traffic congestion on
highway. Knoop \& Daamen (2017) on Dutch legal traffic regulations which states that; 'a lane changing vehicle has to give priority to vehicles in the target lane' noted that vehicles on the target lane sometimes show courtesy to the entering vehicle by either changing lane towards the middle lane or by reducing speed to create a larger gap for the entering vehicle to merge. However, Long et al. (2018) found that the spacing of successive ramps is determined by the length of motorway over which the level of turbulence is raised. Mohan \& Ramadurai (2017) showed that the capacity of heterogeneous traffic is higher than homogeneous traffic due to the gap filling behavior but the former may not follow lanes and parameters such as velocity distribution and time. Moreover entering vehicles are sometimes willing to accept very short gaps as they join the highway but relax to more comfortable values shortly thereafter, Marczak et al. (2015).
A new macroscopic traffic flow model was proposed by H Khan et al. (2019) which considered the driver presumption based on driver reaction and traffic stimuli. The model results showed that drivers react to frontal stimuli which results in changes in vehicle density and speed unlike in Pane-Whitman model which showed unrealistic driver behavior due to the use of a constant speed. van Beinum et al. (2018) introduced a new method for detecting the start and end distances of traffic turbulence. They compared the traffic operations near the ramps to those on regular highway section and were able to estimate the turbulence influence area from the model results. Earlier traffic flow theories and models missed the discontinuous character of probability of passing introduced in the three-phase traffic theory of Kerner (2010). Thus they could not explain the traffic breakdown at the highway bottleneck as observed in real traffic data.

In this research the kinetic traffic flow model of Klar \& Wegener (1998) which expresses the lane-change term explicitly from pure anticipation term is used to establish macroscopic traffic flow model equations. Moreover lane changes can produce acceleration and deceleration resulting in stop-and-go traffic flow behavior Stern et al. (2017). According to Helbing (2001), a well-defined criteria for a good traffic flow model should contain only a few parameters and variables which are easy to observe, and the measured values are realistic to suit the macroscopic traffic flow model. Zhang et al. (2020) showed that most of the macroscopic fundamental diagram (MFD) traffic models are more suitable in estimation, prediction and development of control algorithms for macroscopic traffic flow on large scale networks in real time applications. This is why the study of MFD has received a lot of interest in recent traffic flow studies. Tilg
et al. (2018) studied a multi-class hybrid model which included lane change positions as exogenous input and the simulated results showed that by optimizing the distribution of the desired lane change positions, the capacity of the weaving increased by a significant number. They concluded that there is great potential of automated vehicle technology for increasing the capacity of weaving sections. Furthermore a good traffic model should reproduce all known features of traffic flow like localized jams and all transition states of traffic congestion. Thus vehicle lane-change manoeuvres can either maintain free traffic flow or lead to traffic breakdown at the bottlenecks and this is what our research is based on. This study provides an insight into real traffic lane change behaviors at the bottlenecks and contributes to a better understanding of the nature of traffic phase processes during congestion near the ramps.

In the next chapter the general equations governing the kinetic and macroscopic traffic flow models are established.

## CHAPTER THREE

## DERIVATION OF THE MACROSCOPIC TRAFFIC MODEL OF AW-RASCLE TYPE

### 3.1 Introduction

In this chapter kinetic traffic flow interaction operators (gain and loss terms) equations are obtained. These kinetic traffic flow model equations describe the traffic in a cummulative way averaging over all lanes and are the basis of the development of the macroscopic traffic flow models in this study. To derive the corresponding macroscopic model equations from the kinetic equations, we apply the method of moments to obtain the equations for the dynamics of density and flow rate.

### 3.2 Assumptions to the Study

The following assumptions are made when deriving the traffic flow model equations:

1. All vehicles are classified the same and therefore their length are neglected i.e. vehicles are considered to be points.
2. Motorcycles and bicycles have their special lane to use and do not interfere or interact with traffic flow in the highway.
3. Pedestrians use designated foot bridges and paths for either crossing or movement along the highway respectively.

### 3.3 The Kinetic Traffic Flow Model

The kinetic traffic flow model is described by use of the distribution functions of velocity of vehicles in traffic flow. We consider a highway with $N$ lanes numbered by $\alpha=1, \ldots, N$. Letting $f_{\alpha}(x, v)$ denote a single car distribution function which describes the number of cars at location $x$ with velocity $v$ on lane $\alpha$, and $f_{\alpha}^{(2)}\left(x, v, h, v_{+}\right)$denote the leading vehicle distribution function which describes the number of pairs of cars at $x$ with velocity $v$ and leading cars at $x+h$ with velocity $v_{+}$, where $h>0$ is the distance headway between the two considered locations of vehicles. According to Klar \& Wegener (1998), we approximate the leading vehicle distribution $f_{\alpha}^{(2)}\left(x, v, h, v_{+}\right)$by using $f_{\alpha}(x, v)$
and a correlation function to obtain a closed equation which describe the influence of the vehicle interactions. If $F_{\alpha}(x, v)$ denotes the probability distribution function in speed $v$ of vehicles at $x, F_{\alpha}^{+}\left(v_{+} ; h, v, x\right)$ denotes the probability distribution in $v_{+}$of the leading vehicless at distance $h$ for cars at $x$ with speed $v$, and $Q_{\alpha}(h ; v, x)$ denotes the probability distribution of leading cars in $h$ for a car at $x$ with velocity $v$, then;

$$
\begin{gather*}
f_{\alpha}(x, v)=\rho_{\alpha}(x) F_{\alpha}(x, v)  \tag{3.1}\\
f_{\alpha}^{(2)}\left(x, v, h, v_{+}\right)=F_{\alpha}^{+}\left(v_{+} ; h, v, x\right) Q_{\alpha}(h ; v, x) f_{\alpha}(x, v) \tag{3.2}
\end{gather*}
$$

Assuming the leading vehicles are distributed according to the probability $F_{\alpha}(x, v) \quad$ at $\quad x+h, \quad$ i.e $\quad F_{\alpha}^{+}\left(v_{+} ; h, v, x\right)=F_{\alpha}\left(x+h, v_{+}\right) \quad$ and $Q_{\alpha}(h ; v, x)=q_{\alpha}\left(h ; v, f_{\alpha}(x, v)\right)$ then

$$
\begin{equation*}
f_{\alpha}^{(2)}\left(x, v, h, v_{+}\right) \sim q_{\alpha}\left(h ; v, f_{\alpha}(x, v)\right) F_{\alpha}\left(x+h, v_{+}\right) f_{\alpha}(x, v) \tag{3.3}
\end{equation*}
$$

Here the kinetic equation for the distribution functions $\left(f_{1}, \ldots, f_{N}\right)$ on $N$ lanes is obtained by finding the kinetic interaction operators i.e the Gain $(G)$ and Loss $(L)$ operators. The gain terms or loss terms account for the increase (decrease) of $f_{\alpha}(x, v)$ respectively when a vehicle interacts with a leading vehicle and emerges with a certain speed as a result of the interaction. Therefore,

$$
\begin{gather*}
\partial_{t} f_{\alpha}+v \partial_{x} f_{\alpha}=C_{\alpha}^{+}\left(f_{1}, \ldots, f_{N}\right)=\left(G_{B}^{+}-L_{B}^{+}\right)\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) \\
+\left(G_{A}^{+}-L_{A}^{+}\right)\left(f_{\alpha}\right)+\left[G_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}, f_{\alpha+2}\right)-L_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}\right)\right] \\
+\left[G_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}\right)-L_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right)\right] \tag{3.4}
\end{gather*}
$$

Taking $\rho_{\alpha}=\int_{0}^{w} f_{\alpha}(x, v) d v, f_{\alpha}=\rho_{\alpha} F_{\alpha}$ and $q_{X}\left(v, f_{\alpha}\right)=q\left(H_{X}(v), v, f_{\alpha}\right)$ where $H_{X}(v) ; X=B, A$ is the threshold for braking and acceleration respectively.
The LHS of the kinetic equation (3.4) describes the continuous dynamics of the phase-space density (PSD) due to the motions of traffic flow while the RHS describes the discontinuous changes of this function due to lane-changing, acceleration and deceleration. It is assumed that when a driver intends to change lane, he first select the target lane. The probability that a lane is selected depends on many factors such as speed of the concerned vehicle, traffic conditions in the current lane and the target lanes. However if either the left lane or the right lane is selected, then the vehicle seeks for an acceptable space gap (distance between the front ends of two successive vehicles in the same lane) in the selected lane, which depends on the traffic regulations.

In this study, lane $\alpha$ is taken as the considered lane and the traffic flow regulation is based on keep left lane rule for slow moving vehicles unless overtaking.
Figure (3.1) shows the multi-lane highway under consideration in the traffic flow modeling. The arrows indicate the gain and loss interaction terms due to vehicle changing lanes to their target lanes.


Figure 3.1: Section of the multi-lane highway showing the kinetic traffic interaction operators due to lane-changing manoeuvres.

If $P_{L}$ and $P_{R}$ denote the probability of a vehicle lane change to either left or right respectively and using the convention $P_{R}\left(v, f_{N+1}\right)=P_{L}\left(v, f_{0}\right)=0$, then the gain and loss terms in equation (3.4) are approximated using (3.3).

In the following subsections we focus on derivation of gain and loss interaction terms due to vehicle's lane changing manoeuvre.

### 3.3.1 Gain and Loss Terms Due to Lane Changing to the Right

A vehicle will change lane to the right if the braking line is reached and a lane change is possible with a (probability $P_{R}$ ), resulting to the following vehicles interaction:
(a) Gain term from the right $\left(G_{R}^{+}\right)$is given by;

$$
\begin{gathered}
G_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}^{(2)}, f_{\alpha+2}\right) \\
=\int_{\hat{v}_{-}>v} P_{L}\left(v, f_{\alpha}(x)\right)\left|v-\hat{v}_{-}\right|\left[1-P_{R}\left(\hat{v}_{-}, f_{\alpha+2}\left(x-H_{B}\left(\hat{v}_{+}\right)\right)\right)\right]
\end{gathered}
$$

$$
\begin{equation*}
\times f_{\alpha+1}^{(2)}\left(x-H_{B}\left(\hat{v}_{-}\right), \hat{v}_{-}, H_{B}\left(\hat{v}_{-}\right), v\right) d \hat{v}_{-} \tag{3.5}
\end{equation*}
$$

But using the equation (3.3), we have:

$$
\begin{gather*}
f_{\alpha+1}^{(2)}\left(x-H_{B}\left(\hat{v}_{-}\right), \hat{v}_{-}, H_{B}\left(\hat{v}_{-}\right), v\right) \\
\approx q_{B}\left(\hat{v}_{-}, f_{\alpha+1}\left(x-H_{B}\left(\hat{v}_{-}\right)\right)\right) f_{\alpha+1}\left(x-H_{B}\left(\hat{v}_{-}\right), \hat{v}_{-}\right) F_{\alpha+1}(x, v) \tag{3.6}
\end{gather*}
$$

such that (3.5) reduces to:

$$
\begin{gather*}
G_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}^{(2)}, f_{\alpha+2}\right) \simeq G_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}, f_{\alpha+2}\right) \\
=\int_{\hat{v}_{-}>v} P_{L}\left(v, f_{\alpha}(x)\right)\left[1-P_{R}\left(\hat{v}_{-}, f_{\alpha+2}\left(x-H_{B}\left(\hat{v}_{-}\right)\right)\right)\right]\left|v-\hat{v}_{-}\right| \\
\times q_{B}\left(\hat{v}_{-}, f_{\alpha+1}\left(x-H_{B}\left(\hat{v}_{-}\right)\right)\right) f_{\alpha+1}\left(x-H_{B}\left(\hat{v}_{-}\right), \hat{v}_{-}\right) F_{\alpha+1}(x, v) d \hat{v}_{-} \tag{3.7}
\end{gather*}
$$

Setting $f_{\alpha+1}\left(x-H_{B}\left(\hat{v}_{-}\right), \hat{v}_{-}\right)=\rho_{\alpha+1} F_{\alpha+1}\left(x-H_{B}\left(\hat{v}_{-}\right), \hat{v}_{-}\right)=\rho_{\alpha+1} \delta_{u_{\alpha+1}^{-}}\left(\hat{v}_{-}\right)$ and $F_{\alpha+1}(x, v)=\delta_{u_{\alpha+1}}(v)$ then equation (3.7) becomes;

$$
\begin{gather*}
G_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}, f_{\alpha+2}\right) \\
=\int_{\hat{v}_{-}>v} P_{L}\left(v, \rho_{\alpha}\right)\left[1-P_{R}\left(\hat{v}_{-}, \rho_{\alpha+2}\right)\right]\left|v-\hat{v}_{-}\right| q_{B}\left(\hat{v}_{-}, \rho_{\alpha+1}\right) \\
\times \rho_{\alpha+1} \delta_{u_{\alpha+1}^{-}}\left(\hat{v}_{-}\right) \delta_{u_{\alpha+1}}(v) d \hat{v}_{-} \tag{3.8}
\end{gather*}
$$

(b) Loss term to the right $\left(L_{R}^{+}\right)$is given by:

$$
\begin{equation*}
L_{R}^{+}\left(f_{\alpha}^{(2)}, f_{\alpha+1}\right)=\int_{v>\hat{v}_{+}} P_{R}\left(v, f_{\alpha+1}(x)\right)\left|v-\hat{v}_{+}\right| f_{\alpha}^{(2)}\left(x, v, H_{B}(v), \hat{v}_{+}\right) d \hat{v}_{+} \tag{3.9}
\end{equation*}
$$

since $f_{\alpha}^{(2)}\left(x, v, H_{B}(v), \hat{v}_{+}\right) \simeq q_{B}\left(H_{B}(v), f_{\alpha}(x, v)\right) f_{\alpha}(x, v) F_{\alpha}\left(x+H_{B}(v), \hat{v}_{+}\right)$ then,

$$
\begin{gather*}
L_{R}^{+}\left(f_{\alpha}^{(2)}, f_{\alpha+1}\right) \simeq L_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}\right) \\
=\int_{v>\hat{v}_{+}} P_{R}\left(v, f_{\alpha+1}(x, v)\right)\left|v-\hat{v}_{+}\right| q_{B}\left(H_{B}(v), f_{\alpha}(x, v)\right) f_{\alpha}(x, v) F_{\alpha}\left(x+H_{B}(v), \hat{v}_{+}\right) d \hat{v}_{+} \tag{3.10}
\end{gather*}
$$

setting $f_{\alpha}(x, v)=\rho_{\alpha} F_{\alpha}(x, v)=\rho_{\alpha} \delta_{u_{\alpha}}(v)$ and $F_{\alpha}\left(x+H_{B}(v), \hat{v}_{+}\right)=\delta_{u_{\alpha}^{+}}\left(\hat{v}_{+}\right)$then equation (3.10) reduces to,

$$
\begin{gather*}
L_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}\right) \\
=\int_{v>\hat{v}_{+}} P_{R}\left(v, f_{\alpha+1}(x, v)\right)\left|v-\hat{v}_{+}\right| q_{B}\left(H_{B}(v), f_{\alpha}(x, v)\right) \rho_{\alpha} \delta_{u_{\alpha}}(v) \delta_{u_{\alpha}^{+}}\left(\hat{v}_{+}\right) d \hat{v}_{+} \\
=\int_{v>\hat{v}_{+}} P_{R}\left(v, \rho_{\alpha+1}\right)\left|v-\hat{v}_{+}\right| q_{B}\left(H_{B}(v), \rho_{\alpha}\right) \rho_{\alpha} \delta_{u_{\alpha}}(v) \delta_{u_{\alpha}^{+}}\left(\hat{v}_{+}\right) d \hat{v}_{+} \tag{3.11}
\end{gather*}
$$

### 3.3.2 Gain and Loss Terms Due to Lane Changing to the Left

A vehicle will change lane to the left if it reaches the braking line and is not able to overtake using the right lane. This yields the following vehicle interactions:
(a) Gain term from the left lane $\left(G_{L}^{+}\right)$is given by:

$$
\begin{equation*}
G_{L}^{+}\left(f_{\alpha-1}^{(2)}, f_{\alpha}\right)=\int_{v>\hat{v}_{+}} P_{R}\left(v, f_{\alpha}(x, v)\right)\left|v-\hat{v}_{+}\right| f_{\alpha-1}^{(2)}\left(x, v, H_{B}(v), \hat{v}_{+}\right) d \hat{v}_{+} \tag{3.12}
\end{equation*}
$$

since
$f_{\alpha-1}^{(2)}\left(x, v, H_{B}(v), \hat{v}_{+}\right) \simeq q_{B}\left(H_{B}(v), f_{\alpha-1}(x, v)\right) f_{\alpha-1}(x, v) F_{\alpha-1}\left(x+H_{B}(v), \hat{v}_{+}\right)$ then equation (3.12) reduces to;

$$
\begin{gather*}
G_{L}^{+}\left(f_{\alpha-1}^{(2)}, f_{\alpha}\right) \simeq G_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}\right) \\
=\int_{v>\hat{v}_{+}} P_{R}\left(v, f_{\alpha}(x, v)\right)\left|v-\hat{v}_{+}\right| q_{B}\left(H_{B}(v), f_{\alpha-1}(x, v)\right) f_{\alpha-1}(x, v) F_{\alpha-1}\left(x+H_{B}, \hat{v}_{+}\right) d \hat{v}_{+} \tag{3.13}
\end{gather*}
$$

using $f_{\alpha-1}(x, v)=\rho_{\alpha-1} F_{\alpha-1}(x, v)$ and $F_{\alpha-1}\left(x+H_{B}, \hat{v}_{+}\right)=\delta_{u_{\alpha-1}^{+}}\left(\hat{v}_{+}\right)$equation (3.13) reduces to;

$$
\begin{gather*}
G_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}\right) \\
=\int_{v>\hat{v}_{+}} P_{R}\left(v, f_{\alpha}(x, v)\right)\left|v-\hat{v}_{+}\right| q_{B}\left(H_{B}(v), f_{\alpha-1}(x, v)\right) \rho_{\alpha-1} \delta_{u_{\alpha-1}}(v) \delta_{u_{\alpha-1}^{+}}\left(\hat{v}_{+}\right) d \hat{v}_{+} \\
=\int_{v>\hat{v}_{+}} P_{R}\left(v, \rho_{\alpha}\right)\left|v-\hat{v}_{+}\right| q_{B}\left(H_{B}(v), \rho_{\alpha-1}\right) \rho_{\alpha-1} \delta_{u_{\alpha-1}}(v) \delta_{u_{\alpha-1}^{+}}\left(\hat{v}_{+}\right) d \hat{v}_{+} \tag{3.14}
\end{gather*}
$$

(b) Loss term to the left $\left(L_{L}^{+}\right)$is given by:

$$
\begin{gather*}
L_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}^{(2)}, f_{\alpha+1}\right) \\
=\int_{\hat{v}_{-}>v} P_{L}\left(v, f_{\alpha-1}(x)\right)\left[1-P_{R}\left(\hat{v}_{-}, f_{\alpha+1}\left(x-H_{B}\left(\hat{v}_{-}\right)\right)\right)\right]\left|v-\hat{v}_{-}\right| \\
\times f_{\alpha}^{(2)}\left(x-H_{B}\left(\hat{v}_{-}\right), \hat{v}_{-}, H_{B}\left(\hat{v}_{-}\right), v\right) d \hat{v}_{-} \tag{3.15}
\end{gather*}
$$

Since

$$
\begin{gathered}
f_{\alpha}^{(2)}\left(x-H_{B}\left(\hat{v}_{-}, \hat{v}_{-}, H_{B}\left(\hat{v}_{-}\right), v\right)\right) \\
\simeq q_{B}\left(\hat{v}_{-}, f_{\alpha}\left(x-H_{B}\left(\hat{v}_{-}\right)\right)\right) f_{\alpha}\left(x-H_{B}\left(\left(\hat{v}_{-}\right), \hat{v}_{-}\right)\right) F_{\alpha}(x, v)
\end{gathered}
$$

then equation (3.15) becomes;

$$
\begin{gather*}
L_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}^{(2)}, f_{\alpha+1}\right)=L_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) \\
=\int_{\hat{v}_{-}>v} P_{L}\left(v, f_{\alpha-1}(x)\right)\left[1-P_{R}\left(\hat{v}_{-}, f_{\alpha+1}\left(x-H_{B}\left(\hat{v}_{-}\right)\right)\right)\right]\left|v-\hat{v}_{-}\right| \\
\times q_{B}\left(\hat{v}_{-}, f_{\alpha}\left(x-H_{B}\left(\hat{v}_{-}\right)\right)\right) f_{\alpha}\left(x-H_{B}\left(\hat{v}_{-}\right), \hat{v}_{-}\right) F_{\alpha}(x, v) d \hat{v}_{-} \tag{3.16}
\end{gather*}
$$

Setting $f_{\alpha}\left(x-H_{B}\left(\hat{v}_{-}\right), \hat{v}_{-}\right)=\rho_{\alpha} F_{\alpha}\left(x-H_{B}\left(\hat{v}_{-}\right), \hat{v}_{-}\right)=\rho_{\alpha} \delta_{u_{\alpha}^{-}}\left(\hat{v}_{-}\right) \quad$ and $F_{\alpha}(x, v)=\delta_{u_{\alpha}}(v)$ equation (3.16) reduces to,

$$
\begin{gather*}
L_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) \\
=\int_{\hat{v}_{-}>v} P_{L}\left(v, \rho_{\alpha-1}\right)\left[1-P_{R}\left(\hat{v}_{-}, \rho_{\alpha+1}\right)\right]\left|v-\hat{v}_{-}\right| \\
\times q_{B}\left(\hat{v}_{-}, \rho_{\alpha}\right) \rho_{\alpha} \delta_{u_{\alpha}^{-}}\left(\hat{v}_{-}\right) \delta_{u_{\alpha}}(v) d \hat{v}_{-} \tag{3.17}
\end{gather*}
$$

### 3.3.3 Gain and Loss terms Due to Acceleration

A car will accelerate if the acceleration line is reached. Therefore:
(a) Gain term from acceleration $\left(G_{A}^{+}\right)$is given by:

$$
\begin{equation*}
G_{A}^{+}\left(f_{\alpha}^{(2)}\right)=\iint_{\hat{v}<\hat{v}_{+}}\left|\hat{v}-\hat{v}_{+}\right| \sigma_{A}(v, \hat{v}) f_{\alpha}^{(2)}\left(x, v, \hat{H}_{A}(\hat{v}), \hat{v}_{+}\right) d \hat{v} d \hat{v}_{+} \tag{3.18}
\end{equation*}
$$

since $f_{\alpha}^{(2)}\left(x, \hat{v}, H_{A}(\hat{v}), \hat{v}_{+}\right) \simeq q_{A}\left(H_{A}(\hat{v}), f_{\alpha}(x, v)\right) f_{\alpha}(x, \hat{v}) F_{\alpha}\left(x+H_{A}(\hat{v}), \hat{v}_{+}\right)$ then equation (3.18) becomes,

$$
\begin{gather*}
G_{A}^{+}\left(f_{\alpha}^{(2)}\right) \simeq G_{A}^{+}\left(f_{\alpha}\right) \\
=\iint_{\hat{v}<\hat{v}_{+}}\left|\hat{v}-\hat{v}_{+}\right| \sigma_{A}(v, \hat{v}) q_{A}\left(H_{A}(\hat{v}), f_{\alpha}(x, \hat{v})\right) f_{\alpha}(x, \hat{v}) F_{\alpha}\left(x+H_{A}(\hat{v}), \hat{v}_{+}\right) d \hat{v} d \hat{v}_{+} \tag{3.19}
\end{gather*}
$$

Setting $f_{\alpha}(x, \hat{v})=\rho_{\alpha} F_{\alpha}(x, \hat{v})=\rho_{\alpha} \delta_{u_{\alpha}}(\hat{v})$ and $F_{\alpha}\left(x+H_{A}(\hat{v}), \hat{v}_{+}\right)=\delta_{u_{\alpha}^{+}}\left(\hat{v}_{+}\right)$ equation (3.19) reduces to;

$$
\begin{align*}
& G_{A}^{+}\left(f_{\alpha}\right)=\iint_{\hat{v}<\hat{v}_{+}}\left|\hat{v}-\hat{v}_{+}\right| \sigma_{A}(v, \hat{v}) q_{A}\left(H_{A}(\hat{v}), f_{\alpha}(x, \hat{v})\right) \rho_{\alpha} \delta_{u_{\alpha}}(\hat{v}) \delta_{u_{\alpha}^{+}}\left(\hat{v}_{+}\right) d \hat{v} d \hat{v}_{+} \\
& \quad=\iint_{\hat{v}<^{+}}\left|\hat{v}-\hat{v}_{+}\right| \sigma_{A}\left(v, \hat{v}_{+}\right) q_{A}\left(H_{A}(\hat{v}), \rho_{\alpha}\right) \rho_{\alpha} \delta_{u_{\alpha}}(\hat{v}) \delta_{u_{\alpha}^{+}}\left(\hat{v}_{+}\right) d \hat{v} d \hat{v}_{+}^{‘} \tag{3.20}
\end{align*}
$$

(b) Loss term from acceleration $\left(L_{A}^{+}\right)$is given by:

$$
\begin{equation*}
L_{A}^{+}\left(f_{\alpha}^{(2)}\right)=\int_{v<\hat{v}_{+}}\left|v-\hat{v}_{+}\right| f_{\alpha}^{(2)}\left(x, v, H_{A}(v), \hat{v}_{+}\right) d \hat{v}_{+} \tag{3.21}
\end{equation*}
$$

since $f_{\alpha}^{(2)}\left(x, v, H_{A}(v), \hat{v}_{+}\right) \simeq q_{A}\left(H_{A}(v), f_{\alpha}(x, v)\right) f_{\alpha}(x, v) F_{\alpha}\left(x+H_{A}(v), \hat{v}_{+}\right)$ then equation (3.21) becomes,

$$
\begin{gather*}
L_{A}^{+}\left(f_{\alpha}^{(2)}\right) \simeq L_{A}^{+}\left(f_{\alpha}\right) \\
=\int_{v<\hat{v}_{+}}\left|v-\hat{v}_{+}\right| q_{A}\left(H_{A}(v), f_{\alpha}(x, v)\right) f_{\alpha}(x, v) F_{\alpha}\left(x+H_{A}(v), \hat{v}_{+}\right) d \hat{v}_{+} \tag{3.22}
\end{gather*}
$$

Setting $f_{\alpha}(x, v)=\rho_{\alpha} F_{\alpha}(x, v)=\rho_{\alpha} \delta_{u_{\alpha}}(v)$ and $F_{\alpha}\left(x+H_{A}(v), \hat{v}_{+}\right)=\delta_{u_{\alpha}^{+}}\left(\hat{v}_{+}\right)$ equation (3.22) reduces to,

$$
\begin{align*}
& L_{A}^{+}\left(f_{\alpha}\right)=\int_{v<\hat{v}_{+}}\left|v-\hat{v}_{+}\right| q_{A}\left(H_{A}(v), f_{\alpha}(x, v)\right) \rho_{\alpha} \delta_{u_{\alpha}}(v) \delta_{u_{\alpha}^{+}}\left(\hat{v}_{+}\right) d \hat{v}_{+} \\
& =\int_{v<\hat{v}_{+}}\left|v-\hat{v}_{+}\right| q_{A}\left(H_{A}(v), f_{\alpha}(x, v)\right) \rho_{\alpha} \delta_{u_{\alpha}}(v) \delta_{u_{\alpha}^{+}}\left(\hat{v}_{+}\right) d \hat{v}_{+} \tag{3.23}
\end{align*}
$$

### 3.3.4 Gain and Loss Terms Due to Deceleration

A vehicle will brake if it reaches the braking line and the driver is not able to change to the right or left lane. Thus;
(a) Gain term from braking interaction $\left(G_{B}^{+}\right)$is given by:

$$
\begin{gather*}
G_{B}^{+}\left(f_{\alpha-1}, f_{\alpha}^{(2)}, f_{\alpha+1}\right)= \\
\iint_{\hat{v}>\hat{v}_{+}} P_{B}\left(\hat{v}, \hat{v}_{+}, f_{\alpha-1}\left(x+H_{B}(v)\right), f_{\alpha+1}(x)\right)\left|\hat{v}-\hat{v}_{+}\right| \sigma_{B}(v, \hat{v}) f_{\alpha}^{(2)}\left(x, \hat{v}, H_{B}(\hat{v}), \hat{v}_{+}\right) d \hat{v} d \hat{v}_{+} \tag{3.24}
\end{gather*}
$$

where the probability of braking $\left(P_{B}\right)$ on lane $\alpha$ is given by the equation:

$$
\begin{gather*}
P_{B}\left(v, v_{+}, f_{\alpha-1}\left(x+H_{B}(v)\right), f_{\alpha+1}(x)\right) \\
=\left[1-P_{L}\left(v, f_{\alpha+1}(x)\right)\right]\left[1-P_{R}\left(v_{+}, f_{\alpha-1}\left(x+H_{B}(v)\right)\right)\right] \tag{3.25}
\end{gather*}
$$

using the convention

$$
\begin{equation*}
P_{R}\left(v, f_{N+1}\right)=P_{L}\left(v, f_{0}\right)=0 \tag{3.26}
\end{equation*}
$$

and

$$
\begin{equation*}
f_{\alpha}^{(2)}\left(x, \hat{v}, H_{B}(\hat{v}), \hat{v}_{+}\right)=q_{B}\left(H_{B}(\hat{v}), f_{\alpha}(x, \hat{v})\right) f_{\alpha}(x, \hat{v}) F_{\alpha}\left(x+H_{B}(\hat{v})\right) \tag{3.27}
\end{equation*}
$$

Then equation (3.24) becomes,

$$
\begin{gather*}
G_{B}^{+}\left(f_{\alpha-1}, f_{\alpha}^{(2)}, f_{\alpha+1}\right) \simeq G_{B}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) \\
=\iint_{\hat{v}>\hat{v}_{+}} P_{B}\left(\hat{v}, \hat{v}_{+}, f_{\alpha-1}\left(x+H_{B}(v)\right), f_{\alpha+1}(x)\right)\left|\hat{v}-\hat{v}_{+}\right| \sigma_{B}(v, \hat{v}) \\
\times q_{B}\left(H_{B}(\hat{v}), f_{\alpha}(x, \hat{v})\right) f_{\alpha}(x, \hat{v}) F_{\alpha}\left(x+H_{B}(\hat{v}), \hat{v}_{+}\right) d \hat{v} d \hat{v}_{+} \tag{3.28}
\end{gather*}
$$

With $f_{\alpha}(x, \hat{v})=\rho_{\alpha} F_{\alpha}(x, \hat{v})=\rho_{\alpha} \delta_{u_{\alpha}}(\hat{v})$ and $F_{\alpha}\left(x+H_{B}(\hat{v}), \hat{v}_{+}\right)=\delta_{u_{\alpha}^{+}}\left(\hat{v}_{+}\right)$ equation (3.28) reduces to

$$
\begin{gather*}
G_{B}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) \\
=\iint_{\hat{v}>\hat{v}_{+}} P_{B}\left(\hat{v}, \hat{v}_{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right)\left|\hat{v}-\hat{v}_{+}\right| \sigma_{B}(v, \hat{v}) \\
\times q_{B}\left(H_{B}(\hat{v}), \rho_{\alpha}\right) \rho_{\alpha} \delta_{u_{\alpha}}(\hat{v}) \delta_{u_{\alpha}^{+}}\left(\hat{v}_{+}\right) d \hat{v} d \hat{v}_{+} \tag{3.29}
\end{gather*}
$$

(b) Loss term from braking interaction $\left(L_{B}^{+}\right)$is given as:

$$
L_{B}^{+}\left(f_{\alpha-1}, f_{\alpha}^{(2)}, f_{\alpha+1}\right) \simeq L_{B}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right)
$$

$$
\begin{equation*}
=\int_{v>\hat{v}_{+}} P_{B}\left(v, \hat{v}_{+}, f_{\alpha-1}\left(x+H_{B}(v), f_{\alpha+1}(x)\right)\right)\left|v-\hat{v}_{+}\right| f_{\alpha}^{(2)}\left(x, v, H_{B}(v), \hat{v}_{+}\right) d \hat{v}_{+} \tag{3.30}
\end{equation*}
$$

But $f_{\alpha}^{(2)}\left(x, v, H_{B}(v), \hat{v}_{+}\right) \simeq q_{B}\left(H_{B}(v), f_{\alpha}(x, v)\right) f_{\alpha}(x, v) F_{\alpha}\left(x+H_{B}(v), \hat{v}_{+}\right)$ therefore equation (3.30) becomes,

$$
\begin{gather*}
L_{B}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) \\
=\int_{v>\hat{v}_{+}} P_{B}\left(v, \hat{v}_{+}, f_{\alpha-1}\left(x+H_{B}(v)\right), f_{\alpha+1}(x)\right)\left|v-\hat{v}_{+}\right| q_{B}\left(H_{B}(v), f_{\alpha}(x, v)\right) \\
\times f_{\alpha}(x, v) F_{\alpha}\left(x+H_{B}(v), \hat{v}_{+}\right) d \hat{v}_{+} \tag{3.31}
\end{gather*}
$$

With $f_{\alpha}(x, v)=\rho_{\alpha} F_{\alpha}(x, v)=\rho_{\alpha} \delta_{u_{\alpha}}(v)$ and $F_{\alpha}\left(x+H_{B}(v), \hat{v}_{+}\right)=\delta_{u_{\alpha}^{+}}\left(\hat{v}_{+}\right)$ equation (3.31) reduces to;

$$
\begin{gather*}
L_{B}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) \\
=\int_{v>\hat{v}_{+}} P_{B}\left(v, \hat{v}_{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right)\left|v-\hat{v}_{+}\right| q_{B}\left(H_{B}(v), \rho_{\alpha}\right) \rho_{\alpha} \delta_{u_{\alpha}}(v) \delta_{u_{\alpha}^{+}}\left(\hat{v}_{+}\right) d \hat{v}_{+} \tag{3.32}
\end{gather*}
$$

### 3.4 The Macroscopic Traffic Flow Model Equations

The method of moments is applied to the kinetic model equations to obtain the corresponding macroscopic equations of $k^{t h}$ order. This is achieved by expressing the macroscopic traffic flow variables in terms of the phase-space density. First, the intrinsic macroscopic variables are defined using moments of the distribution functions $f_{\alpha}(x, v, t)$ as;
density:

$$
\begin{equation*}
\rho_{\alpha}(x, t)=\int_{0}^{v_{\max }} f_{\alpha}(x, v, t) d v \tag{3.33}
\end{equation*}
$$

mean velocity:

$$
\begin{equation*}
u_{\alpha}(x, t)=\left(\rho_{\alpha}(x, t)\right)^{-1} \int_{0}^{v_{\max }} v f_{\alpha}(x, v, t) d v \tag{3.34}
\end{equation*}
$$

The macroscopic equations are of certain infinitely order, that is; the macroscopic density equation depends on the mean velocity while the macroscopic equation for the mean velocity depends on the velocity variance given by;

$$
\begin{equation*}
\theta_{\alpha}(x, t)=\left(\rho_{\alpha}(x, t)\right)^{-1} \int_{0}^{v_{\max }}\left(v-u_{\alpha}\right)^{2} f_{\alpha}(x, v, t) d v \tag{3.35}
\end{equation*}
$$

However, Ngoduy (2006b) were able to show that the second order macroscopic equations of density and velocity dynamics are sufficient to describe the flow of traffic. Therefore we focus on the second order macroscopic traffic flow equations and close this system of equations by assuming that the speed variance is a function of the first two moments. In this study, we approximate the distribution function using the one-node ansatz, (Kimathi, 2012) as;

$$
\begin{equation*}
f_{\alpha}(x, v, t) \simeq \rho_{\alpha}(x, t) \delta\left(v-u_{\alpha}(x, t)\right) \tag{3.36}
\end{equation*}
$$

where $\delta($.$) is the Dirac delta in the sense of distribution and is a linear function$ that maps every function to its value at zero.
According to Aw \& Rascle (2000), equation (3.36) represent a situation whereby all the vehicles present at location $x$ at instantaneous time $t$ move at the same average speed $u_{\alpha}$. The advantage of using (3.36) is that, it readily gives an AwRascle type macroscopic traffic flow model and approximates the traffic pressure to zero, that is;

$$
\begin{equation*}
\rho_{\alpha} \theta_{\alpha}=\int_{0}^{v_{\max }}\left(v-u_{\alpha}\right)^{2} f_{\alpha}(x, v, t) d v \simeq \int_{0}^{v_{\max }} \rho_{\alpha}\left(v-u_{\alpha}\right)^{2} \delta\left(v-u_{\alpha}\right) d v=0 \tag{3.37}
\end{equation*}
$$

To this end, we multiply the inhomogeneous kinetic equation (3.4) by $v^{k}$ for $k=0,1$ and integrate it with respect to $v$ in the range of $\left[0, v_{\max }\right]$ to get the following set of macroscopic balance equations;

$$
\begin{align*}
& \partial_{t} \int_{0}^{v_{\max }} v^{k} f_{\alpha}(x, v, t) d v+\partial_{x} \int_{0}^{v_{\max }} v^{k+1} f_{\alpha}(x, v, t) d v \\
& \quad=\int_{0}^{v_{\max }} v^{k} C_{\alpha}^{+}\left(f_{1}^{(2)}, \ldots, f_{N}^{(2)}, f_{1}, \ldots, f_{N}\right)(x, v, t) d v \tag{3.38}
\end{align*}
$$

Using the Gain and Loss terms interactions due to lane changing, acceleration and braking, on substitution equation (3.38) reduces to:

$$
\left.\left.\left.\begin{array}{c}
\partial_{t} \int_{0}^{v_{\max }} v^{k} f_{\alpha}(x, v, t) d v+\partial_{x} \int_{0}^{v_{\max }} v^{k+1} f_{\alpha}(x, v, t) d v \\
=\int_{0}^{v_{\max }} v^{k}\left\{\left(G_{B}^{+}-L_{B}^{+}\right)\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right)+\left(G_{A}^{+}-L_{A}^{+}\right) f_{\alpha}\right. \\
+\left[G_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}, f_{\alpha+2}\right)-L_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}\right)\right] \\
+ \tag{3.39}
\end{array}\right] G_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}\right)-L_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right)\right]\right\} d v .
$$

When $k=0$, we obtain the conservation equation (equation of continuity) and for $k=1$ the momentum equation (equation for dynamics of mean speed) is obtained, therefore:
(a) Multiplying equation (3.29) by $v^{k}$, we have:

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k} G_{B}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) d v \\
=\int_{0}^{v_{\max }} v^{k} \iint_{\hat{v}>\hat{v}_{+}} P_{B}\left(\hat{v}, \hat{v}_{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right)\left|\hat{v}-\hat{v}_{+}\right| \sigma_{B}(v, \hat{v}) q_{B}\left(H_{B}(\hat{v}), \rho_{\alpha}\right) \\
\times \rho_{\alpha} \delta_{u_{\alpha}}(\hat{v}) \delta_{u_{\alpha}^{+}}\left(\hat{v}_{+}\right) d \hat{v} d \hat{v}_{+} d v \tag{3.40}
\end{gather*}
$$

As a consequence of Dirac delta $\delta$ (.) function equation (3.40) becomes:

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k} G_{B}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) d v= \\
\int_{0}^{v_{\max }} v^{k} P_{B}\left(u_{\alpha}, u_{\alpha}^{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right)\left|u_{\alpha}-u_{\alpha}^{+}\right| \sigma_{B}\left(v, u_{\alpha}\right) q_{B}\left(H_{B}\left(u_{\alpha}\right), \rho_{\alpha}\right) \rho_{\alpha} d v \tag{3.41}
\end{gather*}
$$

where $\sigma_{B}\left(v, u_{\alpha}\right)=\frac{1}{u_{\alpha}(1-\beta)} \varkappa\left[\beta u_{\alpha}, u_{\alpha}\right](v)$ is a probability distribution and accounts for the imperfect adaptation of the faster vehicle with speed $u_{\alpha}$ to the speed $u_{\alpha}^{+}$of the slower vehicle ahead during interaction, $\varkappa$ is a characteristic function denoting the effective cross section that cater for the increased number of vehicle interactions in traffic congestion due to vehicular space requirements. Thus equation (3.41) reduces to:

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k} G_{B}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) d v= \\
P_{B}\left(u_{\alpha}, u_{\alpha}^{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right)\left|u_{\alpha}-u_{\alpha}^{+}\right| q_{B}\left(H_{B}\left(u_{\alpha}\right), \rho_{\alpha}\right) \rho_{\alpha}\left[\frac{1}{u_{\alpha}(1-\beta)} \int_{\beta u_{\alpha}}^{u_{\alpha}} v^{k} d v\right] \tag{3.42}
\end{gather*}
$$

Similarly for equation (3.32)

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k} L_{B}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) d v \\
=\int_{0}^{v_{\text {max }}} v^{k} \int_{v>\hat{v}_{-}} P_{B}\left(v, \hat{v}_{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right)\left|v-\hat{v}_{+}\right| q_{B}\left(H_{B}(v), \rho_{\alpha}\right) \\
\times \rho_{\alpha} \delta_{u_{\alpha}}(v) \delta_{u_{\alpha}^{+}}\left(\hat{v}_{+}\right) d \hat{v}_{+} d v \tag{3.43}
\end{gather*}
$$

As a consequence of Dirac delta $\delta($.$) function, equation (3.43) reduces to:$

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k} L_{B}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) d v \\
=\left[P_{B}\left(u_{\alpha}, u_{\alpha}^{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right)\left|u_{\alpha}-u_{\alpha}^{+}\right| q_{B}\left(H_{B}\left(u_{\alpha}\right), \rho_{\alpha}\right) \rho_{\alpha}\right] \int_{0}^{v_{\max }} u_{\alpha}^{k} d u_{\alpha} \\
=P_{B}\left(u_{\alpha}, u_{\alpha}^{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right)\left|u_{\alpha}-u_{\alpha}^{+}\right| q_{B}\left(H_{B}\left(u_{\alpha}\right), \rho_{\alpha}\right) \rho_{\alpha} u_{\alpha}^{k} \tag{3.44}
\end{gather*}
$$

Combining equation (3.42) and (3.44) as in equation (3.4) to get;

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k}\left(G_{B}^{+}-L_{B}^{+}\right)\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) d v \\
=\rho_{\alpha}\left|u_{\alpha}-u_{\alpha}^{+}\right| P_{B}\left(u_{\alpha}, u_{\alpha}^{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right) q_{B}\left(H_{B}\left(u_{\alpha}\right), \rho_{\alpha}\right)\left[\frac{1}{u_{\alpha}-u_{\alpha} \beta} \int_{\beta u_{\alpha}}^{u_{\alpha}} v^{k} d v-u_{\alpha}^{k}\right] \\
=\rho_{\alpha}\left|u_{\alpha}-u_{\alpha}^{+}\right| P_{B}\left(u_{\alpha}, u_{\alpha}^{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right) q_{B}\left(H_{B}\left(u_{\alpha}\right), \rho_{\alpha}\right)\left[\frac{u_{\alpha}^{k+1}-\left(\beta u_{\alpha}\right)^{k+1}}{(k+1)\left(u_{\alpha}-\beta u_{\alpha}\right)}-u_{\alpha}^{k}\right] \tag{3.45}
\end{gather*}
$$

But braking occurs when $u_{\alpha}>u_{\alpha}^{+}$, i.e $\left|u_{\alpha}-u_{\alpha}^{+}\right|=-\left(u_{\alpha}^{+}-u_{\alpha}\right)$
Therefore for $k=0$, equation (3.45) reduces to zero but for $k=1$ it becomes;

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k}\left(G_{B}^{+}-L_{B}^{+}\right)\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) d v \\
=-\rho_{\alpha}\left(u_{\alpha}^{+}-u_{\alpha}\right) P_{B}\left(u_{\alpha}, u_{\alpha}^{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right) q_{B}\left(H_{B}\left(u_{\alpha}\right), \rho_{\alpha}\right)\left[\frac{u_{\alpha}(\beta-1)}{2}\right] \tag{3.46}
\end{gather*}
$$

Using the Taylor's approximation, $u_{\alpha}^{+}(x+h)-u_{\alpha}(x) \simeq h \partial_{x} u_{\alpha}$, equation (3.46) reduces to;

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k}\left(G_{B}^{+}-L_{B}^{+}\right)\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) d v \\
\simeq \rho_{\alpha} h_{B} P_{B}\left(u_{\alpha}, u_{\alpha}^{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right) q_{B}\left(H_{B}\left(u_{\alpha}\right), \rho_{\alpha}\right) \frac{1-\beta}{2} u_{\alpha} \partial_{x} u_{\alpha} \tag{3.47}
\end{gather*}
$$

(b) Multiplying equation (3.8) by $v^{k}$ to get,

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k} G_{A}^{+}\left(f_{\alpha}\right) d v \\
=\int_{0}^{v_{\max }} v^{k} \iint_{\hat{v}<\hat{v}_{+}}\left|\hat{v}-\hat{v}_{+}\right| \sigma_{A}(v, \hat{v}) q_{A}\left(H_{A}(\hat{v}), \rho_{\alpha}\right) \rho_{\alpha} \delta_{u_{\alpha}}(\hat{v}) \delta_{u_{\alpha}^{+}}\left(\hat{v}_{+}\right) d \hat{v} d \hat{v}_{+} d v \tag{3.48}
\end{gather*}
$$

As a consequence of Dirac delta $\delta$ (.) function, equation (3.48) reduces to:

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k} G_{A}^{+}\left(f_{\alpha}\right) d v= \\
\rho_{\alpha}\left|u_{\alpha}-u_{\alpha}^{+}\right| q_{A}\left(H_{A}\left(u_{\alpha}\right), \rho_{\alpha}\right) \int_{0}^{v_{\max }} v^{k} \sigma_{A}\left(v, u_{\alpha}\right) d v \tag{3.49}
\end{gather*}
$$

with $\sigma_{A}\left(v, u_{\alpha}\right)=\frac{\varkappa}{\min \left(v_{\max }, u_{\alpha}\right)-u_{\alpha}}\left[u_{\alpha}, \min \left(v_{\max }, u_{\alpha}\right)\right] v=\frac{\varkappa\left[u_{\alpha}-\hat{u}_{\alpha}\right]}{\hat{u}_{\alpha}-u_{\alpha}} v$, where $\hat{u}_{\alpha}=\min \left(v_{\max }, \eta u_{\alpha}\right)$ and $\varkappa$ is the characteristics function, then equation (3.49) reduces to:

$$
\begin{equation*}
\int_{0}^{v_{\max }} v^{k} G_{A}^{+}\left(f_{\alpha}\right)=\rho_{\alpha}\left|u_{\alpha}-u_{\alpha}^{+}\right| q_{A}\left(H_{A}\left(u_{\alpha}\right), \rho_{\alpha}\right)\left[\frac{1}{\hat{u}_{\alpha}-u_{\alpha}} \int_{u_{\alpha}}^{\hat{u}_{\alpha}} v^{k} d v\right] \tag{3.50}
\end{equation*}
$$

Similarly for equation (3.23) we have:

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k} L_{A}^{+}\left(f_{\alpha}\right) d v \\
=\int_{0}^{v_{\max }} v^{k} \int_{v<\hat{v}_{+}}\left|v-\hat{v}_{+}\right| q_{A}\left(H_{A}(v), \rho_{\alpha}\right) \rho_{\alpha} \delta_{u_{\alpha}}(v) \delta_{u_{\alpha}^{+}}\left(\hat{v}_{+}\right) d \hat{v}_{+} d v \tag{3.51}
\end{gather*}
$$

As a consequence of Dirac delta $\delta$ (.) function, equation (3.51) reduces to:

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k} L_{A}^{+}\left(f_{\alpha}\right) d v=\rho_{\alpha}\left|u_{\alpha}-u_{\alpha}^{+}\right| q_{A}\left(H_{A}\left(u_{\alpha}\right), \rho_{\alpha}\right) \int_{0}^{v_{\max }} u_{\alpha}^{k} d u_{\alpha} \\
=\rho_{\alpha}\left|u_{\alpha}-u_{\alpha}^{+}\right| q_{A}\left(H_{A}\left(u_{\alpha}\right), \rho_{\alpha}\right) u_{\alpha}^{k} \tag{3.52}
\end{gather*}
$$

Combining equations (3.50) and (3.52) as stipulated in (3.4) we have,

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k}\left(G_{A}^{+}-L_{A}^{+}\right) f_{\alpha} d v \\
=\rho_{\alpha}\left|u_{\alpha}-u_{\alpha}^{+}\right| q_{A}\left(H_{A}\left(u_{\alpha}\right), \rho_{\alpha}\right)\left[\frac{1}{\hat{u}_{\alpha}-u_{\alpha}} \int_{u_{\alpha}}^{\hat{u}_{\alpha}} v^{k} d v-u_{\alpha}^{k}\right] \\
=\rho_{\alpha}\left|u_{\alpha}-u_{\alpha}^{+}\right| q_{A}\left(H_{A}\left(u_{\alpha}\right), \rho_{\alpha}\right)\left[\frac{\hat{u}_{\alpha}^{k+1}-u_{\alpha}^{k+1}}{(k+1)\left(\hat{u}_{\alpha}-u_{\alpha}\right)}-u_{\alpha}^{k}\right] \tag{3.53}
\end{gather*}
$$

But acceleration occurs when $u_{\alpha}<u_{\alpha}^{+}$i.e, $\left|u_{\alpha}-u_{\alpha}^{+}\right|=\left(u_{\alpha}^{+}-u_{\alpha}\right)$
therefore for $k=0$, equation (3.53) vanishes but for $k=1$ with $\hat{u}_{\alpha}=\eta u_{\alpha}$ it reduces to:

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k}\left(G_{A}^{+}-L_{A}^{+}\right) f_{\alpha} d v \\
=\rho_{\alpha}\left(u_{\alpha}^{+}-u_{\alpha}\right) q_{A}\left(H_{A}\left(u_{\alpha}\right), \rho_{\alpha}\right) \frac{u_{\alpha}(\eta-1)}{2} \tag{3.54}
\end{gather*}
$$

Again using Taylor's approximation, $u_{\alpha}^{+}(x+h)-u_{\alpha}(x) \simeq h \partial_{x} u_{\alpha}$ on equation (3.54) we get;

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k}\left(G_{A}^{+}-L_{A}^{+}\right) f_{\alpha} d v \\
\simeq \rho_{\alpha} h_{A} q_{A}\left(H_{A}\left(u_{\alpha}\right), \rho_{\alpha}\right) \frac{\eta-1}{2} u_{\alpha} \partial_{x} u_{\alpha} \tag{3.55}
\end{gather*}
$$

For equations (3.47) and (3.55), we assume that the leading vehicles are distributed in such a way that;

$$
\begin{equation*}
h_{B} q_{B}\left(H_{B}\left(u_{\alpha}\right), \rho_{\alpha}\right)=h_{A} q_{A}\left(H_{A}\left(u_{\alpha}\right), \rho_{\alpha}\right)=\frac{d b\left(\rho_{\alpha}\right)}{d \rho_{\alpha}} \tag{3.56}
\end{equation*}
$$

with $b\left(\rho_{\alpha}\right)$ being some increasing function of density $\rho_{\alpha}$.
Therefore the anticipation term is deduced from (3.47) and (3.55) using (3.56) to get;

$$
a\left(\rho_{\alpha}, u_{\alpha}\right)=\left\{\begin{array}{cc}
\rho_{\alpha} \frac{d b\left(\rho_{\alpha}\right)}{d \rho_{\alpha}} \varphi_{B}\left(u_{\alpha}, u_{\alpha}^{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right), & \partial_{x} u_{\alpha}<0  \tag{3.57}\\
\rho_{\alpha} \frac{d b\left(\rho_{\alpha}\right)}{d \rho_{\alpha}} \varphi_{A}\left(u_{\alpha}\right), & \partial_{x} u_{\alpha}>0
\end{array}\right.
$$

where

$$
\begin{equation*}
\varphi_{A}\left(u_{\alpha}\right)=\frac{(\eta-1)}{2} u_{\alpha} \tag{3.58}
\end{equation*}
$$

and

$$
\begin{equation*}
\varphi_{B}\left(u_{\alpha}, u_{\alpha}^{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right)=P_{B}\left(u_{\alpha}, u_{\alpha}^{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right)\left(\frac{1-\beta}{2}\right) u_{\alpha} \tag{3.59}
\end{equation*}
$$

Since the probability of braking $\left(P_{B}\left(u_{\alpha}, u_{\alpha}^{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right)\right)$ decreases from one to zero, then $P_{B}\left(u_{\alpha}, u_{\alpha}^{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right)=e^{-u_{\alpha}^{+}}\left(k_{1} \rho_{\alpha+1}+k_{2} \rho_{\alpha-1}\right)$ with $k_{1}=1-k_{2}$, $\varphi_{A}\left(u_{\alpha}\right)>\varphi_{B}\left(u_{\alpha}, u_{\alpha}^{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right) \equiv P_{B}\left(u_{\alpha}, u_{\alpha}^{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right) \varphi_{A}\left(u_{\alpha}\right), \quad$ and suppose that $\varphi_{A}\left(u_{\alpha}\right) \simeq \operatorname{constant}(C)$, then equation (3.57) becomes;

$$
a\left(\rho_{\alpha}, u_{\alpha}\right)=\left\{\begin{array}{cc}
\rho_{\alpha} \frac{d b\left(\rho_{\alpha}\right)}{d \rho_{\alpha}} C P_{B}\left(u_{\alpha}, u_{\alpha}^{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right), & \partial_{x} u_{\alpha}<0  \tag{3.60}\\
\rho_{\alpha} \frac{d b\left(\rho_{\alpha}\right)}{d \rho_{\alpha}} C, & \partial_{x} u_{\alpha}>0
\end{array}\right.
$$

Assuming that when $\partial_{x} u_{\alpha}<0$, braking is inevitable i.e. $P_{B}\left(u_{\alpha}, u_{\alpha}^{+}, \rho_{\alpha-1}, \rho_{\alpha+1}\right)$ approaches one, then;

$$
\begin{equation*}
a\left(\rho_{\alpha}, u_{\alpha}\right)=\rho_{\alpha} \frac{d b\left(\rho_{\alpha}\right)}{d \rho_{\alpha}} C \tag{3.61}
\end{equation*}
$$

where $b\left(\rho_{\alpha}\right)$ takes the form, (Kimathi, 2012) as;

$$
\begin{equation*}
b\left(\rho_{\alpha}\right)=-\ln \left(1-\rho_{\alpha}\right) \tag{3.62}
\end{equation*}
$$

(c) For equation (3.8) we have,

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k} G_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}, f_{\alpha+2}\right) d v \\
=\int_{0}^{v_{\max }} v^{k} \int_{\hat{v}_{-}>v} P_{L}\left(v, \rho_{\alpha}\right)\left[1-P_{R}\left(\hat{v}_{-}, \rho_{\alpha+2}\right)\right]\left|v-\hat{v}_{-}\right| q_{B}\left(H_{B}\left(\hat{v}_{-}\right), \rho_{\alpha+1}\right) \\
\times \rho_{\alpha+1} \delta_{u_{\alpha+1}^{-}}\left(\hat{v}_{-}\right) \delta_{u_{\alpha+1}}(v) d \hat{v}_{-} d v \tag{3.63}
\end{gather*}
$$

As a consequence of Dirac delta $\delta($.$) function, equation (3.63) reduces to;$

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k} G_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}, f_{\alpha+2}\right) d v \\
=P_{L}\left(u_{\alpha+1}, \rho_{\alpha}\right)\left[1-P_{R}\left(u_{\alpha+1}^{-}, \rho_{\alpha+2}\right)\right]\left|u_{\alpha+1}-u_{\alpha+1}^{-}\right| \\
q_{B}\left(H_{B}\left(u_{\alpha+1}^{-}\right), \rho_{\alpha+1}\right) \rho_{\alpha+1} \int_{0}^{v_{\max }} u_{\alpha+1}^{k} d u_{\alpha+1} \\
=\rho_{\alpha+1} u_{\alpha+1}^{k}\left|u_{\alpha+1}-u_{\alpha+1}^{-}\right| P_{L}\left(u_{\alpha+1}, \rho_{\alpha}\right)\left[1-P_{R}\left(u_{\alpha+1}^{-}, \rho_{\alpha+2}\right)\right] q_{B}\left(H_{B}\left(u_{\alpha+1}^{-}\right), \rho_{\alpha+1}\right) \tag{3.64}
\end{gather*}
$$

Similarly for equation (3.11) we have,

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k} L_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}\right) d v \\
=\int_{0}^{v_{\max }} v^{k} \int_{\hat{v}_{+}>v} P_{R}\left(v, \rho_{\alpha+1}\right)\left|v-\hat{v}_{+}\right| q_{B}\left(H_{B}(v), \rho_{\alpha}\right) \rho_{\alpha} \delta_{u_{\alpha}}(v) \delta_{u_{\alpha}^{+}}\left(\hat{v}_{+}\right) d \hat{v}_{+} d v \tag{3.65}
\end{gather*}
$$

As a consequence of Dirac delta $\delta$ (.) function, equation (3.65) reduces to:

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k} L_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}\right) d v \\
=P_{R}\left(u_{\alpha}, \rho_{\alpha+1}\right)\left|u_{\alpha}-u_{\alpha}^{+}\right| q_{B}\left(H_{B}\left(u_{\alpha}\right), \rho_{\alpha}\right) \rho_{\alpha} \int_{0}^{v_{\max }} u_{\alpha}^{k} d u_{\alpha} \\
=\rho_{\alpha} u_{\alpha}^{k}\left|u_{\alpha}-u_{\alpha}^{+}\right| P_{R}\left(u_{\alpha}, \rho_{\alpha+1}\right) q_{B}\left(H_{B}\left(u_{\alpha}\right), \rho_{\alpha}\right) \tag{3.66}
\end{gather*}
$$

Combining equation (3.64) and (3.66), to get:

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k}\left[G_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}, f_{\alpha+2}\right)-L_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}\right)\right] d v \\
=\rho_{\alpha+1} u_{\alpha+1}^{k}\left|u_{\alpha+1}-u_{\alpha+1}^{-}\right| P_{L}\left(u_{\alpha+1}, \rho_{\alpha}\right)\left[1-P_{R}\left(u_{\alpha+1}^{-}, \rho_{\alpha+2}\right)\right] q_{B}\left(H_{B}\left(u_{\alpha+1}^{-}\right), \rho_{\alpha+1}\right) \\
-\rho_{\alpha} u_{\alpha}^{k}\left|u_{\alpha}-u_{\alpha}^{+}\right| P_{R}\left(u_{\alpha}, \rho_{\alpha+1}\right) q_{B}\left(H_{B}\left(u_{\alpha}\right), \rho_{\alpha}\right) \tag{3.67}
\end{gather*}
$$

Since the probability of lane changing to either left or right can be described by use of negative exponents which have values ranging from zero to one, then the following probabilities are approximated by:

$$
\begin{array}{r}
P_{R}\left(u_{\alpha+1}^{-}, \rho_{\alpha+2}\right) \sim \exp \left(-\rho_{\alpha+2} \phi\left(u_{\alpha+1}^{-}\right)\right)=e^{-\rho_{\alpha+2} \phi\left(u_{\alpha+1}^{-}\right)} \\
P_{R}\left(u_{\alpha}, \rho_{\alpha+1}\right) \sim \exp \left(-\rho_{\alpha+1} \phi\left(u_{\alpha}\right)\right)=e^{-\rho_{\alpha+1} \phi\left(u_{\alpha}\right)} \\
P_{L}\left(u_{\alpha+1}, \rho_{\alpha}\right) \sim \exp \left(-\rho_{\alpha} \phi\left(u_{\alpha+1}\right)\right)=e^{\left(-\rho_{\alpha} \phi\left(u_{\alpha+1}\right)\right)} \tag{3.68}
\end{array}
$$

where $\phi\left(u_{\alpha+1}^{-}\right), \phi\left(u_{\alpha}\right)$ and $\phi\left(u_{\alpha+1}\right)$ are the rates at which the vehicles are changing lanes. Equation (3.62) is differentiated with respect to $\rho_{\alpha}$ to approximate the probability distribution function of the leading vehicles as;

$$
\begin{align*}
q_{B}\left(H_{B}\left(u_{\alpha+1}^{-}\right), \rho_{\alpha+1}\right) & \sim \frac{1}{1-\rho_{\alpha+1}} \\
q_{B}\left(H_{B}\left(u_{\alpha}\right), \rho_{\alpha}\right) & \sim \frac{1}{1-\rho_{\alpha}} \tag{3.69}
\end{align*}
$$

Letting $\phi\left(u_{\alpha+1}^{-}\right)=\phi\left(u_{\alpha}\right)=\phi\left(u_{\alpha+1}\right)=C_{0}$, equation (3.67) on substitution reduces to:

$$
\begin{align*}
& \int_{0}^{v_{\max }} v^{k}\left[G_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}, f_{\alpha+2}\right)-L_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}\right)\right] d v \\
= & \rho_{\alpha+1} u_{\alpha+1}^{k}\left|u_{\alpha+1}-u_{\alpha+1}^{-}\right|\left(1-e^{-\rho_{\alpha+2} C_{0}}\right)\left(\frac{1}{1-\rho_{\alpha+1}}\right) \\
& \times\left(e^{-\rho_{\alpha} C_{0}}\right)-\rho_{\alpha} u_{\alpha}^{k}\left|u_{\alpha}-u_{\alpha}^{+}\right| e^{-\rho_{\alpha+1} C_{0}}\left(\frac{1}{1-\rho_{\alpha}}\right) \tag{3.70}
\end{align*}
$$

(d) For equation (3.14) we have:

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k} G_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}\right) d v \\
=\int_{0}^{v_{\max }} v^{k} \int_{v>\hat{v}_{+}} P_{R}\left(v, \rho_{\alpha}\right)\left|v-\hat{v}_{+}\right| q_{B}\left(H_{B}(v), \rho_{\alpha-1}\right) \rho_{\alpha-1} \delta_{u_{\alpha-1}}(v) \delta_{u_{\alpha-1}^{+}}\left(\hat{v}_{+}\right) d \hat{v}_{+} d v \tag{3.71}
\end{gather*}
$$

As a consequence of Dirac delta $\delta$ (.) function, equation (3.71) reduces to;

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k} G_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}\right) d v \\
=\left|u_{\alpha-1}-u_{\alpha-1}^{+}\right| P_{R}\left(u_{\alpha-1}, \rho_{\alpha}\right) q_{B}\left(H_{B}\left(u_{\alpha-1}\right), \rho_{\alpha-1}\right) \rho_{\alpha-1} \int_{0}^{v_{\max }} u_{\alpha-1}^{k} d u_{\alpha-1} \\
=\left|u_{\alpha-1}-u_{\alpha-1}^{+}\right| P_{R}\left(u_{\alpha-1}, \rho_{\alpha}\right) q_{B}\left(H_{B}\left(u_{\alpha-1}\right), \rho_{\alpha-1}\right) \rho_{\alpha-1} u_{\alpha-1}^{k} \tag{3.72}
\end{gather*}
$$

Similarly equation (3.17) becomes;

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k} L_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) d v \\
=\int_{0}^{v_{\max }} v^{k} \int_{\hat{v}_{-}>v} P_{L}\left(v, \rho_{\alpha-1}\right)\left[1-P_{R}\left(\hat{v}_{-}, \rho_{\alpha+1}\right)\right]\left|v-\hat{v}_{-}\right| q_{B}\left(\left(\hat{v}_{-}\right), \rho_{\alpha}\right) \\
\times \rho_{\alpha} \delta_{u_{\alpha}^{-}}\left(\hat{v}_{-}\right) \delta_{u_{\alpha}}(v) d \hat{v}_{-} d v \tag{3.73}
\end{gather*}
$$

As a consequence of Dirac delta $\delta$ (.) function, equation (3.73) reduces to:

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k} L_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) d v \\
=P_{L}\left(u_{\alpha}, \rho_{\alpha-1}\right)\left[1-P_{R}\left(u_{\alpha}^{-}, \rho_{\alpha+1}\right)\right]\left|u_{\alpha}-u_{\alpha}^{-}\right| q_{B}\left(H_{B}\left(u_{\alpha}^{-}\right), \rho_{\alpha}\right) \rho_{\alpha} d u_{\alpha} \int_{0}^{v_{\max }} u_{\alpha}^{k} d u_{\alpha} \\
=\rho_{\alpha} u_{\alpha}^{k}\left|u_{\alpha}-u_{\alpha}^{-}\right| P_{L}\left(u_{\alpha}, \rho_{\alpha-1}\right)\left(1-P_{R}\left(u_{\alpha}^{-}, \rho_{\alpha+1}\right)\right) q_{B}\left(H_{B}\left(u_{\alpha}^{-}\right), \rho_{\alpha}\right) \tag{3.74}
\end{gather*}
$$

Combining equations (3.72) and (3.74) as stipulated in equation (3.4) to have:

$$
\begin{gather*}
\int_{0}^{v_{\max }} v^{k}\left[G_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}\right)-L_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right)\right] d v \\
=\left[\rho_{\alpha-1} u_{\alpha-1}^{k}\left|u_{\alpha-1}-u_{\alpha-1}^{+}\right| P_{R}\left(u_{\alpha-1}, \rho_{\alpha}\right) q_{B}\left(H_{B}\left(u_{\alpha-1}\right), \rho_{\alpha-1}\right)\right] \\
-\rho_{\alpha} u_{\alpha}^{k}\left|u_{\alpha}-u_{\alpha}^{-}\right| P_{L}\left(u_{\alpha}, \rho_{\alpha-1}\right)\left(1-P_{R}\left(u_{\alpha}^{-}, \rho_{\alpha+1}\right)\right) q_{B}\left(H_{B}\left(u_{\alpha}^{-}\right), \rho_{\alpha}\right) \tag{3.75}
\end{gather*}
$$

The following approximations of the probabilities are taken as stipulated in (3.68):

$$
\begin{gather*}
P_{R}\left(u_{\alpha-1}, \rho_{\alpha}\right) \sim \exp \left(-\rho_{\alpha} \phi\left(u_{\alpha-1}\right)\right)=e^{\left(-\rho_{\alpha} \phi\left(u_{\alpha-1}\right)\right)} \\
P_{R}\left(u_{\alpha}^{-}, \rho_{\alpha+1}\right) \sim \exp \left(-\rho_{\alpha+1} \phi\left(u_{\alpha}^{-}\right)\right)=e^{\left(-\rho_{\alpha+1} \phi\left(u_{\alpha}^{-}\right)\right)} \\
P_{L}\left(u_{\alpha}, \rho_{\alpha-1}\right) \sim \exp \left(-\rho_{\alpha-1} \phi\left(u_{\alpha}\right)\right)=e^{\left(-\rho_{\alpha-1} \phi\left(u_{\alpha}\right)\right)} \tag{3.76}
\end{gather*}
$$

Where $\phi\left(u_{\alpha-1}\right), \phi\left(u_{\alpha}^{-}\right)$and $\phi\left(u_{\alpha}\right)$ are the rates at which the vehicles are changing lanes.

Approximating the probability distribution function of the leading vehicles as in equation (3.69) by;

$$
\begin{align*}
q_{B}\left(H_{B}\left(u_{\alpha-1}\right), \rho_{\alpha-1}\right) & \sim \frac{1}{1-\rho_{\alpha-1}} \\
q_{B}\left(H_{B}\left(u_{\alpha}^{-}\right), \rho_{\alpha}\right) & \sim \frac{1}{1-\rho_{\alpha}} \tag{3.77}
\end{align*}
$$

Taking $\phi\left(u_{\alpha-1}\right)=\phi\left(u_{\alpha}^{-}\right)=\phi\left(u_{\alpha}\right) \sim C_{0}$, a constant and on substitution, equation (3.75) becomes:

$$
\begin{align*}
& \int_{0}^{v_{\max }} v^{k}\left[G_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}\right)-L_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right)\right] d v \\
& \quad=\rho_{\alpha-1} u_{\alpha-1}^{k}\left|u_{\alpha-1}-u_{\alpha-1}^{+}\right| e^{-\rho_{\alpha} C_{0}}\left(\frac{1}{1-\rho_{\alpha-1}}\right) \\
& -\rho_{\alpha} u_{\alpha}^{k}\left|u_{\alpha}-u_{\alpha}^{-}\right| e^{-\rho_{\alpha-1} C_{0}}\left(1-e^{-\rho_{\alpha+1} C_{0}}\right)\left(\frac{1}{1-\rho_{\alpha}}\right) \tag{3.78}
\end{align*}
$$

For $k=0$ and $k=1$, the RHS of equation (3.78) respectively is denoted by:

$$
\begin{gather*}
\Phi_{L}^{0}(\alpha-1, \alpha, \alpha+1) \\
=\rho_{\alpha-1}\left|u_{\alpha-1}-u_{\alpha-1}^{+}\right| e^{-\rho_{\alpha} C_{0}}\left(\frac{1}{1-\rho_{\alpha-1}}\right) \\
-\rho_{\alpha}\left|u_{\alpha}-u_{\alpha}^{-}\right| e^{-\rho_{\alpha-1} C_{0}}\left(1-e^{-\rho_{\alpha+1} C_{0}}\right)\left(\frac{1}{1-\rho_{\alpha}}\right)  \tag{3.79}\\
\Phi_{L}^{1}(\alpha-1, \alpha, \alpha+1) \\
=\rho_{\alpha-1} u_{\alpha-1}\left|u_{\alpha-1}-u_{\alpha-1}^{+}\right| e^{-\rho_{\alpha} C_{0}}\left(\frac{1}{1-\rho_{\alpha-1}}\right) \\
-\rho_{\alpha} u_{\alpha}\left|u_{\alpha}-u_{\alpha}^{-}\right| e^{-\rho_{\alpha-1} C_{0}}\left(1-e^{-\rho_{\alpha+1} C_{0}}\right)\left(\frac{1}{1-\rho_{\alpha}}\right) \tag{3.80}
\end{gather*}
$$

Where $\Phi_{L}^{0}(\alpha-1, \alpha, \alpha+1)$ and $\Phi_{L}^{1}(\alpha-1, \alpha, \alpha+1)$ represent the vehicles interaction terms due to lane changing to the left.
Similarly for $k=0$ and $k=1$ the RHS of equation (3.70) respectively is denoted by:

$$
\begin{gather*}
\Phi_{R}^{0}(\alpha, \alpha+1, \alpha+2)=\rho_{\alpha+1}\left|u_{\alpha+1}-u_{\alpha+1}^{-}\right| e^{-\rho_{\alpha} C_{0}}\left(\frac{1}{1-\rho_{\alpha+1}}\right) \\
\times\left(1-e^{-\rho_{\alpha+2} C_{0}}\right)-\rho_{\alpha}\left|u_{\alpha}-u_{\alpha}^{+}\right| e^{-\rho_{\alpha+1} C_{0}}\left(\frac{1}{1-\rho_{\alpha}}\right)  \tag{3.81}\\
\Phi_{R}^{1}(\alpha, \alpha+1, \alpha+2)=\rho_{\alpha+1} u_{\alpha+1}\left|u_{\alpha+1}-u_{\alpha+1}^{-}\right| e^{-\rho_{\alpha} C_{0}}\left(\frac{1}{1-\rho_{\alpha+1}}\right) \\
\times\left(1-e^{-\rho_{\alpha+2} C_{0}}\right)-\rho_{\alpha} u_{\alpha}\left|u_{\alpha}-u_{\alpha}^{+}\right| e^{-\rho_{\alpha+1} C_{0}}\left(\frac{1}{1-\rho_{\alpha}}\right) \tag{3.82}
\end{gather*}
$$

where $\Phi_{R}^{0}(\alpha, \alpha+1, \alpha+2)$ and $\Phi_{R}^{1}(\alpha, \alpha+1, \alpha+2)$ represent the vehicles interaction terms due to lane changing to the right.
Therefore equation (3.38) becomes:

$$
\begin{gather*}
\partial_{t} \int_{0}^{v_{\max }} v^{k} f_{\alpha}(x, v, t) d v+\partial_{x} \int_{0}^{v_{\max }} v^{k+1} f_{\alpha}(x, v, t) d v \\
\quad=\Phi_{L}^{k}(\alpha-1, \alpha, \alpha+1)+\Phi_{R}^{k}(\alpha-1, \alpha, \alpha+1) \tag{3.83}
\end{gather*}
$$

Since $f_{\alpha}(x, v, t)=\rho_{\alpha} \delta_{u_{\alpha}}(v)$ from equation (3.1) then (3.83) reduces to:

$$
\begin{gather*}
\partial_{t} \int_{0}^{v_{\max }} v^{k} \rho_{\alpha} \delta_{u_{\alpha}}(v) d v+\partial_{x} \int_{0}^{v_{\max }} v^{k+1} \rho_{\alpha} \delta_{u_{\alpha}}(v) d v \\
\quad=\Phi_{L}^{k}(\alpha-1, \alpha, \alpha+1)+\Phi_{R}^{k}(\alpha-1, \alpha, \alpha+1) \tag{3.84}
\end{gather*}
$$

For $k=0$, we obtain the conservation equation in terms of the numbers of vehicles flowing in and out of a highway cell as:

$$
\begin{equation*}
\partial_{t} \rho_{\alpha}+\partial_{x}\left(\rho_{\alpha} u_{\alpha}\right)=\Phi_{L}^{0}(\alpha-1, \alpha, \alpha+1)+\Phi_{R}^{0}(\alpha, \alpha+1, \alpha+2) \tag{3.85}
\end{equation*}
$$

And for $k=1$ we get the equation of momentum as;

$$
\begin{gather*}
\partial_{t}\left(\rho_{\alpha} u_{\alpha}\right)+\partial_{x}\left(\rho_{\alpha} u_{\alpha}^{2}\right)-a\left(\rho_{\alpha}\right) \partial_{x} u_{\alpha} \\
=\Phi_{L}^{1}(\alpha-1, \alpha, \alpha+1)+\Phi_{R}^{1}(\alpha, \alpha+1, \alpha+2) \tag{3.86}
\end{gather*}
$$

Where $a\left(\rho_{\alpha}\right)$ is the anticipation term from drivers due to speed adaptation effect given by (3.61) and deduced from (3.62) as;

$$
\begin{equation*}
a\left(\rho_{\alpha}\right)=\frac{\rho_{\alpha}}{1-\rho_{\alpha}} C \tag{3.87}
\end{equation*}
$$

Rewriting equations (3.85) and (3.86) in two forms based on how (3.86) is
formulated, and first dealing with the homogeneous part of both equations which represent the continuous flow of the traffic in the highway, thus;

1. None conservative form (primitive) in terms of primitive variables $\rho_{\alpha}$ and $u_{\alpha}$, equation (3.85) and (3.86) becomes;

$$
\begin{gather*}
\partial_{t} \rho_{\alpha}+\partial_{x}\left(\rho_{\alpha} u_{\alpha}\right)=0  \tag{3.88}\\
\partial_{t} u_{\alpha}+\left(u_{\alpha}-\frac{a\left(\rho_{\alpha}\right)}{\rho_{\alpha}}\right) \partial_{x} u_{\alpha}=0 \tag{3.89}
\end{gather*}
$$

Equation (3.89) describes the dynamics rate of velocity $u_{\alpha}$ in space and time while (3.88) is the equation of continuity.
2. Conservative form: To determine the conservative form of equations (3.88) and (3.89), the following equation is introduced from (3.87):

$$
\begin{equation*}
\frac{a\left(\rho_{\alpha}\right)}{\rho_{\alpha}}=\rho_{\alpha} \frac{C}{\rho_{\alpha}\left(1-\rho_{\alpha}\right)}=\rho_{\alpha} p^{\prime}\left(\rho_{\alpha}\right) \tag{3.90}
\end{equation*}
$$

where $p \prime\left(\rho_{\alpha}\right)$ denotes differentiation with respect to $\rho_{\alpha}$, see equation (3.61). Multiplying (3.88) throughout by $p^{\prime}\left(\rho_{\alpha}\right)$ to get;

$$
\begin{equation*}
p^{\prime}\left(\rho_{\alpha}\right)\left(\partial_{t} \rho_{\alpha}+u_{\alpha} \partial_{x} \rho_{\alpha}+\rho_{\alpha} \partial_{x} u_{\alpha}\right)=0 \tag{3.91}
\end{equation*}
$$

or

$$
\begin{equation*}
\rho_{\alpha} p \prime\left(\rho_{\alpha}\right) \partial_{x} u_{\alpha}=-p \prime\left(\rho_{\alpha}\right)\left(\partial_{t} \rho_{\alpha}+u_{\alpha} \partial_{x} \rho_{\alpha}\right) \tag{3.92}
\end{equation*}
$$

substituting equation (3.92) into (3.89) to get;

$$
\begin{equation*}
\partial_{t} u_{\alpha}+u_{\alpha} \partial_{x} u_{\alpha}+p^{\prime}\left(\rho_{\alpha}\right)\left(\partial_{t} \rho_{\alpha}+u_{\alpha} \partial_{x} \rho_{\alpha}\right)=0 \tag{3.93}
\end{equation*}
$$

or

$$
\begin{equation*}
\partial_{t}\left(u_{\alpha}+p\left(\rho_{\alpha}\right)\right)+u_{\alpha} \partial_{x}\left(u_{\alpha}+p\left(\rho_{\alpha}\right)\right)=0 \tag{3.94}
\end{equation*}
$$

Multiplying (3.88) by the term $u_{\alpha}+p\left(\rho_{\alpha}\right)$, and (3.94) by $\rho_{\alpha}$ and adding the resulting equations to get;

$$
\begin{gather*}
\left(u_{\alpha}+p\left(\rho_{\alpha}\right)\right) \partial_{t} \rho_{\alpha}+\left(u_{\alpha}+p\left(\rho_{\alpha}\right)\right) \partial_{x} \rho_{\alpha} u_{\alpha} \\
+\rho_{\alpha} \partial_{t}\left(u_{\alpha}+p\left(\rho_{\alpha}\right)\right)+\rho_{\alpha} u_{\alpha} \partial_{x}\left(u_{\alpha}+p\left(\rho_{\alpha}\right)\right)=0 \tag{3.95}
\end{gather*}
$$

Equation (3.95) on re-arranging and combining with (3.88) give the following
conservative form of the system of equations;

$$
\begin{gather*}
\partial_{t} \rho_{\alpha}+\partial_{x}\left(\rho_{\alpha} u_{\alpha}\right)=0  \tag{3.96}\\
\partial_{t}\left[\rho_{\alpha}\left(u_{\alpha}+p\left(\rho_{\alpha}\right)\right)\right]+\partial_{x}\left[\rho_{\alpha} u_{\alpha}\left(u_{\alpha}+p\left(\rho_{\alpha}\right)\right)\right]=0 \tag{3.97}
\end{gather*}
$$

where the conservative variables are traffic density $\rho_{\alpha}$ which describe the number of vehicles on a lane of a roadway segment and the traffic momentum is denoted by $\gamma_{\alpha}$ which describe the aggregate momentum of all vehicles present given by;

$$
\begin{equation*}
\gamma_{\alpha}=\rho_{\alpha} u_{\alpha}+\rho_{\alpha} p\left(\rho_{\alpha}\right) \tag{3.98}
\end{equation*}
$$

### 3.5 Specification of the Relaxation Term

In order to show the appearance of the three traffic phase transitions $(F \rightarrow S \rightarrow J)$ that occur at the bottlenecks such as on-ramps, off-ramps and weaving areas, a relaxation term is introduced to the velocity dynamics equation (3.86) as specified in (Kimathi, 2012). It is through this relaxation term which describes the tendency of drivers to relax to the equilibrium situation that Kerner (2010) hypothesis of 3-phase traffic flow is incorporated into our model for multi-lane traffic flow. In this study, the relaxation term depends on the speed of the vehicles and traffic density on the particular lanes, thus it is specified as:

$$
\begin{equation*}
R\left(\rho_{\alpha}, u_{\alpha}\right)=\frac{1}{T}\left(U_{\alpha}^{e}\left(\rho_{\alpha}, u_{\alpha}\right)-u_{\alpha}\right) \tag{3.99}
\end{equation*}
$$

where $T$ is the relaxation time and
$U_{\alpha}^{e}\left(\rho_{\alpha}, u_{\alpha}\right)=\left\{\begin{array}{lll}u_{1}^{e}\left(\rho_{\alpha}\right), & \rho_{\alpha}<\rho_{\alpha, \text { min }}^{\text {syn }}, & \text { or } u_{\alpha}>R\left(\rho_{\alpha}\right), \\ \rho_{\alpha, \text { min }}^{\text {syn }}<\rho_{\alpha}<\rho_{\alpha, \text { max }}^{\text {free }} \\ u_{2}^{e}\left(\rho_{\alpha}\right), & u_{\alpha}<R\left(\rho_{\alpha}\right), & \rho_{\alpha, \text { min }}^{\text {syn }}<\rho_{\alpha}<\rho_{\alpha, \text { max }}^{\text {free }}, \text { or } \rho_{\alpha}>\rho_{\alpha, \text { max }}^{\text {free }}\end{array}\right.$
$u_{1}^{e}\left(\rho_{\alpha}\right)$ and $u_{2}^{e}\left(\rho_{\alpha}\right)$ are two optimal velocity curves defined in Kimathi (2012) such that $u_{2}^{e}\left(\rho_{\alpha}\right)<u_{1}^{e}\left(\rho_{\alpha}\right), 0 \leq \rho_{\alpha}<\rho_{\alpha, \max }$ with $u_{1}\left(\rho_{\alpha, \max }\right)=u_{2}\left(\rho_{\alpha}, \max \right)=0$ The two optimal velocity are monotone decreasing functions of density satisfying the property;
$u_{1}^{e}\left(\rho_{\alpha}\right)>R\left(\rho_{\alpha}\right)>u_{2}^{e}\left(\rho_{\alpha}\right), \rho_{\alpha, \text { min }}^{\text {syn }}<\rho_{\alpha}<\rho_{\alpha, \text { max }}^{\text {free }}$
with $u_{2}^{e}\left(\rho_{\alpha, \text { min }}^{\text {syn }}\right)=R\left(\rho_{\alpha, \text { min }}^{\text {syn }}\right), u_{1}^{e}\left(\rho_{\alpha, \text { max }}^{\text {free }}\right)=R\left(\rho_{\alpha, \text { max }}^{\text {free }}\right)$.
Here $\rho_{\alpha, \text { min }}^{\text {syn }}$ is the minimum density below which synchronized flow cannot occur, $\rho_{\alpha, \text { max }}^{\text {free }}$ is the limit density for free flow existence and $R\left(\rho_{\alpha}\right)$ is a switching curve
viewing the traffic dynamics from the density perspective.
The curve $u_{1}^{e}\left(\rho_{\alpha}\right)$ characterize the fast mode where the traffic is less dense and allow easy lane change and overtaking manoeuvres.
The curve $u_{2}^{e}\left(\rho_{\alpha}\right)$ characterize the slower mode, where the traffic is more dense and give a lesser chance of lane change and overtaking manoeuvres.
Therefore equation (3.86) becomes;

$$
\begin{gather*}
\partial_{t}\left(\rho_{\alpha} u_{\alpha}\right)+\partial_{x}\left(\rho_{\alpha} u_{\alpha}^{2}\right)-a\left(\rho_{\alpha}\right) \partial_{x} u_{\alpha} \\
=\rho_{\alpha} R\left(\rho_{\alpha}, u_{\alpha}\right)+\Phi_{L}^{1}(\alpha-1, \alpha, \alpha+1)+\Phi_{R}^{1}(\alpha, \alpha+1, \alpha+2) \tag{3.101}
\end{gather*}
$$

In the next section, features of the macroscopic traffic flow model are presented.

### 3.6 The Features of the Derived Macroscopic Traffic Flow Model

For a proper numerical approximation equations (3.85) and (3.86) can be cast into:

$$
\begin{equation*}
\partial_{t} U_{\alpha}+\partial_{x} F\left(U_{\alpha}\right)=S\left(U_{\alpha}\right) \tag{3.102}
\end{equation*}
$$

where $\quad U_{\alpha}=\left(\rho_{\alpha}, \gamma_{\alpha}\right)^{T}, \quad F\left(U_{\alpha}\right)=\left(\rho_{\alpha} u_{\alpha}, \gamma_{\alpha} u_{\alpha}\right)^{T} \quad$ and $S\left(U_{\alpha}\right)=\left(\rho_{\alpha} u_{\alpha}, \rho_{\alpha} u_{\alpha}\left(u_{\alpha}+p\left(\rho_{\alpha}\right)\right)\right)^{T}$ are vectors of conserved variables, fluxes and the source term respectively.
To develop the numerical method, the source term $S\left(U_{\alpha}\right)$ is negleted because of its discontinuous character and first deal with the homogeneous part of equation (3.102) which represents a hyperbolic type of wave given as;

$$
\begin{equation*}
\partial_{t} U_{\alpha}+\partial_{x} F\left(U_{\alpha}\right)=0 \tag{3.103}
\end{equation*}
$$

Since the homogeneous part of the systems (3.85)-(3.86) and (3.88)- (3.89) are identical for smooth solutions, then the later system is used to show the hyperbolic features (wave form nature) of the derived macroscopic traffic flow model equations by expressing (3.88)- (3.89) in vector form of primitive variables $\rho_{\alpha}$ and $u_{\alpha}$. To approximate the flux in the road segment, we decompose the Jacobian matrix into its eigen values and eigen vectors. Thus,
the system of equations (3.103) is hyperbolic if the Jacobian matrix;

$$
J\left(U_{\alpha}\right)=\frac{\partial F\left(U_{\alpha}\right)}{\partial U_{\alpha}}=\left(\begin{array}{ll}
\frac{\partial f_{1}}{\partial u_{1}} & \frac{\partial f_{1}}{\partial u_{2}} \\
\frac{\partial f_{2}}{\partial u_{1}} & \frac{\partial f_{2}}{\partial u_{2}}
\end{array}\right)
$$

has real eigenvalues $\lambda_{1}\left(U_{\alpha}\right)$ and $\lambda_{2}\left(U_{\alpha}\right)$, where $\lambda_{1}\left(U_{\alpha}\right) \leq \lambda_{2}\left(U_{\alpha}\right)$.
To compute the eigenvalues of $J\left(U_{\alpha}\right)$, equations (3.88) and (3.89) are expanded to yield;

$$
\begin{gather*}
\partial_{t} \rho_{\alpha}+u_{\alpha} \partial_{x} \rho_{\alpha}+\rho_{\alpha} \partial_{x} u_{\alpha}=0  \tag{3.104}\\
\partial_{t} u_{\alpha}+0 \partial_{x} \rho_{\alpha}+\left(u_{\alpha}-\frac{a\left(\rho_{\alpha}\right)}{\rho_{\alpha}}\right) \partial_{x} u_{\alpha}=0 \tag{3.105}
\end{gather*}
$$

therefore;

$$
J\left(U_{\alpha}\right)=\left(\begin{array}{cc}
u_{\alpha} & \rho_{\alpha}  \tag{3.106}\\
0 & u_{\alpha}-\rho_{\alpha} p^{\prime}\left(\rho_{\alpha}\right)
\end{array}\right)
$$

and the eigenvalues $\lambda_{i}$ corresponding to $J\left(U_{\alpha}\right)$ are computed as;

$$
\left|J\left(U_{\alpha}\right)-\lambda_{i} I\right|=\left|\begin{array}{cc}
u_{\alpha}-\lambda_{i} & \rho_{\alpha}  \tag{3.107}\\
0 & u_{\alpha}-\rho_{\alpha} p^{\prime}\left(\rho_{\alpha}\right)-\lambda_{i}
\end{array}\right|=0
$$

Solving the equation (3.107), to get $\lambda_{1}=u_{\alpha}-\rho_{\alpha} p^{\prime}\left(\rho_{\alpha}\right)<u_{\alpha}, \lambda_{2}=u_{\alpha}$ which are real and distinct. Therefore the system (3.102) and the macroscopic traffic flow model equations (3.96) and (3.97) are strictly hyperbolic. These eigen values represent the characteristics speed that govern the propagation of information in the traffic stream. From Aw-Rascle model, $\lambda_{2}$ being the largest eigen value equals to the traffic flow velocity $u_{\alpha}$ and is the contact wave (single jump discontinuity) while $\lambda_{1}$ is either a shock or a rarefaction wave. This implies that no traffic information travels faster than the traffic and therefore the anisotropic character of vehicular traffic flow is preserved. With reference to the waves associated with $\lambda_{1}$ as 1 -waves and to those associated with $\lambda_{2}$ as 2 -waves, we determine the right eigenvectors $R^{1}$ and $R^{2}$ of the matrix $J\left(U_{\alpha}\right)$ corresponding to the eigenvalues $\lambda_{i}$, $i=1,2$ respectively as follows:
$R^{1}\left(U_{\alpha}\right)=\left[\begin{array}{c}1 \\ -p \prime\left(\rho_{\alpha}\right)\end{array}\right]$ and $R^{2}\left(U_{\alpha}\right)=\left[\begin{array}{l}1 \\ 0\end{array}\right]$
Letting $\nabla \lambda_{i}\left(U_{\alpha}\right)$ to be the gradient of the eigenvalue $\lambda_{i}\left(U_{\alpha}\right), i=1,2$, we determine the kind of waves associated with each eigenvalue by checking
whether the dot product $\nabla \lambda_{i}\left(U_{\alpha}\right) \cdot R^{i}\left(U_{\alpha}\right)$ is zero or not. That is for $\lambda_{1}\left(U_{\alpha}\right)=u_{\alpha}-\rho_{\alpha} p^{\prime}\left(\rho_{\alpha}\right)$, we have;

$$
\begin{equation*}
\binom{\partial_{\rho_{\alpha}}\left(u_{\alpha}-\rho_{\alpha} p^{\prime}\left(\rho_{\alpha}\right)\right)}{\partial_{u_{\alpha}}\left(u_{\alpha}-\rho_{\alpha} p^{\prime}\left(\rho_{\alpha}\right)\right)} \cdot\binom{1}{-p^{\prime}\left(\rho_{\alpha}\right)}=-\partial_{\rho_{\alpha}}\left(\rho_{\alpha} p^{\prime}\left(\rho_{\alpha}\right)\right)-p^{\prime}\left(\rho_{\alpha}\right) \neq 0 \tag{3.108}
\end{equation*}
$$

and for $\lambda_{2}\left(U_{\alpha}\right)=u_{\alpha}$, we have;

$$
\begin{equation*}
\binom{\partial_{\rho_{\alpha}} u_{\alpha}}{\partial_{u_{\alpha}} u_{\alpha}} \cdot\binom{1}{0}=0 \tag{3.109}
\end{equation*}
$$

implying that $1^{\text {st }}$ characteristics field is genuinely nonlinear and $2^{\text {nd }}$ characteristic field is linearly degenerate. Therefore the 1 -waves are either a rarefaction (smooth) or shock wave (jump discontinuity) and the 2 -waves are contact discontinuities.

In the next subsection, the schemetic diagram of the part of the highway under study and the specific lanes model equations of the bottlenecks are presented.

### 3.6.1 Outline of the Lanes Specific Model Equations

We consider a three lanes highway with an on and off-ramp bottlenecks in the traffic flow simulations. Figure 3.2 show part of the highway with the botllenecks under consideration, the on-ramp merging and the off-ramp diverging zone are as indicated.


Figure 3.2: On ramp and Off ramp interchange

### 3.6.1.1 On-ramp Scenario

(i) When $\alpha=1$, the vehicles on lane 1 can only change to lane 2 otherwise the vehicles continue moving in lane 1.

Since the probability of lane-change from lane 1 to lane 0 is zero, then equations (3.79) to (3.82) respectively reduces to;

$$
\begin{align*}
& \Phi_{L}^{0}(0,1,2)=\rho_{0}\left|u_{0}-u_{0}^{+}\right| e^{-\rho_{1} C_{0}}\left(\frac{1}{1-\rho_{0}}\right)  \tag{3.110}\\
& \Phi_{L}^{1}(0,1,2)=\rho_{0} u_{0}\left|u_{0}-u_{0}^{+}\right| e^{-\rho_{1} C_{0}}\left(\frac{1}{1-\rho_{0}}\right)  \tag{3.111}\\
& \Phi_{R}^{0}(1,2,3)=\rho_{2}\left|u_{2}-u_{2}^{-}\right| e^{-\rho_{1} C_{0}}\left(\frac{1}{1-\rho_{2}}\right) \\
& \times\left(1-e^{-\rho_{3} C_{0}}\right)-\rho_{1}\left|u_{1}-u_{1}^{+}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{1}}\right)  \tag{3.112}\\
& \Phi_{R}^{1}(1,2,3)=\rho_{2} u_{2}\left|u_{2}-u_{2}^{-}\right| e^{-\rho_{1} C_{0}}\left(\frac{1}{1-\rho_{2}}\right) \\
& \times\left(1-e^{-\rho_{3} C_{0}}\right)-\rho_{1} u_{1}\left|u_{1}-u_{1}^{+}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{1}}\right) \tag{3.113}
\end{align*}
$$

and equations (3.85) and (3.86) for lane 1 respectively reduce to;

$$
\begin{gather*}
\partial_{t} \rho_{1}+\partial_{x}\left(\rho_{1} u_{1}\right)=\Phi_{R}^{0}(1,2,3)+\Phi_{L}^{0}(0,1,2)  \tag{3.114}\\
\partial_{t}\left(\rho_{1} u_{1}\right)+\partial_{x}\left(\rho_{1} u_{1}^{2}\right)-a\left(\rho_{1}\right) \partial_{x} u_{1}=\Phi_{R}^{1}(1,2,3)+\Phi_{L}^{1}(0,1,2) \tag{3.115}
\end{gather*}
$$

(ii) When $\alpha=2$, the vehicles on lane 2 can change lanes to right lane or left lane otherwise vehicles stay in lane 2 .
Therefore equations (3.79) to (3.82) respectively reduces to;

$$
\begin{gather*}
\Phi_{L}^{0}(1,2,3)=\rho_{1}\left|u_{1}-u_{1}^{+}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{1}}\right) \\
-\rho_{2}\left|u_{2}-u_{2}^{-}\right| e^{-\rho_{1} C_{0}}\left(1-e^{-\rho_{3} C_{0}}\right)\left(\frac{1}{1-\rho_{2}}\right)  \tag{3.116}\\
\Phi_{L}^{1}(1,2,3)=\rho_{1} u_{1}\left|u_{1}-u_{1}^{+}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{1}}\right) \\
-\rho_{2} u_{2}\left|u_{2}-u_{2}^{-}\right| e^{-\rho_{1} C_{0}}\left(1-e^{-\rho_{3} C_{0}}\right)\left(\frac{1}{1-\rho_{2}}\right)  \tag{3.117}\\
\Phi_{R}^{0}(2,3)=\rho_{3}\left|u_{3}-u_{3}^{+}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{3}}\right) \\
-\rho_{2}\left|u_{2}-u_{2}^{-}\right| e^{-\rho_{3} C_{0}}\left(\frac{1}{1-\rho_{2}}\right) \tag{3.118}
\end{gather*}
$$

$$
\begin{gather*}
\Phi_{R}^{1}(2,3)=\rho_{3} u_{3}\left|u_{3}-u_{3}^{+}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{3}}\right) \\
\quad-\rho_{2} u_{2}\left|u_{2}-u_{2}^{-}\right| e^{-\rho_{3} C_{0}}\left(\frac{1}{1-\rho_{2}}\right) \tag{3.119}
\end{gather*}
$$

Thus equations (3.85) and (3.86) is given by;

$$
\begin{gather*}
\partial_{t} \rho_{2}+\partial_{x}\left(\rho_{2} u_{2}\right)=\Phi_{R}^{0}(2,3)+\Phi_{L}^{0}(1,2,3)  \tag{3.120}\\
\partial_{t}\left(\rho_{2} u_{2}\right)+\partial_{x}\left(\rho_{2} u_{2}^{2}\right)-a\left(\rho_{2}\right) \partial_{x} u_{2}=\Phi_{R}^{1}(2,3)+\Phi_{L}^{1}(1,2,3) \tag{3.121}
\end{gather*}
$$

(iii) When $\alpha=3$, the vehicles can only change lane to the left lane otherwise vehicles remain in lane 3 and the probability of lane-change from lane 3 to lane 4 is zero, equations (3.79) to (3.82) respectively reduce to;

$$
\begin{align*}
& \Phi_{L}^{0}(2,3)=\rho_{2}\left|u_{2}-u_{2}^{+}\right| e^{-\rho_{3} C_{0}}\left(\frac{1}{1-\rho_{2}}\right)-\rho_{3}\left|u_{3}-u_{3}^{-}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{3}}\right)  \tag{3.122}\\
& \Phi_{L}^{1}(2,3)=\rho_{2} u_{2}\left|u_{2}-u_{2}^{+}\right| e^{-\rho_{3} C_{0}}\left(\frac{1}{1-\rho_{2}}\right)-\rho_{3} u_{3}\left|u_{3}-u_{3}^{-}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{3}}\right) \tag{3.123}
\end{align*}
$$

Therefore equations (3.85) and (3.86) becomes;

$$
\begin{gather*}
\partial_{t} \rho_{3}+\partial_{x}\left(\rho_{3} u_{3}\right)=\Phi_{L}^{0}(2,3)  \tag{3.124}\\
\partial_{t}\left(\rho_{3} u_{3}\right)+\partial_{x}\left(\rho_{3} u_{3}^{2}\right)-a\left(\rho_{3}\right) \partial_{x} u_{3}=\Phi_{L}^{1}(2,3) \tag{3.125}
\end{gather*}
$$

### 3.6.1.2 Off-ramp Scenario

(i) When $\alpha=1$, the vehicles on lane 1 can diverge to lane 0 or change to lane 2 otherwise the vehicles continue moving in lane 1.
Therefore equations (3.79) to (3.82) reduces to;

$$
\begin{gather*}
\Phi_{L}^{0}(0,1,2)=-\rho_{1}\left|u_{1}-u_{1}^{-}\right| e^{-\rho_{0} c_{0}}\left(1-e^{-\rho_{2} c_{0}}\right) \frac{1}{1-\rho_{1}}  \tag{3.126}\\
\Phi_{L}^{1}(0,1,2)=-\rho_{1} u_{1}\left|u_{1}-u_{1}^{-}\right| e^{-\rho_{0} c_{0}}\left(1-e^{-\rho_{2} c_{0}}\right) \frac{1}{1-\rho_{1}}  \tag{3.127}\\
\Phi_{R}^{0}(1,2,3)=\rho_{2}\left|u_{2}-u_{2}^{-}\right| e^{-\rho_{1} C_{0}}\left(\frac{1}{1-\rho_{2}}\right) \\
\times\left(1-e^{-\rho_{3} C_{0}}\right)-\rho_{1}\left|u_{1}-u_{1}^{+}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{1}}\right) \tag{3.128}
\end{gather*}
$$

$$
\begin{align*}
& \Phi_{R}^{1}(1,2,3)=\rho_{2} u_{2}\left|u_{2}-u_{2}^{-}\right| e^{-\rho_{1} C_{0}}\left(\frac{1}{1-\rho_{2}}\right) \\
\times & \left(1-e^{-\rho_{3} C_{0}}\right)-\rho_{1} u_{1}\left|u_{1}-u_{1}^{+}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{1}}\right) \tag{3.129}
\end{align*}
$$

Equations (3.85) and (3.86) for lane 1 respectively reduces to;

$$
\begin{gather*}
\partial_{t} \rho_{1}+\partial_{x}\left(\rho_{1} u_{1}\right)=\Phi_{R}^{0}(1,2,3)+\Phi_{L}^{0}(0,1,2)  \tag{3.130}\\
\partial_{t}\left(\rho_{1} u_{1}\right)+\partial_{x}\left(\rho_{1} u_{1}^{2}\right)-a\left(\rho_{1}\right) \partial_{x} u_{1}=\Phi_{R}^{1}(1,2,3)+\Phi_{L}^{1}(0,1,2) \tag{3.131}
\end{gather*}
$$

(ii) When $\alpha=2$, the vehicles on lane 2 can change lanes to right lane or left lane otherwise vehicles stay in lane 2 .
Therefore equations (3.79) to (3.82) respectively reduces to;

$$
\begin{gather*}
\Phi_{L}^{0}(1,2,3)=\rho_{1}\left|u_{1}-u_{1}^{+}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{1}}\right) \\
-\rho_{2}\left|u_{2}-u_{2}^{-}\right| e^{-\rho_{1} C_{0}}\left(1-e^{-\rho_{3} C_{0}}\right)\left(\frac{1}{1-\rho_{2}}\right)  \tag{3.132}\\
\Phi_{L}^{1}(1,2,3)=\rho_{1} u_{1}\left|u_{1}-u_{1}^{+}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{1}}\right) \\
-\rho_{2} u_{2}\left|u_{2}-u_{2}^{-}\right| e^{-\rho_{1} C_{0}}\left(1-e^{-\rho_{3} C_{0}}\right)\left(\frac{1}{1-\rho_{2}}\right)  \tag{3.133}\\
\Phi_{R}^{0}(2,3)=\rho_{3}\left|u_{3}-u_{3}^{+}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{3}}\right) \\
-\rho_{2}\left|u_{2}-u_{2}^{-}\right| e^{-\rho_{3} C_{0}}\left(\frac{1}{1-\rho_{2}}\right)  \tag{3.134}\\
\Phi_{R}^{1}(2,3)=\rho_{3} u_{3}\left|u_{3}-u_{3}^{+}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{3}}\right) \\
-\rho_{2} u_{2}\left|u_{2}-u_{2}^{-}\right| e^{-\rho_{3} C_{0}}\left(\frac{1}{1-\rho_{2}}\right) \tag{3.135}
\end{gather*}
$$

Thus equations (3.85) and (3.86) is given by;

$$
\begin{gather*}
\partial_{t} \rho_{2}+\partial_{x}\left(\rho_{2} u_{2}\right)=\Phi_{R}^{0}(2,3)+\Phi_{L}^{0}(1,2,3)  \tag{3.136}\\
\partial_{t}\left(\rho_{2} u_{2}\right)+\partial_{x}\left(\rho_{2} u_{2}^{2}\right)-a\left(\rho_{2}\right) \partial_{x} u_{2}=\Phi_{R}^{1}(2,3)+\Phi_{L}^{1}(1,2,3) \tag{3.137}
\end{gather*}
$$

(iii) When $\alpha=3$, the vehicles can only change lane to the left lane otherwise vehicles remain in lane 3 and the probability of lane-change from lane 3 to lane

4 is zero.
Therefore equations (3.79) to (3.82) respectively reduce to;

$$
\begin{align*}
& \Phi_{L}^{0}(2,3)=\rho_{2}\left|u_{2}-u_{2}^{+}\right| e^{-\rho_{3} C_{0}}\left(\frac{1}{1-\rho_{2}}\right)-\rho_{3}\left|u_{3}-u_{3}^{-}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{3}}\right)  \tag{3.138}\\
& \Phi_{L}^{1}(2,3)=\rho_{2} u_{2}\left|u_{2}-u_{2}^{+}\right| e^{-\rho_{3} C_{0}}\left(\frac{1}{1-\rho_{2}}\right)-\rho_{3} u_{3}\left|u_{3}-u_{3}^{-}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{3}}\right) \tag{3.139}
\end{align*}
$$

Thus equations (3.85) and (3.86) becomes;

$$
\begin{gather*}
\partial_{t} \rho_{3}+\partial_{x}\left(\rho_{3} u_{3}\right)=\Phi_{L}^{0}(2,3)  \tag{3.140}\\
\partial_{t}\left(\rho_{3} u_{3}\right)+\partial_{x}\left(\rho_{3} u_{3}^{2}\right)-a\left(\rho_{3}\right) \partial_{x} u_{3}=\Phi_{L}^{1}(2,3) \tag{3.141}
\end{gather*}
$$

### 3.6.1.3 Weaving Area Scenario

When the on-ramp and off-ramp are too close to one another on the highway, they form a freeway weaving section. The two ramps are joined by an auxiliary lane where two merges and two diverges are superposed. Thus traffic weaving is the crossing of two or more traffic streams traveling in the same direction in a limited segment length. Since the lane change interactions occur mainly between auxiliary lane and lane 1 of the highway then all the weaving vehicles must perform mandatory lane change (either merge to or diverge from the highway) within the length of the auxiliary lane. Therefore;
(i) When $\alpha=1$, the vehicles on this lane can either join the auxiliary lane targeting to exit the highway through the off-ramp or change to lane 2 to avoid congestion in the current lane, otherwise the vehicles continue moving in lane 1. At the same time the vehicles from the on-ramp are aiming to merge into the highway as soon as a space gap is available.
Thus equations (3.110) and (3.126), (3.111) and (3.127), (3.112) and (3.128), (3.113) and (3.129) are combined respectively to give;

$$
\begin{gather*}
\Phi_{L}^{0}(0,1,2)=\rho_{0}\left|u_{0}-u_{0}^{+}\right| e^{-\rho_{1} C_{0}}\left(\frac{1}{1-\rho_{0}}\right)-\rho_{1}\left|u_{1}-u_{1}^{-}\right| e^{-\rho_{0} c_{0}}\left(1-e^{-\rho_{2} c_{0}}\right) \frac{1}{1-\rho_{1}} \\
\Phi_{L}^{1}(0,1,2)=\rho_{0} u_{0}\left|u_{0}-u_{0}^{+}\right| e^{-\rho_{1} C_{0}}\left(\frac{1}{1-\rho_{0}}\right)  \tag{3.142}\\
\quad-\rho_{1} u_{1}\left|u_{1}-u_{1}^{-}\right| e^{-\rho_{0} c_{0}}\left(1-e^{-\rho_{2} c_{0}}\right) \frac{1}{1-\rho_{1}} \tag{3.143}
\end{gather*}
$$

$$
\begin{align*}
& \quad \Phi_{R}^{0}(1,2,3)=\rho_{2}\left|u_{2}-u_{2}^{-}\right| e^{-\rho_{1} C_{0}}\left(\frac{1}{1-\rho_{2}}\right) \\
& \times\left(1-e^{-\rho_{3} C_{0}}\right)-\rho_{1}\left|u_{1}-u_{1}^{+}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{1}}\right)  \tag{3.144}\\
& \Phi_{R}^{1}(1,2,3)=\rho_{2} u_{2}\left|u_{2}-u_{2}^{-}\right| e^{-\rho_{1} C_{0}}\left(\frac{1}{1-\rho_{2}}\right) \\
& \times\left(1-e^{-\rho_{3} C_{0}}\right)-\rho_{1} u_{1}\left|u_{1}-u_{1}^{+}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{1}}\right) \tag{3.145}
\end{align*}
$$

Therefore, equations (3.85) and (3.86) for lane 1 respectively reduce to;

$$
\begin{gather*}
\partial_{t} \rho_{1}+\partial_{x}\left(\rho_{1} u_{1}\right)=\Phi_{R}^{0}(1,2,3)+\Phi_{L}^{0}(0,1,2)  \tag{3.146}\\
\partial_{t}\left(\rho_{1} u_{1}\right)+\partial_{x}\left(\rho_{1} u_{1}^{2}\right)-a\left(\rho_{1}\right) \partial_{x} u_{1}=\Phi_{R}^{1}(1,2,3)+\Phi_{L}^{1}(0,1,2) \tag{3.147}
\end{gather*}
$$

(ii) When $\alpha=2$, the vehicles on this lane can change lanes to lane 1 or lane 3 otherwise the vehicles continue moving in lane 2 .
Therefore, equations (3.79) to (3.82) respectively reduces to;

$$
\begin{gather*}
\Phi_{L}^{0}(1,2,3)=\rho_{1}\left|u_{1}-u_{1}^{+}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{1}}\right) \\
-\rho_{2}\left|u_{2}-u_{2}^{-}\right| e^{-\rho_{1} C_{0}}\left(1-e^{-\rho_{3} C_{0}}\right)\left(\frac{1}{1-\rho_{2}}\right)  \tag{3.148}\\
\Phi_{L}^{1}(1,2,3)=\rho_{1} u_{1}\left|u_{1}-u_{1}^{+}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{1}}\right) \\
-\rho_{2} u_{2}\left|u_{2}-u_{2}^{-}\right| e^{-\rho_{1} C_{0}}\left(1-e^{-\rho_{3} C_{0}}\right)\left(\frac{1}{1-\rho_{2}}\right)  \tag{3.149}\\
\Phi_{R}^{0}(2,3)=\rho_{3}\left|u_{3}-u_{3}^{+}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{3}}\right) \\
-\rho_{2}\left|u_{2}-u_{2}^{-}\right| e^{-\rho_{3} C_{0}}\left(\frac{1}{1-\rho_{2}}\right)  \tag{3.150}\\
\Phi_{R}^{1}(2,3)=\rho_{3} u_{3}\left|u_{3}-u_{3}^{+}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{3}}\right) \\
-\rho_{2} u_{2}\left|u_{2}-u_{2}^{-}\right| e^{-\rho_{3} C_{0}}\left(\frac{1}{1-\rho_{2}}\right) \tag{3.151}
\end{gather*}
$$

Thus equations (3.85) and (3.86) is given by;

$$
\begin{gather*}
\partial_{t} \rho_{2}+\partial_{x}\left(\rho_{2} u_{2}\right)=\Phi_{R}^{0}(2,3)+\Phi_{L}^{0}(1,2,3)  \tag{3.152}\\
\partial_{t}\left(\rho_{2} u_{2}\right)+\partial_{x}\left(\rho_{2} u_{2}^{2}\right)-a\left(\rho_{2}\right) \partial_{x} u_{2}=\Phi_{R}^{1}(2,3)+\Phi_{L}^{1}(1,2,3) \tag{3.153}
\end{gather*}
$$

(iii) When $\alpha=3$, the vehicles can only change lane to the left lane otherwise vehicles remain in lane 3 since the probability of lane-change from lane 3 to lane 4 is zero.
Thus equations (3.79) to (3.82) respectively reduce to;

$$
\begin{align*}
& \Phi_{L}^{0}(2,3)=\rho_{2}\left|u_{2}-u_{2}^{+}\right| e^{-\rho_{3} C_{0}}\left(\frac{1}{1-\rho_{2}}\right)-\rho_{3}\left|u_{3}-u_{3}^{-}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{3}}\right)  \tag{3.154}\\
& \Phi_{L}^{1}(2,3)=\rho_{2} u_{2}\left|u_{2}-u_{2}^{+}\right| e^{-\rho_{3} C_{0}}\left(\frac{1}{1-\rho_{2}}\right)-\rho_{3} u_{3}\left|u_{3}-u_{3}^{-}\right| e^{-\rho_{2} C_{0}}\left(\frac{1}{1-\rho_{3}}\right) \tag{3.155}
\end{align*}
$$

Therefore equations (3.85) and (3.86) becomes;

$$
\begin{gather*}
\partial_{t} \rho_{3}+\partial_{x}\left(\rho_{3} u_{3}\right)=\Phi_{L}^{0}(2,3)  \tag{3.156}\\
\partial_{t}\left(\rho_{3} u_{3}\right)+\partial_{x}\left(\rho_{3} u_{3}^{2}\right)-a\left(\rho_{3}\right) \partial_{x} u_{3}=\Phi_{L}^{1}(2,3) \tag{3.157}
\end{gather*}
$$

It should be noted that the lane changing processes when $\alpha=2$ and $\alpha=3$ for the three type of bottlenecks scenarios are similar as stipulated in subsections (3.6.1.1), (3.6.1.2) and (3.6.1.2) respectively.

The discretization and numerical techniques for the solution of the derived macroscopic traffic flow model equations (3.85) and (3.86) are presented in the next chapter.

## CHAPTER FOUR

## METHODS OF SOLUTION

## Introduction

In this chapter we discretize the homogeneous part of the traffic flow model equations (3.102) and the source term (4.18). Godunov scheme ( finite volume method ) is used to solve the macroscopic traffic flow models equations (3.85) and (3.86) via a computer algorithm. We prefer the integral form of conservation laws to differential form because of the following two reasons:
(i) the governing equations are derived based on physical conservation principles expressed as integral relations on control volumes.
(ii) the integral formulation requires less smoothness of the solution, which paves the way to extend the class of admissible solutions to include discontinuous characters (weak solutions), ?.
This method utilizes the Riemann problem which is basically a combination of the governing conservation law, the initial and boundary conditions. We simulate a multi-lane road that have stationary bottlenecks by obtaining a numerical solution to the conservative system (4.19). First, the Riemann problem is solved using the 1-and 2-waves and the jump discontinuities regarded in the equations of conservative form (3.96) and (3.97).

### 4.1 The Riemann Problem and its Solution

To solve numerically the discontinuous (source term) part of the equation (3.102), Riemann problem and its solution is introduced. The conservative law forms of the derived macroscopic traffic flow model equations (3.96) and (3.97) of the AwRascle type is used to set up the Riemann problem with piecewise constant initial data as follows;

$$
\left\{\begin{array}{c}
\partial_{t} U_{\alpha}+\partial_{x} F\left(U_{\alpha}\right)=0  \tag{4.1}\\
U_{\alpha}(x, 0)=\left\{\begin{array}{lll}
U_{\alpha, L} & \text { if } & x<0 \\
U_{\alpha, R} & \text { if } & x>0
\end{array}\right.
\end{array}\right.
$$

where $U_{\alpha}=\left(\rho_{\alpha}, \gamma_{\alpha}\right)^{T}, F\left(U_{\alpha}\right)=\left(\rho_{\alpha} u_{\alpha}, \gamma_{\alpha} u_{\alpha}\right)^{T}, U_{\alpha, L}$ and $U_{\alpha, R}$ are the piecewise constant traffic state on the left or right of the jump discontinuity at $x=0$.
Since $\lambda_{1}\left(\rho_{\alpha}, u_{\alpha}\right)<\lambda_{2}\left(\rho_{\alpha}, u_{\alpha}\right)$ for all $U_{\alpha}$, then the general solution of the Riemann problem will have 1- waves connecting the left state $U_{\alpha, L}$ to an intermediate state $U_{\alpha, M}$ and a 2- waves connecting $U_{\alpha, M}$ to the right state
$U_{\alpha, R}$. The 1-waves will either be shock waves when $U_{\alpha, L}>U_{\alpha, R}$ or rarefaction waves when $U_{\alpha, L}<U_{\alpha, R}$. Therefore the following types of solutions exist;
(i) 1-shock wave connecting $U_{\alpha, L}$ to an intermediate state $\left(U_{\alpha, M}\right)$ followed by a 2 -contact wave discontinuity connecting $U_{\alpha, M}$ to $U_{\alpha, R}$. In this case the shock speed $S_{\alpha, 1}$ can either be a backward or a forward propagating wave, see Figure 4.1(a) and (b) respectively. It is noted that $S_{\alpha, T}=S_{\alpha, H}=S_{\alpha, 1}$ for a 1-shock.
(ii) 1-rarefaction wave connecting $U_{\alpha, L}$ to $U_{\alpha, M}$ followed by a 2-contact discontinuity connecting $U_{\alpha, M}$ to $U_{\alpha, R}$. The right edge of the rarefaction fan is denoted by $S_{\alpha, H}$ (rarefaction head) and the left edge of the fan by $S_{\alpha, T}$ (rarefaction tail) as shown in Figure 4.2.

Therefore the two constants states $U_{\alpha, L}$ and $U_{\alpha, R}$ are connected through a

(b)

Figure 4.1: (a and b) Possible shock solutions
single jump discontinuity in a genuinely non-linear field. To determine the intermediate state $U_{\alpha, M}$, we use $i$-Lax curves associated to the $i$-waves,


Figure 4.2: Possible rarefaction fan solution
$i=1,2$ and compute the Riemann invariants so as to represent the solution on ( $\rho_{\alpha}, \rho_{\alpha} u_{\alpha}$ ) phase plane. Here the 1-Lax (2-Lax) is the set of points in the ( $\rho_{\alpha}, \rho_{\alpha} u_{\alpha}$ ) phase plane, which can be connected to a given state by 1-wave or 2 -wave. Phase transition is referred as the event of traffic changing from one state to another. According to Aw \& Rascle (2000), the shock and rarefaction curves coincide for Aw-Rascle system. The 1-Lax curves are determined by a situation where a given left state $U_{\alpha, L}$ is connected to an arbitrary state $U_{\alpha, \star}$ on the right state by a 1 -shock of speed $S_{\alpha, 1}$.
This is only possible if the following entropy condition is satisfied;

$$
\lambda_{\alpha, 1}\left(\rho_{\alpha, \star}, u_{\alpha, \star}\right)<S_{\alpha, 1}<\lambda_{\alpha, 1}\left(\rho_{\alpha, L}, u_{\alpha, L}\right)
$$

Moreover any discontinuity propagating with speed $S_{\alpha, 1}$ must satisfy the RankineHugoniot condition, (?);

$$
\begin{equation*}
F\left(U_{\alpha, \star}\right)-F\left(U_{\alpha, L}\right)=S_{\alpha, 1}\left(U_{\alpha, \star}-U_{\alpha, L}\right) \tag{4.2}
\end{equation*}
$$

Therefore equation (4.2) on substitution reduces to (4.3) and (4.4);

$$
\begin{align*}
& \rho_{\alpha, \star} u_{\alpha, \star}-\rho_{\alpha, L} u_{\alpha, L}=S_{\alpha, 1}\left(\rho_{\alpha, \star}-\rho_{\alpha, L}\right)  \tag{4.3}\\
& \gamma_{\alpha, \star} u_{\alpha, \star}-\gamma_{\alpha, L} u_{\alpha, L}=S_{\alpha, 1}\left(\gamma_{\alpha, \star}-\gamma_{\alpha, L}\right) \tag{4.4}
\end{align*}
$$

Eliminating $S_{\alpha, 1}$ from equations (4.3) and (4.4) gives;

$$
\begin{equation*}
\frac{\gamma_{\alpha, \star} u_{\alpha, \star}-\gamma_{\alpha, L} u_{\alpha, L}}{\rho_{\alpha, \star} u_{\alpha, \star}-\rho_{\alpha, L} u_{\alpha, L}}=\frac{\gamma_{\alpha, \star}-\gamma_{\alpha, L}}{\rho_{\alpha, \star}-\rho_{\alpha, L}} \tag{4.5}
\end{equation*}
$$

on cross multiplication and simplifying equation (4.5) reduces to;

$$
\begin{equation*}
\frac{\gamma_{\alpha, \star}}{\rho_{\alpha, \star}}=\frac{\gamma_{\alpha, L}}{\rho_{\alpha, L}} \tag{4.6}
\end{equation*}
$$

Since the state $U_{\alpha, \star}$ is arbitrary, the 1-lax curves passing through $U_{\alpha, L}$ are derived from equation (4.6) using equation (3.98) in terms of the primitive variables as;

$$
\begin{equation*}
\frac{\gamma_{\alpha, \star}}{\rho_{\alpha, \star}}=u_{\alpha, \star}+p\left(\rho_{\alpha, \star}\right)=u_{\alpha, L}+p\left(\rho_{\alpha, L}\right) \tag{4.7}
\end{equation*}
$$

or

$$
\begin{equation*}
L_{\alpha, 1}\left(\rho_{\alpha} ; \rho_{\alpha, L}, u_{\alpha, L}\right)=u_{\alpha, L}+p\left(\rho_{\alpha, L}\right)-p\left(\rho_{\alpha}\right) \tag{4.8}
\end{equation*}
$$

with the 1-wave being a shock wave when $\rho_{\alpha, L}<\rho_{\alpha, \star}$ and a rarefaction when $\rho_{\alpha, L}>\rho_{\alpha, \star}$.
The 2-lax curves are obtained from one of the Rankine-Hugoniot condition (4.3) or (4.4) and imposing that an arbitrary left state $U_{\alpha, \star}$ can be connected to a given right state $U_{\alpha, R}$ by a contact discontinuity of speed $S_{\alpha, 2}$ with the following parallel characteristics condition, (?) satisfied;

$$
\lambda_{\alpha, 2}\left(\rho_{\alpha, \star}, u_{\alpha, \star}\right)=\lambda_{\alpha, 2}\left(\rho_{\alpha, R}, u_{\alpha, R}\right)=S_{\alpha, 2}
$$

and using equation (3.105) we have;

$$
\begin{equation*}
\rho_{\alpha, \star} u_{\alpha, \star}-\rho_{\alpha, R} u_{\alpha, R}=u_{\alpha, R}\left(\rho_{\alpha, \star}-\rho_{\alpha, R}\right) \tag{4.9}
\end{equation*}
$$

which reduces to $u_{\alpha, \star}=u_{\alpha, R}$. With $U_{\alpha, \star}$ being arbitrary, then 2-lax curves pass through $U_{\alpha, R}$ and are straight lines from origin in the ( $\rho_{\alpha}, \rho_{\alpha} u_{\alpha}$ ) plane given by;

$$
\begin{equation*}
L_{\alpha, 2}\left(\rho_{\alpha} ; \rho_{\alpha, R}, u_{\alpha, R}\right)=u_{\alpha, R} \tag{4.10}
\end{equation*}
$$

Thus the Riemann invariants $w_{\alpha, 1}$ and $w_{\alpha, 2}$ associated with the characteristic are stated as $\lambda_{\alpha, 1}$ and $\lambda_{\alpha, 2}$ respectively;

$$
\begin{equation*}
w_{\alpha, 1}=u_{\alpha}+p\left(\rho_{\alpha}\right), w_{\alpha}=u_{\alpha} \tag{4.11}
\end{equation*}
$$

The solution to the Riemann problem (4.1), say $U_{\alpha, G}$ is given by the set at $x=0$ as follows;

$$
U_{\alpha, G}=\left\{\begin{array}{ccc}
U_{\alpha, M} & \text { if } & S_{\alpha, 1}<0  \tag{4.12}\\
U_{\alpha, L} & \text { if } & S_{\alpha, 1}>0 \\
\tilde{U}_{\alpha} & \text { if } & S_{\alpha, T}<0<S_{\alpha . H}
\end{array}\right.
$$

where $U_{\alpha, M}=\left(\rho_{\alpha, M}, \gamma_{\alpha, M}\right)^{T}$ is the intermediate state computed from the lax curves given by;

$$
\begin{equation*}
u_{\alpha, M}+p\left(\rho_{\alpha, M}\right)=u_{\alpha, L}+p\left(\rho_{\alpha, L}\right) \tag{4.13}
\end{equation*}
$$

Since $u_{\alpha, M}=u_{\alpha, R}$ and $\gamma_{\alpha}=\rho_{\alpha} u_{\alpha}+\rho_{\alpha} p\left(\rho_{\alpha}\right)$ then from (4.13) the following expressions are obtained;

$$
\begin{gather*}
\rho_{\alpha, M}=p^{-1}\left(u_{\alpha, L}+p\left(\rho_{\alpha, L}\right)-u_{\alpha, R}\right) \\
\gamma_{\alpha, M}=\rho_{\alpha, M} \frac{\gamma_{\alpha, L}}{\rho_{\alpha, L}} \tag{4.14}
\end{gather*}
$$

Therefore $U_{\alpha, M}$ is obtained explicitly in terms of $U_{\alpha, L}$ and $U_{\alpha, R}$. Next we obtain the solution $\tilde{U}_{\alpha}=\left(\tilde{\rho}_{\alpha}, \tilde{\gamma}_{\alpha}\right)^{T}$ inside the rarefaction fan by considering the speed of the characteristic rays and the lax curves to get;

$$
\begin{align*}
& \tilde{u}_{\alpha}+p\left(\tilde{\rho}_{\alpha}\right)=u_{\alpha, L}+p\left(\rho_{\alpha, L}\right) \\
& \tilde{u}_{\alpha}-\tilde{\rho}_{\alpha} p\left(\tilde{\rho}_{\alpha}\right)=0 \quad \because \frac{x}{t}=0 \tag{4.15}
\end{align*}
$$

Solving the system of equations (4.15) simultaneously for $\tilde{\rho}_{\alpha}$ and $\tilde{u}_{\alpha}$ gives the desired form of the Riemann solution $\tilde{U}_{\alpha}$. For the speeds $S_{\alpha, 1}, S_{\alpha, T}$ and $S_{\alpha, H}$ , we note that $U_{\alpha, L}$ and $U_{\alpha, M}$ can be connected by a 1 -rarefaction as long as they lie on the same integral curve and that they must satisfy the condition $\lambda_{\alpha, 1}\left(\rho_{\alpha, L}, u_{\alpha, L}\right)<\lambda_{\alpha, 1}\left(\rho_{\alpha, M}, u_{\alpha, M}\right),(?)$.
From the Rankine Hugonoit condition (4.2), the two states should be connected by a 1 -shock if $\lambda_{\alpha, 1}\left(\rho_{\alpha, L}, u_{\alpha, L}\right)>\lambda_{\alpha, 1}\left(\rho_{\alpha, M}, u_{\alpha, M}\right)$ to give;

$$
\begin{equation*}
S_{\alpha, 1}=\frac{\rho_{\alpha, L} u_{\alpha, L}-\rho_{\alpha, M} u_{\alpha, M}}{\rho_{\alpha, L}-\rho_{\alpha, M}} \tag{4.16}
\end{equation*}
$$

Also from (?), the right and left edge of the 1-rarefaction wave carry the value $U_{\alpha, M}$ and $U_{\alpha, L}$ respectively. Using the $\lambda_{1}=U_{\alpha}-\rho_{\alpha} p \prime\left(\rho_{\alpha}\right)$ to obtain;

$$
\begin{gather*}
S_{\alpha, T}=u_{\alpha, L}-\rho_{\alpha, L} p^{\prime}\left(\rho_{\alpha, L}\right) \\
S_{\alpha, H}=u_{\alpha, M}-\rho_{\alpha, M} p^{\prime}\left(\rho_{\alpha, M}\right) \tag{4.17}
\end{gather*}
$$

With the aid of the above i-Lax curves ( for $i=1,2$ ), the following two cases illustrate how the derived Aw-Rascle type traffic flow model (3.96) and (3.97) handle transitions from a left state $U_{L}$ to the right state $U_{R}$ on the ( $\rho_{\alpha}, \rho_{\alpha} u_{\alpha}$ ) phase plane, (Kimathi, 2012).
Case 1: $\rho_{L}=\rho_{R}, u_{L}>u_{R}$. To reach $U_{R}$ from $U_{L}$, the model predicts that the traffic should first decelerate through 1-shock to state $U_{M}$ along the outer 1-Lax curve and then transit from $U_{M}$ to $U_{R}$ along the lower 2-Lax ray while maintaining its average speed.
Case 2: $\rho_{L}>\rho_{R}, u_{L}<u_{R}$. To reach $U_{R}$ from $U_{L}$, the model predicts that traffic first accelerates through 1-rarefaction fan to a new state $U_{\widetilde{M}}$ situated along the inner 1-Lax curve and on the upper 2-Lax curve (ray). Then transits from $U_{\widetilde{M}}$ to $U_{R}$ along the upper 2-Lax curve (ray).
We conclude by defining the source term $S\left(U_{\alpha}\right)$ as the vector:

$$
\begin{equation*}
S\left(U_{\alpha}\right)=\binom{\Phi_{L}^{0}(\alpha-1, \alpha, \alpha+1)+\Phi_{R}^{0}(\alpha, \alpha+1, \alpha+2)}{\Phi_{L}^{1}(\alpha-1, \alpha, \alpha+1)+\Phi_{R}^{1}(\alpha, \alpha+1, \alpha+2)} \tag{4.18}
\end{equation*}
$$

where $\quad \Phi_{L}^{0}(\alpha-1, \alpha, \alpha), \quad \Phi_{L}^{1}(\alpha-1, \alpha, \alpha+1), \quad \Phi_{R}^{0}(\alpha, \alpha+1, \alpha+2) \quad$ and $\Phi_{R}^{1}(\alpha, \alpha+1, \alpha+2)$ are given by the equations (3.79) to (3.82) respectively.
Therefore writing the conservative system (3.96) and (3.97) in vector form as earlier introduced when setting up the Riemann problem in (4.1) and using (4.18), we obtain the following system;

$$
\begin{equation*}
\partial_{t} U_{\alpha}+\partial_{x} F\left(U_{\alpha}\right)=S\left(U_{\alpha}\right) \tag{4.19}
\end{equation*}
$$

which is the conservative form of the macroscopic traffic flow model.

In the next sections the finite volume method and discretized traffic flow model equations of the source terms are presented.

### 4.2 The Godunov Scheme

This is a finite volume method (FVM) which allow a direct discretization of variables in the physical space for arbitrary mesh configuration without necessary an explicit computational of metric coefficient. That is, a discretization method fitted for the numerical simulations of various types of conservation laws. The scheme is proven to satisfy all the positivity constraints of conservative traffic variables. Moreover, the proposed scheme is more robust, fast and reliable than the others for accurate representation of shocks and able to incorporate the flow-
density relations. The main advantage of the finite volume method (FVM) over the finite difference method (FDM) is that the spatial discretization is carried out directly in the physical space and does not need any transformation between the physical and the computational co-ordinate system. Furthermore, the scheme works with control volumes rather than the grid intersection points and therefore has the capacity to accommodate any type of grid. The FVM is based on the solution of the Riemann problem which is defined by an initial value problem equation (4.1).
Given that the general data for the homogeneous system of equation:

$$
\begin{equation*}
\partial_{t} U_{\alpha}+\partial_{x} F\left(U_{\alpha}\right)=0 \tag{4.20}
\end{equation*}
$$

is $\widetilde{U}_{\alpha}\left(x, t^{n}\right)$, the solution is evolved to a time step $t^{n+1}=t^{n}+\Delta t$ by use of the Godunov method in the following steps:
(i) At first the highway is divided into small cells $i$ which have a fixed length $x_{i}-x_{i+1}$ and assume a piecewise constant distribution of data by defining cell averages as;

$$
\begin{equation*}
U_{\alpha, i}^{n}=\frac{1}{\triangle x} \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \widetilde{U}_{\alpha}\left(x, t^{n}\right) d x \tag{4.21}
\end{equation*}
$$

The spatial domain is discretized into M cells, $C_{i}=\left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$ for $i=1 \ldots . M$ of the same size $\triangle x$. These cell averages produce the required piecewise constant distribution $U_{\alpha}\left(x, t^{n}\right)=U_{\alpha, i}^{n}$ for all $x \in C_{i}, i=1 \ldots . M, n \in \mathbb{N}$. As such the data now consist of the set of values $\left\{U_{\alpha, i}^{n}\right\}$.
(ii) For the rectangular control volume $\left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right] \times\left[t^{n}, t^{n+1}\right]$ shown in Figure 4.3, the integral form of the conservative law equation (3.102) can be expressed as;

$$
\begin{gather*}
\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \widetilde{U}_{\alpha}\left(x, t^{n+1}\right) d x=\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \tilde{U}_{\alpha}\left(x, t^{n}\right) d x \\
+\int_{t^{n}}^{t^{n+1}} F\left(\widetilde{U}_{\alpha}\left(x_{i-\frac{1}{2}}, t\right)\right) d t-\int_{t^{n}}^{t^{n+1}} F\left(\widetilde{U}_{\alpha}\left(x_{i+\frac{1}{2}}, t\right)\right) d t \tag{4.22}
\end{gather*}
$$

Now for a time step size $\Delta t=t^{n+1}-t^{n}$ that is sufficiently small (such that there is no wave interaction in the cell $C_{i}$, see Figure 4.3), the solution $\widetilde{U}_{\alpha}(x, t)$ for $t \in\left[t^{n}, t^{n+1}\right]$ and $x \in\left[x_{i}, x_{i+1}\right]$ is defined as follows;

$$
\begin{equation*}
\tilde{U}_{\alpha}(x, t)=U_{\alpha, i+\frac{1}{2}}(\bar{x} / \bar{t}) \tag{4.23}
\end{equation*}
$$



Figure 4.3: Typical rectangular control volume
where $(\bar{x}, \bar{t})$ are local co-ordinates given by;

$$
\begin{array}{cc}
\bar{x}=x-x_{i+\frac{1}{2}}, & \bar{t}=t-t^{n}, \\
x \in\left[x_{i}, x_{i+1}\right], & \left.t \in\left[t^{n}, t^{n+1}\right],\right\}  \tag{4.24}\\
\bar{x} \in\left[-\frac{\Delta x}{2}, \frac{\Delta x}{2}\right], & \bar{t} \in[0, \Delta t]
\end{array}
$$

In terms of the solutions specified in equation (4.23), we have;

$$
\begin{gather*}
\widetilde{U}_{\alpha}\left(x_{i-\frac{1}{2}}, t\right)=U_{\alpha, i-\frac{1}{2}}(0)  \tag{4.25}\\
\tilde{U}_{\alpha}\left(x_{i+\frac{1}{2}}, t\right)=U_{\alpha, i+\frac{1}{2}}(0) \tag{4.26}
\end{gather*}
$$

where $U_{\alpha, i+\frac{1}{2}}(0)$ is a constant state and a solution of the Riemann problem $R P\left(U_{\alpha, i}^{n}, U_{\alpha, i+1}^{n}\right)$ along a ray of constant slope, see Figure (4.3) and likewise $U_{\alpha, i-\frac{1}{2}}(0)$ is the solution of $R P\left(U_{\alpha, i-1}^{n}, U_{\alpha, i}^{n}\right)$.
Dividing equation (4.22) by $\triangle x$ throughout yields;

$$
\begin{align*}
\frac{1}{\triangle x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \widetilde{U}_{\alpha}\left(x, t^{n+1}\right) d x & =\frac{1}{\triangle x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \widetilde{U}_{\alpha}\left(x, t^{n}\right) d x+\frac{1}{\triangle x} \int_{t^{n}}^{t^{n+1}} F\left(\widetilde{U}_{\alpha}\left(x_{i-\frac{1}{2}}, t\right)\right) d t \\
- & \frac{1}{\triangle x} \int_{t^{n}}^{t^{n+1}} F\left(\widetilde{U}_{\alpha}\left(x_{i+\frac{1}{2}}, t\right)\right) d t \tag{4.27}
\end{align*}
$$

which reduces to;

$$
\begin{align*}
\frac{1}{\triangle x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \widetilde{U}_{\alpha}\left(x, t^{n+1}\right) d x & =\frac{1}{\triangle x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \widetilde{U}_{\alpha}\left(x, t^{n}\right) d x+\frac{F}{\triangle x}\left(U_{\alpha, i-\frac{1}{2}}(0)\right)\left[t^{n+1}-t^{n}\right] \\
- & \frac{F}{\triangle x}\left(U_{\alpha, i+\frac{1}{2}}(0)\right)\left[t^{n+1}-t^{n}\right]  \tag{4.28}\\
\frac{1}{\triangle x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \widetilde{U}_{\alpha}\left(x, t^{n+1}\right) d x & =\frac{1}{\triangle x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \widetilde{U}_{\alpha}\left(x, t^{n}\right) d x+\frac{\Delta t}{\triangle x} F\left(U_{\alpha, i-\frac{1}{2}}(0)\right) \\
& -\frac{\triangle t}{\triangle x} F\left(U_{\alpha, i+\frac{1}{2}}(0)\right) \tag{4.29}
\end{align*}
$$

Using the definition (4.21) in equation (4.29), gives the desired Godunov method as:

$$
\begin{equation*}
U_{\alpha, i}^{n+1}=U_{\alpha, i}^{n}+\frac{\Delta t}{\Delta x}\left[F\left(U_{\alpha, i-\frac{1}{2}}^{n}(0)\right)-F\left(U_{\alpha, i+\frac{1}{2}}^{n}(0)\right)\right] \tag{4.30}
\end{equation*}
$$

In order to contain the interactions of the waves within the cell $C_{i}$ during the calculations, Courant-Friedrichs-Lewy restriction is imposed (CFL condition) on time step size;

$$
\begin{equation*}
\Delta t \leq \frac{C_{c f l} \triangle x}{\operatorname{Max}\left\{\left|\lambda_{i}\left(U_{\alpha}\right)\right|, \quad i=1,2\right\}} \tag{4.31}
\end{equation*}
$$

where $C_{c f l} \leq 1$, a constant called the Courant number. This is a condition for numerical stability where the numerical solution is unstable if the errors grow exponentially, which in turn may lead to oscillation of traffic variables with very short wave length.

### 4.3 The Discrete Form of the Source Term

In order to proceed with simulation of the traffic flow features, the source term $S\left(U_{\alpha}\right)$ is introduced to the right hand side of the conservative system of equation (4.20).

The following approximations are used to obtain the required discrete equations. $u_{\alpha}^{+}(x, t) \simeq u_{\alpha}(i+1, k)$ and $u_{\alpha}^{-}(x, t) \simeq u_{\alpha}(i-1, k)$

### 4.3.1 The Discrete Source Terms for an On-ramp

A highway with three lanes and an on-ramp is considered as the bottleneck for the traffic simulations as shown in Figure 4.4. The on-ramp lane is denoted by lane 0 while the other three lanes on the highway are labeled 1,2 and 3 from the
left most. The on-ramp merging zone is the region between the entrance into the highway and the end of the acceleration lane.


Figure 4.4: Section of the highway with three lanes and an on-ramp.
( $i$ ) For lane 1, i.e when $\alpha=1$, equations (3.110), (3.111), (3.112) and (3.113) respectively become;

$$
\begin{gather*}
\Phi_{L}^{0}(0,1,2) \simeq \rho_{0}(i, k)\left|u_{0}(i, k)-u_{0}(i+1, k)\right| e^{-\rho_{1}(i, k) C_{0}}\left(\frac{1}{1-\rho_{0}(i, k)}\right) \\
\Phi_{L}^{1}(0,1,2) \simeq \rho_{0}(i, k) u_{0}(i, k)\left|u_{0}(i, k)-u_{0}(i+1, k)\right| e^{-\rho_{1}(i, k) C_{0}}\left(\frac{1}{1-\rho_{0}(i, k)}\right) \\
\Phi_{R}^{0}(1,2,3) \simeq \rho_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i-1, k)\right| e^{-\rho_{1}(i, k) C_{0}}\left(\frac{1}{1-\rho_{2}(i, k)}\right)  \tag{4.33}\\
\times\left(1-e^{-\rho_{3}(i, k) C_{0}}\right)-\rho_{1}(i, k)\left|u_{1}(i, k)-u_{1}(i+1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{1}(i, k)}\right) \\
\Phi_{R}^{1}(1,2,3) \simeq \rho_{2}(i, k) u_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i-1, k)\right| e^{-\rho_{1}(i, k) C_{0}}\left(\frac{1}{1-\rho_{2}(i, k)}\right) \times  \tag{4.34}\\
\left(1-e^{-\rho_{3}(i, k) C_{0}}\right)-\rho_{1}(i, k) u_{1}(i, k)\left|u_{1}(i, k)-u_{1}(i+1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{1}(i, k)}\right) \tag{4.35}
\end{gather*}
$$

(ii) For lane 2, i.e when $\alpha=2$, equations (3.116) to (3.119) respectively becomes;

$$
\begin{gather*}
\Phi_{L}^{0}(1,2,3) \simeq \rho_{1}(i, k)\left|u_{1}(i, k)-u_{1}(i+1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{1}(i, k)}\right) \\
-\rho_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i-1, k)\right| e^{-\rho_{1}(i, k) C_{0}}\left(1-e^{-\rho_{3}(i, k) C_{0}}\right)\left(\frac{1}{1-\rho_{2}(i, k)}\right) \\
\Phi_{L}^{1}(1,2,3) \simeq \rho_{1}(i, k) u_{1}(i, k)\left|u_{1}(i, k)-u_{1}(i+1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{1}(i, k)}\right) \\
-\rho_{2}(i, k) u_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i-1, k)\right| e^{-\rho_{1}(i, k) C_{0}}\left(1-e^{-\rho_{3}(i, k) C_{0}}\right)\left(\frac{1}{1-\rho_{2}(i, k)}\right)  \tag{4.37}\\
\Phi_{R}^{0}(2,3) \simeq \rho_{3}(i, k)\left|u_{3}(i, k)-u_{3}(i+1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{3}(i, k)}\right) \\
-\rho_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i-1, k)\right| e^{-\rho_{3}(i, k) C_{0}}\left(\frac{1}{1-\rho_{2}(i, k)}\right)  \tag{4.38}\\
\Phi_{R}^{1}(2,3) \simeq \rho_{3}(i, k) u_{3}(i, k)\left|u_{3}(i, k)-u_{3}(i+1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{3}(i, k)}\right) \\
-\rho_{2}(i, k) u_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i-1, k)\right| e^{-\rho_{3}(i, k) C_{0}}\left(\frac{1}{1-\rho_{2}(i, k)}\right) \tag{4.39}
\end{gather*}
$$

(iii) For lane 3, i.e when $\alpha=3$, equations (3.122) and (3.123) respectively becomes;

$$
\begin{gather*}
\Phi_{L}^{0}(2,3) \simeq \rho_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i+1, k)\right| e^{-\rho_{3}(i, k) C_{0}}\left(\frac{1}{1-\rho_{2}(i, k)}\right) \\
-\rho_{3}(i, k)\left|u_{3}(i, k)-u_{3}(i-1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{3}(i, k)}\right)  \tag{4.40}\\
\Phi_{L}^{1}(2,3) \simeq \rho_{2}(i, k) u_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i+1, k)\right| e^{-\rho_{3}(i, k) C_{0}}\left(\frac{1}{1-\rho_{2}(i, k)}\right) \\
-\rho_{3}(i, k) u_{3}(i, k)\left|u_{3}(i, k)-u_{3}(i-1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{3}(i, k)}\right) \tag{4.41}
\end{gather*}
$$

### 4.3.2 The Discrete Source Terms for an Off-ramp

Figure 4.3 shows part of the highway with three lanes and an off-ramp considered for the traffic simulation. Lane 0 denotes the off-ramp lane while the lanes on the highway are labeled as lanes 1,2 and 3 respectively from left. The off-ramp merging zone is part of the highway from the start of deceleration lane to the exit
lane 0 .


Figure 4.5: Section of the highway with three lanes and an off-ramp.
(i) For lane 1, i.e when $\alpha=1$, equations (3.126) to (3.129) respectively becomes;

$$
\begin{gather*}
\Phi_{L}^{0}(0,1,2) \simeq-\rho_{1}(i, k)\left|u_{1}(i, k)-u_{1}(i-1, k)\right| \\
\times\left(e^{-\rho_{0}(i, k) C_{0}}\left(\frac{1}{1-\rho_{1}(i, k)}\right)\left(1-e^{-\rho_{2}(i, k) C_{0}}\right)\right) \\
\Phi_{L}^{1}(0,1,2) \simeq-\rho_{1}(i, k) u_{1}(i, k)\left|u_{1}(i, k)-u_{1}(i-1, k)\right| \\
\times\left(e^{-\rho_{0}(i, k) C_{0}}\left(\frac{1}{1-\rho_{1}(i, k)}\right)\left(1-e^{-\rho_{2}(i, k) C_{0}}\right)\right) \\
\Phi_{R}^{0}(1,2,3) \simeq \rho_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i-1, k)\right| e^{-\rho_{1}(i, k) C_{0}}\left(\frac{1}{1-\rho_{2}(i, k)}\right) \\
\times\left(1-e^{-\rho_{3}(i, k) C_{0}}\right)-\rho_{1}(i, k)\left|u_{1}(i, k)-u_{1}(i+1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{1}(i, k)}\right)  \tag{4.44}\\
\Phi_{R}^{1}(1,2,3) \simeq \rho_{2}(i, k) u_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i-1, k)\right| e^{-\rho_{1}(i, k) C_{0}}\left(\frac{1}{1-\rho_{2}(i, k)}\right) \times \\
\left(1-e^{-\rho_{3}(i, k) C_{0}}\right)-\rho_{1}(i, k) u_{1}(i, k)\left|u_{1}(i, k)-u_{1}(i+1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{1}(i, k)}\right) \tag{4.45}
\end{gather*}
$$

(ii) For lane 2, i.e when $\alpha=2$, equations (3.132) to (3.135) respectively becomes;

$$
\begin{gather*}
\Phi_{L}^{0}(1,2,3) \simeq \rho_{1}(i, k)\left|u_{1}(i, k)-u_{1}(i+1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{1}(i, k)}\right) \\
-\rho_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i-1, k)\right| e^{-\rho_{1}(i, k) C_{0}}\left(1-e^{-\rho_{3}(i, k) C_{0}}\right)\left(\frac{1}{1-\rho_{2}(i, k)}\right) \\
\Phi_{L}^{1}(1,2,3) \simeq \rho_{1}(i, k) u_{1}(i, k)\left|u_{1}(i, k)-u_{1}(i+1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{1}(i, k)}\right) \\
-\rho_{2}(i, k) u_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i-1, k)\right| e^{-\rho_{1}(i, k) C_{0}}\left(1-e^{-\rho_{3}(i, k) C_{0}}\right)\left(\frac{1}{1-\rho_{2}(i, k)}\right)  \tag{4.47}\\
\Phi_{R}^{0}(2,3) \simeq \rho_{3}(i, k)\left|u_{3}(i, k)-u_{3}(i+1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{3}(i, k)}\right) \\
-\rho_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i-1, k)\right| e^{-\rho_{3}(i, k) C_{0}}\left(\frac{1}{1-\rho_{2}(i, k)}\right)  \tag{4.48}\\
\Phi_{R}^{1}(2,3) \simeq \rho_{3}(i, k) u_{3}(i, k)\left|u_{3}(i, k)-u_{3}(i+1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{3}(i, k)}\right) \\
-\rho_{2}(i, k) u_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i-1, k)\right| e^{-\rho_{3}(i, k) C_{0}}\left(\frac{1}{1-\rho_{2}(i, k)}\right) \tag{4.49}
\end{gather*}
$$

(iii) For lane 3, i.e when $\alpha=3$, equations (3.138) and (3.139) respectively becomes;

$$
\begin{gather*}
\Phi_{L}^{0}(2,3) \simeq \rho_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i+1, k)\right| e^{-\rho_{3}(i, k) C_{0}}\left(\frac{1}{1-\rho_{2}(i, k)}\right) \\
-\rho_{3}(i, k)\left|u_{3}(i, k)-u_{3}(i-1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{3}(i, k)}\right)  \tag{4.50}\\
\Phi_{L}^{1}(2,3) \simeq \rho_{2}(i, k) u_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i+1, k)\right| e^{-\rho_{3}(i, k) C_{0}}\left(\frac{1}{1-\rho_{2}(i, k)}\right) \\
-\rho_{3}(i, k) u_{3}(i, k)\left|u_{3}(i, k)-u_{3}(i-1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{3}(i, k)}\right) \tag{4.51}
\end{gather*}
$$

### 4.3.3 The Discrete Source Terms for weaving section

Figure 4.6 shows the section of the weaving area within the three lanes highway where the vehicles diverge from or merge to the highway indicated by the mergingdiverging zone. The arrows indicate direction of the traffic flow.


Figure 4.6: A typical weaving section of the highway showing the merging and diverging traffic manoeuvres.
(i) For lane 1 i.e when $\alpha=1$, equations (3.142) to (3.145) respectively becomes;

$$
\begin{gather*}
\Phi_{L}^{0}(0,1,2) \simeq \rho_{0}(i, k)\left|u_{0}(i, k)-u_{0}(i+1, k)\right| e^{-\rho_{1}(i, k) C_{0}}\left(\frac{1}{1-\rho_{0}(i, k)}\right) \\
-\rho_{1}(i, k)\left|u_{1}(i, k)-u_{1}(i-1, k)\right| e^{-\rho_{0}(i, k) C_{0}}\left(\frac{1}{1-\rho_{1}(i, k)}\right)\left(1-e^{-\rho_{2}(i, k) C_{0}}\right) \\
\Phi_{L}^{1}(0,1,2) \simeq \rho_{0}(i, k) u_{0}(i, k)\left|u_{0}(i, k)-u_{0}(i+1, k)\right| e^{-\rho_{1}(i, k) C_{0}}\left(\frac{1}{1-\rho_{0}(i, k)}\right) \\
-\rho_{1}(i, k) u_{1}(i, k)\left|u_{1}(i, k)-u_{1}(i-1, k)\right| e^{-\rho_{0}(i, k) C_{0}}\left(\frac{1}{1-\rho_{1}(i, k)}\right)\left(1-e^{-\rho_{2}(i, k) C_{0}}\right) \\
\times\left(1-e^{-\rho_{3}(i, k) C_{0}}\right)-\rho_{1}(i, k)\left|u_{1}(i, k)-u_{1}(i+1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{1}(i, k)}\right)  \tag{4.53}\\
\Phi_{R}^{0}(1,2,3) \simeq \rho_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i-1, k)\right| e^{-\rho_{1}(i, k) C_{0}}\left(\frac{1}{1-\rho_{2}(i, k)}\right) \\
\Phi_{R}^{1}(1,2,3) \simeq \rho_{2}(i, k) u_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i-1, k)\right| e^{-\rho_{1}(i, k) C_{0}}\left(\frac{1}{1-\rho_{2}(i, k)}\right)  \tag{4.54}\\
\times\left(1-e^{-\rho_{3}(i, k) C_{0}}\right)-\rho_{1}(i, k) u_{1}(i, k)\left|u_{1}(i, k)-u_{1}(i+1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{1}(i, k)}\right) \tag{4.55}
\end{gather*}
$$

(ii) For lane 2, i.e when $\alpha=2$, equations (3.148) to (3.151) respectively becomes;

$$
\begin{gather*}
\Phi_{L}^{0}(1,2,3) \simeq \rho_{1}(i, k)\left|u_{1}(i, k)-u_{1}(i+1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{1}(i, k)}\right) \\
-\rho_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i-1, k)\right| e^{-\rho_{1}(i, k) C_{0}}\left(1-e^{-\rho_{3}(i, k) C_{0}}\right)\left(\frac{1}{1-\rho_{2}(i, k)}\right) \\
\Phi_{L}^{1}(1,2,3) \simeq \rho_{1}(i, k) u_{1}(i, k)\left|u_{1}(i, k)-u_{1}(i+1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{1}(i, k)}\right) \\
-\rho_{2}(i, k) u_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i-1, k)\right| e^{-\rho_{1}(i, k) C_{0}}\left(1-e^{-\rho_{3}(i, k) C_{0}}\right)\left(\frac{1}{1-\rho_{2}(i, k)}\right)  \tag{4.57}\\
\quad(4.57) \\
\Phi_{R}^{0}(2,3) \simeq \rho_{3}(i, k)\left|u_{3}(i, k)-u_{3}(i+1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{3}(i, k)}\right)  \tag{4.58}\\
\Phi_{R}^{1}(2,3) \simeq \rho_{3}(i, k) u_{3}(i, k)\left|u_{3}(i, k)-u_{3}(i+1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{3}(i, k)}\right) \\
-\rho_{2}(i, k) u_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i-1, k)\right| e^{-\rho_{3}(i, k) C_{0}}\left(\frac{1}{1-\rho_{2}(i, k)}\right) \tag{4.59}
\end{gather*}
$$

(iii) For lane 3, i.e when $\alpha=3$, equations (3.154) and (3.155) respectively becomes;

$$
\begin{gather*}
\Phi_{L}^{0}(2,3,) \simeq \rho_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i+1, k)\right| e^{-\rho_{3}(i, k) C_{0}}\left(\frac{1}{1-\rho_{2}(i, k)}\right) \\
-\rho_{3}(i, k)\left|u_{3}(i, k)-u_{3}(i-1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{3}(i, k)}\right)  \tag{4.60}\\
\Phi_{L}^{1}(2,3) \simeq \rho_{2}(i, k) u_{2}(i, k)\left|u_{2}(i, k)-u_{2}(i+1, k)\right| e^{-\rho_{3}(i, k) C_{0}}\left(\frac{1}{1-\rho_{2}(i, k)}\right) \\
-\rho_{3}(i, k) u_{3}(i, k)\left|u_{3}(i, k)-u_{3}(i-1, k)\right| e^{-\rho_{2}(i, k) C_{0}}\left(\frac{1}{1-\rho_{3}(i, k)}\right) \tag{4.61}
\end{gather*}
$$

In the next section we simulate the traffic congestion near the bottlenecks on a 3 -lanes highway using the discretized equations obtained.

### 4.4 Numerical Simulations

In this study we simulate the macroscopic traffic flow model for a 3-lanes highway with an on-ramp, off-ramp and a weaving section.
For simulations, the highway segment near each of the bottleneck is divided into certain number of cells with a fixed length $x_{i}-x_{i+1}$ and time step $\Delta t=t^{n+1}-t^{n}$.

Let the highway under consideration be along the $x$-axis; where $x=-30$ is the distance upstream of the bottleneck, $x=0$ is the location of the bottleneck and $x=10$ is the distance downstream of the bottleneck. Let the flow of traffic be in the direction of increasing $x$ along the axis and $t \in[0,500]$ be the time interval of the traffic simulation.

Table 4.1 shows the values of the various parameters used in the coded algorithm.

Table 4.1: Model parameters used in simulations

| Parameters | Values | Parameters | Values |
| :---: | :---: | :---: | :---: |
| $C=C_{0}$ | 0.45 | $\rho_{\alpha, \text { min }}^{\text {syn }}$ | 0.5 |
| $C_{c f l}$ | 0.5 | $\rho_{\alpha, \text { max }}^{\text {free }}$ | 0.3 |
| $\rho_{\alpha, j a m}$ | 0.9 | $u_{\alpha, \text { syn }}$ | 0.28 |

In order to compute the solutions of the model equations, we use a mathematical and graphical computer software package (MAT-LAB) which has numerical, graphical and programming capabilities. The discretized macroscopic model equations for the homogeneous part and the source terms of equation (4.19) are coded in MAT-LAB software under the initial condition (4.1) and boundary condition (3.26). These model equations are then solved through the computer coded algorithm attached in appendix (7.2).

The results obtained are presented and discussed in the next chapter.

## CHAPTER FIVE

## RESULTS AND DISCUSSIONS

### 5.1 Introduction

The discretized form of the model equations presented in chapter 4 which are coded in MATLAB software package were run under the given initial and boundary conditions. The results produced are presented in form of graphs and space-time plots.
It is worth noting that in all the spatiotemporal traffic congestion patterns produced at the bottlenecks, the velocity and density trajectories are observed to lean towards left because the vehicles are moving downstream while the congestion propagates upstream and this takes time to build up.

### 5.2 Traffic Breakdown at the On-ramp

At these locations, the deterministic disturbance of free flow at the bottleneck is caused by merging of an on-ramp inflow rate $\left(q_{o n}\right)$ and a flow rate $\left(q_{1, i n}\right)$ on the lane adjacent to the bottleneck.

### 5.2.1 Spatiotemporal Congested Traffic Patterns

Simulations of the observed features of spatiotemporal congested traffic patterns that occur in the vicinity of the bottleneck are shown in the Figure 5.1 (a,b,...f). When traffic breakdown is experienced at the on-ramp, various congested patterns of synchronized flow are observed upstream and downstream of the bottleneck, see Figure 5.1 (a,b,...,f).
From these patterns, it is observed that congestion in lane 1 starts immediately the vehicles from the on-ramp merge on to the highway and propagates upstream. The congestion is set into lane 1 because the drivers in this lane sometimes show courtesy by slowing down to give way to those merging from the on-ramp. Furthermore, numerous unreasonable forced traffic merging from on-ramp to the highway by aggressive drivers are also in progress simultaneously. This happens frequently during rush hours, causing traffic to build up in lane 1 and spreads toward upstream of the on-ramp. Hence stop-and-go traffic patterns are formed as depicted in Figure 5.1 (a) indicated by the region of blue spikes (low velocity) intercepted by yellow spikes (high velocity). This shows that a wide moving jam (J) occurs on the highway upstream of the on-ramp as indicated by region of low velocity and average


Figure 5.1: (a) to (f), respective velocity and density space - time traffic patterns of lanes 1, 2 and 3 near the on-ramp.
density (over-deceleration effect). This is because of the fact that, the first vehicles to slow down are the ones traveling in the lane adjacent to the on-ramp. After sometimes, the traffic flow in lanes 2 and 3 are also affected by the aggressive drivers in lane 1 who upon experiencing or anticipating the constrainedness opt to improve or maintain their driving conditions by changing
lane to these inner lanes. Figure 5.1 (c to f) show that in lane 2 and 3, there is a tendency towards synchronization of vehicles' speeds on the highway at the bottleneck indicated by region of fluctuating average low traffic velocities and high densities upstream of the on-ramp. This is due to the traffic disturbance caused by the on-ramp inflow rate ( $q_{o n}$ ) and more vehicles in lane 1 tend to change lanes to the inner lanes. This implies that a moving synchronized pattern (MSP) indicated by region of decreasing velocity and increasing density in (Figure 5.1, c to f) appears on the highway upstream of the on-ramp. Thus the two lanes (2 and 3) experience traffic congestion upstream of the bottleneck where the traffic queue grow at the tail while the vehicles at the head of the queue accelerate to move out of the jam. However, in the three lanes downstream of the on-ramp, there is an immediate decrease in both velocity and density showing that few vehicles are able to manoeuvre through the traffic merging region. Therefore, free flow is maintained downstream of the highway after the merging zone in the three lanes where the vehicles have to over-accelerate and move with their desired speed.

### 5.2.2 Flow-Density Plane of the Congested Traffic

Figure 5.2 (a) shows the flow-density relationship for lane 1 , where there is a decrease in flow rate within the deterministic disturbance as the vehicle density increases at the on-ramp $(x=0)$. Initially, the disturbance occurs only in lane 1 when the vehicles from on-ramp lane 0 merge with the vehicles in that lane. This disturbance grows and lead to free-synchronized flow $(F \rightarrow S)$ transition in lane 1 near the on-ramp.
Consequently, the aggressive drivers in lane 1 opt to change lane to the faster ones immediately they experience the traffic turbulence.

The flow rate in lanes 2 and 3 is sustained at the bottleneck ( $x=0$ ), see Figure 5.3 (b) and 5.4 (b) implying that most vehicles on the highway prefer to move in lanes 2 and 3 than in lane 1 as long as possible to avoid the vehicles merging onto the highway from on-ramp. At location $x=-10$ upstream of the bottleneck, there is a random fluctuation in flow rate with increase of traffic density as shown in Figure 5.2 (a), 5.3 (a) and 5.4 (a). These show that maximum flow rate is attained at low density and vice versa. However, this traffic flow situation is short lived since vehicles are interacting by changing lanes from lane 1 to the right lanes in the vicinity of the on-ramp. Therefore a transition of free flow to synchronized flow $(F \rightarrow S)$ occurs (where the flow rate is high and the average velocity is low). This ( $F \rightarrow S$ ) phase transition last for only a short period before
a transition from synchronized to free flow $(S \rightarrow F)$ occurs. This shows that the traffic phases transition exchange are continuous near the on-ramp and complete a traffic hysteresis loop, in which the lower part of the loop represents the vehicle's acceleration branch in $F \rightarrow S$ transition while the upper part of the loop is the deceleration branch associated with $S \rightarrow F$ transition.


Figure 5.2: Traffic flow rate-density relationship in lane 1 at location $x$ is -10 and $x$ is 0


Figure 5.3: Traffic flow rate-density relationship in lane 2 at location $x$ is -10 and $x$ is 0


Figure 5.4: Traffic flow rate-density relationship in lane 3 at location $x$ is -10 and $x$ is 0

### 5.3 Traffic Breakdown at the Off-ramp

Here the deterministic disturbance is localized upstream of the off-ramp merging zone where the vehicles exit from the main-road through the off-ramp.

### 5.3.1 Spatiotemporal Congested Traffic Patterns

Simulations of the observed features of spatiotemporal congested traffic patterns that occur in the vicinity of the off-ramp are as shown in Figures 5.5 (a,b,..., f). The various congested traffic patterns of synchronized flow in the upstream and downstream of the off-ramp are also shown in the Figure 5.5. It is observed that when the distance to the beginning of the off-ramp merging zone decreases, more vehicles attempt to make lane-change to lane 1 . This show that most vehicles aiming to exit through the off-ramp try indeed to move in lane 2 and 3 as long as possible before changing to lane 1.
Thus, when vehicles on lane 1 approach the off-ramp, they decelerate and perform mandatory lane-change if their target is to exit the highway or continue moving in the same lane otherwise discretionary lane-change to the right lanes are conducted. That is, a wide moving jam (J) appears on lane 1 upstream of the off-ramp due to the vehicles changing lanes from lane 2 and 3 to lane 1 aiming to exit the highway. Therefore, traffic congestion is initially set on lane 1 upstream of the beginning of the off-ramp but later this disturbance affect the traffic free flow in lane 2 and 3 near the bottleneck. This happens frequently during rush hours and causes heavy traffic to build up on lane 1 upstream of the off-ramp.


Figure 5.5: (a) to (f), respective velocity and density space - time traffic patterns of the lanes 1, 2 and 3 near the off-ramp.

Hence a stop-and-go traffic pattern is formed near the off-ramp due to vehicles braking as they enter the deceleration lane as depicted in Figure 5.5(a).
Due to the traffic disturbance caused by the off-ramp inflow rate ( $q_{i n}$ ) and some vehicles in lane 1 going through the highway opting to change lane to lanes 2 and 3 , there is an increase in vehicles' density and a decrease in vehicle velocity in the three lanes upstream of the bottleneck as shown in Figure 5.5 (a and b). Thus there is a tendency towards synchronization of vehicles speeds on the highway upstream of the bottleneck indicated by region of fluctuating average low velocities caused by lane changing behavior, see Figure 5.5 (c and e). This shows that a moving synchronized pattern (MSP) appears on the highway upstream of the off-ramp in lane 2 and 3. Therefore lane 1 experiences traffic congestion upstream of the bottleneck where the traffic queue grow at the tail while the vehicles at the head of the queue exit through the off-ramp. However, in the three lanes downstream of the off-ramp, there is an immediate decrease in both velocity and density showing that few vehicles are able to manoeuvre out of the traffic merging region. The results show that the capacity of the local road connecting to the highway through the off-ramp determine the congestion in the diverging section.

### 5.3.2 Flow-Density Plane of the Congested Traffic

Figure 5.6, 5.7 and 5.8 show the results of the traffic flow-density relationship in the three lanes at location $x=-10$ and $x=0$ of the ramp. In lane 1 , there is a decrease in flow rate within the deterministic disturbance as the vehicle density increases at the off-ramp $(x=0)$. That is, the congestion in lane 1 occurs when the vehicles in lane 2 and 3 change lanes to lane 1 upstream of the off-ramp aiming to exit the highway and the congestion in the local road connected by the off-ramp can also extend to the main-road through lane 1. Consequently, the aggressive drivers in lane 1 going through the highway may opt to change lanes to the faster ones immediately they approach the traffic merging region. Moreover, the flow rate in lanes 2 and 3 is sustained at the bottleneck $(x=0)$, (see Figure 5.7 (b) and 5.8 (b)) implying that at the bottleneck, most vehicles on the highway prefer to move in lanes 2 and 3 than in lane 1 as long as possible to avoid the vehicles diverging from the highway to the off-ramp.
At location $(x=-10)$ of the bottleneck, there is a random fluctuation in flow rate with increase of traffic density as shown in Figure 5.6 (a), 5.7 (a) and 5.8 (a), thus maximum flow rate is attained at low density and vice versa.


Figure 5.6: Traffic flow rate-density relationship in lane 1 at location $x$ is -10 and $x$ is 0


Figure 5.7: Traffic flow rate-density relationship in lane 2 at location $x$ is -10 and $x$ is 0


Figure 5.8: Traffic flow rate-density relationship in lane 3 at location $x$ is -10 and $x$ is 0

This traffic flow situation is short lived since vehicles are interacting by changing lanes continuously in the vicinity of an off-ramp. Therefore, a phase transition of free flow to synchronized flow $(F \rightarrow S$ ) occurs (where the flow rate is high and the average velocity is low). This $(F \rightarrow S)$ transition last for only a short period and transition from synchronized to free flow $(S \rightarrow F)$ appear.

Thus, the traffic phase transition exchange is continuous at this location and complete a traffic hysteresis loop, in which the upper part of the loop represents the vehicle deceleration branch in $(F \rightarrow S)$ transition while the lower part of the loop is the acceleration branch associated with $(S \rightarrow F)$.

### 5.4 Traffic Breakdown at the Weaving Section

This type of weaving is whereby the vehicles weave from on-ramp to the highway and from freeway to off-ramp using a single lane. Therefore the deterministic disturbance is localized within the given auxiliary lane segment.

### 5.4.1 Spatiotemporal Congested Traffic Patterns

Figure 5.9 (a to f) show the features of the simulated traffic congestion which propagates upstream of the weaving section of the freeway. Once the highway traffic free flow experience traffic breakdown, various synchronized patterns (SP) are observed within and upstream of the weaving section as depicted in figure 5.9. These patterns are classified as localized (LSP), moving (MSP) and widening (WSP). Figure 5.9 (c to f) show moving synchronized traffic flow patterns (MSP) observed in lanes 2 and 3 upstream of the weaving section. These MSPs grow because of the velocity decrease and density increase within them as they move upstream of the weaving area.
The patterns show that lane changes at these sections occur mainly between the auxiliary lane and lane 1 of the highway.

It is observed that when the vehicles moving in lane 1 targeting to exit through the off-ramp approach the weaving area, they decelerate and weave into the auxiliary lane. On the other hand vehicles from the on-ramp aiming to merge into the highway join the auxiliary lane and perform mandatory lane-changes by weaving. Figure 5.9 ( a and b ) show a wide moving jam ( J ) appearing in lane 1 where the velocity fluctuate from minimum to maximum value while the traffic is operating on average density. These changes in both velocity and density indicate the braking processes within the vicinity of the weaving section. Therefore lane 1 experiences most of the traffic congestion upstream of the weaving region compared to the other two lanes especially during rush hours.


Figure 5.9: (a) to (f), respective velocity and density space - time traffic patterns of the lanes $(1,2,3)$ within and near the weaving section.

This is due to aggressive drivers aiming to exit highway through the weaving section opting to stay in the inner lanes as long as possible before making forced lane-change to lane 1. This shows that most of the expressway vehicles prefer to move in lane 2 and 3 to avoid the localized disturbance near the weaving
section. However, there is free flow downstream of the weaving region since the jam propagate upstream and the traffic disturbance is within the bottleneck.

### 5.4.2 Flow-Density Plane of the Congested Traffic

Figure 5.10, 5.11 and 5.12 show the traffic flow-density planes of the three lanes where the flow rate decreases with an increase of density within the deterministic disturbance. Since most of the lane-changes are performed at the weaving section then lane 1 experiences wide moving jam (J) at $x=0$, see Figure 5.10 (b). As a result of these lane-changes, the increase in disturbance reaches a limit upon which the traffic velocity decreases as the density increases abruptly and lead to free-synchronized flow $(F \rightarrow S)$ transition upstream of the bottleneck. Therefore, a moving synchronized pattern (MSP) emerges in lanes 2 and 3 as shown in Figure 5.11 (b) and $5.12(\mathrm{~b})$. This $(F \rightarrow S)$ transition last for only a short period since vehicles are changing lanes continuously in the vicinity of the weaving area and a transition from synchronized to free flow $(S \rightarrow F)$ appears. Thus, the traffic phase transition exchange is continuous at weaving sections and complete a traffic hysteresis loop where the upper part of the loop represents the vehicle deceleration branch in $(F \rightarrow S)$ transition while the lower part of the loop is the acceleration branch associated with $(S \rightarrow F)$.
At location $(x=-10)$ of the bottleneck, there is a random fluctuation in flow rate with increase of traffic density at location $(x=-10)$ of the bottleneck as shown in Figure 5.10 (a), 5.11 (a) and 5.12 (a). At these locations wide moving jam (J) propagate through the free flow and maintain the mean velocity of the downstream front. Furthermore when traffic is saturated, vehicles change lanes at the start of the weaving section independently of their direction.


Figure 5.10: Traffic flow rate-density relationship in lane 1 at location x is -10 and x is 0


Figure 5.11: Traffic flow rate-density relationship in lane 2 at location x is -10 and $x$ is 0


Figure 5.12: Traffic flow rate-density relationship in lane 3 at location x is -10 and x is 0

### 5.5 Discussion of the results

The results show that lane-change manoeuvre is frequently performed near the ramps and weaving areas. This indicate that traffic congestion is mostly generated at ramps and weaving sections where it is propagated to the next section of the highway upstream. High traffic density accompanied by low velocity indicate phase transition from free flow to synchronized flow and vice versa.

At the on-ramps, the lane-change process start at a relatively short distance upstream of the ramp where there is slight movement of traffic from the left lane towards the right lane. However, more vehicles are able to enter the merging section when the merge rate increases and interact with the vehicles on the highway. That is, more traffic breakdown is generated in the traffic flow due to unreasonable driving behaviors such as forced merging and lane changing from on-ramp. Thus there is frequent traffic congestion in the merging section of the highway.
Elsewhere at off-ramps, the lane-change process start at a far distance upstream of the ramp where there is slight movement of traffic from the right to the left lane. The capacity of the local road connecting the highway determine the traffic capacity of the off-ramp. Therefore traffic congestion occurring in the local road can quickly spill to the highway main lanes and adversely affect traffic flow in the designated express lanes.

It is noted that changes in vehicles speed is higher around off-ramps than around on-ramps due to difference in the type of lane-change manouevres. That is, when vehicles are entering the highway they require acceleration but when exiting, they decelerate to a safe speed before turning to the off-ramp after the deceleration lane. Therefore most traffic congestion originates at merge, diverge and weaving areas where it propagates to the next section upstream. The results produced by the model are consistent with the empirical data and shows good agreement with the real life situation near the highway bottlenecks. Thus lane-change manoeuvre is a macroscopic indicator for traffic congestion around the bottlenecks.

The next chapter gives conclusions and recommendations for further studies.

## CHAPTER SIX

## CONCLUSIONS AND RECOMMENDATIONS

### 6.1 Conclusions

A multi-lane macroscopic traffic flow model of Aw-Rascle type within the framework of the 3-phase traffic flow theory of Kerner has been developed. This was achieved by applying the method of moments on the kinetic traffic flow model equations to obtain the corresponding macroscopic traffic flow model equations.
The hyperbolic nature of the derived macroscopic model equations has been discussed. A relaxation term was incorporated into the macroscopic equations to constitute the three phase traffic theory. The specific lanes model equations of an on-ramp, an off-ramp and a weaving section are obtained for a three-lanes highway.
The numerical method (finite volume method) for solving the conservative form of the discretized macroscopic model equations has been presented while the Riemann problem and its solution was constructed to solve the source terms using Euler's method.
Simulations of the traffic congestions near the on-ramp, off-ramp and the weaving section of a three-lanes highway were done through coding of the discretized form of conservative equations in MATLAB software package. The computer coded algorithm was run under the given boundary conditions and the results were presented in form of graphs and space-time plots.
The simulations show that the initial traffic flow disturbance occurs first on the lane adjacent to the bottleneck due to the merging of vehicles from on-ramp and vehicles changing lanes from highway onto the off-ramp merging zone. However the disturbance can grow and affect the traffic flow in the inner lanes leading to a transition from free flow $(F)$ to synchronized $(S)$, in particular when the lane change leads to deceleration of the following vehicles in the target lanes.
With these simulations near the bottlenecks, the derived macroscopic traffic flow model is able to reproduce the spatiotemporal features of real traffic flow patterns observed at the bottlenecks.
Therefore the model can be used to:
(i) improve the optimal design of highways geometry and safety of roads by identifying the effective location of the bottlenecks to evaluate the impact of new road infrastructure.
(ii) solve road congestion problems by either erecting traffic control lights or
develop an advanced multi-agent control strategies for training of the traffic marshals and policemen.
(iii) evaluate and recommend the use of a new part of road infrastructure.
(iv) recommend the minimum length of auxiliary lanes for ramps and weaving sections.

All these will improve the quality of travel by providing prior information about traffic congestion to road users and develop the dynamic management strategies in highways within the real traffic framework.

### 6.2 Recommendations for Further Research

In this work, a macroscopic traffic flow model has been developed with traffic simulations done in the vicinity of an on-ramp, off-ramp and a weaving section as the bottlenecks.
It is recommended that:
(i) further investigation on traffic breakdown and congestion due to weaving section (where the merge and a diverge are in close proximity) with consideration of various length of the auxiliary lanes to be carried out and a comparison of these effects with those of the study be done.
(ii) simulation of traffic congestion in weaving sections with more than one lane where the on-ramps and off-ramps are on opposite sides of the highway, to be carried out.

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## APPENDICES

## Publications

Part of this work has been published in two international journals, i.e:

Ndungu, W. K., Kimathi, M. E. M., \& Theuri, D. M. (2018). Modeling Traffic Flow on Multi-Lane Road: Effects of Lane-Change Manoeuvre Due to an On-ramp. Global Journal of Pure and Applied Mathematics, 14 (10), 1389-1406. https://www.ripublication.com/gjpam.htm

Ndungu, W. K., Kimathi, M. E. M., \& Theuri, D. M. (2019). Modeling Traffic Flow on Multi-Lane Road: Effects of Lane-Change Manoeuvre Due to an Off-ramp. International Journal of Advances in Applied Mathematics and Mechanics, 6(4), 1-13.
https://www.ijaamm.com

## MATLAB Codes

```
    function laneChanges_Ndungu()
    clear all;clc;
    tend=500.0;
    plotting=1; plotFD=1; plotFD1=0; plotFD2=5; plotFD3=-10;
pausing=0;
    storeFD1=1;
    lanes1=1; lanes2=2; lanes3=3; Trelax=5; rhofree=0.3; rhojam=0.9;
rhosyn=0.5; Usyn=0.28; boundaryL=0;%0=Neumann,1=Dirichlet boundaryR=0;
boundaryRflux=0.01; boundaryLflux=0.25; contraction_x=0; contraction_flux=in
solver=2;Ao=.75;Vmax=0;
AwRascle.name='AwRascle'; AwRascle.FundamentalDiagram=0;
AwRascle.plotmarker='k'; AwRascle.pausing=pausing;
AwRascle.Solver=solver; % AwRascle.Pressure=pressure;
AwRascle.nu=2; AwRascle.C=.45;% for Aggressive model
AwRascle.C1=.45;AwRascle.Co=0.45;AwRascle.Vref=.45;
AwRascle.Ao=Ao;AwRascle.Vmax=Vmax; AwRascle.h=1;
AwRascle.Lanes1=lanes1; AwRascle.Lanes2=lanes2;
AwRascle.Plot=plotting; AwRascle.plotvelocity=1;
AwRascle.boundaryR=boundaryR;
AwRascle.boundaryL=boundaryL;
AwRascle.boundaryRflux=boundaryRflux;
AwRascle.boundaryLflux=boundaryLflux;
AwRascle.contraction_flux=contraction_flux;
AwRascle.contraction_x=contraction_x;
AwRascle.contraction_x_smooth=contraction_x_smooth;
AwRascle.lanefactor=lanefactor;
AwRascle.plotFD=plotFD;
% KIN={};
KIN=AwRascle;
KIN.name='AwRascle model'; KIN.FundamentalDiagram=1;
KIN.plotmarker='m'; % [xKIN,t,uKIN,istoreUKIN,FD1KIN,UsurfKIN,tKIN]=Solve_L
[xxKIN,t,xuKIN,yuKIN, zuKIN, istoreUKIN , xFD1KIN, xUsurfKIN , yFD1KIN , yUsurfKIN , zF
    xpos=KIN.plotFD3;
[dummy,ipos]=min(abs(xxKIN-xpos));
xrpos=xuKIN(ipos,1);
xvel=xuKIN(:,2)./xuKIN(:,1)-pressNL(KIN,xuKIN(:,1));
```

```
    xvpos=xvel(ipos);
    [r V1 Rline Jline]=plotFundamentalDiagram(KIN,xpos,xrpos,xvpos);
    %
save('KIN.mat','xKIN','tKIN','uKIN','FD1KIN','UsurfKIN','tend','KIN');
    if(plotting==1)
    print -depsc Kinetic
    end
    plotFD=0;
    KIN.plotFD=plotFD;
    clf;
    % plotTraffic(KIN,xKIN,uKIN,tend,1,'m');
    %
    % legend(KIN.name)
    % legend('location','SouthWest')
    print -depsc Compare
    instant=min(istoreUKIN)
    c1=2;%floor(instant)
    c2=2;%floor(instant2)
    figure(1)
B=1.5;
subplot(3,1,1)
    RE= plot(xxKIN,xUsurfKIN(instant-c1,:,3)','r');set(RE, 'LineWidth', B);
title('Velocity');%%xlabel('x')
ylabel('u_1');axis([-30 10 0.0 0.7])
    subplot(3,1,2)
    RE= plot(xxKIN,yUsurfKIN(instant-c1,:,3)','r');set(RE, 'LineWidth', B);
%title('Velocity');%%xlabel('x')
ylabel('u_2');axis([-30 10 0.0 0.7])
subplot(3,1,3)
RE= plot(xxKIN,zUsurfKIN(instant-c1,:,3)','r');set(RE, 'LineWidth', B);
%title('Velocity');%%xlabel('x')
ylabel('u_3');axis([[-30 10 0.0 0.7])
    figure(2)
    subplot(3,1,1)
    RE= plot(xxKIN,xUsurfKIN(instant-c1,:,1)','r');set(RE, 'LineWidth', B);
    title('Density');%%xlabel('x');
    ylabel('\rho_1');axis([-30 10 0.0 0.7])
    subplot(3,1,2)
```

```
RE= plot(xxKIN,yUsurfKIN(instant-c1,:,1)','r');set(RE, 'LineWidth', B);
%title('Density');xlabel('x');
ylabel('\rho_2');axis([[-30 10 0.0 0.7])
subplot(3,1,3)
    RE= plot(xxKIN,zUsurfKIN(instant-c1,:,1)','r');set(RE, 'LineWidth', B); %ti
xlabel('x');ylabel('\rho_3');axis([-30 10 0.0 0.7])
figure(3)
%%subplot (2, 2, 1)
%surf(tKIN,xxKIN,xUsurfKIN(:,:,3)')
%title('lane1 velocity')
surf(tKIN,xxKIN,xUsurfKIN(:,:,1)')
title('lane1 density')
xlabel('time(t)')
ylabel('distance(x)')
axis([0 tend -30 10])
colorbar;
colorbar('location','EastOutside')
caxis([0.23 0.58]);
view(2)
shading flat
figure(4)
%%subplot(2,2,2)
%surf(tKIN,xxKIN,yUsurfKIN(:, :,3)')
%title('lane2 velocity')
surf(tKIN,xxKIN,yUsurfKIN(:,:,1)')
title('lane2 density')
xlabel('time(t)')
ylabel('distance(x)')
axis([0 tend -30 10])
colorbar;
colorbar('location','EastOutside')
caxis([0.23 0.58]);
view(2)
shading flat
figure(5)
%%subplot (2, 2, 3)
%surf(tKIN,xxKIN,zUsurfKIN(:,:,3)')
%title('lane3 velocity')
```

```
surf(tKIN,xxKIN,zUsurfKIN(:,:,1)')
title('lane3 density')
xlabel('time(t)')
ylabel('distance(x)')
axis([0 tend -30 10])
colorbar;
colorbar('location','EastOutside')
caxis([0.23 0.58]);
view(2)
shading flat
figure(6)
%%subplot(2, 2, 1)
lf=1;plot(lf*r,lf*(r.*V1),'-b')
hold on;
plot(lf*r,lf*(r.*Rline),'-.m',lf*r,lf*(r.*Jline),'--r'); [dummy,iFD1]=min(abs
set(RE, 'LineWidth', B);
legend('Q_1^e(rho)','rho*R(rho)','Q_2^e(rho)','AwRascle');legend('location',
figure(7)
%%subplot(2,2,3)
lf=1;plot(lf*r,lf*(r.*V1),'-b')
hold on;
plot(lf*r,lf*(r.*Rline),'-.m',lf*r,lf*(r.*Jline),'--r');
[dummy,iFD1]=min(abs(xxKIN-KIN.plotFD1));%=xpos
aqqDensity=(xUsurfKIN(:,iFD1,1)+yUsurfKIN(:,iFD1,1)+zUsurfKIN(:,iFD1,1))/3;
aqqFlow=(xUsurfKIN(:,iFD1,1).*xUsurfKIN(:,iFD1,3)+yUsurfKIN(:,iFD1,1).*yUsur
+zUsurfKIN(:,iFD1,1).*zUsurfKIN(:,iFD1,3))/3;
RE=plot(aqqDensity,aqqFlow, 'k');
set(RE, 'LineWidth', B);
legend('Q_1^e(rho)','rho*R(rho)','Q_2^e(rho)','AwRascle');legend('location',
hold off;title('lane-averaged phase transitions at x=0')
xlabel('density');ylabel('flow rate');axis([00 0.7 0 0.2])
    end
function
    [xx,t,xu,yu,zu,istoreU,xFD1stored,xUsurf , yFD1stored, yUsurf,zFD1stored,zUsurf
display(Problem.name)
    nbc=2;inx=250;nx=inx+2*nbc;nu=2;
xlow=-30.0; xup=10.0;dx=(xup-xlow)/(inx-1);
xx=xlow-nbc*dx:dx:xup+nbc*dx;
```

Inital Conditions Problem.Lanes=ones(nx,1)*Problem.Lanes1; xu=zeros(nx,nu);yu=2 icont=find(xx>Problem.contraction_x); xr=0.3027*ones(nx,1);xr(icont)=0.27*on $\mathrm{yv}=\mathrm{V} 1 \mathrm{func}(\mathrm{yr}) ; \% \mathrm{yv}$ (icont) $=\mathrm{V} 1 \mathrm{func}(\mathrm{yr}$ (icont)) ;
zv=V1func(zr);\%zv(icont)=V1func(zr(icont));
type=Problem.FundamentalDiagram;
if (type==-1)

elseif(type==1)\%AW-RASCLE MODEL
$\mathrm{xy}=\mathrm{xr} . * \mathrm{xv}+\mathrm{xr} . * \mathrm{press}($ Problem $, \mathrm{xr}, 1: \mathrm{nx}) ; \mathrm{yy}=\mathrm{yr} . * \mathrm{yv}+\mathrm{yr} . * \mathrm{press}($ Problem, yr, $1: \mathrm{nx}) ; \mathrm{z}$. end
$\mathrm{xu}(:, 1)=\mathrm{xr} ; \mathrm{xu}(:, 2)=\mathrm{xy}$; yu(: , 1) $=\mathrm{yr} ; \mathrm{yu}(:, 2)=\mathrm{yy} ; \mathrm{zu}(:, 1)=\mathrm{zr} ; \mathrm{zu}(:, 2)=\mathrm{zy}$;
if (Problem.storeFD1==1)
[dummy,iFD1]=min(abs(xx-Problem.plotFD3));
istoreFD1=1;
xFD 1 stored (istoreFD1, 1) $=\mathrm{xu}(\mathrm{iFD} 1,1)$; yFD 1 stored (istoreFD1, 1) $=\mathrm{yu}(\mathrm{iFD} 1,1) ; \mathrm{zFD} 1$ st xFD1stored (istoreFD1,2)=xu(iFD1,2);yFD1stored(istoreFD1,2)=yu(iFD1,2);zFD1st istoreFD1=istoreFD1+1;
istoreU=1;
$x U s u r f(i s t o r e U, 1: n x, 1)=x u(1: n x, 1) ; y U s u r f(i s t o r e U, 1: n x, 1)=y u(1: n x, 1) ; z U s u r f(i$ $x U s u r f(i s t o r e U, 1: n x, 3)=0 *(x u(1: n x, 2) . / x u(1: n x, 1)-p r e s s(P r o b l e m, x u(1: n x, 1), 1:$ elseif (type==1)\%AW-RASCLE MODEL
$x \operatorname{USurf}($ istoreU, $1: n \mathrm{x}, 3)=(\mathrm{xu}(1: \mathrm{nx}, 2) . / \mathrm{xu}(1: n \mathrm{x}, 1))-\mathrm{press}(\operatorname{Problem}, \mathrm{xu}(1: \mathrm{nx}, 1), 1: \mathrm{n}$ $y \operatorname{Usurf}(i s t o r e U, 1: n x, 3)=(y u(1: n x, 2) . / y u(1: n x, 1))-\operatorname{press}(\operatorname{Problem}, \mathrm{yu}(1: n x, 1), 1: n$ zUsurf(istoreU, $1: n x, 3)=(z u(1: n x, 2) . / z u(1: n x, 1))-\operatorname{press}(\operatorname{Problem}, z u(1: n x, 1), 1: n$ end
istoreU=istoreU+1;
end
$\mathrm{t}=0.0$;itvector=1; tvector(itvector)=t;itvector=itvector +1 ;
iteration $=0$;maxiteration=10000; dt=1.e-8;cfl=0.499; desiredcfl=0.49;
while ( $(\mathrm{t}<\mathrm{tend}) \& \&(i t e r a t i o n<m a x i t e r a t i o n))$
iteration=iteration+1;
$d t=\min (d t, t e n d-t)$;
xustore=xu;yustore=yu;zustore=zu;
\%compute conservation law \%LF
[xu, xspeed, yu, yspeed, $\mathrm{zu}, \mathrm{zspeed}]=$ Lanes_LF_Step(Problem, $\mathrm{nx}, \mathrm{nbc}, \mathrm{nu}, \mathrm{dt}, \mathrm{dx}, \mathrm{xu}, \mathrm{yu}$, \%update boundary
[xu,yu,zu]=Lanes_boundary (Problem, nx, nbc, xu, yu, zu);
\%update source terms
[xu, xsourcedt , yu, ysourcedt, zu,zsourcedt]=source_step (Problem, xx , dt , xu , yu, zu , msourcedt=max (max (xsourcedt, ysourcedt), zsourcedt) ;
\%timestep control
if $($ smax $<=0) \%$ set $d t=0.1$ if no speed available
smax=dx/0.1*desiredcfl;
end
if $(((d t<(d x / s m a x * c f l))) \& \&(d t<m s o u r c e d t)) \%$ accepted timestep $t=t+d t ;$ tvector (it xFD1stored (istoreFD1, 1) =xu(iFD1,1) ; yFD1stored (istoreFD1, 1) =yu(iFD1, 1) ; zFD1st xFD1stored (istoreFD1 , 2) =xu(iFD1, 2) ; yFD1stored (istoreFD1, 2) =yu(iFD1, 2) ; zFD1st istoreFD1=istoreFD1+1;
$x \operatorname{USurf}($ istoreU, $1: n x, 1)=x u(1: n x, 1) ; y \operatorname{surf}(i s t o r e U, 1: n x, 1)=y u(1: n x, 1) ; z U s u r f(i$ if (type==-1)
$x \operatorname{dsurf}($ istoreU $, 1: n x, 3)=0 *(x u(1: n x, 2) . / x u(1: n x, 1)-p r e s s(P r o b l e m, x u(1: n x, 1), 1: 1$
yUsurf (istoreU, $1: n x, 3)=0 *(y u(1: n x, 2) . / y u(1: n x, 1)-p r e s s(P r o b l e m, y u(1: n x, 1), 1: 1$
zUsurf (istoreU, $1: n \mathrm{n}, 3)=0 *(\mathrm{zu}(1: \mathrm{nx}, 2) . / \mathrm{zu}(1: \mathrm{nx}, 1)-\mathrm{press}($ Problem, $\mathrm{zu}(1: \mathrm{nx}, 1), 1: 1$ elseif(type==1)\%AW-RASCLE MODEL
$x \operatorname{Usurf}($ istoreU, $1: n x, 3)=(x u(1: n x, 2) . / x u(1: n x, 1))-\operatorname{press}(P r o b l e m, x u(1: n x, 1), 1: n$
$y \operatorname{Usurf}(i s t o r e U, 1: n x, 3)=(y u(1: n x, 2) . / y u(1: n x, 1))-\operatorname{press}(\operatorname{Problem}, y u(1: n x, 1), 1: n z$
zUsurf (istoreU, $1: n x, 3)=(z u(1: n x, 2) . / z u(1: n x, 1))-p r e s s(P r o b l e m, z u(1: n x, 1), 1: n z$
end
istoreU=istoreU+1;
end
else\%restore
xu=xustore;yu=yustore;zu=zustore;
end
\%set new timestep
$\mathrm{dt}=\mathrm{dx} / \mathrm{smax} * d e s i r e d c f l$;
$d t=m i n(d t, m s o u r c e d t)$;
if (Problem.Plot==1)
plotTraffic(Problem, xx, xu, yu,zu,t,0, Problem.plotmarker);
drawnow;
if (Problem.pausing==1)
pause();
end
end
end
plotTraffic(Problem, $x x, x u, y u, z u, t, 1$, Problem.plotmarker);
if (iteration>=maxiteration)
disp('out of iterations')
pause;
end
end
function
[xunew, xspeed, yunew, yspeed, zunew, zspeed] =Lanes_LF_Step(Problem, nx , nbc , nu , dt , \%generate the random numbers
nrand=100; ran=randi ([001], nx-1, nrand) ; k=1:nrand; an=ran*2. (-k)'; type=Probler
$x v=0 *(x u(1: n x-1,2) . / x u(1: n x-1,1)-\operatorname{press}(\operatorname{Problem}, x u(1: n x-1,1), 1: n x-1))$;
$y v=0 *(y u(1: n x-1,2) . / y u(1: n x-1,1)-\operatorname{press}(\operatorname{Problem}, y u(1: n x-1,1), 1: n x-1))$;
$z v=0 *(z u(1: n x-1,2) . / z u(1: n x-1,1)-\operatorname{press}(P r o b l e m, z u(1: n x-1,1), 1: n x-1))$;
elseif (type==1) \%AW-RASCLE MODEL
$x v=(x u(1: n x-1,2) . / x u(1: n x-1,1))-p r e s s(P r o b l e m, x u(1: n x-1,1), 1: n x-1) ;$
$y v=(y u(1: n x-1,2) . / y u(1: n x-1,1))-\operatorname{press}^{(P r o b l e m, y u(1: n x-1,1), 1: n x-1) ; ~}$
$z v=(z u(1: n x-1,2) . / z u(1: n x-1,1))-\operatorname{press}(\operatorname{Problem}, z u(1: n x-1,1), 1: n x-1) ;$
end
xian=find(an<dt/dx*xv);yian=find(an<dt/dx*yv);zian=find(an<dt/dx*zv); \%
solution with only the 2 wave u12 xustar=zeros (size (xu)) ; yustar=zeros(size(yu) [xustar (2:nx,$:$ ) , yustar $(2: n x,:), \operatorname{zustar}(2: n x,:)]=$ starstate (Problem, xu(1:nx-1, : yu(1:nx-1,:), yu(2:nx,:), zu(1:nx-1,:), zu(2:nx,:), $: n x-1,2: n x) ; x u 12=z e r o s(s i z$
$\%$ adding the 1 wave
\% solver=solverchoice(v,nx) ;
if (Problem.Solver==1)
xfluxm=zeros (size(xu));xfluxp=zeros (size(xu));yfluxm=zeros (size (yu));yfluxp= xfluxp2=zeros (size(xu)) ;xfluxp1=zeros (size(xu)) ;yfluxp2=zeros(size (yu)) ;yflu eros(size(zu));zfluxp1=zeros(size(zu));
$[x f \operatorname{luxm}(2: n x-1,:), x s 1, y f \operatorname{luxm}(2: n x-1,:), y s 1, \operatorname{zfluxm}(2: n x-1,:), z s 1]=G o d u n o v \_f l u z$ $1,:), x u(3: n x,:), \ldots$

$[x f l u x p 1(2: n x-1,:), x s 21, y f l u x p 1(2: n x-1,:), y s 21, z f l u x p 1(2: n x-1,:), z s 21]=$ Godun yu(1:nx-2,: ), yu12 (2:nx-1, : ) , zu (1:nx-2,: ), zu12 (2:nx-1, : ), $1: n \mathrm{n}-2,2: n \mathrm{n}-1)$;
$[\mathrm{xfluxp} 2(2: \mathrm{nx}-1,:), \mathrm{xs} 22, \mathrm{yfluxp} 2(2: \mathrm{nx}-1,:), \mathrm{ys} 22, \mathrm{zf} \operatorname{luxp} 2(2: \mathrm{nx}-1,:), \mathrm{zs} 22]=$ AwRas yu12(2:nx-1,:), zu12(2:nx-1,:), 2:nx-1);
$\mathrm{xs} 2=\max (\mathrm{xs} 21, \mathrm{xs} 22)$; ys2=max (ys21, ys22) ; zs2=max (zs21,zs22) ;
[xieq, yieq, zieq] =uequal (xustar ( $2: n \mathrm{x}-1,:$ ) , xu12 ( $2: n \mathrm{n}-1,:$ ) , yustar $(2: n \mathrm{x}-1,:$ ) , yu1
$\operatorname{xfluxp}(2: n x-1,:)=x f l u x p 1(2: n x-1,:) ; x f l u x p(x i e q,:)=x f l u x p 2(x i e q,:) ; x f l u x p(1: 1$

```
xfluxm(2:nx-1,:)=xfluxm(2:nx-1,:)+dx/dt*(xu(2:nx-1,:)-xu12(2:nx-1,:)) ;yfluxm
zfluxm(2:nx-1,:)=zfluxm(2:nx-1,:)+dx/dt*(zu(2:nx-1,:)-zu12(2:nx-1,:)) ; % cfl
    xspeed=2*\operatorname{max}(\operatorname{max}(\operatorname{max}(\operatorname{abs}(xs1))),\operatorname{max}(\operatorname{max}(\textrm{abs}(\textrm{xs}2))));yspeed=2*\operatorname{max}(\operatorname{max}(\operatorname{max}(ab
bs(ys2))));
zspeed=2*max (max (max (abs(zs1))) , max (max (abs(zs2)))) ;
xunew=zeros(size(xu)) ; yunew=zeros(size(yu));zunew=zeros(size(zu));
xin=nbc+1:nx-nbc;one=ones(size(xin));
xunew (xin,:)=xu(xin,:)-dt/dx*(xfluxm(xin,:)-xfluxp(xin-one, :));yunew(xin, :)=
dt/dx*(yfluxm(xin,:)-yfluxp(xin-one,:)); zunew(xin,:)=zu(xin,:)-dt/dx*(zflux
else
disp('No Solver specified');
pause;
end
end
function [xunew, yunew, zunew]=Lanes_boundary(Problem,nx,nbc, xu,yu,zu)
xunew=xu;yunew=yu;zunew=zu;
%left
bcL=Problem.boundaryL;
if(bcL==1)
%Dirichlet
xubc=xu(nbc+1,:);yubc=yu(nbc+1,:);zubc=zu(nbc+1,:);
xflux=Problem.boundaryLflux;yflux=Problem.boundaryLflux;zflux=Problem.bounda
yubc (1, 2)=yflux+yubc(1,1)*press(Problem, yubc (1, 1), 1);zubc (1, 2)=zflux+zubc(1,
for ibc=1:nbc
xunew(ibc,:)=xubc;yunew(ibc,:)=yubc;zunew(ibc,:)=zubc;
end
else
%zero Neumann at the left end
xubc=xu(nbc+1,:);yubc=yu(nbc+1,:);zubc=zu(nbc+1,:);
for ibc=1:nbc
xunew(ibc,:)=xubc;yunew(ibc,:)=yubc;zunew(ibc,:)=zubc;
end
end
    %right
bcR=Problem.boundaryR;
%zero Neumann at the right end
if(bcR==1)
%Dirichlet
```

```
    xubc=xu(nx-nbc,:);yubc=yu(nx-nbc,:);zubc=zu(nx-nbc,:);
```

    xflux=Problem.boundaryRflux;yflux=Problem.boundaryRflux;zflux=Problem.boundaj
    xubc (1,2)=xflux+xubc (1,1)*press (Problem,xubc (1,1),1);\%VERIFY THIS!!!!!!!!!
    \(\operatorname{yubc}(1,2)=y f l u x+y u b c(1,1) * \operatorname{press}(\operatorname{Problem}, \operatorname{yubc}(1,1), 1) ; z u b c(1,2)=z f l u x+z u b c(1\),
    for \(\mathrm{ibc}=\mathrm{nx}-\mathrm{nbc}+1: \mathrm{nx}\)
    xunew (ibc,: )=xubc; yunew(ibc,:)=yubc;zunew(ibc,:)=zubc;
    end
    else
    xubc=xu(nx-nbc,:);yubc=yu(nx-nbc,:);zubc=zu(nx-nbc,:);
    for \(i b c=n x-n b c+1: n x\)
    xunew (ibc,: )=xubc; yunew(ibc,:)=yubc; zunew (ibc,: )=zubc; end
    end
    end
    function plotTraffic(Problem, \(x x, x u, y u, z u, t, h o l d o n, c o l o r) \%\) visualization of tr
    type=Problem.FundamentalDiagram;
    if (Problem.plotFD==1)
    ncol=3;
    else
    ncol=2;
    end
    nline=3;icol=1;iline=1;subplot(nline,ncol, [(iline-1)*ncol+icol (iline-1)*nco
    if (holdon==1)
    hold on;
    end
    \%
    plot( $\mathrm{xx}, \mathrm{xu}(:, 1)$, color); ylabel('rho'); ax=axis; axis([ax(1) ax(2) $0 \max (x u(:, 1))+($
hold off; end if (Problem.plotFD==1)
subplot(nline,ncol,(iline-1)*ncol+icol); xpos=Problem.plotFD1; [dummy,ipos]=m:
if (type==-1)
xvel=0*(xu(:,2)./xu(:, 1)-pressNL(Problem, xu(: , 1))) ;yvel=0*(yu(:,2)./yu(: , 1)-
elseif(type==1)\%AW-RASCLE MODEL
xvel=(xu(:,2)./xu(:,1))-pressNL(Problem, xu(:,1)) ;yvel=(yu(:, 2)./yu(: , 1))-pre
end
xvpos=xvel(ipos); plotFundamentalDiagram(Problem, xpos,xrpos, xvpos);
yvpos=yvel(ipos);plotFundamentalDiagram(Problem,xpos,yrpos,yvpos); end
iline=iline +1 ;icol=1; subplot(nline, ncol, [(iline-1)*ncol+icol (iline-1)*ncol+
hold on;
end
if(Problem.plotvelocity==1)
if (type==-1)
xvel=0*(xu(:,2)./xu(:,1)-pressNL(Problem, xu(:,1)));yvel=0*(yu(:,2)./yu(:,1) elseif (type==1) \%AW-RASCLE MODEL
xvel=(xu(:,2)./xu(:,1))-pressNL(Problem, xu(:,1)); yvel=(yu(:, 2)./yu(:, 1))-pre end
\%plot(xx,xvel, color); ylabel('v'); ax=axis;axis([ax(1) ax(2) $0 \max (x v e l)+0.1])$ $\%$ plot (xx, xu(: , 2) , color) ; ylabel ('y');
plot(xx,yu(:, 2), color); ylabel('y');
end
title(['time= , num2str(t)]);
if (holdon==1)
hold off;
end
if (Problem.plotFD==1)
subplot(nline,ncol, (iline-1)*ncol+icol); xpos=Problem.plotFD2; [dummy,ipos]=m: xpos)) ;yrpos=yu(ipos,1);
if (type==-1)
xvel $=0 *(x u(:, 2) . / x u(:, 1)-\operatorname{pressNL}(\operatorname{Problem}, x u(:, 1))) ; y v e l=0 *(y u(:, 2) . / y u(:, 1)-$
xvel=(xu(:,2)./xu(:,1))-pressNL(Problem, xu(:,1)); yvel=(yu(:,2)./yu(:,1))-pre \%xvpos=xvel(ipos) ; plotFundamentalDiagram(Problem, xpos,xrpos, xvpos) ;
yvpos=yvel(ipos); plotFundamentalDiagram(Problem,xpos,yrpos,yvpos); end if (nline>2)
iline=iline+1;icol=1; subplot(nline, ncol, [(iline-1)*ncol+icol (iline-1)*ncol+ if (holdon==1)
hold on;
end
if (Problem.plotvelocity==1)
if (type==-1)

xvel=(xu(:,2)./xu(:,1))-pressNL(Problem, xu(:,1)); yvel=(yu(:,2)./yu(:,1))-pre $\%$ plot (xx,xvel.*xu(:,1), color); ylabel('Q');ax=axis;axis([ax(1) ax(2) $0 \max (x v$
end
if (holdon==1)
hold off;
end
if (Problem.plotFD==1)
subplot(nline, ncol, (iline-1)*ncol+icol); xpos=Problem.plotFD3; [dummy,ipos]=m
if (type==-1)
xvel=0*(xu(:,2)./xu(:,1)-pressNL(Problem, xu(:,1))); yvel=0*(yu(:,2)./yu(:,1) xvel=(xu(:,2)./xu(:,1))-pressNL(Problem,xu(:,1)); yvel=(yu(:,2)./yu(:,1))-pre \%xvpos=xvel(ipos);plotFundamentalDiagram(Problem, xpos,xrpos,xvpos); yvpos=yvel(ipos);plotFundamentalDiagram(Problem,xpos,yrpos,yvpos); end end
drawnow;
end
function [r,V1,Rline,Jline]=plotFundamentalDiagram(Prob, xpos,rpos,vpos) $\mathrm{nr}=500$; $\mathrm{r}=0: 1 /(\mathrm{nr}-1): 1$;
rhofree=Prob.rhofree; rhojam=Prob.rhojam;rhosyn=Prob.rhosyn;
irfree=find(r<rhofree);irsync=find((r>=rhofree)\&(r<=rhosyn));
irsync2=find((r>rhosyn)\&(r<=rhojam));irjam=find(r>rhojam);rsync=r(irsync); \% V1
$\mathrm{V} 0=0.85 ; \mathrm{c}=2.9$;V1func=@(rho) V0*tanh((.45./rho-0.02)/(c*V0));V1=V1func(r); \%Jamline

V0s=0.5;cs=2.9;Jlinefunc=@(rho)V0s*tanh((.45*(1./rho-1.1))/(cs*V0s)); Jline=J Jline (irfree) =V1 (irfree) ; V1 (irsync2)=Jline (irsync2) ; Jline (irjam)=max (zeros (s V1 (irjam) $=\max (z e r o s(s i z e(i r j a m)), J l i n e(i r j a m)) ;$
\%Rline
Rline=zeros(size (r)) ; slope=(V1func(rhosyn)-Jlinefunc (rhofree)) /(rhosyn-rhofr Rline (irsync) $=(r$ (irsync) - rhofree*ones (size (r (irsync)) )) *slope+Jlinefunc (rhof
Rline(irsync2)=ones(size(irsync2));Rline(irjam)=ones(size(irjam));
if (Prob.contraction_x<xpos)
lanefactor=Prob.lanefactor; r=r/lanefactor;
end
plot(r,1.*V1)
hold on;
plot(r,1.*Rline,'g');plot(r,1.*Jline,'m') ;plot(rpos,1.*vpos, 'k*', 'Markersize hold off;
xlabel(['rho at ' num2str(xpos)]);ylabel('v');axis([0 1000.995$])$
end
function [xieq,yieq, zieq]=uequal(xu1,xu2,yu1,yu2,zu1,zu2,diff) xieq=1:length
xieq1=find(abs(xu1(:,xiu)-xu2(:,xiu))<diff);xieq=intersect(xieq,xieq1); end for $y i u=1: l e n g t h(y u 1(1,:))$
yieq1=find(abs(yu1(:,yiu)-yu2(:,yiu))<diff);yieq=intersect(yieq,yieq1);
end
for $z i u=1: l e n g t h(z u 1(1,:))$
zieq1=find(abs(zu1(:,ziu)-zu2(:,ziu))<diff);zieq=intersect(zieq,zieq1);
end
end
function [xfG,xs,yfG,ys,zfG,zs]=Godunov_flux(Problem, xul, xur, yul, yur, zul, zur \%find the new state
xspeed=zeros(length (xul) , 3) ; yspeed=zeros (length(yul) ,3) ; zspeed=zeros (length (2 [xustar, yustar, zustar]=starstate(Problem, xul , xur , yul , yur , zul , zur, intervall, il intervalm=intervall;
$\operatorname{xrr}=\operatorname{xur}(:, 1) ; \operatorname{xrl}=\operatorname{xul}(:, 1) ; \operatorname{xrm=xustar}(:, 1) ; \operatorname{xyr}=\operatorname{xur}(:, 2) ; \operatorname{xyl=xul}(:, 2) ; \operatorname{xym=xust}$ $\operatorname{yrr}=\operatorname{yur}(:, 1) ; \operatorname{yrl=yul}(:, 1) ; \operatorname{yrm=yustar}(:, 1) ; \operatorname{yyr}=\operatorname{yur}(:, 2) ; \operatorname{yyl=yul}(:, 2) ; y y m=y u s t$ $\operatorname{zrr}=\operatorname{zur}(:, 1) ; \operatorname{zrl}=\operatorname{zul}(:, 1) ; \operatorname{zrm=zustar}(:, 1) ; \operatorname{zyr}=\operatorname{zur}(:, 2) ; \operatorname{zyl=zul}(:, 2) ; z y m=z u s t$ type=Problem.FundamentalDiagram;
if (type==1) \%AW-RASCLE MODEL
xvl=xyl./xrl-press(Problem, xrl,intervall) ; xvr=xyr./xrr-press (Problem, xrr,int press (Problem, xrm, intervalm);
yvl=yyl./yrl-press (Problem, yrl, intervall) ; yvr=yyr./yrr-press (Problem, yrr, int press (Problem, yrm, intervalm) ;
zvl=zyl./zrl-press (Problem,zrl, intervall) ; zvr=zyr./zrr-press (Problem, zrr, int press (Problem, zrm, intervalm);
xizerol=find (xrl<=0) ; xizeror=find (xrr<=0) ; xvl (xizerol)=zeros (length (xizerol)
yizerol=find (yrl<=0) ; yizeror=find (yrr<=0) ; yvl (yizerol)=zeros (length (yizerol)
zizerol=find (zrl<=0) ; zizeror=find (zrr<=0) ; zvl (zizerol)=zeros (length (zizerol)
xizerom=find $(x r m<=0) ;$ xrm (xizerom) $=$ zeros $(l e n g t h(x i z e r o m), 1) ; x v m(x i z e r o m)=x v l($
yizerom=find $(y r m<=0) ;$ yrm (yizerom) $=$ zeros (length (yizerom) , 1) ; yvm (yizerom) =yvl (
zizerom=find (zrm<=0) ; zrm (zizerom) =zeros (length (zizerom) , 1) ; zvm (zizerom) =zvl ( $\operatorname{xspeed}(:, 3)=x v r ; y s p e e d(:, 3)=y v r ; \operatorname{zspeed}(:, 3)=z v r$;
xitype1=find((xvl-xvm-(xrl.*pressprime (Problem, xrl, intervall)-xrm.*pressprime yitype1=find((yvl-yvm-(yrl.*pressprime (Problem, yrl, intervall)-yrm.*pressprime zitype1=find((zvl-zvm-(zrl.*pressprime(Problem,zrl,intervall)-zrm.*pressprim xitype2=setdiff $((1: l e n g t h(x u l)), x i t y p e 1) ; y i t y p e 2=s e t d i f f((1: l e n g t h(y u l)), y i t)$ xspeed (xitype1, 1) $=($ xvl (xitype1) . $* x r l(x i t y p e 1)-x v m(x i t y p e 1) . * x r m(x i t y p e 1)) . /(2$ yspeed (yitype1, 1) = (yvl (yitype1). *yrl (yitype1) -yvm(yitype1).*yrm(yitype1))./ yspeed $(y i t y p e 1,2)=(y v l(y i t y p e 1) . * y r l(y i t y p e 1)-y v m(y i t y p e 1) . * y r m(y i t y p e 1)) . /($ yspeed (yitype2,1)=yvl (yitype2) -yrl (yitype2) .*pressprime (Problem, yrl (yitype2) zspeed (zitype1,1) = (zvl (zitype1).*zrl (zitype1)-zvm (zitype1).*zrm (zitype1))./(2 \%find correct 'Godunov'-states according to the speeds xil=find(xspeed(:,1)> xuG(xil,: ) =xul (xil,: ) ;yuG(yil,: )=yul (yil,:); zuG(zil,:)=zul(zil,:);
xuG (xistar, : ) =xustar (xistar, : ) ; yuG (yistar, : ) =yustar (yistar, : ) ; zuG (zistar, : )=

options=optimset('Display','off');
for xi=xirare
xqinit=[xrl(xi) xvl(xi)];xrhs(1)=xvl(xi)+press(Problem, xrl(xi),xi); xrare=fs for $y i=y i r a r e$
yqinit=[yrl(yi) yvl(yi)];yrhs(1)=yvl(yi)+press(Problem,yrl(yi),yi); yrare=fs $\operatorname{yuG}(\mathrm{yi}, 1)=\mathrm{yrare}(1)$; yuG(yi,2)=yrare(1)*yrare(2)+yrare(1)*press(Problem, yrare( for zi=zirare
zqinit=[zrl(zi) zvl(zi)];zrhs(1)=zvl(zi)+press(Problem,zrl(zi),zi);
zrare=fsolve(@(zq) zrarefactionshape(Problem,zq,zrhs,zi),zqinit,options);
$\operatorname{zuG}(z i, 1)=\operatorname{zrare}(1) ; \operatorname{zuG}(z i, 2)=z r a r e(1) * z r a r e(2)+z r a r e(1) * p r e s s($ Problem ,zrare (
end
end
[xfG,xs,yfG,ys,zfG,zs]=AwRascle_flux(Problem,xuG,yuG,zuG,intervall); end
function xres=xrarefactionshape(Problem,xq,xrhs,interval) type=Problem.Fundar \%riemann invariant
xres(1) $=x q(2)+$ press (Problem, xq(1), interval)-xrhs(1);
$\%$ slope of the characteristic $=$ first eigenvalue
$\mathrm{xres}(2)=\mathrm{xq}(2)-\mathrm{xq}(1) * \mathrm{pressprime}$ (Problem , xq(1), interval)-xrhs (2);
else
\%
end
end
function [xf,xs,yf,ys,zf,zs]=AwRascle_flux(Problem,xu,yu,zu,interval) xf=zer xpp=pressprime(Problem,xr,interval) ;ypp=pressprime(Problem, yr,interval) ; zpp= type=Problem. FundamentalDiagram;
if (type==-1)
$x f=z e r o s(s i z e(x u)) ; x s=z e r o s(\operatorname{size}(x u)) ; y f=z e r o s(s i z e(y u)) ; y s=z e r o s(s i z e(y u)) ;$
zu));
elseif(type==1)\%AW-RASCLE MODEL
xv=xy./xr-xp;yv=yy./yr-yp;zv=zy./zr-zp;
$x f(:, 1)=x v . * x r ; x f(:, 2)=x v . * x y ; y f(:, 1)=y v . * y r ; y f(:, 2)=y v . * y y ; z f(:, 1)=z v . * z r ; z:$
xs=zeros(size(xu)); ys=zeros(size(yu)); zs=zeros (size(zu));
xs $(:, 1)=x v-x r . * x p p ; y s(:, 1)=y v-y r . * y p p ; z s(:, 1)=z v-z r . * z p p ;$
$x s(:, 2)=x v ; y s(:, 2)=y v ; z s(:, 2)=z v$;
end
end
function $\mathrm{p}=$ pressNL(Problem,rho)

```
C=Problem.C;
C1=Problem.C1;
type=Problem.FundamentalDiagram;
if(type==-1)
p=C*ones(size(rho));
elseif(type==1)%AW-RASCLE MODEL
p=C1*log(rho./(ones(size(rho))-rho));
end
end
function p=press(Problem,rho,interval)
C=Problem.C;
C1=Problem.C1;
type=Problem.FundamentalDiagram;
if(type==-1)
p=C*ones(size(rho));
elseif(type==1)%AW-RASCLE MODEL
p=C1*log(rho./(ones(size(rho))-rho));
end
end
function rho=invpress(Problem,p,Vel,interval)
C=Problem.C;
C1=Problem.C1;
lanes=Problem.Lanes(interval);
type=Problem.FundamentalDiagram;
if(type==-1)
rho=C*lanes.*ones(size(p));
elseif(type==1)%AW-RASCLE MODEL
rho=exp(p/C1)./(ones(size(p))+exp(p/C1));
end
end
function p=pressprime(Problem,rho,interval)
C=Problem.C;
C1=Problem.C1;
type=Problem.FundamentalDiagram;
if(type==-1)
p=C*ones(size(rho));
elseif(type==1)%AW-RASCLE MODEL
p=C1*ones(size(rho))./(rho.*(ones(size(rho))-rho));
```

end
end
\%the right hand side for the relaxation term and lane-change term
function [xunew, xmaxdt, yunew, ymaxdt, zunew, zmaxdt]=source_step (Problem, xx, dt,
xunew $(:, 1)=x u(:, 1)+1 * d t *(x s o u r c e(:, 1)+L * x P H I(:, 1))$;
xunew $(:, 2)=x u(:, 2)+1 * d t *(x s o u r c e(:, 2)+L * x P H I(:, 2))$;
yunew $(:, 1)=y u(:, 1)+1 * d t *(y s o u r c e(:, 1)+L * y P H I(:, 1))$;
yunew $(:, 2)=y u(:, 2)+1 * d t *(y s o u r c e(:, 2)+L * y P H I(:, 2))$;
zunew $(:, 1)=z u(:, 1)+1 * d t *(z s o u r c e(:, 1)+L * z P H I(:, 1))$;
zunew $(:, 2)=z u(:, 2)+1 * d t *(z s o u r c e(:, 2)+L * z P H I(:, 2))$;
xmaxdt=inf; ymaxdt=inf; zmaxdt=inf;
if $(\max ($ abs $(x s o u r c e))>0)$
xmaxdt=xT;
end
if $(\max ($ abs $($ ysource $))>0)$
ymaxdt=yT;
end
if (max (abs (zsource) $)>0$ )
zmaxdt=zT;
end
end
function [xsource, xT,ysource, yT, zsource, zT]=righthandside(Problem, xx, xu, yu, zl
\%the fundamental diagram
type=Problem.FundamentalDiagram;
\% T=Problem.Trelax;
$x T=z e r o s(\operatorname{size}(x u(:, 1))) ; y T=z \operatorname{eros}(\operatorname{size}(y u(:, 1))) ; z T=z \operatorname{cros}(\operatorname{size}(z u(:, 1)))$; for
rhofree=Problem.rhofree;
rhojam=Problem.rhojam;
rhosyn=Problem.rhosyn;
xsource=zeros(size(xu)); ysource=zeros(size(yu));zsource=zeros(size(zu));
xr=xu(:,1);yr=yu(:,1);zr=zu(:,1);
if (type==-1)
$x v=x u(:, 2) . / x r-p r e s s(P r o b l e m, x r, 1: l e n g t h(x u)) ; y v=y u(:, 2) . / y r-p r e s s(P r o b l e m, y)$
zv=zu(:,2)./zr-press(Problem,zr,1:length(zu));
elseif(type==1)\%AW-RASCLE MODEL

zv=zu(:,2)./zr-press(Problem,zr,1:length(zu));
end
x_cont=Problem.contraction_x;
x_cont_dist=Problem.contraction_x_smooth;
lanefactor=Problem.lanefactor;
itrans=find((xx>=x_cont-x_cont_dist)\&(xx<x_cont+x_cont_dist));
$i_{-}$cont=find(xx>=x_cont+x_cont_dist);
xr(i_cont)=lanefactor*xr(i_cont);
\% lanefactor if(abs(lanefactor-1)>1.e-5)
$\mathrm{xa}=1 /((1-\mathrm{length}(i t r a n s))) *(1-1 /$ lanefactor) $) \mathrm{xb}=1-\mathrm{xa}$; rmaxtrans=xa*(1:length
else rmaxtrans=ones(size(itrans));
end
xr(itrans)=xr(itrans)./rmaxtrans';
irfree=find(xr<rhofree);
irsync=find((xr>=rhofree)\&(xr<=rhosyn));
irsync2=find((xr>rhosyn)\&(xr<=rhojam));
irjam=find(xr>rhojam);
$\mathrm{V} 0=0.85$; c=2.9;
V1func=@(rho)V0*tanh((.45./rho-0.02)/(c*V0));
$\mathrm{xV} 1=\mathrm{V} 1 \mathrm{func}(\mathrm{xr})$; $\mathrm{yV} 1=\mathrm{V} 1 \mathrm{func}(\mathrm{yr}) ; \mathrm{zV} 1=\mathrm{V} 1 \mathrm{func}(\mathrm{zr})$;
\%Jamline
V0s=0.5;cs=2.9;
Jlinefunc=@(rho)V0s*tanh((.45*(1./rho-1.1))/(cs*V0s));
xJline=Jlinefunc(xr);yJline=Jlinefunc(yr);zJline=Jlinefunc(zr);
$x J l i n e(i r f r e e)=x V 1(i r f r e e) ; y J l i n e(i r f r e e)=y V 1(i r f r e e) ; ~ z J l i n e(i r f r e e)=z V 1(i r ~$ \%Rline
xRline=zeros(size(xr)); yRline=zeros(size(yr)); zRline=zeros(size(zr)); xslo rhofree*ones(size(xr(irsync))))*xslope+Jlinefunc(rhofree)*ones(size(xr(irsyn rhofree*ones(size(yr(irsync))))*yslope+Jlinefunc(rhofree)*ones(size(yr(irsyn rhofree*ones(size(zr(irsync))))*zslope+Jlinefunc(rhofree)*ones(size(zr(irsyn xRline(irsync2)=ones(size(irsync2)) ;yRline(irsync2)=ones(size(irsync2));zRli xRline (irjam)=ones(size(irjam));yRline (irjam)=ones(size(irjam));zRline(irjam if (type==-1)
xve=@(xr,xv)(1-xr)-xv;yve=@(yr,yv)(1-yr)-yv;zve=@(zr,zv)(1-zr)-zv;
xrel=xve(xu(:,1),xv);yrel=yve(yu(:,1),yv);zrel=zve(zu(:,1),zv);
xsource(:,2)=xu(:,1).*xrel./xT(:,1);ysource(:,2)=yu(:,1).*yrel./yT(:,1);zsou elseif(type==1)\%AW RASCLE MODEL
$x V==z e r o s(l e n g t h(x u), 1) ; y V e=z e r o s(l e n g t h(y u), 1) ; z V e=z e r o s(l e n g t h(z u), 1) ;$ $x V e(i r f r e e)=x V 1$ (irfree); $y V e(i r f r e e)=y V 1(i r f r e e) ; z V e(i r f r e e)=z V 1$ (irfree); \%parameter for thr linear interpolation
alpha=0.7;
alpha=min(alpha, alpha-1.e-2);
xkink1=xV1 (irsync) +alpha*(xRline(irsync)-xV1 (irsync)) ; ykink1=yV1 (irsync) +al xabove=find (xv(irsync) $>x k i n k 1)$; yabove=find (yv(irsync) >ykink1) ; zabove=find (zv $x V e(i r s y n c(x a b o v e))=x V 1(i r s y n c(x a b o v e)) ; y V e(i r s y n c(y a b o v e))=y V 1$ (irsync (yabov xkink2=xJline(irsync)+alpha*(xRline(irsync)-xJline(irsync)); ykink2=yJline(i xbelow=find (xv(irsync) <=xkink2) ; ybelow=find (yv (irsync) <=ykink2) ; zbelow=find ( $x \operatorname{xVe}(i r s y n c(x b e l o w))=x J l i n e(i r s y n c(x b e l o w)) ; y V e(i r s y n c(y b e l o w))=y J l i n e(i r s y n c$ xmiddle=find $((\operatorname{xv}(i r s y n c)<=x k i n k 1) \&(x v(i r s y n c)>x k i n k 2)) ;$ ymiddle=find((yv(irsync)<=ykink1) \& (yv(irsync)>ykink2));
zmiddle=find((zv(irsync)<=zkink1)\&(zv(irsync)>zkink2));
xa=(xV1 (irsync(xmiddle)) -xkink1 (xmiddle) $-(x J l i n e(i r s y n c(x m i d d l e))-x k i n k 2(x m i$ ya=(yV1 (irsync(ymiddle)) -ykink1 (ymiddle)-(yJline (irsync (ymiddle)) -ykink2 (ymi za=(zV1 (irsync(zmiddle))-zkink1 (zmiddle)-(zJline(irsync (zmiddle))-zkink2(zmi xb=-xa.*xRline (irsync(xmiddle)) ; yb=-ya.*yRline (irsync (ymiddle)) ; zb=-za.*zRli $x \mathrm{xVe}(\operatorname{irsync}(x m i d d l e))=x a . * x v(i r s y n c(x m i d d l e))+x b+x v(i r s y n c(x m i d d l e)) ; ~ y V e(i r s$ xVe(irsync2)=xJline (irsync2) ; yVe(irsync2) =yJline(irsync2) ; zVe(irsync2)=zJlin xsource $(:, 2)=x u(:, 1) . *((x V e-x v) . / x T(:, 1)) ; \operatorname{ysource}(:, 2)=y u(:, 1) . *((y V e-y v) . / y$ end
end
function [xPHI, yPHI, zPHI]=lanechanges(Problem, $x x, x u, y u, z u, n x$ )
xPHI1=zeros (size (xu)) ; xPHI2=zeros (size (xu)) ; yPHI1=zeros (size (yu)) ;yPHI2=zero e(zu)); zPHI2=zeros(size(zu));
xPHI=zeros(size(xu)) ; yPHI=zeros(size(yu)) ; zPHI=zeros (size(zu));
$\mathrm{xr}=\mathrm{xu}(:, 1) ; \mathrm{yr}=\mathrm{yu}(:, 1) ; \mathrm{zr}=\mathrm{zu}(:, 1)$;
$\mathrm{wr}=0.3 *$ ones $(\operatorname{size}(\mathrm{xu}(:, 1))) ; \mathrm{V} 0=0.85 ; \mathrm{c}=2.9 ; \mathrm{V} 1$ func=@(rho)V0*tanh((.45./rho-0. type=Problem.FundamentalDiagram;
if (type==-1)
xv=xu(:,2)./xr-press (Problem, xr, 1:length(xu)) ; yv=yu(:, 2)./yr-press (Problem, y zv=zu(:,2)./zr-press(Problem,zr,1:length(zu));

Co=Problem.Co;
else(type==1); \%AW-RASCLE MODEL
$x v=x u(:, 2) . / x r-p r e s s(P r o b l e m, x r, 1: l e n g t h(x u)) ; y v=y u(:, 2) . / y r-p r e s s(P r o b l e m, y$
zv=zu(:,2)./zr-press(Problem, zr, 1:length(zu));
Co=Problem.Co;
end
\%expression for PHI~0_1 (0,1,2) THE ON-RAMP SITUATION
$x$ PHI1 $(2: n \mathrm{x}-1,1)=\mathrm{wr}(2: \mathrm{nx}-1) . * \operatorname{abs}(\mathrm{wv}(2: n \mathrm{x}-1)-\mathrm{wv}(3: \mathrm{nx})) . * \exp (-\mathrm{Co} * \mathrm{xr}(2: \mathrm{nx}-1)) . *($
\%expression for PHI~0_2(1,2,3) LANE 1

```
xPHI2(2:nx-1,1)=yr(2:nx-1).*abs(yv(2:nx-1)-yv(1:nx-2)).*exp(-Co*xr(2:nx-1))
```

$\exp (-C o * z r(2: n x-1))) \ldots$
$-\mathrm{xr}(2: \mathrm{nx}-1) . * \operatorname{abs}(\mathrm{xv}(2: \mathrm{nx}-1)-\mathrm{xv}(3: \mathrm{nx})) . * \exp (-\mathrm{Co} * \mathrm{yr}(2: \mathrm{nx}-1)) . *(1 . /(1-\mathrm{xr}(2: \mathrm{nx}-1$
$\mathrm{xPHI}(2: n \mathrm{x}-1,1)=\mathrm{xPHI} 1(2: \mathrm{nx}-1,1)+\mathrm{xPHI2}(2: \mathrm{nx}-1,1)$;
\%expression for PHI^1_1(0,1,2) THE ON-RAMP SITUATION xPHI1 (2:nx-1,2)=wr (2:nx \%expression for PHI^1_2(1,2,3) LANE 1
$x$ xHI2 $(2: n x-1,2)=y r(2: n x-1) \cdot * y v(2: n x-1) . * \operatorname{abs}(y v(2: n x-1)-y v(1: n x-2)) . * \exp (-C o *$ 1)) ).*(1-exp(-Co*zr(2:nx-1)))...
$-\mathrm{xr}(2: \mathrm{nx}-1) . * \mathrm{xv}(2: \mathrm{nx}-1) . * \operatorname{abs}(\mathrm{xv}(2: \mathrm{nx}-1)-\mathrm{xv}(3: \mathrm{nx})) . * \exp (-\operatorname{Co} * \mathrm{yr}(2: \mathrm{nx}-1)) . *(1 . /$
$\mathrm{xPHI}(2: n \mathrm{x}-1,2)=\mathrm{xPHI} 1(2: \mathrm{nx}-1,2)+\mathrm{xPHI2}(2: \mathrm{nx}-1,2)$;
\%expression for PHI~0_1 $(1,2,3)$ LANE 2
yPHI1 (2:nx-1,1) $=x r(2: n x-1) . * \operatorname{abs}(x v(2: n x-1)-x v(3: n x)) . * \exp (-C o * y r(2: n x-1)) . *($
$-\mathrm{yr}(2: \mathrm{nx}-1) . * \operatorname{abs}(\mathrm{yv}(2: \mathrm{nx}-1)-\mathrm{yv}(1: \mathrm{nx}-2)) . * \exp (-\mathrm{Co} * \mathrm{xr}(2: \mathrm{nx}-1)) . *(1 . /(1-\mathrm{yr}(2: \mathrm{nx}$
\%expression for PHI~0_2(2,3) LANE 2
$\operatorname{yPHI2}(2: n \mathrm{x}-1,1)=\mathrm{zr}(2: \mathrm{nx}-1) . * \operatorname{abs}(\mathrm{zv}(2: n \mathrm{x}-1)-\mathrm{zv}(3: \mathrm{nx})) . * \exp (-\operatorname{Co} \operatorname{tyr}(2: n \mathrm{x}-1)) . *($
$-y r(2: n x-1) . * \operatorname{abs}(y v(2: n x-1)-y v(1: n x-2)) . * \exp (-\operatorname{Co} * \operatorname{zr}(2: n x-1)) . *(1 . /(1-y r(2: n x$
$\mathrm{yPHI}(2: n \mathrm{x}-1,1)=\mathrm{yPHI} 1(2: n \mathrm{x}-1,1)+\mathrm{yPHI2}(2: \mathrm{nx}-1,1)$;
\%expression for PHI^1_1 $(1,2,3)$ LANE 2
yPHI1 $(2: n x-1,2)=x r(2: n x-1) . * x v(2: n x-1) . * a b s(x v(2: n x-1)-x v(3: n x)) . * \exp (-C o * y r$
1)))...
$-\mathrm{yr}(2: n \mathrm{x}-1) . * \mathrm{yv}(2: \mathrm{nx}-1) . * \operatorname{abs}(\mathrm{yv}(2: \mathrm{nx}-1)-\mathrm{yv}(1: \mathrm{nx}-2)) . * \exp (-\mathrm{Co} * \mathrm{xr}(2: n \mathrm{x}-1)) . *(1$
\%expression for PHI^1_2 2,3 ) LANE 2
yPHI2 $(2: n x-1,2)=z r(2: n x-1) . * z v(2: n x-1) . * a b s(z v(2: n x-1)-z v(3: n x)) . * \exp (-C o * y r$
1)))...
$-\mathrm{yr}(2: \mathrm{nx}-1) \cdot * \mathrm{yv}(2: \mathrm{nx}-1) . * \operatorname{abs}(\mathrm{yv}(2: \mathrm{nx}-1)-\mathrm{yv}(1: \mathrm{nx}-2)) . * \exp (-\mathrm{Co} * \mathrm{zr}(2: \mathrm{nx}-1)) . *(1$
yPHI (2:nx-1,2)=yPHI1(2:nx-1,2)+yPHI2(2:nx-1,2);
\%expression for PHI~0_1 $(2,3)$ LANE 3
zPHI1 (2:nx-1,1) $=\mathrm{yr}(2: n \mathrm{x}-1) . * \operatorname{abs}(\mathrm{yv}(2: n \mathrm{x}-1)-\mathrm{yv}(3: \mathrm{nx})) . * \exp (-\operatorname{Co} * \mathrm{zr}(2: n \mathrm{x}-1)) . *($
$-\mathrm{zr}(2: \mathrm{nx}-1) . * \operatorname{abs}(\mathrm{zv}(2: \mathrm{nx}-1)-\mathrm{zv}(1: \mathrm{nx}-2)) . * \exp (-\mathrm{Co} * \mathrm{yr}(2: \mathrm{nx}-1)) . *(1 . /(1-\mathrm{zr}(2: \mathrm{nx}$
\%expression for PHI^1_1 $(2,3)$ LANE 3
zPHI1 $(2: n x-1,2)=y r(2: n x-1) . * y v(2: n x-1) . * \operatorname{abs}(y v(2: n x-1)-y v(3: n x)) . * \exp (-C o * z r$
1)))...
$-\mathrm{zr}(2: \mathrm{nx}-1) \cdot * \mathrm{zv}(2: \mathrm{nx}-1) \cdot * \operatorname{abs}(\mathrm{zv}(2: \mathrm{nx}-1)-\mathrm{zv}(1: \mathrm{nx}-2)) . * \exp (-\mathrm{Co} * \mathrm{yr}(2: \mathrm{nx}-1)) . *(1$
end

