MODELLING CREDIT RISKS USING UNBOUNDED TRANSITIONAL MATRICES

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Modelling Credit Risks Using Unbounded Transitional Matrices

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DECLARATION

This thesis is my original work and has not been presented for a degree in any University.

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DEDICATION

I dedicate this work to my Dad Christopher Monari and Mom Genevieve Nyanchoka.

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This research though rigorous, it has been enriching academically as well as fullfilling. It was not possible doing it alone and there are a number of people who contributed to its completion.

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LIST OF ABBREVIATIONS

BCBS-Basel Committee of Banking Supervision

CDO-Collateral Debt Obligation

CSFP-Credit Swiss Financial Products

CTMC- Continuous Time Markov Chain

CTMs-Credit Transition Matrices

DD-Distance-to-Default

DTMC- Discrete Time Markov Chain

EAD-Exposure at Default

EDF-Expected Default Frequency

ELGD- Expected Loss Given Default

EM- Expectation Maximization

GM- Generator Matrix

IRB-I nternal Rating Based

L-Debtvalue at maturity $t = T$

MLE- Maximum Likelihood Estimator

PD- Probability of Default

PGF-Probability generating Function

r-risk-free rate(constant)

SMEs-Small and Medium Enterprises

S&P- Standard and Poor's

TM-Transitional Matrix

ABSTRACT

Credit Risk management are ways of mitigating losses by considering the Bank's capital adequacy and reserves for loan losses and it is a challenging process for most banking institutions. In many inputs of Risk management, Credit Migration matrices or Transition Matrices are the main inputs. In this Thesis, conditions for existence of a true generator in instances where the transition matrix is unbounded is identified for a Markov transition matrix empirically observed. The Thesis comes up with generators which are valid and singles out the correct one compatible with the Credit rating behaviours and demonstrates how to obtain a generator which is approximate when a true generator is non existence especially in unbounded transitional matrices. Illustrations are given using secondary data gotten the standard and Poors website. The main challenge in transition matrices is in obtaining the generator matrix $\hat{Q}$ for $\hat{P}$ such that the exponential of $\hat{Q}$ will yield $\hat{P}$. This challenge is known as embedding problem and is mostly experienced in Matrices higher than 3 by 3 square matrix. This problem is addressed where four statistical methods that use generator matrices to generate transitional matrices are proposed. They are the Diagonal and Weighted adjustment method, the Generator Quasi-Optimization method, the EM algorithm method and finally the Gibbs sampler also known as the Markov Chain Monte Carlo method. The Credit data is analysed using the four methods and the best perfoming method gotten from comparison using the L-norm.

Chapter One

Introduction

Credit Risk is a very Critical area in Financial Institutions. Stakeholders, regulators, consumers and institutions have a lot of concern in Credit Risk and it is a subject of research interest not limited to statistical research. Wikipedia defines Credit risk as, "A credit risk is risk of default on a debt that may arise from a borrower failing to make required payments. In the first resort, the risk is that of the lender and includes lost principal and interest, disruption to cash flows, and increased collection costs. The loss may be complete or partial." (Al-Zahrani & Tayachi, 2021). Default event is the main concern of Credit Risk and mostly occurs when a debtor fails to meet its financial obligation as stipilated in the Credit Contract. Some of the examples of Credit default events are bond default, Credit Card Charge off, corporate bankruptcy, personal loans default and mortgage foreclosure. This chapter will discuss the background of the study, statement of the problem, objectives of the study, justification of the study, scope of the study and and

organization of the thesis.

1.1 Background of the study

Credit risk management is a method of mitigating losses adopted by banks by taking into consideration their Capital at any given time as well as their corresponding loan reserves (Trueck & Rachev, 2009). Transitional matrices are characterized by changes in the quality of credit and are the major inputs found in most of Risk Management applications, assessment of Risk Management portfolio, Credit structuring and pricing. For instance, Credit Migration or transition matrices are the main inputs of the New Basel Accord (BIS(2001)) and therefore the estimation of transitional Matrices are very critical(Wright, Sheedy, & Magee, 2018). Conditions for which a true generator exists or not are outlined and a model which explores the non negativity condition in cases where the generator matrix is unbounded is explored in this study. Banks worldwide have adopted what is known as Credit Risk strategy which is basically a process which adopts the development of a score card (Bülbül, Hakenes, & Lambert, 2019). Before implementation of a Credit Risk policy, the Credit Risk strategy informs the Bank's Credit personell on how to interpret the score and what are the adequate actions to be taken following the interpretation. In addition to Credit Risk strategy, Banks nowadays use Credit ratings obtained from Credit rating agencies. Globally, the big three agencies are Moody's, Standard and Poor's and Fitch (Brusov, Filatova, & Orekhova, 2021). The role of these agencies is to provide an assessment of the

relative Credit risk of debt securities or financial instruments which are structured. They also assess in some cases goverment's Creditworthiness as well as their securities. In order to evaluate borrower's insolvency, Credit ratings are issued by the rating agencies which corresponds to the borrower's Credit Risk which is the risk that the borrower will default. The Credit agency's final rating represents the evaluation of the Credit risk of a borrower at a given time. In Kenya, Credit information is collected and kept by two consumer reporting agencies namely Transunion and Metropol(Mokaya, 2019). Transition Matrices rating are receiving increased attention in the financial sector and the two main rating services i.e Moody's and Standard and Poor's publish annually on the Credit ratings using transition Matrices as well as other information related. One year is the shortest time interval in which one can estimate a transition matrix. A shorter period than 1 year will be too small for one to make a reliable estimation of transitional Matrix except in cases where valuation needs to be done for example valuation of a default swap. This research will propose a model whereby one can be able to estimate a Credit transition matrix given any period. If a generator can be obtained from a transitional matrix P**, i.e a Matrix** Q **whose sums of row add to** 0 **and the off diagonal entries are non negative such that** $exp\left(Q\right) = P$**, then** $P\left(t\right) = exp\left(tQ\right)$ **and the matrices can be obtained for any particular time** $t \geq 0$**. However, the major problem is to find out whether generator** Q **exists or not and how to obtain generator** Q **in situations where the transitional matrices is unbounded. The issue of identification and existence of transitional matrix generators has**

not been addressed in Financial literature. Historically, to default in Credit was treated as a crime and in various times and places, the punishment was through torture, imprisonment, mutilation or death (Wikipedia contributors, 2021a). The next subsection will discuss briefly the definition and history of Credit Risk.

1.1.1 Credit Risk History and Definition

Credit is more ancient than writing. A code known as Hammurabi's code (Wikipedia contributors, 2021a) had the legal thinking codified in Mesopotamic 4000 years ago but the basic rules of borrowing were not captured or outlined in the code. Aspects such as collateral, default and interest rate were not addressed in the code but the code emphasized that failure to honour a debt repayment was to be treated as a crime and was categorized together with theft and fraud. The code also had limits to penalties e.g Creditors could seize a defaulter and sell him into slavery but the wife and children could be sold to only a three year term. The Bible also has records of debt enslavement without disapproval where for instance in the story of Elisha and the widow `https://www.biblegateway.com/passage/?search=2+Kings+4&version=NIV` whereby she faced enslavement of her two children because the husband had died before paying his debts. The Bible even goes further in providing a limit for the Creditor's collection rights which is mercy aspect unlike Hammurabi's code. In modern world, protection aspects of bankruptcy and from the Creditors are absent entirely from both the Bible and Hammurabi. In

most cases from history, any default of Credit was treated as a crime and at some places, default was punishable by imprisonment or enslavement, torture or even death and sometimes the punishment could be meted on the debtors as well as their dependants. Financial Institutions have been facing difficulties every year for a number of reasons (DeHaan, 2017). The main cause of major Banking problems is related directly to weak Credit Standards for counterparties and borrowers, weak portfolio risk management or lax in attending to economic changes or circumstances leading to deteriorating in Credit standards of bank counterparties. Credit risk is a key concern in Kenyan Market (2021, Mutua, Munda, Mutai, & Omulo, 2021) as experienced from the recent closure of major retail stores chains due to debt. Credit Risk is defined as the potential a borrower or counterparty borrowing from a bank fails to honour their obligations according to the agreed terms. Credit Risk Modelling is very important in Banking. Any Loan facility requested from a Bank or a Credit Institution has to be decided by a Credit analyst whose job is to either approve or decline. For example if a Bank receives a Mortgage Loan application from a building company whose directors are well known to the Bank's Credit Analyst. If there has been a slump in the buiding industry in recent times and a lot of default has been accrued from Mortgage Loan facilities held by the Bank. Despite the Company's directors being well known by the Credit Analyst, he will naturally decline the loan basing from the historical default data of similar Loan facilities. Another option the Credit Analyst will do is to approve the Mortgage facility on condition that it is insured so as

to protect the Bank against any potential loss. Since any Customer applying for a Credit facility in a financial institution is a potential defaulter, there is a need for a robust Credit Risk management and many approaches have been fronted by various scholars. The Basel accord was introduced by the Bank of International settlement (BIS) in the year 2001 whose purpose was to improve the overall stability of the Banking sector by enabling them to absorb shocks arising from stress related to financial and economic conditions from any source as well as improving governance and risk management. The next subsection will discuss in detail the Basel Accord.

1.1.2 Basel Accord

The Basel accords are Banking regulations categorized into three series and are set by the Basel Committee of Banking Supervision(BCBS) (Goodhart, 2011). The Accords designs are to ensure that enough Capital is provided by banking institutions in order for them to meet the obligations and to ensure that unexpected losses are absorbed. Basel III (Fidrmuc & Lind, 2020) is the latest accord that was set in November 2010. BCBS consists of a supervisory committee established by governors of Central bank in ten Countries in the year 1974. The membership of the committee was expanded in 2009 and in 2014. BCBS provides an avenue of Banking supervisory cooperation and interaction on various matters. The main objective of BCBS is to ensure that supervisory issues which are key are understood and to ensure that Banking quality is improved worldwide. The Basel III guidelines have three major as-

pects also known as the three pillars namely Minimum Capital requirement, Supervisory review process and Market discipline. Basel Accord developed a model where a Corporate Credit Hazard model was utilized to figure the administrative capital for a wide range of advances(Basel Advisory group of 2005 on banking Supervision) despite the fact that the fundamental thought of such a model, that default happens when obligations surpass resources is not the motivation behind why consumers default. Collateral Debt Obligations (CDO's) comprise loans and debt instruments which have different Credit Ratings. Each firm is believed to migrate between Credit ratings according to a Markovian Transition Matrix, an example is the Standard and Poor's One-Year Transition Matrix. Credit Transition (Migration) Matrices portray the past changes in Credit nature of obligors (commonly, firms in customary Corporate Finance or pools of benefits, in organized Finance). Such frameworks are cardinal contributions to many hazard risk management applications, including portfolio risk appraisal, displaying the term structure of Credit risk premia, and the valuing of Credit subsidiaries. Additionally, in the New Basel Accord capital necessities are driven partially by rating relocation. In such applications their precise evaluations are critical. There are several methods of estimating migration matrices and the widely used frequency methods are directly based on historical perfomance. The next subsection will introduce the Markov Chain which forms the basis of Transitional matrices.

Table 1.1: Types of Markov Chains

	Countable state space	Continuous/General state space
Discrete time	Discrete time markov Chain(DTMC) on finite or countable state space	Markov chain is on a state space which is measurable

1.1.3 Markov Chain

A markov chain is a type of stochastic model which describes a sequence of events which have the probability of each event depending on the state which was attained in the previous event(Wikipedia contributors, 2021c). In time which is continuous, it is also called a Markov process named after a Russian Mathematician known as Andrey Markov. Markov processes are also the basis for methods of stochastic simulation called Markov chain Monte carlo and is commonly used for simulation of samples from probability distributions which are complex. It is also applied in Bayesian statistics as well as artificial intelligence. A markov process is defined as a stochastic process which satisfies or fullfills the Markov property. In other terms, a Markov process is a process in which can be used to make predictions basing on only its present state and knowing the full history of the process. In other words, the present state condition, the future and past states of a process are independent.

There is need for specification of the state space of the system and the index of the time parameter. The table below outlines a summary of the two types of markov chains i.e discrete time and continuous time markov chain (Hanks, 2017). A state space is a set of all configurations possible in a system. The

next section will briefly provide an overview of the rating agencies, rating process and categories. It will provide a rough overview of procedures of rating as implemented by standard & Poor's(S&P), one of the most used Credit rating agency.

1.1.4 Credit Rating

Rating agencies have a long tradition and started in the United States of America.(Kerwer, 2002) S&P history is traced back to the year 1860 and it began rating of corporate and government issuers more than 80 years ago. S&P ratings are input data of several Credit risk softwares eg Credit Metrics and they generally provide two different types of ratings; Issue-Specific Credit ratings and Issuer Credit ratings. Obligors are divided into categories ranging from AAA which reflects the strongest quality of Credit to D which reflects default occurence(Omstedt, 2020). The four highest categories AAA, AA, A and BBB are recognized as investment grades while BB and below ratings are regarded to be speculative grades or non investment grades. The table K.1 in appendix shows a summary of how the different categories of rating ought to be interpreted.

1.2 Statement of the Problem

Over a decade ago, guidelines for Bank regulatory Capital were provided by the Basel committee. The objective for these guidelines was to provide protection to all financial system risks and to level the playing field globally for all financial institutions. The Capital Accord of 1988 provided regulations regarding the Capital amount that should be held by Banks against the risk of Credit or Credit risk. The final Capital Accord was published and implemented in 2006. Statistically, there has been a significant Credit Risk increase in countries like United States of America, Korea, Japan and not limited to Kenya . The committee approach in the past was a "one size for all" but now it provides an allowance for all Banks to use internal models for their Credit Risks assessment. As a result, there has been an emergence of new technologies and models for analysing Credit Risk but the unbounded generator matrices in analysing Credit risks is not being utilized. Today, Credit risk analysis which is accurate is very important considering the volatility of the market and fluctuating economies. Although there has been a development of Credit risk models, no consideration was made as regarding to Bank Loans Credit Risks. In the Banking industry, Credit risk is associated with the Loans quality and likelihood or probability of default. Potentially, Credit risk is a variation in market value and net income of loans or assets resulting from delayed payment or non payment. On the other hand, Cash flow of the bank assets which are mainly loans may be altered by changes in Bank's operating environment or changes in economic conditions for example the recent bill on capping of interest in Kenya. The purpose of this thesis is to provide a model which is very effective to measure Credit risk by use of transitional matrices especially in cases where the transitional matrices are unbounded. Most studies ignored the time varying risk premium and Credit cycle. A square matrix A is called a transitional Matrix if all the entries of A are non-negative and the sum of the entries in any given row is 1. In all instances, the transitional Matrix must be unbounded. This thesis aims to bring in the aspect of using statistical analysis in Credit assessment especially in Kenyan banks since the current methods employed are not statistical based but rather employ assessment using historical data which may at times due to lack of borrower's trends. This study proposes the Gibbs sampler method for generator matrix. This method is compared with three other methods; the diagonal and weighted adjustment method, the generator quasi optimization method and the Expectation-Maximization algorithm as would be seen in chapter four.

1.3 Objectives of the study

1.3.1 General Objective

To Model Credit Risks using Unbounded Transitional Matrices.

1.3.2 Specific Objectives

1. To develop an unbounded transitional matrix from a Credit ratings data.
2. To derive the risk premium properties from the developed transitional matrix.
3. To convert the Discrete Time Transitional Matrix obtained into Continuous Time transitional matrix in order to get a generator Matrix.

4. To get the most suitable generator matrix for the developed Continuous Time Transitional Matrix especially in cases where the matrix is embeddable.

1.4 Research Questions

This thesis sought to answer the following research questions.

1. How do we develop an unbounded transitional matrix from a Credit ratings data?

2. How do we derive the risk premium properties from the developed transitional matrix?

3. How do we convert the Discrete Time Transitional Matrix developed into Continuous Time transitional matrix in order to get a generator Matrix?

4. How do we get the most suitable generator matrix for the developed Continuous Time Transitional Matrix especially in cases where the matrix is embeddable?

1.5 Justification of the study

In the new Basel Capital Accord(Basel II) (Goodhart, 2011), the Internal Ratings Based (IRB) approach provides an allowance for banks to make use of internal Credit rating developed on their own. Banks need to provide an

estimation of the matrix entirely of transition probabilities between classes of rating and the Accord empasizes that the role of these probabilities should be essential in regulatory capital calculation, approval of Credit, allocation of internal capital, risk management and bank functions in Corporate finance (Merikas, Merika, Penikas, & Surkov, 2020). For the purposes of regulation, the Accord provides a requirement that Financial Institutions should establish procedures which are rigorous for statistical models validation to be used in internal ratings. This requirement is in Basel committee on Banking 2005. These procedures include tests which are out of sample and historical data over a long period of time is used. Technically, these procedures are a challenge to many Financial Institutions globally not limited to Kenyan Banks especially to those that have a large number of business lines that are of high quality and of whose default data are unavailable. Regulators expect that portfolios which have low default rates should also follow the minimum Internal ratings based (IRB) standards for probability conservation and accuracy of estimates of default despite them having data limitations. There are two technical challenges which are related to portfolios with low default rates.

The first technical challenge is the default probability estimation where no history has been recorded as default. According to Hamilton *et al* (2007), (Cantor & Hamilton, 2007) report, there were sixteen years with no issues of default over the period of years 1980-2006. However none of the portfolios or assets sampled over the years were default free completely hence the need for a model which should assign a probability default which is positive.

The second technical challenge is the assessment of the low default portfolios predictive perfomances. The procedures of testing the samples (Shumway 2001) (Shumway, 2001) cannot be applied in such cases because the zero default frequencies do not have a benchmark reasonable for comparison with the predictions of the models. Several models have been reviewed in this thesis and three contributions have been made. The first contribution is to develop a model that provides a description which Kenyan banks employ. A Credit rating is assigned to a Credit facility based on the Credit worthiness current assessment. The assessment depends on specific and systematic firm's variables. The model has macroeconomic unobserved effects which transitional probabilities depend on for different classes of Credit in any given period. The specification of the model provides an allowance of auto-correlation across transition time probabilities from any class of Credit.

Lastly, the heterogeneity of the model is taken into account of its Credit worthiness of Credit class which are the same and can have effects which are significant on diversification of Credit risk. The second contribution addresses the difficulty in predictive perfomance difficulty of a model with sparse data like in low default portfolios. Bayesian technique of predictive power was used in the first approach. The second approach in analysing the model predictive performance is a variant of out of sample testing which takes into account the transition probabilities estimates with their credible intervals corresponding at 95 percent.

The third contribution applies the methodology to the secondary data set

of Standard and Poor's between 2010-2018 and contributing to the Basel II policy. Different specifications are calibrated of the Credit rating model and estimated transtion matrices are shown to exhibit a behaviour which is non-Markovian. The macroeconomic shock randomly observed improves significantly the predictive power of the model and also accounts for the dependence observed on the probabilities of transition for Credit classes which are different in any given period.
Credit Risk is normally assessed in two different ways; qualitative or quantitative methods. However, in some cases, information may not be available for the Bank to apply the two ways to assess Credit Risk. In this Thesis, the main focus will be on the probability of Credit Rating Transition from one level to another using unbounded Transitional Matrix Models.

1.6 Scope of the Study

In this Thesis, the existence of a true generator conditions are identified for an empirically observed Markov transition matrix. This Thesis identifies conditions for a true generator matrix to exist in case where the Candidate matrix is unbounded. This Thesis analyses various models proposed by various scholars on how transition matrices can be used to manage Credit Risks in banks. The Thesis sought to identify the best model of deriving a generator matrix which can be used in predicting Credit Transition matrices that can be applied in Kenya in order to reduce non-performing loans in situations where the transition matrix is unbounded i.e matrix which is not a scalar of multiple identity matrix and does not obey the properties of a Hilbert Matrix as explained later in section 3.8.

Migration or transition matrices are major inputs for risk management in Credit Risk Management, Credit Value-at-Risk or pricing using derivatives. This Thesis investigates migration in Credit ratings in structured finance and compare ratings mobility across different sectors/products and traditional corporate finance, utilizing the ratings data available in Standard and Poor's website. Credit migration or transition matrices are characterized by the past changes in Credit quality of obligors (typically firms), are cardinal inputs to many risk management applications, not limited to portfolio risk assessment, modelling the term structure of Credit risk premia, and pricing of Credit derivatives. Credit Transition Matrix has received attention since the seminal work done by Jarrow, lando and Turnbull (1997). Examples include Kijima and Komoribayashi (1998) who improved on the procedure of estimation proposed by Jarrow et al (1997), a Markov process using one factor to model Credit transitions was also proposed by Belkin Suchower and Forest Jr (1998). In the New Basel Accord BIS (2001), capital requirements are driven in part by migration ratings. Accurate estimation of the same is therefore critical. The simplest use of a transition or migration matrix is for the bond valuation or loan portfolio which may be used by a portfolio or risk manager. Given a Credit Grade today, say BBB, the value of that Credit Asset for One Year hence will depend on the probability that it will be BBB, migrate to a better or worse Credit grade, or default by the time year ends. This can range from an increase in value of 1-2%. In case of upgrade to a decline in value of 30-50% in case of default. Sophisticated examples of risky bond pricing methods outlined by Jarrow and Turnbull (1995), Jarrow, Lando and Turnbull, (1997), require these matrices as a cardinal input, as do Credit derivatives such as the model by Kijima and Komoribayashi (1998). Credit portfolio models such as Credit Metrics (JP Morgan (1997)) used in risk management make use of this matrix for simulation of the value distribution of a portfolio of Credit assets. Credit Risk Models are used for justification of spreads in interest rates and to quantify the inherent risk in Credit contracts and Credit portfolios. Rating institutions have huge data bases for calibration of their statistical models. The behaviour of one single debtor is more or less easy to model and the more challenging problem is to find appropriate models for the joint behaviour of many debtors in a Credit portfolio.

Credit ratings assess relative expected loss. they don't intend to have particular probability default captured over a particular horizon. Data inspected simply indicates that within a rating category, default rates may rise or fall over time and sometimes significantly. Ratings have been proved to be effective when used as ordinal measure of Credit risk. Sometimes, default is not the only interest of a Credit event, upgrades and downgrades can occur in an investment. Moody's Credit transitional model allows expected default rate to be assigned to a rated Credit and generates transition forecasts for Credits which can be extended over from one quarter to 5 years or more. Credit ratings are intended as relative assessments of expected loss and they are not

intended to capture a particular default probability over a horizon which is particular . Doing Simple inspection of the data indicates that within a category rating, there is rise and fall of default rates over time, and sometimes quite significantly. Furthermore, their cycle again, conditional on rating is strongly correlated with the economic cycle. Credit risk remains the dominant problem confronting banks. Nevertheless, there is a need for banks to identify, monitor and control Credit risk and ensuring capital adequacy to anticipate the risk (Basel Committee on Banking Supervision (1999)). Basel II confirmed that financial institutions must have the ability to analyse Credit models and Internal Ratings to ensure the model is calibrated to measure Credit Risk consistently and meaningfully. Furthermore, Credit Risk is the main risk faced by Financial Institutions. Van Deventer and Imai (2003) specifically mentioned that Credit Risk is the major reason for bank default. BIS (2005) confirmed that the main reason for bank failure is low Credit quality and poor Credit Risk evaluation. Credit risk evaluation which is done poorly tends to neglect the use of capital requirements to expedite an accurate valuation and tight control of Credit risk exposure to a bank. Credit ratings created or produced by agencies and offices like Moodys, Standard and Poors and Fitch furnish or provide financial market members with educated and well informed feelings or opinions, of a standardized or institutionalized nature, on the likelihood or probability that Credit issues will be adjusted in a precised manner. The significance of ratings or appraisals as a source of information or wellspring of data to investors or financial specialists has

increased and expanded in recent years, as Credit Markets have developed and grown more Global and come to include and incorporate a wider and extensive range/scope of obligors, items and structures. After some time, Credit products are at risk to move starting from one rating category or class to another. Accordingly, more emphasis has been put under or placed on understanding not just default hazard as well as rating progress transition risk, for example the adjustments in Credit quality surveyed by rating offices when news affecting an obligors Credit quality is uncovered or revealed; this is also alluded to as Credit Rating Migration. Ratings or evaluations, volatility or unpredictability and default risk chance are probably going to change over the appraisal range or ratings spectrum, for example increasing or expanding with each consecutive or back to back movement or development down the ratings scale especially when moving from venture evaluation or investment grade to theoretical evaluation or speculative grade. The need and requirements for powerful and robust models of the Credit risk of portfolios of Customer or Consumer Loans has been brought into sharp focus and concentration by the failure and disappointment of the ratings evaluation agencies/organisations to accurately and precisely survey the Credit risks and dangers of Mortgage or Home loan Backed Securities (MBS) and collateralized debt obligations (CDO) which are based or in the light of such portfolios. There are numerous reasons advanced and put forward for the subprime home loan or Mortgage crisis and the subsequent or ensuing Credit crunch (Hull 2009, Demyanyk and van Hemert 2008) however plainly and clearly one reason that the previous prompted to the latter was the absence of an easily and effectively updatable model of the Credit danger or risk of portfolios of Customer loans.

1.7 Organization of the Thesis

The first chapter of this Thesis gives a prologue or introduction to the Credit hazard or Risk Models in the Banking industry. The statement of the problem describes the probability of a Transitional Matrix moving from one state to another and sought to define what an unbounded matrix is. It also describes General objectives, Specific objectives, Research Questions and outlines Justification of the study where the traditional banking methods of assesing Credit Risk are explained i.e internal ratings based models. The chapter also describes the main focus of the thesis which is the probability of Credit Rating transition moving from one level to another in cases where the transition Matrix models are unbounded.

The second Chapter provides the Literature Review where the existing models for analysing Credit risks are outlined together with their weaknesses. The Chapter defines Credit risks and describes the importance of Credit Risk Modelling to Banking. The chapter reviews previous studies done in modelling credit risks using migration matrices and their contributions. The Key objectives of this Thesis is to identify the most appropriate generator matrix for a Markov Transition Matrix which is unbounded.

In the third Chapter, Methodology is described where techniques on identifying a Matrix generator is well outlined. Bounded matrices, Markov chains, Generator Matrices and their estimation, ways of estimating CTMC from DTMC and embeddable Markov chains are well analysed and defined. The problem of embeddability in 2×2 and 3×3 Transition Matrices is discussed and various ways of correcting the Matrix Q is well outlined.

The fourth Chapter outlines the results and discussion . The Data is based on some empirical observation from Standard and Poors data got from its website. Incase the transition matrix P is not embeddable, four methods namely the Diagonal adjustment, the Weighted adjustment, the EM algorithm and the Markov Chain Monte Carlo Method also known as the Gibbs sampler are studied and the best method proposed.

The Fifth Chapter which is the last chapter provides a summary of all the objectives and contributions of the Thesis and considers possible extensions to this Thesis. The main feature of this PHD thesis which is original involves application to Data which is real and covers details to a maximum level. The same model presented is innovative and the level of the detail of the Data Analysis is rare and unique. Even though the equations are not detailed due to confidentiality reasons, they provide a sufficient insight and appreciation of all the concepts outlined. The equations and the analysis described are not the pillar elements of this thesis since they depend on the secondary Standard and Poors Data from the website and their Credit products. The Key elements are the concepts behind them which are the pillar assets of this Thesis.

Chapter Two

Literature Review

2.1 Introduction

About two decades ago, guidelines were provided by the Basel committee on Banking Supervision and Capital regulatory on Banks. The objective of this was to level the playing field globally for all financial institution and provide a protection from all financial system risks. Though models of Credit Risks and dangers have been created and developed by researchers and scholars such as Wei(2003), Lando and Turnbull(1997) and Kijima and Komoribayashi(1998)(Kijima & Komoribayashi, 1998), none has considered Bank Loans Credit Risks. This chapter will review the models advanced by various scholars.

2.2 Conceptual Review

2.2.1 Credit Risk

Credit Risk is a loss possibility resulting from a failure on the part of a borrower to make a loan repayment or to meet his or her contractual obligation. It is also a risk on the part of a lender incase they do not receive their principal and interest due to them which may result in cash flow interuption as well as debt collection increase costs.

2.2.2 Transition Matrix

A matrix(plural matrices) is an array of rectangular numbers or a table of numbers, expressions or symbols arranged in rows and columns (Wikipedia contributors, 2021d). Matrices are majorly applied in linear transformations for example in vector rotation where the three dimensional space is a linear transformation. They can also be applied in most scientific fields like physics, computer graphics, calculus just to mention a few. In business field like economics, they are used to describe economic relationships.

A transition matrix may refer to a matrix associated with basis change or a stochastic matrix or a state transition matrix. A stochastic matrix is a square matrix describing a Markov chain with non negative entries which are probabilities (Wikipedia contributors, 2021e).

2.3 Theoretical Review

2.3.1 Estimation of transition matrices for sovereign Credit ratings

(Hu, Kiesel, & Perraudin, 2002) studied the estimation of transition matrices for sovereign Credit ratings. In their paper, they showed how one may combine sovereign default information studied over a longer period from a number of countries in order to derive estimates of the credit transition matrices. Their approach consisted mostly of modelling sovereign defaults using a common maximum likelihood.

2.3.2 One parameter presentation of Credit risk

(Belkin, Suchower, & Forest Jr, 1998) in their paper, presented a one parameter presentation of Credit risk and transitional matrices. They started with the credit metrics perception that transition matrices rating are as a result of "binning" of a standard normal variable X which provides a measure of credit worthiness changes. They assumed that X splits into two parts namely; First, a component Y which is idiosyncratic and unique to each borrower. then secondly, a component Z of the system which is shared by all borrowers. Z provides a measure of credit cycle which means that the default rates valued at the end of period risk rating are not predicted using the average historical transition rates by the credit grade initial matrix. In good credit years, Z will be positive meaning that for each initial credit rating, a default rate which is a ratio of lower than average and higher than average upgrades to downgrade. the reverse will be true in bad years. (Belkin et al., 1998) described a method of getting an estimate of Z from the tabulated TM got from Standard and Poor's (SP) and Moody's. He described a method of evaluating a TM using an assumed value of Z.

2.3.3 Cross Sector volatility

(Collet & Ielpo, 2018) studied cross sector volatility which they found could threaten the financial muscle of credit markets and the portfolio of credit bond. In their study, they measured spill overs of cross sector volatilities focusing mainly on the US investment grade bonds. They found out that the volatility spill overs were high in the US investment Credit market and the net contributors were goods, insurance and energy sectors in the shock experienced over the $1996-2017$ period. They applied structural analysis of the spill over history using multivariate VAR Markov switching model over a three regime and found out that having the three different regimes will result in different structures of volatility spill over. They concluded that goods and insurance sectors are the sources of volatility spill overs during crisis periods and their estimates formed a large spill over portion difficult to anticipate.

2.3.4 Mover-Stayers approach

(Ferretti, Gabbi, Ganugi, Sist, & Vozzella, 2019) in their paper proposed that the banks may misestimate transition probability and as a result may lend

money incoherently with borrowers default trajectory hence causing system distress and asset quality deterioration. They applied a Mover-stayer model to estimate the risk of migration of SMEs and found that banks were overestimating their credit risk hence resulting in excessive regulating capital. Their study has vital macroeconomic implication since holding a big capital buffer is expensive for banks and this affects their lending ability in the economy at large. Their conclusion is valid especially during economic downturns with the results exacerbating the risk capital cyclicaly hence worsening the economic condition. In their study, they also explained the misevaluation of borrowers and the true relevant weight of non performing loans in banking portfolios. The mover-stayers approach assists in reducing calculation inaccuracy when especially analysing the historical migrations of borrowers' ratings hence improving the resource efficacy allocation process and stability of the banking industry. (Landini, Uberti, & Casellina, 2018) proposed a not-standard approach which considered a portfolio on an open sample which allows entries, stayers migrations and exits. While it is consistent with observations, the open sample approach contrasted with the method of standard closed sample. In their paper, they proposed a methodology for integrating the outcome of the standard closed method from the open-sample perspective while sieving out some of the standard method assumption. They concluded by basing on the Markovian hypothesis that with a priori absorbing default state, the method of standard closed sample is supposed to be dropped for not being able to predict the lender's bankruptcy. Secondly, in order to satisfy many re-

liable approximation of new regulatory standards, the portion to approximate migration rates matrices for Credit risks should incorporate either entries or exits. Lastly, the static eigen-decomposition required procedure to forecast rates of migration should be substituted with a stochastic process methodology while at the same time conditioning the forecasts to macroeconomic scenarios.

2.3.5 Novel substitute of Credit Transitional Matrices Study

(Štěpánková, 2021) in their study analysed a novel substitute of CTMs got from Bank sourced CTMs. They provided an insight into approximation of bank sourced CTMs by analysing the extent to which the dependency of CTms are on the credit risks datasets and the aggregation method outlines that inform its choice. They showed that bank sourced CTMs are more viable than those got from Credit rating agencies with larger off-diagonal rates of transition and higher upgrade propensity. Finally, they created a set of particular CTMs for the industry which previously were impossible to obtain due to the sparsity of data suffered by credit agencies and showed the consequences of their differences, signalling the emerging of business cycles which are specific to the industry. They proposed that their approach using large scale Monte Carlo simulation be implemented by the regulators and financial organizations interested to improve their models of Credit risk.

2.3.6 Study on estimates of Banks credit risk

(Máková, 2019) studied banks' credit risk models by investigating features estimates of their credit risk and evaluating uses of transition matrix estimates as well as related assumptions. They used a specific dataset of internal credit risk estimates got from global A-IRB banks, exploring monthly observations of EU large corporates and $20,000$ North American Corporates over the $2015-2018$ time period. Their study empirically tests the most used Markovian property assumptions and time homogeneity at a larger scale than most previous studies. The results showed that estimates of internal credit risk do not fullfill these assumptions as they demonstrated evidence of both time heterogeneity and path-dependency. Lastly, their findings contradicted previous findings on data from credit rating agency and banks are prone to revert to actioning their rating.

(Kreps, 2019) in their paper titled The Black-Scholes-Merton Model as an Idealization of Discrete-time Economies came up with assumptions that Credit Risk models are categorized into two classes; Structural or asset value models and Reduced form or default rate models. Structural models have origins with the famous Merton Model (Bharath & Shumway, 2004) where the firm's default is modelled as per its assets and liabilities relationship at the end of the given period. The value of the debt of a firm at maturity will equal the liabilities nominal value minus the European pay-off put option of the value of the firm. The asset value is processed as a geometric Brownian motion model and default occurs when the maturity level asset value is lower than the li-

abilities. In such cases, the debtor exercises the put option and hands over the firm. The Merton's model assumption are in tandem with Black Scholes model hence the pricing of a risky Credit can be done with the pricing theory option of the Black-scholes. The Merton's model has further been developed by various authors to have a bankruptcy inclusion possibility before maturity or the evolution of stochastic of the risk free rate. These structural models are referred to as the latent variable or threshold models. The Reduced form model also known as the Jarrow turnbull model was published in the year 1995 by Robert A. Jarrow and Stuart Turnbull (Wikipedia contributors, 2021b). However most approaches reviewed do not investigate whether the Matrices are unbounded and how best to get the generator matrices for the transitional matrices.

2.3.7 Markov Generator Matrices Estimation

The problem commonly encountered in Credit Risk management is how to estimate the probabilities of default (PD) in the high investment grades when given insufficient data. To address this issue, there is need to model the transition matrices using Continuous Time markov Chain(CTMC). The approach of CTMC uses the probability of successive downgrades which leads to default in a manner that very small PD can be captured. In most Banking applications, the method faces a data limitation problem since it makes use of data which is continuously observed in order to estimate the transition matrices intensities. In reality, the internal rating systems data in all individual banks are annual or biannual. In order to apply this method, the methods of estimating basing on discretely observed rating data from Banks need to be analysed for practical purposes hence there is need to estimate CTMC from DTMC. This research will define and discuss Bounded matrices in detail with relevant examples, it will compare the functions for estimating the Markov generator matrices from discrete time observations, the discrete time log likelihood function, the generator matrix exponential function, the generator matrix estimation and its confidence interval in a bid to model Credit riks using unbounded transition Matrices.

2.3.8 The Credit Risk+ approach

The Credit Risk+ model was developed by Credit Swiss Financial Products(CSFP) and is currently one of the benchmark models of the financial industry in Credit Risk management area(Gundlach & Lehrbass, 2013). It is also used widely in the community of supervisory since it uses the same data as the basic input which is a requirement of Basel II Internal Ratings Based (IRB) approach. The CreditRisk+ model is a reduced form model for Credit Risk portfolio. In comparison to the asset value models, it processes Credit defaults by modelling it directly instead of having a stochastic process definition of the asset value of the firm which will indirectly lead to defaults. The default probabilities can vary depending on factors underlying hence rates of default are not constant like the case of Credit Metrics or KMV models

but rather are stochastic variables model. It has an important property that instead of portfolio loss distribution property, an analytical solution can be derived for the loss distribution of a Credit portfolio given by approximation means of the portfolio loss probability generating function. It also seems easier for data calibration to the model than for multifactor asset value models case and very importantly, concentration risk driver is revealed in the Credit risk.

A private company named after its founders namely Kealhofer, McQuown and Vacisek came up with the KMV model in 1989 and is presently maintained by Moody's KMV(Valášková, Gavláková, & Dengov, 2014). It is based on Merton's approach but in a slightly varied manner in order to gauge a Credit portfolio risk. The KMV model main contribution is in its calibration which makes it to achieve the correspondence of default probabilities to the empirically observed one and not its theoretical model. The calibration makes use of a huge data base. The Expected Default Frequency (EDF) is computed within the KMV model and is the Capital structure of the firm based together with its asset volatility and asset current value of the firm. This is done in three stages; First, iterative procedure for asset value estimation is used by the KMV and asset returns volatility analysed. This method is Merton's model based in modelling equity as the firm's underlying assets call option with the liabilities of the firm as the strike price. In Merton's model, the probability of default of a given firm is evaluated by the asset value probabulity V_1 in one year which is lying below the threshold value B, a representation of the debt

of the firm. Hence the default probability PD_{Merton} in Merton's model is a current asset value V_0, the asset value annual mean μ_v and volatility δ_v together with the threshold B function. An estimated probability is represented by the EDF that a given firm within one year will default. In the model of KMV, the EDF is different slightly but has astructure which is similar to the Merton's Model probability of default. The $1 - \phi$ function above is replaced by some function which is decreasing and is empirically estimated in the KMV model. The firms are assumed to be homogeneous in the KMV model in probability of defaults for equal DDs. The mapping between EDF and DD is empirically determined basing on a database of events of historical default.

2.3.9 The CreditMetrics Model

The Creditmetrics approach departs from the assumption that the Bond market value or its default probability is derived by using the firm's assets value as its main input variable. As a tool for managing risk, the model ought to be applicable to all types of financial instruments having inherent Credit risk. Also the procedure of valuation must be consistent with the market prices.(Guptin et al 1994). Hence the Credit metrics utilizes for the valuation the Company's rating, historical transition matrices and bond prices which are empirically derived. The assumption is that all variables other than the current issuer rating over time behave deterministically. Hence the bond or loan valueat the risk time horizon T is dependent essentially on the state of rating of the issuer at this time i. CreditMetrics makes the assumption that if the issuer

is not in default state at the risk time horizon, the bond or loan value is determined by outstanding cash flow discounting by making use of Credit spreads over the interest rate r which are riskless. The spread correspond with the rating state i of the T issuer. Hence the bond distribution or loan values in T are given by the probabilities $P(X = i)$ of the rating states which are different in T together with the values corresponding to the bond $V_{i,T}$. In the CreditMetrics model, for one to obtain the possible ratings distribution at t, the initial vector is multiplied with a t-step transition matrix. If the risk horizon is greater than one year, the suggestion is to compute the transition probabilities required vector $p_i \cdot (t)$ either with a multiple of transition matrix P which is one year, $p_i \cdot (t) = \delta_i \cdot p^t$ or with direct estimates t-year transition matrix $p_i \cdot (t) = \delta_i \cdot p(t)$. Hence all possible future ratings at time t are obtained with the corresponding transition probabilities;

Rating at t	1	2	...	k-1	k
Migration probability	$\mathbf{p}_{i1}(t)$	$\mathbf{p}_{i2}(t)$	...	$\mathbf{p}_{i(k-1)}(t)$	$\mathbf{p}_{ik}(t)$

2.4 Theoretical Framework

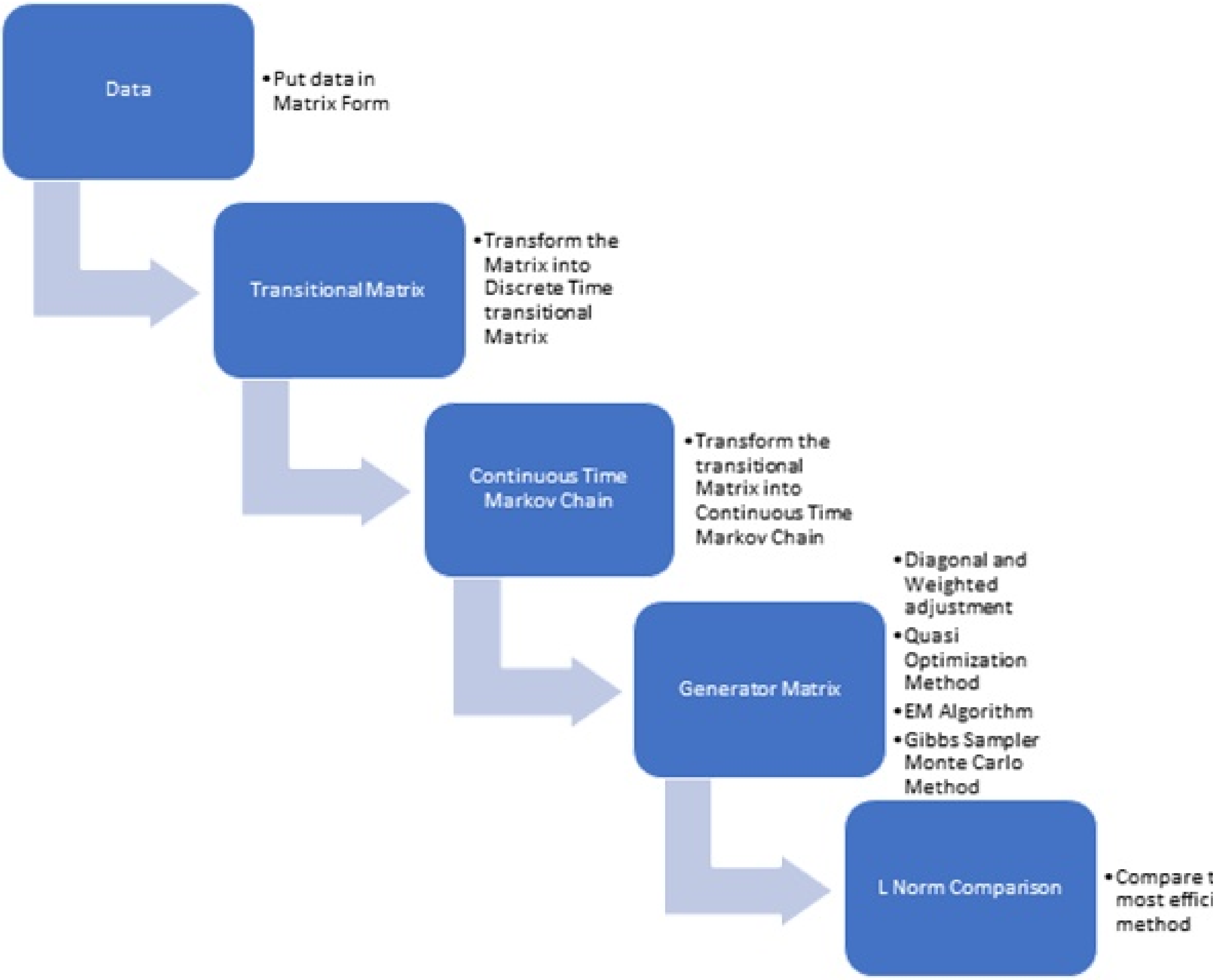

Figure 2.1: Theoretical framework

2.5 Summary

The reviewed studies explore the estimation of transition matrices for sovereign Credit ratings, One parameter presentation of Credit risk, Cross Sector volatility, Mover-Stayers approach, Novel substitute of Credit Transitional Matrices, estimates of Banks credit risk, Credit Risk+ approach and CreditMetrics Model. While studies have been done extensively on Credit risks, none so far has been done on cases of unbounded matrices and embeddability problem. Credit risk models have the assumption that over a period of time, there is a migration between different Credit states in a counterparties Credit rating (Ferretti et al., 2019). This process can be viewed as a mathematically finite Markov chain and has the assumption that the Credit rating successively changes under time intervals given from one state to another with a specific probability. These probabilities of the Credit migration are what forms a Credit migration or transition matrix. The yearly transition matrices specifying the probability of state changes over a period of one year can be obtained from agencies specializing in the Credit data. The risk management practises and pricing of instruments which are in reduced form are modelled on the finite markov chain model. For instance, Jarrow et al (Jarrow, Lando, & Turnbull, 1997) and Das and Tufano(1996) (Tufano, 1996) did some discussion on the prices and hedging of counter party risk and corporate debt with some embedded options making use of Markov chain model of Credit risk migration. In both discrete and continuous time process models, the knowledge of the yearly transition matrices is not sufficient for pricing since

many instruments mature or have cash flows occuring on a non yearly basis. For instance, pricing a risky instrument of Credit maturing in six months will require a transition matrix which is six months. In general, there is need to get transition matrices over time horizon which are imaginary. If the process of transition is time homogeneous, then the transition matrix obtained in n-period can be evaluated by raising the transition matrix of the single period to power n. Although the model feature is very appealing, it has some problems which can be conceptual. For instance a six-months transition matrix ought to be a square root of the yearly transition matrix but this is not possible since it raises the yearly transition matrix to a less than one power hence resulting in a matrix having elements which are negative.This negative elements matrix is not a transition matrix. Moreover, even when it is non existent, a transition matrix which is six months may not be unique (Pfeuffer, Möstel, & Fischer, 2019). When one is confronted with this problem, one of the possible solution is to reconstruct it in a way that allows it to be solved easily. This research will attempt to use the regularization process in order to obtain time horizons which are arbitrary and approximate the roots of a yearly transition matrix. One way that transition matrix are obtained for periods that have arbitrary length is to use embedding of the DTMC into CTMC (Pfeuffer et al., 2019). For a CTMC, any transition matrix of any period length can be evaluated as the exponential of the generator matrix. Hence solving and evaluating the embedding problem allows one essentially to compute a generator consistent with the yearly transition matrix of the DTMC. But the generator

computing of a transition matrix in existence using its logarithms has issues of uniqueness and existence. As we'll note in this thesis, transition matrices which are empirically observed have typical properties precluding the generator existence (Israel, Rosenthal, & Wei, 2001). Conversely, more than one generator gives rise to transition matrix which is the same (Pfeuffer et al., 2019). To evade these difficulties, the suitable approach is to directly estimate the generator and then put it into use by constructing the necessary transition matrices. Unfortunately, data provided by agencies on Credit migration only deal with yearly transition matrices leaving out their generators.

Chapter Three

Methodology

3.1 Introduction

Markov chain is a stochastic process named after Andrey Markov and is based on a memoryless property also called the Markov property(Houag, 2016). This property means that the past and future are independent of the present provided the present is known. Markov chain is used in modelling many processes by making the analysis of the processes easy. There are two types of Markov Chains; Discrete and Continous Markov chain. The Continuous Time Markov Chain (CTMC) having finite states is applied in many real life situations. In finance, it is used to analyze the time series hence assisting in understanding the prices change and predict financial market trends. The estimation of the CTMC parameters when this process has been continuously observed in some specific interval is simple and known. But the estimation of these parameters from Discrete Time Markov Chain (DTMC)

observations closely related to the embedding problem is not an easy process. This latter estimation problem was considered as an important statistical problem in modeling credit risk using transitional matrices with the main problem being getting the best and unique generator matrix estimator. This chapter provides a description of the research design and methodology used in the study. It covers Data Source, Modelling Transition Matrix, Model derivation, Default Probability and Risk Premium Properties, Discrete Time Markov Chain, Continuous Time Markov Chain, Bounded Matrices, Markov chain model, Generator Matrices, Embeddable Markov Chain Matrices, the 2×2 TM embedding problem, the 3×3 TM embedding problem and lastly the Complex Eigen Values.

3.2 Data Source

The data used in the study was got from Standard and Poors website `https://cerep.esma.europa.eu/cerep-web/statistics/transitionMatrice.xhtml`

$$
\begin{array}{c|cccccccc|}
 & AAA & AA & A & BBB & BB & B & CCC & D \\
\hline
AAA & 52 & 30 & 1 & 0 & 0 & 0 & 0 & 0 \\
AA & 5 & 76 & 31 & 3 & 0 & 0 & 0 & 0 \\
A & 0 & 21 & 28 & 19 & 6 & 3 & 0 & 0 \\
BBB & 0 & 0 & 5 & 26 & 8 & 2 & 0 & 1 \\
BB & 0 & 0 & 2 & 13 & 21 & 13 & 3 & 0 \\
B & 0 & 0 & 0 & 0 & 11 & 46 & 0 & 0 \\
C & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\tag{3.1}
$$

3.3 Developing an unbounded transitional matrix from a Credit ratings data

3.3.1 Modelling Transition Matrix

Rating agencies such as Standard and Poor's or Moody's normally publish the Credit ratings of firms periodically. With the introduction of Basell II accord, the Credit rating importance have increased significantly and information used to assess abilities of firms is provided to investors. Ratings are increased or lowered depending on the improvement or decline of the firms

or Company's Credit quality. The present obligor rating provides a prediction for its future which is the main feature of any Credit ratings i.e the past ratings and the present ratings determine the evolution. The use of Markov Chains Model is commonly used to describe the Credit ratings dynamics as illustrated by Jarrow and Turnbull(1995).

Specifically let x, represent at time t, the rating evaluation of a Borrower or Loanee from a bank or Financial institution at time t. Assuming

$$x = \{x_t, t = 0, 1, 2, \ldots\}$$

is a Markov Chain of time homogeneous nature and space

$$S = \{1, 2, \ldots, C, C + 1\}$$

state 1 is the highest class of Credit, state 2 the second highest,..., and state C represents the lowest class of Credit. The default state is represented by state $C + 1$ and is the absorbing state. Let

$$f_{ij} = P\left(x_{t+1} = j | x_t = i\right), \quad i, j \epsilon S, \quad t = 0, 1, 2, \ldots$$

represent the probability state i transists to state j. f_{ij} and P is the one-step probability of transition and measure of probability respectively. The transition class of Credit i to class of Credit j is represented by a transition

matrix which is time homogeneous F where;

$$F = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1c} & f_{1,c+1} \\ f_{21} & f_{22} & \cdots & f_{2c} & f_{2,c+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{c1} & f_{c2} & \cdots & f_{cc} & f_{c,c+1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} \underset{(c \times c)}{A} & \underset{(c \times 1)}{D} \\ \underset{(1 \times c)}{0} & \underset{(1 \times 1)}{1} \end{pmatrix} \tag{3.2}$$

where

$$f_{ij} > 0 \; for \; all \; i, j \; and \sum_{j=1}^{c} f_{ij} = 1 \; for \; all \; i$$

Submatrix $A_{(c \times c)}$ is defined on $S = \{1, 2, ..., C\}$ where S is a non absorbing state.

Submatrix A components indicate the switching of classes of Credit of the Bank's borrower but excludes default state $c + 1$. Components of the column vector $D_{(c \times 1)}$ which are $f_{1,c+1}$ are the probability of transitions of borrowers from banks from any Credit class $i = 1, 2, ..., C$ switching to class of default $j = c + 1$. For sampling sake, it is assumed that bankruptcy given by state $C + 1$ is the absorbing state. After the process enters default state, there will be no return to Credit class state hence

$$f_{c+1,c+1} = 1$$

Therefore it is said that the default state $C + 1$ is also an absorbing state.

3.3.2 Model Derivation

Taking into consideration the Cycle of Credit, there are steps to derive the transition matrix. The first step is to develop a mapping devise through which the probability of transition can be transformed into Credit scores. In this research, normal distribution is used since it is easy to calculate. The sum of row matrix in a transition matrix is 1 hence the cumulative normal distribution function is inverted by beginning from the default function as seen in Belkin, Suchower and Forest(1998), Wei(2003) and Kim(1999). Hence the Transition matrix 3.2 observed above is

$$Y = \begin{pmatrix} y_{12} & y_{13} & y_{14} & \cdots & y_{ic} \\ y_{22} & y_{23} & y_{24} & \cdots & y_{2c} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{c2} & y_{c3} & y_{c4} & \cdots & y_{cc} \end{pmatrix} \tag{3.3}$$

The Matrix 3.3 above is $C \times C$ since there is no need for the rows to be converted into an absorbing state. If Matrix 3.3 is converted into a probability transition matrix, then it results into Matrix 3.2 which is a Transition Matrix. From Belkin, Suchower and Forest(1998), Matrix Y is decomposed into 2 factors of time t as;

$$Y_t = \alpha L_t + \sqrt{1 - \alpha^2 \epsilon_t} \tag{3.4}$$

where L_t is the Credit Cycle. In a good year, the Credit cycle will be positive implying that for the first Credit rating, a lower rate than default average and higher than upgrades average downgrades. On the other hand, a bad Year results to the Credit cycle being negative. The observed transition matrix at any year will have a normal deviation i.e;

$$L_t = 0$$

ϵ_t is non-systematic and every borrower has a unique idiosyncratic factor. L_t and ϵ_t are assumed to be Unit normal variables which are mutually independent. Unknown coefficient α represents the correlation between Credit Cycle L_t and Y_t. In order to minimize the weight, the coefficient of the rows of Matrix 3.3 is found together with the fitted transition probabilities and the discrepancies which are mean squared.

$$P_{i,j} = \Phi\left(y_{i,j+1}\right) - \Phi\left(y_{i,j}\right) \tag{3.5}$$

$$P\left(y_{i,j+1}, y_{i,j}|L_t\right) = \Phi\left(\frac{y_{i,j+1} - \alpha L_t}{\sqrt{1-\alpha^2}}\right) - \Phi\left(\frac{y_{i,j} - \alpha L_t}{\sqrt{1-\alpha^2}}\right) \tag{3.6}$$

$\Phi\left(.\right)$ represents the function of distribution which is standard normal and equation 3.5 and 3.6 represents the fitted and observed transition probability of i state to j state in time t. The least square problem just as Belkan,

Suchower and Forest(1998) takes the form,

$$min_L \sum \sum \frac{n_{t,1} \left[P_t \left(i, j \right) - P_t \left(y_{i,j+1}, y_{i,j} \right) \right]}{P} \tag{3.7}$$

$n_{t,i}$ **represents the borrower's number moving to state** j **from state** i**. The weighting factor is also given as;**

$$\frac{n_{i,j}}{P_t \left(y_{i,j+1}, y_{i,j} L_t \right) \left[1 - P_t \left(y_{i,j+1}, y_{i,j}, L_t \right) \right]}$$

hence the fitted transition matrix can be constructed as;

$$M = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1c} & m_{1,c+1} \\ m_{21} & m_{22} & \cdots & m_{2c} & m_{2,c+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{3.8}$$

3.4 Default probability and Risk Premium properties

The approach of risk-neutral probability is explored to assess the Bank Loans Credit Risk. Default probability tends to zero for higher Credit ratings but in a risk-neutral world, loan rates observed imply a default probability of non-zero. By risk premium estimation, the framework of risk neutral can be controlled. In an ideal situation, loan rates should be matched with the tran-

sition matrices observed so that risk premium will be obtained. A borrower who is likely to default is matched with the stochastic process;

$$\tilde{X} = \left\{ \tilde{X}_t, t = 0, 1, 2, \ldots \right\} \tag{3.9}$$

Credit rating. The Credit falls under the measure or proportion of risk-unbiased probability or likelihood. The fitted change on transition matrix for the reasons or purposes of valuation is changed and transformed into a transition matrix which is risk-neutral under the measure which is equivalent to martiangle where the matrix is denoted as $\tilde{m}$. Under the new measure, the transition matrix does not need to be Markovian if it is a Markov chain which is an absorbing Markov chain. Hence the fitted transition matrix falling under the measure of risk neutral probability is;

$$\tilde{m}(t, t+1) = \begin{pmatrix} m_{11}(t, t+1) & \cdots & m_{1c}(t, t+1) & m_{1,c+1}(t, t+1) \\ m_{21}(t, t+1) & \cdots & m_{2c}(t, t+1) & m_{2,c+1}(t, t+1) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ddots & 0 & 1 \end{pmatrix} = \begin{pmatrix} A(t, t+1) & D(t, t+1) \\ 0 & 1 \end{pmatrix} \tag{3.10}$$

where $m_{ij}(t, t+1) = \tilde{P}\{X_{t+1} = j | X_t = 1\}, \ i, j \epsilon S$

m_{ij} and P denote the probability of risk-neutral transition and the measure of probability of risk-neutral respectively. Equation 3.10 conditions must be met here together with the condition of equivalence that;

$$m_{ij}\left(t,t+1\right)>0$$

if and only if $m_{ij}>0$.

It is noted that for Credit Risk assessment, the risk premium plays a vital role. The bank Loans Credit Risk can be captured using the risky rate i.e loan's rate and the zero risk rate i.e risk-free rates for every class of rating with the measure of risk-neutral probability. Letting $V_0\left(t,T\right)$ be the risk-free Credit price at time t and maturing at time T and its risk which is higher will be $V_1\left(t,T\right)$. However not all Interest and Principal of a Loan is lost when a borrower defaults. In reality, a bank will receive some partial repayment even for a borrower who goes bankrupt. Letting the loan's principal and default to be δ collectable when the borrower defaults. It is also known as recovery rate. In situations where the loan is unsecured i.e has no asset backing or collateral, then $\delta=0$ and the rate of recovery is $0<\delta\leq 1$ on the contrary. Jarrow, Lando and Turnbull(1997) showed that an assumption can be made that

$$m_{i,j}\left(t,t+1\right)=\lambda_{ij}\left(t\right).m_{ij},\ i,j\epsilon S$$

and

$$\lambda_{ij}\left(t\right)=\lambda_{i}\left(t\right)$$

for $j\neq i$

and the risk premium procedure is given as;

$$\lambda_i(0) = \frac{V_0(0,1) - V_i(0,1)}{(1-\delta)\,V_0(0,1)\,m_{i,c+1}} \tag{3.11}$$

Near zeros or zero probability of default i.e $m_{i,c+1} \approx o$ from equation 3.10 causes an explosion of the estimate of risk premium and also an implication that the process of Credit rating and the state of default is independent for every borrower which for bank loans, is irrational and inappropriate. The future default probability is not estimated once a borrower defaults. The assumption of the Credit Class rating of every borrower is modified on the pretext that the Credit class rating is independent. The Risk Premium is redefined as;

$$\rho(t) = \frac{1}{1-m_{i,c+1}} \sum_{j=1}^{c} m_{ij}^{-1}(0,t)\,\frac{V_i(0,t) - \delta V_0(o,t)}{(1-\delta)\,V_0(0,t)} \quad i = 1,2,\dots c \; and \; t = 1\dots T \tag{3.12}$$

$$\tilde{A}(0,t+1) = \tilde{A}(0,1)\,\tilde{A}(t,t+1) \tag{3.13}$$

where $m_{ij}^{-1}(0,t)$ is the inverse components of the inverse matrix $\tilde{A}^{-1}(0,t)$ and $A(0,t)$ is invertible. Equation 3.11 denominator is $(1-m_{i,c+1})$ and not $m_{i,c+1}$ and the problem in equation 3.12 will be amended. $A(t,t+1) = A(t).A$ for equation 3.22 and $A(t)$ is the $(C \times C)$ matrix which is diagonal with risk premium diagonal components modified to $\rho_j(t), j\epsilon S$. Particularly, $t = 0$ risk premium is;

$$\rho_i(0) = \frac{1}{1-m_{i,i+1}} \times \frac{V_0(0,1) - \delta V_i(0,1)}{(1-\delta)\,V_0(0,1)} \quad i = 1,2\dots,C \tag{3.14}$$

The risk premium can be estimated by a recursive method for all loans with

periods $t = 0, 1 \ldots T$. The transition matrix which is risk neutral will vary over time in order of equation 3.12 and 3.14 accompanying the risk premium changes. The indicator function is assumed to be;

$$1_{(I)} = \{1, if I \epsilon \{\tau > T\} \, (3.15)$$

Since the interest rate and Markov processes are independent under the measure of equivalent Martiangle, the loan value is equal to;

$$V_i (t, T) = V_0 (t, T) \left\{ \tilde{E}_t [1_{\tau > T}] + \tilde{E}_t [\delta_{\tau \leq T}] \right\}$$

$$= V_0 (t, T) \left\{ \tilde{Q}_t^i (\tau > T) + \delta \left[1 - \tilde{Q}_t^i (\tau > T) \right] \right\}$$

$$= V_0 (t, T) \left\{ \delta + (1 - \delta) \tilde{Q}_t^i (\tau > T) \right\} \tag{3.16}$$

where $\tilde{Q}_t^i (\tau > T)$ is the loan measure probability that loan rating i will not before time T be in default. Clearly;

$$\tilde{Q}_t^i (\tau > T) = \frac{V_i (t, T) - \delta V_0 (t, T)}{(1 - \delta) V_0 (t, T)}$$

$$= \sum_{j=1}^{c} \tilde{m}_{ij} (t = T) = 1 - \tilde{m}_{i, C+1} (t, T) \tag{3.17}$$

where time $t \leq T$is held including the current $t = 0$.

Before time T, there is an occurence of default probability as;

$$\tilde{Q}_t^i (\tau \leq T) = \frac{V_0 (t, T) - V_i (t, T)}{(1 - \delta) V_0 (t, T)} \tag{3.18}$$

for $i = 1, \ldots, C$ and $T = 1, 2, \ldots$

There are four steps in assessing a Bank loan Credit risk;

1. Construct Transition Matrices observed basing on the rating agencies reports.

2. Using equation 3.5, obtain the fitted transition probabilities and construct the fitted transition matrix.

3. Using equation 3.13 and 3.14, obtain the risk premium.

4. Finally, construct the transition matrix 3.8 and condition on risk premium which is time varying and Credit cycle.

Bank loans default probability is estimated by Markov Chain model which incorporates time-varying risk cycle.

3.5 Converting the Discrete Time Transitional Matrix into Continuous Time Transitional matrix

3.5.1 Discrete Time Markov Chain

Each n stage corresponds with time points given with time constant step between them. For instance between time points which are say two can be 1 year hence $p_{ij}(1)$ is the probability in one year to migrate to state or condition j from state i. Hence $p_{ij}(t)$ during time t, is the likelihood or probability of moving from stage i to stage j.

The discrete time stochastic process $\{X(n), n\epsilon N_0\}$ over space $S = \{1, 2, , \ell\}$ is a DTMC if conditional probability distribution of $X(n+1)$ when given $X(n), \ldots, X(n)$ relies only on $X(n)$.

3.5.2 Continuous Markov Chain

There is need for additional theoretical framework for a continuous non stop time Markov chain. Unlike in discrete framework where transitional probabilities were considered at fixed time points, a stochastic variable T is considered instead which is time spent at each state. Transition rates are also considered instead of fixed time step. If the transition rate is large, the transition takes place sooner hence the time spent T in a continuous Markov chain follows an exponential distribution in each state with the risk parameter as the transition rate. Each possible state will be having a transition rate which is certain. Time $T_1, \ldots, T_N$ is obtained for each possible state $1, \ldots, N$ and the shortest time $min\{T_1, \ldots, T_N\}$ will determine the Markov chain state at transitions and how long the transition will take.
Hence discrete Markov chain in certain state and time will be fixed time points whereas continuous time Markov chain moves at irregular times between the states. The continuous stage transitions are given by;

$$P(t+s) = e^{(t+s)Q} = P(t)\,P(s) \tag{3.19}$$

where Q is the generator Matrix.

3.5.3 Bounded Matrices

A matrix A is referred to as absolutely bounded universally if U^*AU is absolutely bounded for every Unitary matrix U to as A. A Unitary matrix is a complex square matrix which has its conjugate transpose as its inverse. This means that a matrix along its diagonal row is flipped over and its inverse conjugate calculated. An infinite matrix (α_{ij}) is said to be bounded if the matrix (α_{ij}) induces a bounded operator on l^2. An example of a bounded matrix is the Hilbert Matrix. A matrix is absolutely bounded universally if and only if it is the summation of Hilbert-Schmidit matrix and a scalar multiple of the identity matrix. A Hilbert Matrix (named after Hilbert 1894) is a square matrix whose entries are fractions and is given by;

$$H_{ij} = \frac{1}{i+j-1} \tag{3.20}$$

The Hilbert matrix can also be derived using the integral;

$$H_{ij} = \int_0^1 x^{i+j-2} dx \tag{3.21}$$

An example of a 5×5 Hilbert Matrix is given by;

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{bmatrix}$$

The matrices used in data analysis are transition matrices which are not unitary and do not satisfy the Hilbert equation 3.20 hence they are unbounded.

3.5.4 Markov Chain Method

Jarrow et al (1997) (Jarrow et al., 1997) also referred to as JLT was the pioneer in modelling default and transition or migration probabilities using a markov chain on a state space given as $S = \{1, \ldots, k\}$. This section will look into this model and its use in evaluating the risk neutral transition matrices. Discrete and continuous properties of transition matrices will be discussed.

The different rating classes are represented by the state space S where $S = 1$ is the best rating and $S = k$ is the default state. Thus in discrete cases, the $k \times k$ transition matrix which is one period will look like;

$$P = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1k} \\ p_{21} & p_{22} & \dots & p_{2k} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix} \tag{3.22}$$

where $p_{ij} \geq 0$ for all $i, j, i \neq j$

$p_{ii} = 1 - \sum_{j=1}^{k} p_{ij}$ for all i

p_{ij} variable represents the probability of migrating to state j from initial state rating i in a one time step. Hence models which are ratings based can also be seen as a special type of intensity model framework from Deffie and Singleton 1999 where they proposed that Markov Chain can model randomness in default. Alternatively, a Continuous time Markov Chain (CTMC) can be used to model a Credit Migration. The concept of generator matrices and modelling of CTMC of rating transitions will be discussed in the next section.

3.5.5 Homogeneous Continuous Time Markov Chain

A CTMC which is homogeneous is given by a $k \times k$ generator matrix as follows;

$$\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1k} \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2k} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix} \tag{3.23}$$

where $\lambda_{ij} \geq 0$ for all i, j and $\lambda_{ii} = -\sum_{j=1}^{k} \lambda_{ij}$ for $i = 1, \ldots, k$. The elements which are off diagonal represent the probability of migrating from rating i to rating j and absorbing state is represented by default k.

3.6 Generator matrix

3.6.1 Generator matrix estimation

The generator matrix usually denoted by G provides an alternative method of analysing or evaluating continuous time Markov chains. Given a continuous time Markov X_t and assuming $X_0 = i$, the chain will transit to the next state at time T_i, where T_i is given as $T_i \approx \text{Exponential}\left(\lambda_i\right)$. Hence for a very small $\delta > 0$, we have;

$$\begin{aligned} p\left(T_i < \delta\right) &= i - e^{-\lambda_i \delta} \\ &\approx 1 - \left(1 - \lambda_i \delta\right) \\ &= \lambda_i \delta \end{aligned} \tag{3.24}$$

hence in a short interval of length δ, the probability of leaving state i is given as $\lambda_i \delta$. this λ_i is usually referred to as transition rate out of state i and is denoted as;

$$\lambda_i = \lim_{\delta \to 0^+} \left[\frac{p\left(X\left(\delta\right) \neq i | X\left(0\right) = i\right)}{\delta} \right] \tag{3.25}$$

Since probability p_{ij} denotes the probability of going from state i to state j, the quantity is referred to as $g_{ij} = \lambda_i p_{ij}$ i.e the transition rate from state i to state j. The diagonal elements (g_{ii}) of G are chosen in such a way that the rows of G add to 0 i.e

$$\begin{aligned} g_{ii} &= -\sum_{j\neq i} g_{ii} \\ &= -\sum_{j\neq i} \lambda_i p_{ij} \\ &= -\lambda_i \sum_{j\neq i} p_{ij} \\ &= -\lambda_i \end{aligned} \tag{3.26}$$

The last equality in equation 3.27 is as a result of; If $\lambda_i = 0$, then,

$$\lambda_i \sum_{j\neq i} p_{ij} = \lambda_i = 0 \tag{3.27}$$

If $\lambda_i \neq 0$, then $p_{ii} = 0$ i.e no there are no transitions hence;

$$\sum_{j\neq i} p_{ij} = 1 \tag{3.28}$$

3.6.2 Generator Matrix Properties

The properties below should be satisfied by the Generator Matrix Q.

1. $0 \leq -q_{ii} \leq \infty$
2. $q_{ij} \geq 0$ for all $i \neq j$
3. $\sum_i q_{ij} = 0$ for all $i \Longleftrightarrow q_{ii} = -\sum_j q_{ij}$ for all $i \neq j$

3.6.3 Transition matrix and Markov Chains

A Markov chain which is finite in discrete time $S(t)$, $t = 0, 1, 2, \ldots$ has the elements below;

1. A set of states which are finite i.e $S = \{1, 2, \ldots n\}$

2. A primary probability distribution $q(0) = (p\{S(0) = 1\}, p\{S(0) = 2\}, \ldots, p\{S(0) = n\})$ fulfilling $p\{S(n)\} \geq 0, i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} P\{S(0) = i\} = 1$

3. A line of transition matrices $A(t, t+1) = \|a_{ij}(t, t+1)\|$, where $a_i(t, t+1) = P\{S(t+1) = j | S(t) = i\}$

The transition matrices will satisfy;

$$\begin{aligned} \sum_{j=1}^{n} a_{ij}(t, t+1) &= 1 \text{ for } i = 1, 2, \ldots, n \\ a_{ij}(t, t+1) &\geq \ 0 \text{ for } i = 1, 2, \ldots, n \end{aligned} \tag{3.29}$$

The Markov chain $S(t)$ is said to be time homogeneous if the transition probabilities are independent on t, i.e

$$a_{ij}(t, t+1) = a_{ij}(0, 1) = a_{ij} \tag{3.30}$$

In modelling Credit risk, a transition matrix which is one year is the sole information source for transition probabilities. Rating agencies also do not provide transition matrices forecasts which are time dependent. For this reason, it is useful for a Markov chain which is time homogeneous in simulation of Credit events (Evans, 2019). Note that the notation above can be simplified by denoting a single period transition matrix by $A = \|a_{ij}\|$ and a $t-$period generally (for the interval of time $[0, t]$ transition matrix);

$$A(t) = \|a_{ij}(t)\| \tag{3.31}$$

The probabilities of transition $a_{ij}(t)$ will satisfy the expression;

$$a_{ij}(t) = \sum_{k=1}^{n} a_{ik}(m)\, a_{kj}(t-m)\,,\; m = 1, 2, \ldots, t \tag{3.32}$$

or alternatively in the form of matrix

$$A(t) = A(m)\, A(t-m) \tag{3.33}$$

Equation 3.32 is also called the semi group property and it implies;

$$A(t) = A^{t} \tag{3.34}$$

It is easily proved that a transition matrix integer power is a transition matrix too. From equation 3.33, the Markov chain probability of distribution at time t will satisfy;

$$\begin{gathered} q(t) = q(0)\, A^{t}. \\ \text{where} \\ q(t) = (P\{S(t)=0\}\,, P\{S(t)=1\}\,, \ldots, P\{S(t)=n\}) \end{gathered} \tag{3.35}$$

3.6.4 Regularization of Credit Migration Model

Markov Chains which are finite currently in use in modes of Credit risk have a state known as arbitrary state which marks a default event. This absorbing state once arrived at, the Markov chain will indefinitely remain there. Assuming that n represents the default state;

$$a_{nn}\left(t\right) = 1 \, \text{and} \, a_{nj}\left(t\right) = 0 \, \text{for all} \dot{j} \neq n, \; t = 1, 2, \ldots \tag{3.36}$$

To provide a distinction with other general Markov Chains, a Regular Credit Migration Model (RCMM) is a Markov Chain having the properties below;

1. It has only a single absorbing state.
2. It has $t_p \geq 1$ in that $a_{in}\left(t_p\right) > 0$ for all $i = 1, 2, \ldots, n$.
3. The determinant of the yearly transition matrix A is not equal to 0 and has distinct eigen values which alow the computation of logarithm of A.

The vaues of $2 \leq t_p \leq 5$ empirically are common thus a portfolio having the highest Credit rating possesses a positive probability of default in a period of two to five years. Also for RCMM, he migration Matrix A_t satisfies (Snell-Hornby, 1988);

$$A\left(t\right) \longrightarrow D \, \text{as} \, t \longrightarrow \infty \tag{3.37}$$

where;

$$D = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

Equation 3.36 shows that default will occur eventually irrespective of the initial rating of Credit. However, the default average time can be very large for particular initial state of Credit.

3.7 Embeddable Markov Chain Matrices

A CTMC having a finite number of l states is given by $\{p(t), t > 0\}$ given by the relation;

$$p(t) = e^{Q_t} \tag{3.38}$$

given by its generator matrix. Observing the process at discreet time points $t_1 = 0, t_1 = 1, \dots, t_n = T$ having differences between consecutive two discreet time points as 1, a DTMC is obtained giving into consideration homogeneity (Pfeuffer et al., 2019) with time unit 1 given by TM p such that $p(1) = p$. The main problem here is getting an estimator $\hat{Q}$ of the generator matrix Q given this CTMC such that $\hat{p} = e^{\hat{Q}}$, a problem known as embedding problem. To date, only partial results of this embedding problem are in existence and none has been applied in Credit Migration Matrices. The definitions that follow will look on some of the results which are in existence for finite and homogeneous transition matrix p. The next chapter of data analysis will attempt to apply the most suitable model which will solve this embedding problem in unbounded Credit Migration Matrices.

Definition 1 *A TM P is said to be embeddable if it has a generator matrix Q such that;*

$$p = exp(Q) \tag{3.39}$$

Remark 1 *Q is not necessarily unique.*

Definition 2 *A square matrix P is said to be diagonizable if it has an invertible matrix B with a diagonal matrix D such that;*

$$\begin{aligned} B^{-1}PB = D \ or \\ P = BDB^{-1} \end{aligned} \tag{3.40}$$

Remark 2 *Given any matrix A,*

1. *$tr(A) \equiv$ sum of diagonal entries.*
2. *$det(A) \equiv$ determinant of A.*
3. *If A is given by $l \times l$ diagonal matrix (DM) with its diagonal entries given by $a_1, a_2, \ldots, a_l$, it is given by $diag(a_1, a_2, \ldots, a_l)$*
4. *The square matrix P diagonalisation is given by getting B and D which satisfy P.*

Proposition 1 *Given an $l \times l$ matrix D whic is diagonal having $\gamma_i, \forall i = 1, 2, \ldots, l$ which* *are the diagonal entries, then there is;*

$$\begin{cases} e^D & = diag(e^{\gamma_1}, e^{\gamma_2}, \ldots, e^{\gamma_l}) \\ and & \\ logD & = diag(log^{\gamma_1}, log^{\gamma_2}, \ldots, log^{\gamma_l}) \end{cases} \tag{3.41}$$

Proposition 2 *Given $l \times l$ matrix P which is diagonizable, then;*

$$\begin{cases} e^P & = Be^DB^{-1} \\ and & \\ \log P & = B \log DB^{-1} \end{cases} \tag{3.42}$$

3.8 The 2×2 TM embedding problem

A 2×2 TM P is embeddable iff;

$$\begin{cases} det(P) > 0 \\ or \\ tr(P) > 1 \end{cases} \tag{3.43}$$

Proof is given in (Guerry, 2013)

3.9 The 3×3 TM embedding problem

Letting P be a 3×3 TM and given Q as a 3×3 GM. Given $P = e^Q$ having $\gamma \in \mathbb{R}$ will imply that γ is an eigen value of P, $\lambda_1 = e^\gamma$ and $\lambda_2 = e^\gamma$ are also eigen values of P. It is seen that that if there is negative eigen value of P, then it has some multiplicity (Johansen, 1974). Since any TM has one eigen value at least equalling to 1, then given the P eigen values as $(1, \lambda_1, \lambda_2)$, the problem can be segmented into 3 cases;

1. $-\lambda_1 \neq \lambda_2$ with $0 < \lambda_1 < 1, 0 < \lambda_2 < 1$ or $\lambda_1 = e^{\alpha+i\beta}, \lambda_2 = e^{\alpha-i\beta}, 0 < \beta < \pi$.

2. $-\lambda_1 = \lambda_2 = \lambda,\ \ 0 < \lambda < 1$

3. $-\lambda_1 = \lambda_2 = \lambda,\ \ -1 < \lambda < 0$

3.10 The Complex Eigen Values

The case is given by;

Corollary 1 *Given a* 3×3 *matrix* P *which is a TM having eigen values* $(1, \lambda_1, \lambda_2)$ *where* $\lambda_1 \neq \lambda_2$. *If* $0 < \lambda_1 < 1$ *and* $0 < \lambda_2 < 1$, *then* P *will be embeddable iff;*

$$P_{ij}^2 \leq \frac{(\lambda_2^2 - 1)log\lambda_1 - (\lambda_1^2 - 1)log\lambda_2}{(\lambda_2 - 1)log\lambda_1 - (\lambda_1 - 1)log\lambda_2},\ i \neq j \tag{3.44}$$

If $\lambda_1 = e^{\alpha+i\beta}, \lambda_2 = e^{\alpha-i\beta}, 0 < \beta < \pi$, then P will be embeddable iff either;

$$P_{ij}^2(\beta(e^{\alpha}cos\beta - 1) - \alpha e^{\alpha}sin\beta) \geq P_{ij}^2(e^{2\alpha}cos2\beta - 1) - \alpha e^{c\alpha}sin2\beta), i \neq j$$

or

$$\P_{ij}^2((\beta - 2\pi)(e^{\alpha}cos\beta - 1) - \alpha e^{\alpha}sin\beta) \geq \P_{ij}^2((\beta - 2\pi)(e^{2\alpha}cos2\beta - 1) - \alpha e^{2\alpha}sin2\beta), i \neq j \tag{3.45}$$

Chapter Four

Results and Discussion

4.1 Introduction

In the previous chapter, if the approximated transition matrix $\hat{P}$ is embeddable, then the estimator $\hat{Q}$ can be gotten from the generator matrix Q. What of cases when $\hat{P}$ is not embeddable? This chapter will show how to evaluate $\hat{P}$ and find estimator $\hat{P}$ in cases where $\hat{P}$ is embeddable (Pfeuffer et al., 2019). Other methods of estimating $\hat{Q}$ from discrete time data without involving $\hat{P}$ will also be discussed in this chapter.

4.2 The Maximum Likelihood Estimator

Given any observations set, the maximum likelihood estimator is one of the methods for estimating statistical model parameters. It does this by getting the values of the parameter which maximize the likelihood function. The estimates are referred to as Maximum Likehoood function (MLE). The parameter $Q = [q_{ij}]_{i,j \in S}$ is the generator matrix. The likelihood function of Q is given by;

$$L_{(C)}(Q,Y) = \prod_{i=1}^{l}\prod_{j\neq i} q_{ij}^{N_{ij}(T) q_{ij} R_i(T)} \tag{4.1}$$

where;

1. (C) indicates the continuous time observations.

2. $N_{ij}(T)$ is the translation number from state i to state j in $[0,T]$ time interval.

3. $R_i(t) = \int_0^t 1\{Y(s)=i\}\,ds$ is the process time in state t after state i.

Taking the logarithm of equation 4.1 above and then its partial derivative with respect to q_{ij} and equating it to 0 gives the MLE of Q as;

$$\hat{q_{ij}} = \frac{N_{ij}(T)}{R_i(T)} \tag{4.2}$$

The MLE of parameter $\hat{P}$ is given as $\hat{P} = (\hat{p}_{ij})_{ij\in S}$ such that;

$$\begin{aligned} \hat{P}_{ij} &= \frac{K_{ij}(n)}{K_i(n)} \\ \text{where } K_i(n) &= \sum_{j=1}^{l} K_{ij}(n) \end{aligned} \tag{4.3}$$

If the $\hat{P}$ shown by equation 4.3 is embeddable, then the MLE $\hat{Q}$ of the generator matrix exists and is obtained by $\hat{Q} = \log \hat{P}$. If $\hat{P}$ is not embeddable, then there is either existence of Q and other methods of obtaining $\hat{Q}$ using the transition estimator $\hat{P}$ or else there is no existence of MLE $\hat{Q}$.

4.3 The Diagonal and Weighted adjustment

Assuming $l \times l$ transition matrix estimator $\hat{P}$ given by equation 4.3 above is not embeddable. The next step is to obtain a matrix $\hat{Q}$ such that $\hat{P} = exp\left(\hat{Q}\right)$ which is equivalent to as satisfying $\hat{Q} = \log \hat{P}$. The first thing to do is to be sure that the log function exists for $\hat{Q}$ using the theorem below.

Theorem

Let P be $l \times l$ TM and let $F = max\left\{(a-1)^2 + b^2,\ a+b \text{ is an eigen value of } P, a, b\ \in \mathbb{R}\right\}$. Assuming that $F < 1$, then series;

$$\widetilde{Q} = \log \hat{P} = \left(\hat{P} - 1\right) - \frac{\left(\hat{P} - 1\right)^2}{2} - \frac{\left(\hat{P} - 1\right)^3}{3} - \dots \tag{4.4}$$

will converge geometrically giving matrix $\widetilde{Q}$ having row sums equal to 0 and $exp\left(\widetilde{Q}\right) = \hat{P}$. The condition $F < 1$ will not be needed if the series absolutely converges (Israel et al., 2001). Due to embedding problem, it is not definite that the condition of generator matrix $\widetilde{Q}$ to have positive off-diagonal entries though it satisfies $exp\left(\widetilde{Q}\right) = \hat{P}$. This problem can be solved by using two adjustment methods known as the diagonal and Weighted adjustments.

4.3.1 The Diagonal Adjustment

Let generator matrix Q^{DA} elements given as q_{ij}^{DA} be obtained by method of Diagonal Adjustment (DA). The negative value of $\widetilde{Q}$ will be replaced by 0 since they are usually very small and then the difference added to the diagonal entries so as to preserve the property of row sums totalling to 0. The DA method is given by;

$$\begin{aligned} q_{ij}^{DA} &= max\left(\widetilde{q}_{ij}, 0\right), \; i \neq j \\ q_{ii}^{DA} &= -\sum_{j=1\, j\neq i}^{l} q_{ij}^{DA} \end{aligned} \tag{4.5}$$

The Matrix plot of data 3.1 above is given as;

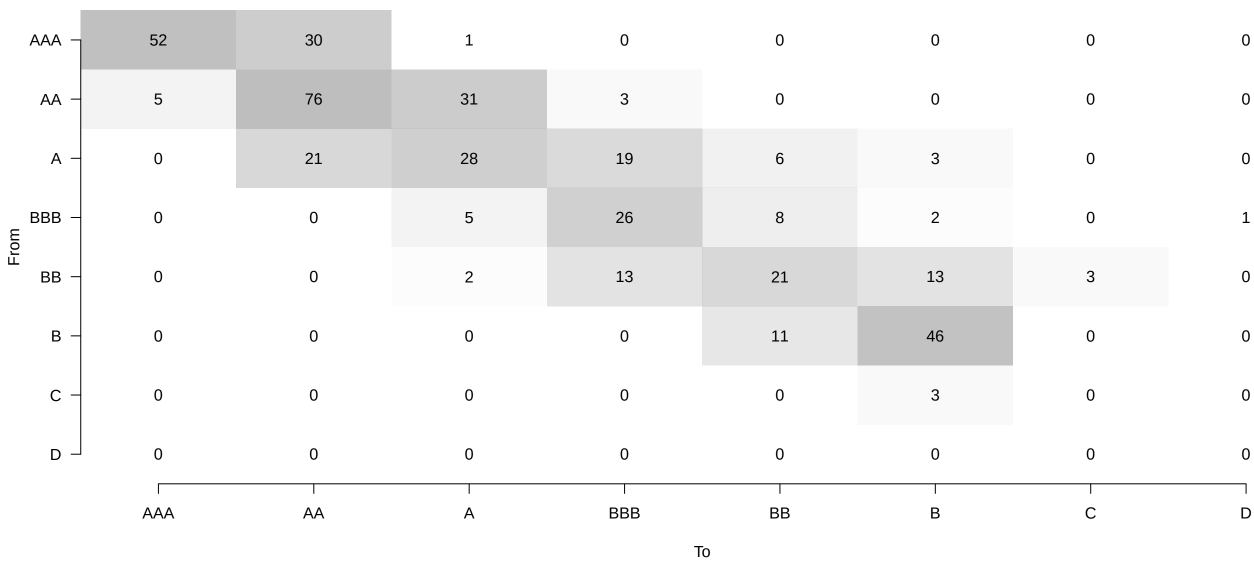

Figure 4.1: Matrix P plot

The transition matrix P for data 3.1 is given as;

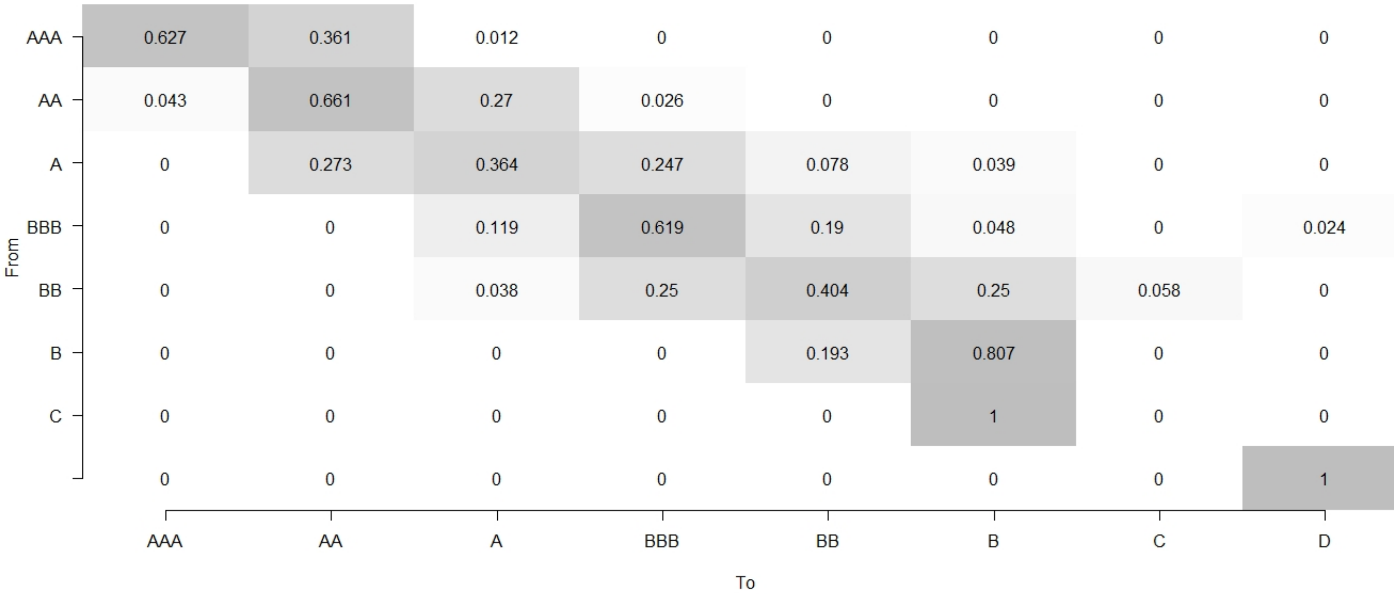

Figure 4.2: Transition matrix P plot

Using equation 4.5, the generator matrix q^{DA} using Diagonal Adjusment method (DA) will be given as;

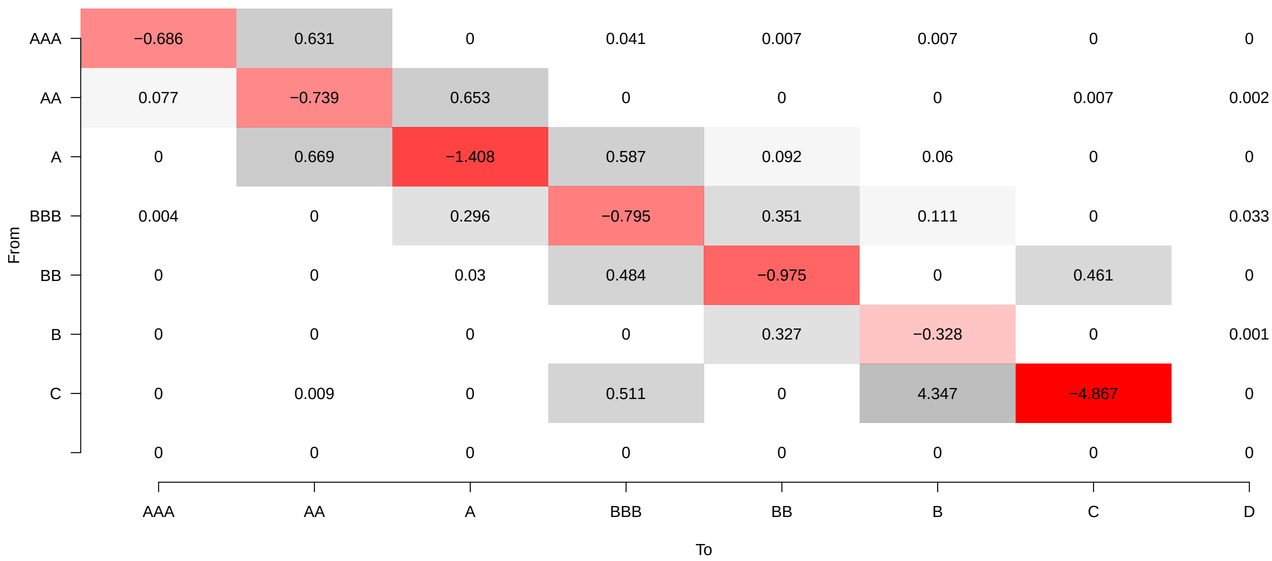

Figure 4.3: Generator Matrix obtained using Dagonal Adjustment Method

4.3.2 The Weighted Adjustment

Let the generator matrix Q^{WA} elements be given as q_{ij}^{WA} which will be obtained using the Weighted Adjustment method. Let;

$$G_i = |\widetilde{q_{ii}}| + \sum_{j \neq i} max\left(\widetilde{q_{ij}}, 0\right), \; B_i = \sum_{j \neq i} max\left(\widetilde{-q_{ij}}, 0\right)$$

$$q_{ij}^{WA} = \begin{cases} 0, \text{ if } i \neq j \text{ and } \widetilde{q_{ij}} \leq 0 \\ \\ \widetilde{q_{ij}} - B_i|\widetilde{q_{ij}}|/G_i \\ \\ \widetilde{q_{ij}}, \text{ otherwise if } \; G_i = 0 \end{cases} \tag{4.6}$$

Since $\sum_{j=1}^{l} \widetilde{q_{ij}} = 0$, then $G_i \geq B_i$, hence that $q_i^{WA} \geq 0 \, \forall i \neq j$. Running equation 4.7 for Weighted Adjustment (WA) method using data 4.6 yields;

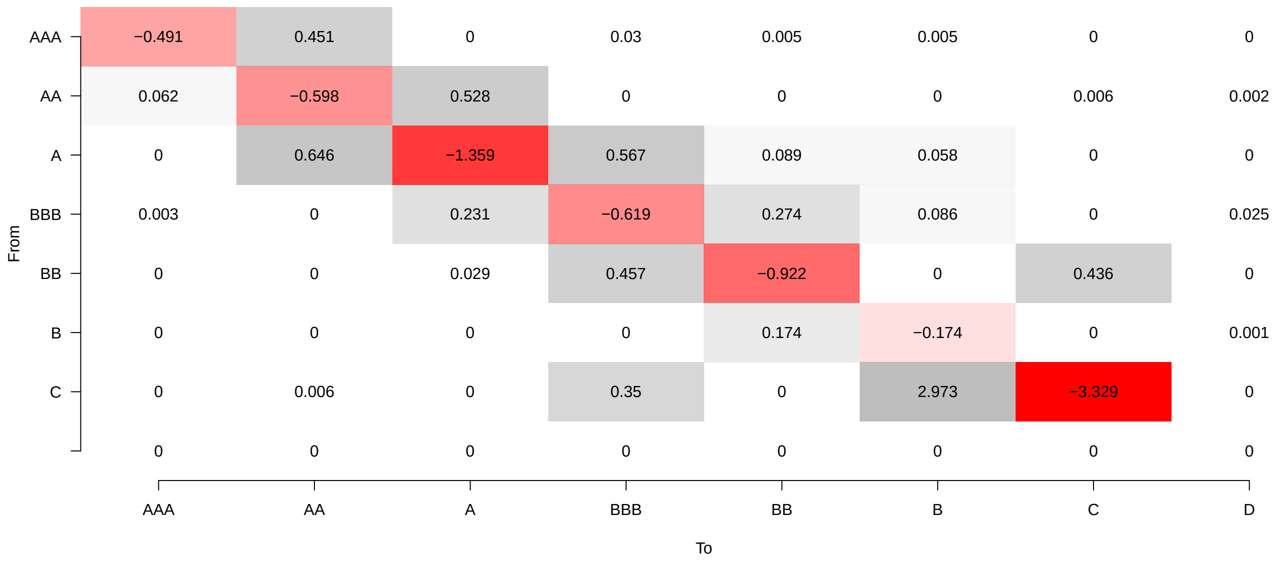

Figure 4.4: Generator Matrix obtained using Weighted Adjustment Method

4.4 Generator Quasi-Optimization method

Using transition matrix P given in figure 4.2 above, $Q = [\tilde{q_{ij}}]_{i,j \in S}$ is set to be solution of $\tilde{Q} = \log \tilde{P}$. This Generator Quasi-Optimization method is used to get the generator matrix approximation similarly to getting a maximization problem solution given as;

$$\underset{Q \in Q}{min} \|Q - \widetilde{Q}\| (4.7)$$

such that the set of all generator Matrices is given by Q and it is also the Eucledian norm $\|.\|$. This problem is solved row by row since the Q conditions () are closed on each row(Carette, 1995). Next solve() which is equal to solving independent minimization problems of l given as;

$$\min_{q \in (l)} \sum_{i}^{l} (p_i - q_i)^2 \tag{4.8}$$

such that $\widetilde{Q}$ has row vector $p = (p_1, \ldots, p_l)$ also permuted as $q_i q_{i+1}$ meaning that the diagonal element in this row is q_i. Also;

$$\rho(l) = \left\{ q \in \mathbb{R} | \sum q_i = 0,\ q_1 \leq 0, q_i \geq 0,\ \text{for}\ i \geq 2 \right\} \tag{4.9}$$

Evaluating equation 4.9 using R programming yield the data below;

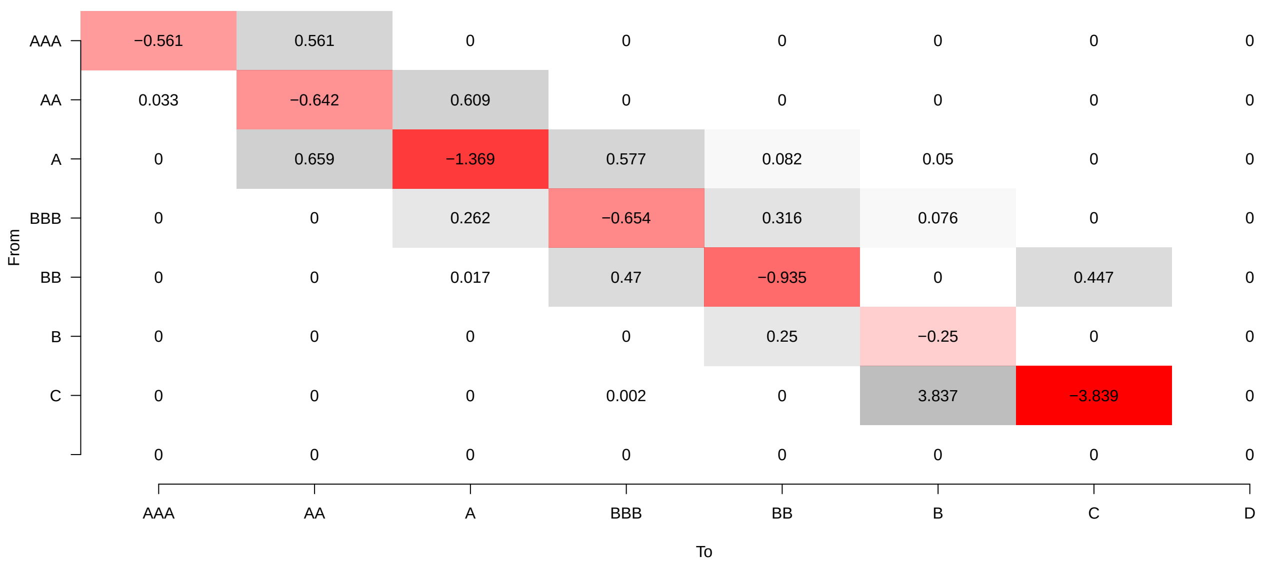

Figure 4.5: Generator Matrix obtained using Quasi Optimization Method

4.5 The EM Logarithm

Let a data set which is complete be denoted by $Y = (X, Z)$. The Expectation Maximization also known as EM is a method of obtaining the MLE when only X observation exists(Pfeuffer, Reis, et al., 2018). It comprises of two steps;

Step 1: The E-step

Involves evaluating the conditional expectation of $log\left(L\left(\theta, Y\right)\right)$ when given X and MLE θ_0 i.e computation of $\mathbb{E}\left[logL\left(\theta, Y\right)|X, \theta_0\right]$.

Step 2: The M-step

Involves obtaining the new MLE θ by getting the maximization of $E\left[L\left(\theta, Y\right)|X, \theta_0\right]$ then setting $\theta_0 = \theta$ and repeating the two steps till the sequence converges. The maximum likelihood function of the data set Y which is continuous is given by equation 4.1.

The E-step: From equation 4.1 and given the initial generator matrix Q_0;

$$\mathbb{E}_{Q_0}[logL_{(c)}\left(Q, Y\right)|X, Q_0] = \sum_{i=1}^{l}\sum_{j\neq i} log\left(q_{ij}\right)\mathbb{E}_{Q_0}[N_{ij}\left(T\right)|X] - \sum_{i=1}^{l}\sum_{j\neq i} q_{ij}\mathbb{E}_{Q_0}[R_i\left(T\right)|X] \tag{4.10}$$

Hence there is need to evaluate $\mathbb{E}_{Q_0}[N_{ij}\left(T\right)|X]$ and $\mathbb{E}_{Q_0}[R_i\left(T\right)|X]$, and since there is;

$$\mathbb{E}_{Q_0}[N_{ij}\left(T\right)|X] = \sum_{k=0}^{n-1}\widetilde{F}^{ij}_{X_k X_{k+1}}\left(t_{k+1} - t_k\right) \tag{4.11}$$

such that

$$\widetilde{F}^{ij}_{kl}\left(t\right) = \mathbb{E}[N_{ij}\left(t\right)|Y\left(t\right) = l, Y\left(0\right) = k] \tag{4.12}$$

and

$$\mathbb{E}_{Q_0}[R_i\left(T\right)|X] = \sum_{k=0}^{n-1}\widetilde{M}_{X_k X_{k+1}}(t_{k+1} - t_k) \tag{4.13}$$

such that;

$$\widetilde{M}^i_{kl} = \mathbb{E}[R_i(t)|Y(t) = l, Y(0) = k] \tag{4.14}$$

is sufficient to evaluate $\widetilde{M}^i_{kl}$ and $F^{ij}_{kl}(t)$.

$\lambda = \underset{i\in S}{max}(-q_{ii}, 0)$ is chosen and $B = I + \lambda^{-1}Q_0$ defined. Letting e_j, e^I_j to denote the unit vector with j^{th} coordinate equalling to 1 and getting its transpose, it is given that;

$$\widetilde{M}^i_{kl}(t) = \frac{M^i_{kl}(t)}{e^t_k e^{Q_0 t} e_l} \tag{4.15}$$

such that;

$$M^i(t) = [M^i_{kl}(l)]_{k,l\in S} = e^{-\lambda t}\lambda^{-1}\sum_{n=1}^{\infty}\frac{(\lambda t)^{n+1}}{(n+1)!}\sum_{s=0}^{n} B^s(e_i e^I_i)B^{n-s} \tag{4.16}$$

and that

$$\widetilde{F}^{ij}_{kl}(t) = \frac{F^{ij}_{kl}(t)}{e^I_k e^{Q_0 t} e_l} \tag{4.17}$$

Such that;

$$F^{ij}(t) = [F^{ij}_{kl}(t)]_{k,l\in S} = q_0 ij e^{-\lambda t}\lambda^{-1}\sum_{n=1}^{\infty}\frac{(\lambda t)^{n+1}}{(n+1)!}\sum_{s=0}^{n} B^s(e_i e^I_j)B^{n-s} \tag{4.18}$$

The M-Step

It is seen that $\hat{Q} = [\hat{q}_{ij}]_{i,j\in S}$ where;

$$\hat{q}_{ij} = \frac{\mathbb{E}_{Q_0}[N_{ij}(T)|X]}{\mathbb{E}_{Q_0}[R_i(T)|X]},\ \forall i \neq j \tag{4.19}$$

maximizes $\mathbb{E}_{Q_0}[logL_{(C)}(Q,Y)|X,Q_0]$ hence the new MLE is $\tilde{Q}$. Putting $\hat{Q} = Q_0$ and repeating this till there is convergence of sequence. Using data 3.1 above and running equation 4.19 using R yields the generator matrix below;

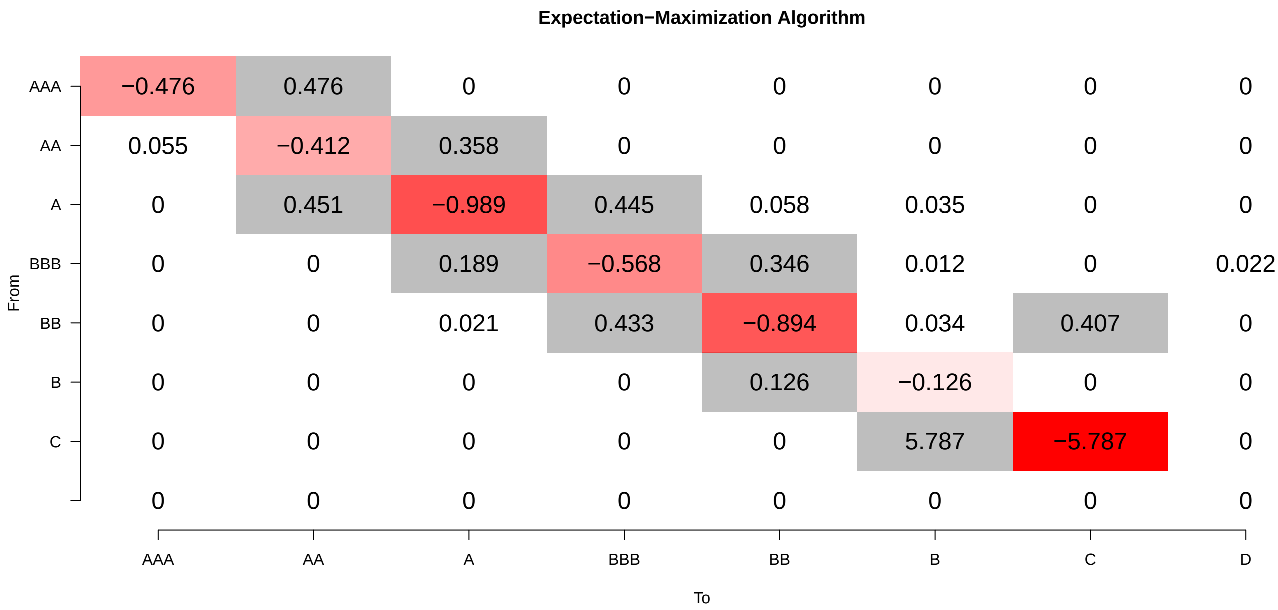

Figure 4.6: Generator Matrix obtained using Expectation-Maximization Algorithm Method

The series of generator matrices $\{Q_k\}_{k=1}^{K}$, obtained will depend on the initial matrix generator Q_0 choice hence it is prudent to select it in a way that the $det(e^{Q_k})$ is greater than 0.

4.6 The Gibbs sampler (Markov Chain Monte Carlo Method)

This section will demonstrate how to make use of Monte Carlo Chain estimation methods. There are various Monte Carlo Markov Chain methods and the Gibbs Sampler is one of them which will be employed here. The Gibbs Sampler method avoids the non-existence problem of MLE and also makes it very easy to compute. The Gibbs sampler is a method that makes use of samples from joint distribution of some conditional distribution . To understand how this algorithm works its steps are outlined as; Suppose K samples are obtained of $X = (x_1, \ldots, x_n)$ from the joint distribution $p(x_1, \ldots, x_n)$. Let the i^{th} sample be given as $X^i = (x_1^{(0)}, \ldots, x_n^{(0)})$ then the algorithm is:

1. Initialize $x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)})$ for $t = 0$

2. For $i = 0, \ldots, k$, sample $x^{(i+1)}$ by sampling each componet of $x_j^{(i+1)}$ $, j = 1, \ldots, n$ and evaluating it using the probability distribution,

$$p(x_j^{(i+1)} | x_1^{(i+1)}, \ldots, x_{j-1}^{(i+1)}, x_{j+1}^{(i+1)}, \ldots, x_n^{(i)})$$

Bayes Theorem

Given equation;

$$p(\phi|Y) = \frac{p(Y|\phi)p(\phi)}{p(Y)} \tag{4.20}$$

ignoring the constant $p(Y)$ and utilizing (Israel et al., 2001);

$$p(\phi|Y) \propto p(Y|\phi)p(\phi) \tag{4.21}$$

where $\propto$ denotes the proportion, $p(\phi)$ te prior distribution and $p(\phi|Y)$ the posterior distribution which is the conditional distribution of parameter ϕ when given data Y. Consider taking a complete data as having discreet time observations $X = \{Y(t_1), \ldots, Y(t_n)\}$ observed at time $(t_1 = 0, \ldots, t_n = T)$. An application of Q and J Gibbs sample will be done by drawing J from (Q, X) and Q from (J, X) hence $\{Q^{(k)}, J^{(k)}\}_{k=1}^{k}$ will be obtained where Q_s denote the generator matrices and J_s denote the simulated Markov Chain samples. Given the prior distribution of Q, the gamma distribution proposed is;

$$p(Q) = \prod_{i=1}^{l} \prod_{j \neq i} q_{ij}^{\alpha_{ij}-1} e^{q_{ij}-\beta_i} \tag{4.22}$$

where $\alpha_{ij} > 0,\ i,j \in S$ and $\beta_i > 0,\ i \in S$. Using the likelihood function, given the complete data in eqaution 1, the posterior of Q is given as;

$$p(Q|J, X) = p(Q|J) \propto p(Q)L_{(c)}(Q, Y) = \prod_{i=1}^{l} \prod_{j \neq i} q_{ij}^{N_{ij}(T)+\alpha_{ij}-1} e^{-q_{ij}(R_i(T)+\beta_i)} \tag{4.23}$$

Using data 3.1 above and running equation 4.24 using R yields the generator matrix using Gibbs Sample3 method s given below;

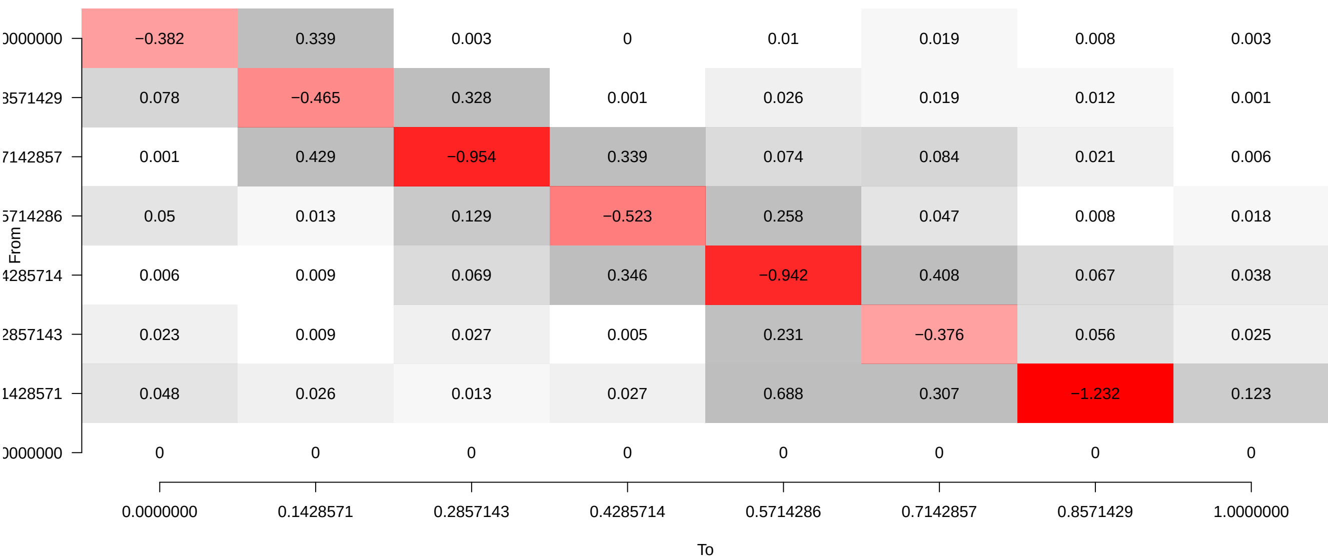

Figure 4.7: Generator Matrix obtained using Gibbs Sampler Method

The Gibbs sampler algorithm is summarized as below;

i) Begin by constructing the initial generator matrix Q_0 be getting $q_{ji,0}$ from the prior distribution having $j \neq i$(Hanks, 2016)

ii) Perform the steps below K times;

   (a) Perform a simulation of continuous Markov Chain sample J having generator matrix Q so that a realization of all observations is achieved.

   (b) Evaluate $N_{ij}(T)$ and $R_i(T)$ from the Markov chain simulated.

   (c) Evaluate a new Q by getting q_{ij} from the posterior distribution $\Gamma(+\alpha_{ij}, 1/(+\beta_i))$

   (d) This new Q is saved and is used in the next simulation.

iii) Let $Q_1, \ldots, Q_k$ be genarator matrices got using Gibbs sampler algorithm and take the mean of N proportions. Hence the estimator $\hat{Q}$ of the generator matrix Q will be given by;

$$\hat{Q} = \frac{1}{K-N} \sum_{i=N+1}^{K} Q_i \tag{4.24}$$

4.7 L Norm comparison between Diagonal Adjustment, Weighted Adjustment, Quasi Optimization, EM Algorithm and Gibbs Sampler Monte Carlo Methods

In order for one to know which Generator approximaton is more suitable or accurate for an 8×8 Credit Generator Matrix, there is need to measure the distance and magnitude between the generator entries. This can be done by using the L norm where it is computed for each Generator matrix obtained by the four methods above. The method with the least distance between the vectors is deemed the best and using R software, it is computed as follows;

$$
\begin{aligned}
&norm[P - exp(Q_{WA})] = 24.40658 \\
&norm[P - exp(Q_{DA})] = 82.1496 \\
&norm[P - exp(Q_{QO})] = 51.21381 \\
&norm[P - exp(Q_{GS})] = 8.207688
\end{aligned}
\tag{4.25}
$$

Hence from norm results 4.25 above, the norm of Gibbs Sampler method given by 8.207688 is the least hence the most suitable for computing the generator for embeddable 8×8 Credit transition Matrix and this thesis proposes the Gibbs sampler method as the most accurate for generator matrix approximation for a Credit Transition Matrix.

Chapter Five

Conclusions and Recommendations

5.1 Introduction

This study aimed at modelling Credit Risks using unbounded transitional matrices. The following objectives guided the study; to develop an unbounded transitional matrix from a Credit ratings data, to derive the risk premium properties from the developed transitional matrix, to derive the risk premium properties from the developed transitional matrix and lastly To get the most suitable generator matrix for the developed Continuous Time Transitional Matrix especially in cases where the matrix is embeddable. Secondary data from standard and poors website was used. This chapter will present a summary of major findings, conclusions and recommendations.Areas for further research are also pointed out.

5.2 Summary

5.2.1 Developing an unbounded transitional matrix from a Credit ratings data

A continuous Time Markov chain (CTMC) is given by its transition rates which also serve as the generator matrix entries. This thesis started with the introduction and application of the Markov Chains in analysing Credit. Various models which apply the Markov chain which are currently in use were reviwed in Chapter 2. None of the four employs the use of unbounded Transition matrices and their Generator matrices. The study converted Credit ratings data 3.1 obtained from Standard and poors website into a transition matrix as seen in fig 4.2 , The thesis has brought out the relation between the transition matrix and the generator matrix outlining both of which are unbounded matrices since the defination and conditions for a Bounded matrix is explained very well in chapter 3 section 3.8.

5.2.2 Deriving the risk premium properties from the developed transitional matrix

The risk properties was explained in section 3.4 in order to assess the Credit riak. The study was guided by the fact that the probability of default tends to zero for higher level Credit ratings in risk neutral world. The credit ratings observed implied a non zero probability of default and by estimation, the risk neutral framework can be controlled. A borrower likely to default was matched with a stochastic process given by equation 3.9 and the rating falls within the proportion or risk unbiased likelihood. This change is then transformed in a risk-neutral transition matrix with a martiangle measure m. The transition matrix falling in the new measure m does not need the requirement to be Markovian and if Markovian, then it exhibits properties of a Markov chain. Lastly, using 3.10, the transition Matrix is fitted in the risk neutral probability.

5.2.3 Converting the Discrete Time Transitional Matrix obtained into Continuous Time transitional matrix in order to get a generator Matrix

The thesis also presents ways of estimating a generator matrix from a continuous time markov chain. As illustrated in chapter four, in cases where we have embeddable matrix $\hat{P}$, it is easy to obtain its generator matrix $\hat{Q}$. In cases where $\hat{P}$ is not embeddable, then obtaining $\hat{P}$ is a complex since there is need to convert the DTMC into CTMC especially in 4×4 matrices and above. This is done by using the four models described; the Diagonal and Weighted adjustment method, the Generator Quasi-optimization method, the EM algorithm method and lastly the Markov chain Monte Carlo method also called the Gibbs sampler method. Using L-norm, Gibbs sampler emerges as the best model for obtaining a generator matrix for non embeddable unbounded transition Matrix. All the models are analysed using R project whose codes are given in the appendix.

5.2.4 Obtaining most ideal Generator matrix from an embeddable Transition Matrix

The embedding problem is discussed for several cases and it's found out that only 2×2 and 3×3 matrices are embeddable. Any higher matrices have to employ various methods to solve the embedding problem in order to get the most suitable Generator matrix. The Credit data got from standard and poors website is an 8×8 transition matrix hence four methods being the Diagonal and Weighted adjustment method, the Generator Quasi-optimization method, the EM algorithm method and lastly the Markov chain Monte Carlo method are applied and compared in order to come up with the most suitable method for deriving the Generator matrix for the data.

5.3 Conclusion

The following conclusions below can be derived from this research; The TM is embeddable under $2 \times 2,\ 3 \times 3$ square matrices which are reversible in necessary and sufficient conditions. The TM P is not embeddable in some other conditions and given a TM P which is embeddable, then the GM Q will satisfy $P = e^{Q}$. From CTMC having a TM $\{P(t),\ t > 1\}$, a homogeneous Markov Chain can be obtained having TM P such that $P = P(1)$. An estimation of MLE of $\hat{P}$ can be made from TM P and given $\hat{P}$ which is embeddable, then MLE of $\hat{Q}$ can be evaluated from GM Q using $\hat{P} = e^{\hat{Q}}$. If the case given is not unique, then there is a way of choosing the best one. Given P which is not embeddable, then Q does not exist or it exists by use of Diagonal Adjustment, Weighted Adjustment, Quasi Optimization, EM algorithm and Gibbs Sampler (Markov

Chain Monte Carlo) methods. The four methods also convert DTMC to CTMC. By use of Gibbs Sampler (MCMC) methods which emerged as the best model, the problem of non existence of Q can be dealt with. All the four models were run using R software and the codes given in the appendix.

5.4 Recommendations

Credit risk modelling and pricing of Credit derivatives of late are the main subjects researched in finance. Transitional matrix rating of late have become a building block in Credit risk research. This Thesis sought to find conditions which a true generator exists for conditions where the transition matrices is unbounded. Most of transitional Matrices observed do not have a valid generator hence a researcher has to either get a generator approximate to the transitional matrix observed or modify the transition Matrix in order to make it to be compatible with Markov process and then seek for true generators. This Thesis recommends deriving a Generator matrix from embeddable transition matrix rather than estimating it using the Gibbs sampler method. The embedding problem is wide and can be applied in many areas hence it is a worthy case to study since this research has applied to only one case of an 8×8 transition matrix.

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Appendix A1

```
{}

#Let P denote the S\&P matrix

install.packages(ctmcd)

install.packages(markovchain)

library(ctmcd)

library(markovchain)

library{'plot.matrix'}

P=c(52,5,0,0,0,0,0,0,30,76,21,0,0,0,0,0,1,31,28,5,2,0,0,0,0,3,19,26,13,0,0,0,0,0,6,8,

P=matrix(P,8)

P[8,]=0

colnames(P)<-c('AAA','AA','A','BBB','BB','B','C','D')

rownames(P)<-c('AAA','AA','A','BBB','BB','B','C','D')

plotM(P)
```

Appendix A2

```
install.packages(ctmcd)

install.packages(markovchain)

library(ctmcd)

library(markovchain)

library{'plot.matrix'}

P=c(52,5,0,0,0,0,0,0,30,76,21,0,0,0,0,0,1,31,28,5,2,0,0,0,0,3,19,26,13,0,0,0,0,0,6,8,

P=matrix(P,8)

P[8,]=0

colnames(P)<-c('AAA','AA','A','BBB','BB','B','C','D')

rownames(P)<-c('AAA','AA','A','BBB','BB','B','C','D')

P=rbind((Y3/rowSums(Y3))[1:7,],c(rep(0,7),1))

 plotM(P)
```

Appendix A3

```
install.packages(ctmcd)

install.packages(markovchain)

library(ctmcd)

library(markovchain)

library{'plot.matrix'}

P=c(52,5,0,0,0,0,0,0,30,76,21,0,0,0,0,0,1,31,28,5,2,0,0,0,0,3,19,26,13,0,0,0,0,0,6,8,

P=matrix(P,8)

P[8,]=0

colnames(P)<-c('AAA','AA','A','BBB','BB','B','C','D')

rownames(P)<-c('AAA','AA','A','BBB','BB','B','C','D')

P=rbind((Y3/rowSums(Y3))[1:7,],c(rep(0,7),1))

gmda=gmDA(P,1)

plotM(gmda)
```

Appendix A4

```
install.packages(ctmcd)

install.packages(markovchain)

library(ctmcd)

library(markovchain)

library{'plot.matrix'}

P=c(52,5,0,0,0,0,0,0,30,76,21,0,0,0,0,0,1,31,28,5,2,0,0,0,0,3,19,26,13,0,0,0,0,0,6,8,

P=matrix(P,8)

P[8,]=0

colnames(P)<-c('AAA','AA','A','BBB','BB','B','C','D')

rownames(P)<-c('AAA','AA','A','BBB','BB','B','C','D')

P=rbind((Y3/rowSums(Y3))[1:7,],c(rep(0,7),1))

gmwa=gmWA(P,1)

QWA=gmWA(P,1)

plotM(QWA)
```

Appendix A5

```
install.packages(ctmcd)

install.packages(markovchain)

library(ctmcd)

library(markovchain)

library{'plot.matrix'}

P=c(52,5,0,0,0,0,0,0,30,76,21,0,0,0,0,0,1,31,28,5,2,0,0,0,0,3,19,26,13,0,0,0,0,0,6,8,

P=matrix(P,8)

P[8,]=0

colnames(P)<-c('AAA','AA','A','BBB','BB','B','C','D')

rownames(P)<-c('AAA','AA','A','BBB','BB','B','C','D')

P=rbind((Y3/rowSums(Y3))[1:7,],c(rep(0,7),1))

gmqo=gmQO(P,1)

plotM(gmqo)
```

Appendix A6

```
install.packages(ctmcd)

install.packages(markovchain)

library(ctmcd)

library(markovchain)

library{'plot.matrix'}

P=c(52,5,0,0,0,0,0,0,30,76,21,0,0,0,0,0,1,31,28,5,2,0,0,0,0,3,19,26,13,0,0,0,0,0,6,8,

P=matrix(P,8)

P[8,]=0

colnames(P)<-c('AAA','AA','A','BBB','BB','B','C','D')

rownames(P)<-c('AAA','AA','A','BBB','BB','B','C','D')

P=rbind((Y3/rowSums(Y3))[1:7,],c(rep(0,7),1))

gmem=gm(P,te=1,method="EM",gmguess=gm0)

plot(gmem)
```

Appendix A7

```
pr=list()

pr[[1]]=matrix(1,8,8)

 pr[[1]][8,]=0

pr[[2]]=c(rep(5,7),Inf)

gmgs=gmGS(P,te=1,sampl_method="Unif",prior=pr,burnin=10,niter=100,verbose=TRUE)

 gmgs
```

Table 1: Summary of different categories of Ratings

Rating	Description
AAA	Extremely strong obligor: Has Capacity to meet its financial obligations.
AA	Very strong obligor: Has Capacity too to meet its financial obligation
A	Obligor is more susceptible to the circumstances changes and economic conditions than the higher rated obligors.
BBB	Obligor exhibits protection parameters which are adequate but changing circumstances or adverse economic conditions will lead to obligor weakened capacity.
BB	Obligor is less vulnerable to speculative issues or non payment but faces major business exposure to adverse economic, financial or business conditions which can lead to the obligor's inadequate ability to meet its financial obligations.
B	Obligor currently has capacity to meet its financial commitments but adverse economic, financial or economic conditions will impair its capacity to meet financial commitments.
CCC	Obligor is vulnerable to nonpayment. It depends on favourable economic, financial or business conditions in order for the obligor to meet its financial commitment.
CC	Obligor is highly vulnerable to non payment.
C	The C rating can be used in situations where bankruptcy petitions have been filed and can be used to cover the situation but payments of the obligor are being honoured.
D	Unlike other ratings, the D rating is not prospective. It is used where a default has occured and not where there is expectation of default.