

## A Dynamic Programming Solution to Solute Transport and Dispersion Equations in Groundwater

M. Mirabzadeh<sup>1</sup> and K. Mohammadi<sup>2\*</sup>

### ABSTRACT

The partial differential equations for water flow and solute transport in a two-dimensional saturated domain are rendered discrete using the finite difference technique; the resulting system of algebraic equations is solved using a dynamic programming (DP) method. The advantage of the DP algorithm is that the problem is converted from solving an algebraic system of order  $NC(NL-1) \times NC(NL-1)$  into one of solving a difference equation of order  $NC \times NC$  over  $NL-1$  steps and involving  $NL-1$  matrix inversions of order  $NC \times NC$ . The accuracy and precision of the solutions are shown by comparing the results with an analytical solution and calculation of mass the balance. In addition, the performance of the DP model was compared with the results of the MOC model developed by US Geological Survey. In all cases, the DP model showed good results with sufficient accuracy.

**Keywords:** Dynamic programming, Groundwater, Numerical model, Solute transport.

### INTRODUCTION

In Iran, groundwater is an important source of irrigation water. In fact, it fulfills more than 50 percent of the country's total irrigation needs. Groundwater has played a major role in increasing food production and achieving food security. Groundwater, a renewable source of water, has the remarkable distinction of being a highly dependable and safe source of water supply for agriculture, domestic and industrial needs. Increased use of chemical fertilizers coupled with improper water management practices has resulted in the deterioration of groundwater quality in several parts of the country. Every year, almost 17.5 BCM of agricultural wastewater is returned to aquifers which account for almost 35% of groundwater extraction (1). This recycling of water causes secondary salinization in irrigated areas.

Solute transport models are being used to

study groundwater quality and water pollution in the subsurface. Several methods have been used to generate a better and faster model, for example: Reddell and Sunada (20); Chatwal *et al.* (6); Cheng and Hodge (7); Gray and Pinder (11); Grove (12); Huyakorn *et al.* (13); Konikow and Bredehoeft (14); and Bouhroum and Bai (5). In addition, certain advances in analytical solutions have been made by some other researchers such as Basha and El Habel (2) and Sim and Chrysikopoulos (21). They have developed one-, two-, and three-dimensional analytical solutions for solute transport in saturated, homogeneous porous media. However, numerical algorithms are still the only solution to complex problems in the field of solute transport in groundwater.

In this paper, a numerical model to solve the solute transport and dispersion equation in groundwater is developed. The purpose of

1. Department of Irrigation Engineering, College of Agriculture, University of Tehran, Karaj, Islamic Republic of Iran.

2. Department of Irrigation and Drainage Engineering, Tarbiat Modarres University, P. O. Box:14115-336, Tehran, Islamic Republic of Iran.

\* Corresponding author, e-mail: kouroshm@modares.ac.ir



the simulation model is to compute the concentration of a non-reactive dissolved chemical species in an aquifer at any given time and place. The partial differential equations are made discrete using the finite difference technique and the resulting system of algebraic equations is solved using a dynamic programming (DP) method which is used here for the first time. Bellman (4), Collins (8) and Collins and Angel (9) applied this technique to the solution of parabolic and elliptic partial differential equations. A dynamic programming method was then used by Mirabzadeh (16, 17, 18, 19) to solve the flow equation in two and three dimensions for heterogeneous and anisotropic aquifers. The higher speed of this algorithm compared with the other matrix solutions such as Jacobi, Gauss-Sidel, and Successive over Relaxation (SOR) has been the motivation for using the method in this research (9, 17).

Convective transport, hydrodynamic dispersion, and chemical reactions are the three main processes in solute transport. Among these, the first two phenomena are more important in most cases and those have been considered in this model. Because convective transport and hydrodynamic dispersion depend on the velocity of groundwater flow, the solute transport equation is a nonlinear equation and must be considered in conjunction with the groundwater flow equation. The computer program solves two simultaneous partial differential equations- one is the groundwater flow equation and the second one is the solute transport equation.

A solution to the solute transport equation in two dimensions has been obtained using a dynamic programming technique. In this paper, the analytical solution (2) and the method of characteristics were used to verify the developed algorithm. The method of characteristics was originally applied to transport in porous media by Garder *et al.* (11) in order to calculate miscible displacement in a reservoir simulation. This method was later implemented in a two-dimensional solute transport model by Konikow and Bredehoeft (14); their code is commonly

referred to as MOC. The method of characteristics uses a conventional particle tracking technique for solving the advection term. For more detail about this method, one can refer to Zheng and Bennett (22).

### Theoretical Development

In a Cartesian coordinates, the governing equation for two dimensional fluid flow in a saturated porous medium by using the continuity equation and Darcy's law, is:

$$\frac{\partial}{\partial x} \left( T_{xx} \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( T_{yy} \frac{\partial h}{\partial y} \right) = S \frac{\partial h}{\partial t} \pm q \quad (1)$$

where  $h$  is hydraulic head [L];  $t$  is time [T];  $q$  is source/sink term [L/T] with a positive sign for the source and negative for sink terms;  $S$  is storage coefficient; and  $T_{xx}$  and  $T_{yy}$  are transmissivity [L<sup>2</sup>/T] in  $x$  and  $y$  direction, respectively. Transmissivity is defined as follows:

$$[T_{ij}] = b[K_{ij}] \quad (2)$$

where  $b$  is the saturated thickness of the aquifer [L]; and  $[K_{ij}]$  is the hydraulic conductivity tensor. It is assumed that the coordinate system is oriented with the conductivity tensor, so that  $K_{ij} = 0$  for  $i \neq j$ .

The solute transport equation describing convection and dispersion of a nonreactive dissolved chemical species in groundwater is as follows (14):

$$\begin{aligned} & \left[ \frac{\partial}{\partial x} \left( b n_e D_{xx} \frac{\partial C}{\partial x} \right) + \frac{\partial}{\partial y} \left( b n_e D_{yy} \frac{\partial C}{\partial y} \right) \right] \\ & + \left[ \frac{\partial}{\partial x} \left( b n_e D_{xy} \frac{\partial C}{\partial y} \right) + \frac{\partial}{\partial y} \left( b n_e D_{yx} \frac{\partial C}{\partial x} \right) \right] \\ & = b n_e \frac{\partial C}{\partial t} + \left( b V_x \frac{\partial C}{\partial x} + b V_y \frac{\partial C}{\partial y} \right) + C \left( n_e \frac{\partial b}{\partial t} - S \frac{\partial h}{\partial t} \right) \\ & \quad + q(C' - C) \end{aligned} \quad (3)$$

where  $C$  is the concentration of the dissolved chemical species [M/L<sup>3</sup>];  $C'$  is the concentration of dissolved chemical in a source or sink fluid [M/L<sup>3</sup>];  $n_e$  is the effective porosity;  $V_x$  and  $V_y$  are Darcy's velocities in  $x$  and  $y$  directions, respectively [L/t]; and  $D_{ij}$  is the coefficient of the hydrodynamic dispersion [L<sup>2</sup>/T]. The dispersion coefficient for

two-dimensional flow in an isotropic aquifer may be written as (4):

$$D_{xx} = \alpha_L \frac{(V_x)^2}{|V|} + \alpha_T \frac{(V_y)^2}{|V|} \quad (4)$$

$$D_{yy} = \alpha_T \frac{(V_x)^2}{|V|} + \alpha_L \frac{(V_y)^2}{|V|} \quad (5)$$

$$D_{xy} = D_{yx} = (\alpha_L - \alpha_T) \frac{V_x V_y}{|V|} \quad (6)$$

where  $\alpha_L$  and  $\alpha_T$  are the longitudinal and transverse dispersivities of the aquifer, respectively [L]; and  $|V|$  is the magnitude of the velocity [L/T].

### Finite Difference Formulation

The two-dimensional model presented here is based on an implicit finite difference scheme using the dynamic programming solution.

### Flow Equation

The compact finite difference form of equation (1) is given by:

$$\begin{aligned} Wx_{(i-1,j)} h_{(i-1,j)}^k + Wx_{(i,j)} h_{(i+1,j)}^k + Wy_{(i,j-1)} h_{(i,j-1)}^k \\ + Wy_{(i,j)} h_{(i,j+1)}^k - Wc_{(i,j)} h_{(i,j)}^k = R_{(i,j)}^{k-1} \end{aligned} \quad (7)$$

$$i = 2, \dots, NL-1$$

$$j = JD(i), \dots, JF(i)$$

where  $i, j, k$  are, respectively, the indices in the  $x, y,$  and time dimensions and  $JD(i), JF(i)$  are the column numbers at the beginning and the end of each row, respectively.

$$Wx_{(i,j)} = T_{xx(i+1/2,j)} \frac{\Delta y_{(j)}}{\Delta x_{(i)}} \quad (8a)$$

$$Wy_{(i,j)} = T_{yy(i+1/2,j)} \frac{\Delta x_{(i)}}{\Delta y_{(j)}} \quad (8b)$$

$$\begin{aligned} Wc_{(i,j)} = Wx_{(i-1,j)} + Wx_{(i,j)} + Wy_{(i,j)} \\ + Wy_{(i,j-1)} + S_{(i,j)} \frac{\Delta x_{(i)} \Delta y_{(j)}}{\Delta t} \end{aligned} \quad (8c)$$

$$R_{(i,j)}^{k-1} = \Delta x_{(i)} \Delta y_{(j)} \left( S_{(i,j)} \frac{h_{(i,j)}^{(k-1)}}{\Delta t} + q_{i,j}^{k-1/2} \right) \quad (8d)$$

### Solute Transport Equation

Using the finite difference discretization approach, equation (3) can be written as:

$$\begin{aligned} [B_{xx(i-1,j)} C_{(i-1,j)} + B_{xx(i,j)} C_{(i+1,j)} \\ + B_{yy(i,j-1)} C_{(i,j-1)} + B_{yy(i,j)} C_{(i,j+1)} - B_{c(i,j)} C_{(i,j)}]^k \\ = R_{c(i,j)}^{k-1} \end{aligned} \quad (9)$$

where

$$B_{xx(i,j)}^k = b_{(i+1/2,j)}^k n_{e(i,j)} D_{xx(i+1/2,j)}^k \quad (10a)$$

$$B_{yy(i,j)}^k = b_{(i,j+1/2)}^k n_{e(i,j)} D_{yy(i,j+1/2)}^k \quad (10b)$$

$$\begin{aligned} B_{c(i,j)}^k = [B_{xx(i-1,j)} + B_{xx(i,j)} + B_{yy(i,j-1)} + B_{yy(i,j)} \\ + \frac{\Delta x_{(i)} \Delta y_{(j)}}{\Delta t} n_{e(i,j)} b_{(i,j)}]^k \end{aligned} \quad (10c)$$

$$\begin{aligned} R_{c(i,j)}^{k-1} = R_{2(i,j)}^{k-1} + R_{3(i,j)}^{k-1} - R_{1(i,j)}^{k-1} - \\ \frac{\Delta x_{(i)} \Delta y_{(j)}}{\Delta t} n_{e(i,j)} b_{(i,j)}^k C_{(i,j)}^{k-1} \end{aligned} \quad (10d)$$

$$\begin{aligned} \frac{R_{1(i,j)}^{k-1}}{\Delta x_{(i)} \Delta y_{(j)}} = \left[ \frac{\partial}{\partial x} \left( b n_e D_{xy} \frac{\partial C}{\partial y} \right) + \frac{\partial}{\partial y} \left( b n_e D_{yx} \frac{\partial C}{\partial x} \right) \right]^{k-1} = \\ \frac{1}{\Delta x_{(i)} \Delta y_{(j)}} \{ B_{xy(i,j)}^k [C_{(i,j-1)} + C_{(i-1,j-1)} - C_{(i,j+1)} - C_{(i-1,j+1)}]^{k-1} \\ + B_{xy(i,j)}^k [C_{(i,j+1)} + C_{(i+1,j+1)} - C_{(i,j-1)} - C_{(i+1,j-1)}]^{k-1} \\ + B_{yx(i,j)}^k [C_{(i-1,j-1)} + C_{(i-1,j)} - C_{(i+1,j-1)} - C_{(i+1,j)}]^{k-1} \\ + B_{yx(i,j)}^k [C_{(i+1,j)} + C_{(i+1,j+1)} - C_{(i-1,j)} - C_{(i-1,j+1)}]^{k-1} \} \end{aligned} \quad (10e)$$

$$B_{xy(i,j)}^k = b_{(i+1/2,j)}^k n_{e(i,j)} D_{xy(i+1/2,j)}^k \quad (10f)$$

$$B_{yy(i,j)}^k = b_{(i,j+1/2)}^k n_{e(i,j)} D_{yy(i,j+1/2)}^k \quad (10g)$$

$$\begin{aligned} \frac{R_{2(i,j)}^{k-1}}{\Delta x_{(i)} \Delta y_{(j)}} = \left[ b V_x \frac{\partial C}{\partial x} + b V_y \frac{\partial C}{\partial y} \right] = \frac{1}{\Delta x_{(i)} \Delta y_{(j)}} \{ \alpha_1 Wx_{(i-1,j)} \\ [h_{(i-1,j)} - h_{(i,j)}]^k [C_{(i-1,j)} - C_{(i,j)}]^{k-1} + \alpha_2 Wx_{(i,j)} \\ [h_{(i+1,j)} - h_{(i,j)}]^k [C_{(i+1,j)} - C_{(i,j)}]^{k-1} + \alpha_3 Wy_{(i,j-1)} [h_{(i,j-1)} \\ - h_{(i,j)}]^k [C_{(i,j-1)} - C_{(i,j)}]^{k-1} + \alpha_4 Wy_{(i,j)} [h_{(i,j+1)} - h_{(i,j)}]^k \\ [C_{(i,j+1)} - C_{(i,j)}]^{k-1} \} \end{aligned} \quad (10h)$$

$$\text{where } \alpha_1 \begin{cases} = 0 & [h_{(i-1,j)} - h_{(i,j)}]^k \leq 0 \\ = 1 & [h_{(i-1,j)} - h_{(i,j)}]^k > 0 \end{cases} \text{ and}$$

also the same thing for  $\alpha_2, \alpha_3,$  and  $\alpha_4.$



$$\frac{R_{3(i,j)}^{k-1}}{\Delta x_{(i)} \Delta y_{(j)}} = \left[ C \left( n_e \frac{\partial b}{\partial t} - S \frac{\partial h}{\partial t} \right) + q(C' - C) \right] = \tag{10i}$$

$$C_{(i,j)}^{k-1} \left[ n_{e(i,j)} \frac{b_{(i,j)}^k - b_{(i,j)}^{k-1}}{\Delta t} - S_{(i,j)} \frac{h_{(i,j)}^k - h_{(i,j)}^{k-1}}{\Delta t} \right]$$

$$+ q [C'_{(i,j)} - C_{(i,j)}^{k-1}]$$

**Dynamic Programming Solution**

In this section, the proposed relationships for solving the resulting algebraic equations are presented. In order to solve equation (7) subject to the boundary conditions, we define the vectors  $H_i$ ,  $W_i$ , and  $R_i^*$  with the dimension of  $N(i) = NC - 2$  for each row:

$$H_i = h_{(i,j)}^{k+1} \quad W_i = [W_{x(i,j)} h_{(i,j)}^k]$$

$$R_i^* = R_{(i,j)}^k$$

$$i = 2, NL - 1 \quad j = JD(i) \dots JF(i)$$

Using the above definitions, equation (7) may be rewritten as follows:

$$F_{(i-1)} H_{(i-1)} + F_{(i)} H_{(i+1)} - G_{(i)} H_{(i)} + W_i = R_i^* \tag{11}$$

$$i = 2, 3, \dots, NL - 1$$

Vectors  $H_{(1)}$ ,  $H_{(NL)}$ , and  $W_i$  are known from the boundary conditions,  $F_{(i)}$  and  $F_{(i-1)}$  are the diagonal matrices, and  $G_{(i)}$  is a tridiagonal matrix with the components defined by the coefficients of the system of equations (7) as follows:

$$F_{(i)} = \text{diag}(f) \quad f = W_x^{(i,j+1)}$$

$$G_{(i)} = [g_{(i,j)}] \quad g_{(i,j)} = \begin{cases} W_c^{(i,j+1)} & i = j \\ -W_y^{(i,j+1)} & i - j = 1 \\ -W_y^{(i,j)} & j - i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Using dynamic programming, equation (11) may be cast in the following form:

$$H_{(i)} = A_{(i-1)} H_{(i-1)} + B_{(i-1)} \tag{12}$$

Solving (11) for  $H_{(i)}$  and using  $H_{(i+1)} = A_{(i)} H_{(i)} + B_{(i)}$  gives the following equation:

$$H_{(i)} = F_{(i-1)} (G_{(i)} - F_{(i)} A_{(i)})^{-1} H_{(i-1)} + (G_{(i)} - F_{(i)} A_{(i)})^{-1} (F_{(i)} B_{(i)} + W_i - R_i^*) \tag{13}$$

Comparing (12) and (13) yield the following recurrence relations:

$$A_{(i-1)} = F_{(i-1)} (G_{(i)} - F_{(i)} A_{(i)})^{-1} \tag{14}$$

$$B_{(i-1)} = A_{(i-1)} F_{(i-1)}^{-1} (F_{(i)} B_{(i)} + W_i - R_i^*) \tag{15}$$

The initial conditions for (14) and (15) are derived from (12) setting  $i = NL$

$$A_{(NL-1)} = 0 \quad B_{(NL-1)} = H_{(NL-1)}$$

Then, the backward solution of (14) and (15) will give the matrices  $A_{(i)}$  and the vectors  $B_{(i)}$ . Equation (12) may now be solved for  $H_{(i)}$  by utilizing  $A_{(i)}$  and  $B_{(i)}$ .

Equation (9) can be solved in the same way. In each time step, after calculating the dispersion coefficient tensor, matrices  $A_{(i)}$  and vectors  $B_{(i)}$  and  $C_{(i)}$  will be computed in a similar procedure to  $A_{(i)}$ , (14),  $B_{(i)}$ , (15), and  $H_{(i)}$ , (12), respectively.

**RESULTS AND DISCUSSION**

In order to verify that the numerical model is solving the governing equations correctly, it is necessary to compare the results to known solutions or results from other models. The solution developed by Basha and El Habel (2) was used to verify the model against the analytical solution. A one-dimensional solute transport in saturated media in x direction was solved analytically. Table 1 shows the parameters that were used in this solution. Figure 1 shows the comparison between the analytical solution and DP model. The mean absolute error was 0.8 percent and the root mean square error between two methods was 0.03 percent which shows good agreement between results.

**Table 1.** Parameters used in the analytical solution.

Parameter	Value
$\Delta x$	25 m
Length (L)	100 m
Initial Concentration ( $C_0$ )	0 mg/lit
Velocity ( $V_x$ )	0.25 m/s
Dispersivity ( $\alpha_L$ )	10 m
Injected Mass ( $C'$ )	1 mg/lit

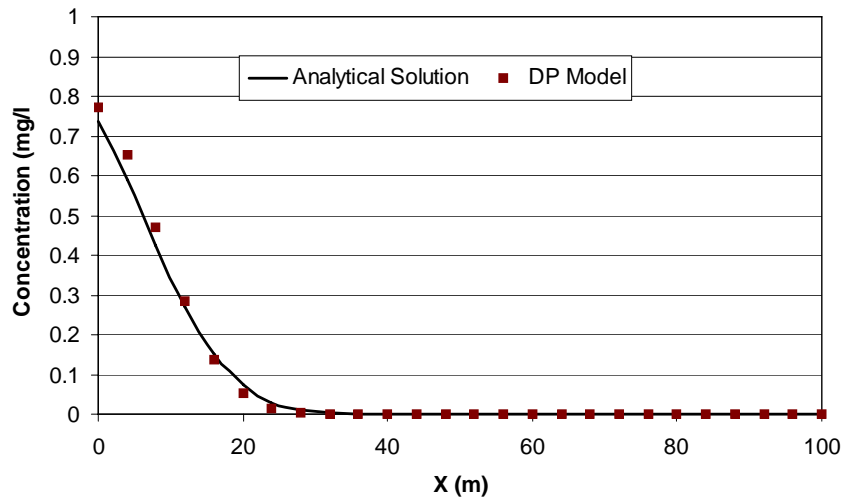


Figure 1. Comparison between the analytical solution and the DP model.

In addition, the model was compared to the results obtained from the MOC model (14). A mass balance test was used as a measure of the numerical accuracy of the solutions in DP and MOC comparisons. Here, the model uses, as the mass residual,  $M_r$ , a criterion compared with the difference between the initial mass,  $M_i$ , and the net mass flux,  $N_f$ .

$$\text{Error} = \frac{100M_r}{M_i - N_f} \quad (16)$$

where  $M_r = \Delta M_s - N_f$  and  $\Delta M_s$  is the change in mass stored in the aquifer [M].

One recharge well and one extraction well were considered in a 2192×2466 m domain. The left and right sides of the domain were impermeable and the northern and southern sides of the domain had constant head boundaries with a 22.5 m and 30 m water elevation, respectively. Figure 2 shows the result for test problem number 1 and the following parameters were used in the compu-

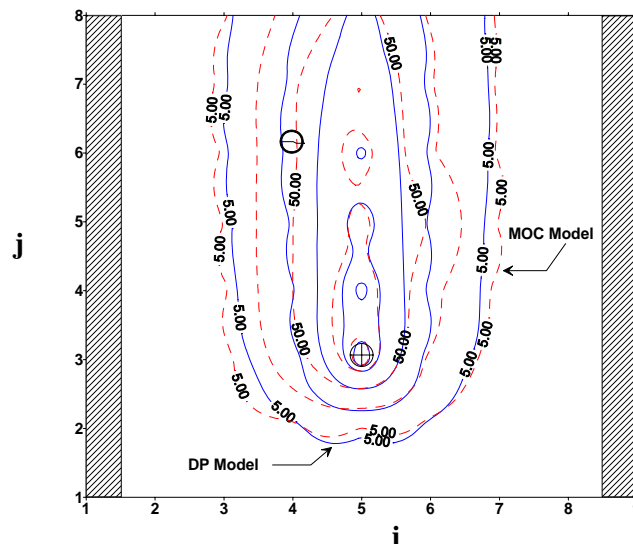


Figure 2. Comparison of the computed concentration between the DP and MOC models after 30 months for test problem number 1.

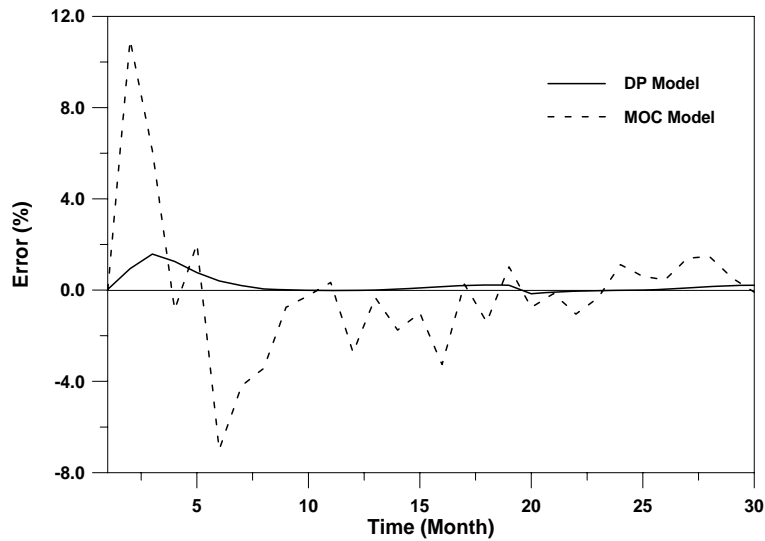


Figure 3. Mass balance error for test problem 1.

tations:

$$K = 1.5 \times 10^{-3} \text{ m/s}$$

$$b = 6.1 \text{ m}$$

$$\alpha_L = 30.5 \text{ m}$$

$$\alpha_T / \alpha_L = 0.3$$

$$q(3, 5) = 0.028 \text{ m}^3/\text{s}$$

$$q(6, 4) = -0.028 \text{ m}^3/\text{s}$$

$$\Delta x = \Delta y = 274 \text{ m}$$

$$S = 0.3$$

$$C' = 220 \text{ mg/lit}$$

$$C_0 = 0.0 \text{ mg/lit}$$

the dynamic programming method and method of characteristic.

In test problem 2, the effect of dispersion was eliminated ( $\alpha_L = 0.0$ ) and other parameters remained the same as in test problem number 1. Again, Figure 4 shows the results of the solute concentration after 2 1/2 years and Figure 5 shows the mass balance errors for this period.

Figure 3 presents the mass balance error for

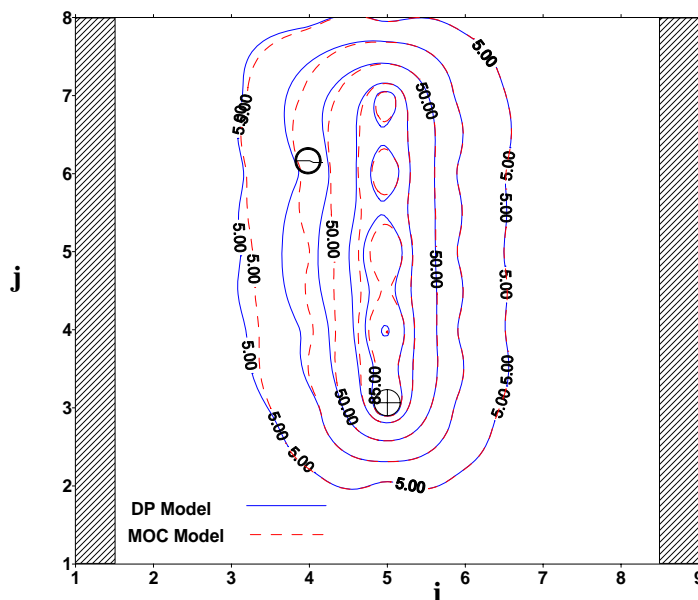


Figure 4. Comparison between the computed concentrations in test problem 2.

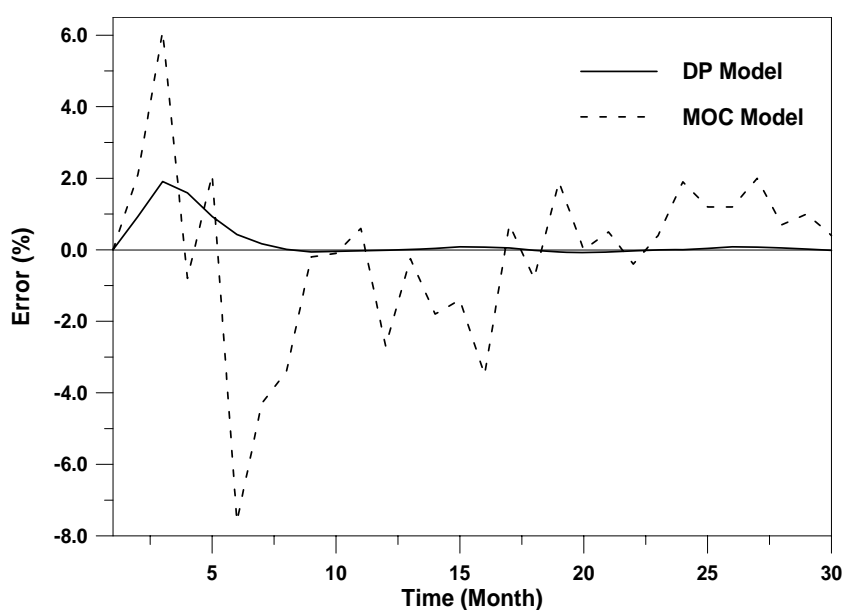


Figure 5. Mass balance error in test problem 2.

## CONCLUSION

A general two-dimensional model has been developed to simulate solute transport through heterogeneous porous media. The agreement between the results from the present model and the analytical solution and the MOC model are good and the mass balance error for the DP model is less than for the MOC model.

Using a dynamic programming method can reduce the time of computation and computer error due to inversion of the large matrices. The application of this method converts the problem from solving an algebraic system of order  $NC(NL-1) \times NC(NL-1)$  into that of solving a difference equation of order  $NC \times NC$  over  $NL-1$  steps and involving  $NL-1$  matrix inversions of order  $NC \times NC$ . In addition, it reduces the storage requirements, since instead of having to store  $(NL \times NC)^2$ , we need to store only  $(NC \times NC)$  numbers at each step.

## REFERENCES

1. Anonymous. 1999. *National Water Master Plan*, Jamab Consulting Engineers Company, Ministry of Power, Tehran, Iran (in Persian).
2. Basha, H.A. and Habel, F. S. El. 1993. Analytical Solution of the One-dimensional Time-dependent Transport Equation. *Water Resour. Res.*, **29(9)**: 3209-3214.
3. Bear, J. 1972. *Dynamics of Fluids in Porous Media*, Elsevier Publications, New York.
4. Bellman, R. 1960. *Introduction to Matrix Analysis*, McGraw Hill, New York.
5. Bouhroum, A. and Bai, M. 1966. Experimental and Numerical Simulation of Solute Transport in Heterogeneous Porous Media, *Int. J. Numer. Analyt. Method. Geomech.*, **20**: 155-171.
6. Chatwal, S. S., Cox, R. L., Green, D. W. and Ghandi, B. 1973. Experimental and Mathematical Modeling of Liquid-liquid Miscible Displacement in Porous Media, *Water Resour. Res.*, **9(5)**: 1369-1377.



7. Cheng, R. T. and Hodge, D. S. 1976. Finite Element Method in Modeling Geologic Transport Processes, *Math. Geol.*, **8(1)**: 43-56.
8. Collins, D. C. 1970. Reduction of Dimensionality in Dynamic Programming via the Method of Diagonal Decomposition, *J. Math. Anal. Appl.*, **30**: 223-235.
9. Collins, D. C. and Angel, E. S. 1971. The Diagonal Decomposition Technique Applied to the Dynamic Programming Solution of Elliptic Partial Differential Equations, *J. Math. Anal. Appl.*, **33**: 467-481.
10. Garder, A. O., Peaceman, Jr., D. W. and A.L. Pozzi, Jr. 1964. Numerical Calculation of Multidimensional Miscible Displacement by the Method of Characteristics. *Soc. Pet. Eng. J.*, **6(2)**: 175-182.
11. Gray, W. G. and Pinder, G. F. 1976. An Analysis of the Numerical Solution of the Transport Equation, *Water Resour. Res.*, **12(3)**: 547-555.
12. Grove, D. B. 1977. The Use of Galerkin Finite Element Methods to Solve Mass-transport Equations. *U.S. Geological Survey Water Resources Inv. 77-49*, 55 pp.
13. Huyakorn, P. S., Lester, B. H. and Mercer, J.W. 1983. An Efficient Finite Element Technique for Modeling Transport in Fractured Porous Media: 1. Single Species Transport, *Water Resour. Res.*, **6(6)**: 1611-1617,
14. Konikow, L. F. and Bredehoeft, J. D. 1984. Computer Model of Two Dimensional Solute Transport and Dispersion in Groundwater. *U. S. Geol. Survey Techniques of Water-Resources Inv.*, Book 7, Chap. C2, 90 pp.
15. Konikow, L. F. and Grove, D. B. 1974. Derivation of Equations Describing Solute Transport in Groundwater, *U.S. Geol. Survey Techniques of Water-Resources Inv. 77-19*, 30 pp.
16. Mirabzadeh, M. 1973. Application de l'Invariant Imbedding a la resolution de l'equation fondamentale des milieux poreux, *Bull. De B. R. G. M. Sec. III*, **3**: 54-58,
17. Mirabzadeh, M. 1974. Application de programmation dynamique aux ecoulement permanents et transitoires en milieux poreux, *Hydrol. Sci. Bull. XX*, **2(6)**: 261-274.
18. Mirabzadeh, M. 1975. Application de la programmation dynamique aux modeles tridimensionnels dans les milieux poreux, *IAHS-AISH Publ.*, **115**: 167-175.
19. Mirabzadeh, M. 1977. The Use of Block Technique in Mathematical Modeling of Ground Water, *Proceedings of the Symposium on Hydrology*, Fort Collins, June 27-29.
20. Reddell, D. L. and Sunada, D. K. 1970. Numerical Simulation of Dispersion in Groundwater Aquifers, *Colorado State Univ. Hydrology Paper 41*, 79 pp.
21. Sim, Y. and Chrysikopoulos, V. Analytical solutions for solute transport in saturated porous media with semi- infinite or finite thickness, *Journal of Advances in Water Resources*, **22(5)**, 507-519, 1999.
22. Zheng, C. and Bennett, G. D. 1995. *Applied Contaminant Transport Modeling*. Van Nostrand Reinhold, New York, pp. 440.



## حل برنامه ریزی پویای معادله انتقال و پخشیدگی املاح در آب زیرزمینی

م . میرابزاده و ک. محمدی

### چکیده

معادلات دیفرانسیل با مشتقات جزئی برای جریان و انتقال املاح در محیط دو بعدی و اشباع آب زیرزمینی با روش تفاضل های محدود بصورت گسسته در آمده و دستگاه معادلات جبری بدست آمده به روش برنامه ریزی پویا و حل گردیدند. مزیت الگوریتم برنامه ریزی پویا در این است که مسئله را از حل یک دستگاه معادلات از مرتبه  $(NL-1) \times (NL-1) \times NC$  را به حل یک دستگاه معادلات از مرتبه  $NC \times NC$  در  $NL-1$  گام تبدیل کرده که شامل  $NL-1$  معکوس کردن ماتریس مرتبه  $NC \times NC$  است. دقت روشهای حل با مقایسه نتایج با حل تحلیلی و محاسبه بیلان جرمی نشان داده شده است. مدل برنامه ریزی پویا همچنین با مدل MOC که توسط موسسه زمین شناسی آمریکا تهیه شده مقایسه گردید. در همه موارد مدل DP نشان داد که می تواند نتایجی با دقت خوب ارائه دهد.