

**A Reduced Form of the Three Factor Commodity Derivative
Valuation Model**

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DECLARATION

This thesis is my original work and has not been presented for a degree in any other university.

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This thesis has been submitted for examination with our approval as university supervisors.

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DEDICATION

To my late mother
JEMAYNESH ABDU HASSAN(Jeme)

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Contents

DECLARATION	i
DEDICATION	ii
ACKNOWLEDGEMENT	iii
ABSTRACT	viii
1 Introduction	1
1.1 Background of the study	1
1.2 Statement of the problem	2
1.3 Objectives of the study	2
1.3.1 General Objective	2
1.3.2 Specific Objectives	2
1.4 Significance of the study	3
1.5 Scope of the study	3
1.6 Limitation of the study	3
1.7 Outline of the thesis	3
2 Literature Review	4
2.1 CIR Model	4
2.2 Discretization Scheme	5
2.3 Theoretical results of Euler and Milstein schemes	5
2.4 Theorem 1- Convergence of Euler scheme	5
2.5 Theorem 2- Convergence of Milstein scheme	6
2.6 Theoretical comparison between Euler and Milstein	7
2.7 Euler-Maruyama discretization scheme	8
2.8 Milstein discretization scheme	8
2.9 Strong Convergence Analysis	8
2.10 Valuation model	9
2.11 Financial market under incomplete information	11
2.12 Valuation of Commodity derivatives	12
2.13 Commodity Futures Prices	13
3 MODEL DISCRETIZATION	18
3.1 Introduction	18
3.2 Financial Market	18

3.3	Discretization of reduced form of the three factor commodity derivative valuation model	23
3.3.1	Euler discretization scheme	23
3.3.2	Milstein discretization scheme	23
3.4	Simulation of valuation model	24
4	Empirical studies of the reduced form of the three factor commodity derivative valuation model	25
4.1	Simulation of valuation model	25
4.1.1	Milstein discretization simulation for $T=1$	25
4.1.2	Euler discretization simulation for $T=1$	26
4.1.3	Simulation results for both Milstein and Euler schemes as maturity expands	27
4.1.4	Simulation results for spot price, convenience yield and interest rate	33
5	Discussions, Conclusions and Recommendations	36
5.1	Discussions	36
5.2	Conclusions	37
5.3	Recommendations	37
	REFERENCES	37
	Appendix 1	40
	Appendix 2	41
	Appendix 3	42
	Appendix 4	46

List of Tables

4.1	Parameters of valuation model for Milstein and Euler discretization scheme . .	25
4.2	Simulation results of Milstein scheme for the first ten simulation paths for $\Delta t = 10^{-5}$ and $T=1$	31
4.3	Simulation results of Milstein scheme for the first ten simulation paths for $\Delta t = 0.1$ and $T=25$	31
4.4	Simulation results of Euler scheme for the first ten simulation paths for $\Delta t = 10^{-5}$ and $T=1$	32
4.5	Simulation results of Euler scheme for the first ten simulation paths for $\Delta t = 0.1$ and $T=25$	32

List of Figures

4.1	Milstein discretization simulation for reduced form three factor valuation model with $\Delta t = 10^{-3}$ and 10^3 simulation paths	25
4.2	Milstein discretization simulation for reduced form three factor valuation model with $\Delta t = 10^{-1}$ and 10^2 simulation paths	26
4.3	Euler discretization simulation for reduced form three factor valuation model with $\Delta t = 10^{-3}$ and 10^3 simulation paths	26
4.4	Euler discretization simulation for reduced form three factor valuation model with $\Delta t = 10^{-1}$ and 10^2 simulation paths	27
4.5	Euler discretization simulation for reduced form three factor valuation model with $\Delta t = 10^{-3}$ and 10^3 simulation paths	28
4.6	Euler discretization simulation for reduced form three factor valuation model with $\Delta t = 10^{-1}$ and 10^2 simulation paths	28
4.7	Milstein discretization simulation for reduced form three factor valuation model with $\Delta t = 10^{-3}$ and 10^3 simulation paths	29
4.8	Milstein discretization simulation for reduced form three factor valuation model with $\Delta t = 10^{-1}$ and 10^2 simulation paths	29
4.9	Simulation for reduced form three factor valuation model with $\Delta t = 0.025$ and 10^4 simulation paths	30
4.10	Simulation for reduced form three factor valuation model with $\Delta t = 0.25$ and 10^3 simulation paths	30
4.11	Simulation of spot price for $\Delta t = 10^{-5}$ and 10^3 simulation paths, $T=1$	33
4.12	Simulation of spot price for $\Delta t = 0.1$ and 10^3 simulation paths, $T=25$	33
4.13	Simulation of convenience yield for $\Delta t = 10^{-5}$ and 10^3 simulation paths, $T=1$	33
4.14	Simulation of convenience yield for $\Delta t = 0.1$ and 10^3 simulation paths, $T=25$	34
4.15	Simulation of interest rate for $\Delta t = 10^{-5}$ and 10^3 simulation paths, $T=1$	34
4.16	Simulation of interest rate for $\Delta t = 0.1$ and 10^3 simulation paths, $T=25$	34

ABSTRACT

This study builds a reduced form of the three factor commodity derivative valuation model by explicitly taking into account the unobservable character of the convenience yield and extends the existing literature on commodity derivative by introducing a new feature, which is Vasicek interest rate model is replaced by CIR interest rate process to prevent the interest rate from going to negative. The spot price process, the instantaneous convenience yield and CIR interest rate process are taken in the reduced form of the three factor commodity derivative valuation model. We study the reduced form of the three factor commodity derivative valuation model based on discretization schemes. We simulate the reduced form the three factor commodity derivative valuation model by using the two known discretization schemes, i.e, Milstein and Euler discretization schemes. We study the performance of Milstein and Euler discretization schemes theoretically and empirically in reduced form the three factor commodity derivative valuation model. The Milstein discretization scheme has better approximation than Euler discretization scheme in reduced form the three factor commodity derivative valuation model. As the time of maturity, T , is less and the time interval decreases the result obtained from the simulation of reduced form the three factor commodity derivative valuation model for spot price process, convenience yield and interest rate process has better approximation. In addition, the data used to test reduced form of the three factor commodity derivative valuation model involves futures contracts from commodity market.

Chapter 1

Introduction

1.1 Background of the study

The study of commodity derivative prices is very important in many application areas in mathematical finance. There have been many commodity derivative models proposed to imitate the stochastic behaviors of commodity derivative prices. This study focuses on the reduced form of the three factor commodity derivative valuation model as proposed in incomplete information model(Ann Ngoc and Constantin Mellios(2015)). Unlike incomplete information model reduced form of the three factor commodity derivative valuation model used CIR interest rate process which has no closed form solution unlike Vasicek interest rate process. Due to this we mainly focus on finding numerical solution rather than finding closed form solution for reduced form of the three factor commodity derivative valuation model using discretization schemes. We present and compare the two common discretization schemes, namely Euler discretization scheme and Milstein discretization scheme. The Milstein discretization scheme is shown to be computationally and theoretically efficient for simulating the stochastic process for reduced form of the three factor commodity derivative valuation model when the discretization time interval and maturity are chosen to be very small. It is clear that many researchers have been explained CIR interest rate process does not have a closed form solution and proposed to use discretization schemes to solve the process numerically by taking smaller time interval Δt . Commodity derivative valuation models are identifying the relevant state variables or factors. A growing number of empirical studies return predictability pointed out the important role of the convenience yield. The spot price and the convenience yield are the two commodity used state variables in pricing commodity derivative valuation models.

A **commodity** is a physical substance, such as food, grains and metals, which is interchangeable with another product of the same type, and which investors buy or sell, usually through futures contracts. A commodity can be produced, consumed, transported or stored. The price of a commodity is subject to supply and demand of the market. More generally, a commodity is a product which trades on a commodity exchange; this would also include foreign currencies and financial instruments and indexes(as defined in Investorwords dictionary). There are many types of commodity. For example,

- **Energy:** crude oil, gasoline, natural gas, electricity, etc.

- **Metals:** copper, silver, gold, aluminum, zinc, etc.
- **Agriculture:** coffee, rice, wheat, salt, beans, etc.

A **Commodity exchange** is an exchange where various commodities are traded. Most commodity markets across the world trade in agricultural products and other raw materials (like sugar, milk, wheat, coffee, oil, metals, etc.) and contracts based on them. These contracts can include legal details regarding spot prices, forwards, futures and options on futures. Commodity exchanges usually trade futures contracts on commodities, basically trading contracts to receive an amount of the commodity in a certain date in the future(as defined in investopedia dictionary).

A **derivative** is financial instrument whose characteristics and value depend upon the characteristics and value of an underlier, typically a commodity, bond, equity or currency. Examples of derivatives include futures and options. Advanced investors sometimes purchase or sell derivatives to manage the risk associated with the underlying security, to protect against fluctuations in value, or to profit from periods of inactivity or decline. These techniques can be quite complicated and quite risky(as defined in investorwords dictionary).

1.2 Statement of the problem

Current studies showed that there are a problem while finding analytical solution of a given stochastic differential equations(Anqi Shao(2012), Akinbo B.J, Faniran T and Ayoola E.O (2015), Aurélien Alfonsi(2005), especially for CIR model since the process has no closed form solution. Hence the researchers are forced to study the process numerically using discretization techniques in order to get best approximation solution for a give process.

1.3 Objectives of the study

1.3.1 General Objective

To construct a suitable reduced form of the three factor commodity derivative valuation model.

1.3.2 Specific Objectives

1. To find a numerical solution of the three factor valuation model due to Milstein and Euler discretization scheme.
2. To study the performance of Milstein and Euler discretization schemes using the found solution in (1).

1.4 Significance of the study

The study develops a reduced form of the three factor commodity derivative valuation model and finds numerical solution for the proposed joint stochastic differential equations using the two known discretization scheme, Euler and Milstein schemes and now useful in many areas like finance to have a good approximation for a joint three factor stochastic differential equations. The study also be an initiation for further studies in this area.

1.5 Scope of the study

This study is basically based on finding numerical solution for three factor valuation model and attempted to find analytical solution like Schwartz(1997) three factor model but unlike Schwartz(1997) model this study used explicitly unsolvable CIR interest rate model hence the proposed valuation model does not solved analytically.

1.6 Limitation of the study

There are many methods which can allow us to find numerical solution for continuous time processes but this study used only Euler and Milstein discretization schemes because of their convergence to the true result. The study limits itself to find analytical solution for the joint stochastic differential equations.

1.7 Outline of the thesis

The thesis has five chapters. In the first chapter the background, significance, the scope and limitation of the study have been done. In the second chapter, review of the related literatures are presented as well. The proposed three factor valuation model and it's over all properties have been presented in chapter three. The empirical study, simulation results of the discretizations have been presented in fourth chapter. In the fifth chapter over all conclusion and recommendation for further studies have been discussed.

Chapter 2

Literature Review

Many researches are attempting to find analytical solution of joint stochastic differential equation. Some of them are able to find analytical solution for a given joint stochastic differential equation(Anh and Mellios(2015), Eduardo S. Schwartz(1997)) and while others are finding numerical solution depends on the continuous time process(Anqi Shao(2012)). In our reduced form three factor valuation model we found numerical solution for the proposed joint stochastic differential equations. Since there is no closed form solution for reduced form three factor valuation model we find numerical solution for proposed model using the two known numerical methods, Milstein and Euler discretization methods(Akinbo B.J, Faniran T. and Ayoola E.O(2015)).

2.1 CIR Model

The Cox-Ingersoll-Ross(CIR) model is a diffusion process suitable for modeling the term structure of interest rate(Anqi Shao(2012)). The dynamics for CIR model is given by,¹

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t \quad (2.1)$$

For $\kappa > 0$, $\theta > 0$, $\sigma > 0$ and Wiener process W . This process has some appealing properties from an practical point of view(Aurélien Alfonsi(2005)) i.e, the interest rate remains positive and the CIR process is mean reverting in nature(Cox, et al.(1981)). The condition $2\kappa\theta > \sigma^2$ would ensure that the origin is inaccessible to the process so that we can grant that the process X_t stays non-negative. One of our challenge, when we are simulating CIR model was explained by Anqi Shao, in his article. One drawback for simulation of CIR model is the process is not explicitly solvable(Aurélien Alfonsi(2005)). Due to this drawback we need to look further and proceed to find the method used to find numerical solution of the process to tackle this problem. The problem can be solved by using Diop A. and Deelstra and Delbaen's approaches i.e,

Diop A. approach

$$X_{t+1} = |X_t + \frac{T}{n}(a - \kappa X_t) + \sigma\sqrt{X_t}(W_{t+1} - W_t)| \quad (2.2)$$

¹refer J.C. Cox, J.E. Ingersoll and S.A. Ross(1981)

Deelstra and Delbaen approach

$$X_{t+1} = X_t + \frac{T}{n}(a - \kappa X_t) + \sigma \sqrt{X_t > 0}(W_{t+1} - W_t) \quad (2.3)$$

2.2 Discretization Scheme

Euler and Milstein schemes can be used to approximate the paths of the interest rate process on discrete time interval and they can also be applicable for continuous time processes(Aurélien Alfonsi(2005)) like spot price process and convenience yield in our case.

2.3 Theoretical results of Euler and Milstein schemes

Consider the i -th component of a general n -dimensional stochastic differential equation, with m -dimensional Wiener process as follows,

$$dX_t^i = a^i(t, X_t)dt + \sum_{j=1}^m b^{i,j}(t, X_t)dW_t^j \quad (2.4)$$

Where $a^i(t, X_t)$ and $b^{i,j}(t, X_t)$ are the drift and the volatility coefficient of the process X respectively. The convergence of Euler and Milstein schemes are discussed in P. E. Kloeden and E. Platen(1999) with theorem as the follows,

2.4 Theorem 1- Convergence of Euler scheme

Let X_0 be the initial state of the true process, Δ be the time interval and Y_0^Δ be the initial state of the simulation process generated by Euler scheme. Suppose that

$$E[|X_0|^2] < \infty \quad (2.5)$$

$$E[|X_0 - Y_0^\Delta|^2]^{\frac{1}{2}} \leq K_1 \Delta^{\frac{1}{2}} \quad (2.6)$$

$$|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \leq K_2|x - y| \quad (2.7)$$

$$|a(t, x)| + |b(t, x)| \leq K_3(1 + |x|) \quad (2.8)$$

$$|a(s, x) - a(t, x)| + |b(s, x) - b(t, x)| \leq K_4(1 + |x|)|s - t|^{\frac{1}{2}} \quad (2.9)$$

$\forall s, t \in [0, T]$ and $x, y \in \mathfrak{R}^d$, where the constants K_1, \dots, K_4 do not depend on the time step Δ . Then, for the Euler approximation Y_Δ , the estimate

$$E \left[|X_T - Y^\Delta(T)| \right] \leq K_5 \Delta^{\frac{1}{2}} \text{ satisfies, where the constant does not depend on } \Delta$$

Illustration of the above conditions,

The first condition (2.5) of theorem 1 implies that the initial state of the true process X must be finite in the mean square sense. The condition (2.6) indicates that the initial state of the simulation Y^Δ must be chosen such that the square root of the mean square error between X_0 and Y_0^Δ is bounded by $K_1 \Delta^{\frac{1}{2}}$, means we must choose the initial state of the simulation such that the difference between it and the initial true state is small enough and bounded by a given Δ . Condition (2.7)(Lipschitz condition) implies that the drift and the diffusion are differentiable everywhere in \mathfrak{R}^d for any $s, t \in [0, T]$ means that the condition guarantees the continuity of the drift and the diffusion coefficients in terms of their second component. Condition (2.8)(linear growth condition) implies that the growths of the drift $a(t, X_t)$ and the diffusion $b(t, X_t)$ must be bounded by the linear growth of $K_3(1 + |x|)$. From the conditions (2.5), (2.6) and (2.7) we can conclude about the existence and uniqueness of the strong solution for the stochastic differential equation (2.4).² Condition (2.9)(Hölder condition³ of order $\frac{1}{2}$) guarantees the continuity and differentiability of the drift and diffusion in terms of their first component.

2.5 Theorem 2- Convergence of Milstein scheme

Let X_0 be the initial state of the true process, Δ be the time interval and Y_0^Δ be the initial state of the simulation process generated by Milstein scheme. Suppose that

$$E[|X_0|^2] < \infty \tag{2.10}$$

$$E[|X_0 - Y_0^\Delta|^2]^{\frac{1}{2}} \leq K_1 \Delta^{\frac{1}{2}} \tag{2.11}$$

$$|\bar{a}(t, x) - \bar{a}(t, y)| \leq K_2 |x - y| \tag{2.12}$$

$$|b^{j1}(t, x) - b^{j1}(t, y)| \leq K_2 |x - y| \tag{2.13}$$

$$|L^{j1} b^{j2}(t, x) - \bar{L}^{j1} b^{j2}(t, y)| \leq K_2 |x - y| \tag{2.14}$$

$$|\bar{a}(t, x)| + |\bar{L}^j \bar{a}(t, x)| \leq K_3(1 + |x|) \tag{2.15}$$

$$|b^{j1}(t, x)| + |\bar{L}^j b^{j1}(t, x)| \leq K_3(1 + |x|) \tag{2.16}$$

$$|\bar{L}^j \bar{L}^{j1} b^{j2}(t, x)| \leq K_3(1 + |x|) \tag{2.17}$$

²see F.C. Klebaner,2005

³every Hölder continuous function is uniformly continuous.

$$|\bar{a}(s, x) - \bar{a}(t, x)| \leq K_4(1 + |x|)|s - t|^{\frac{1}{2}} \quad (2.18)$$

$$|b^{j_1}(s, x) - b^{j_1}(t, x)| \leq K_4(1 + |x|)|s - t|^{\frac{1}{2}} \quad (2.19)$$

$$|\bar{L}^{j_1} b^{j_2}(s, x) - \bar{L}^{j_1} b^{j_2}(t, x)| \leq K_4(1 + |x|)|s - t|^{\frac{1}{2}} \quad (2.20)$$

Where,

$$\bar{a} = a - \frac{1}{2}bb' \text{ and } \bar{L}^j = \sum_{k=1}^d b^{k,j} \frac{\partial}{\partial x^k}$$

$\forall s, t \in [0, T]$ and $x, y \in \mathfrak{R}^d$, $j = 0, \dots, m$, $j_1, j_2 = 1, \dots, m$ where the constants K_1, \dots, K_4 do not depend on Δ . Then, for Milstein approximation Y^Δ , the estimate

$$E \left[|X_T - Y^\Delta(T)| \right] \leq K_5 \Delta$$

satisfies, where the constant K_5 does not depend on Δ .

Illustration of the above conditions

The conditions⁴ (2.10) and (2.12)-(2.17) are to guarantee the existence and uniqueness of the solution for the stochastic differential equation (2.4). The condition (2.11) indicates that the initial state of the simulation Y^Δ must be chosen such that the square root of the mean square error between X_0 and Y_0^Δ is bounded by $K_1 \Delta^{\frac{1}{2}}$. The Lipschitz conditions (2.12)-(2.14) imply that the drift and the diffusion are differentiable everywhere in \mathfrak{R}^d for any $s, t \in [0, T]$. The linear growth conditions (2.15)-(2.17) imply that the growths of the drift $a(t, x)$ and the diffusion $b(t, x)$ must be bounded by a linear growth of $K_3(1 + |x|)$. The Hölder conditions (2.18)-(2.20) guarantee the continuity and differentiability of the drift and diffusion coefficients in terms of their first component.

Remarks

1. From theorem 1 one may bound the discretization error between the true process at time T and the discretization as a function of a constant and the discretization interval. Hence one can get as accurate solution as required by a reduction of Δ only and no transform of the process required.
2. Like theorem 1 from theorem 2 one may bound the discretization error and the discretization at maturity T as a function of a constant and the discretization interval. Therefore one can achieve as accurate solution as required by a reduction of the time interval Δ only.

2.6 Theoretical comparison between Euler and Milstein

The Milstein scheme is an extension of the Euler scheme by simply adding one more term. As a time interval gets smaller and smaller and if we want to improve the accuracy of the simulation, then we must reduce the time discretization step Δ to less than $100\times$ for Euler scheme and we only need to reduce it to $10\times$ for Milstein scheme. Simply this tells us the solution by the Milstein scheme converges to the truth faster than the Euler scheme as $\Delta < 1$.

⁴refer F.C. Klebaner(2005)

2.7 Euler-Maruyama discretization scheme

One of the simplest example of discretization is Euler-Maruyama discretization scheme(Akinbo B.J, Faniran T. and Ayoola E.O(2015)).

We assume stochastic differential equation,

$$dX_t = \mu X_t dt + \sigma X_t dW_t \quad (2.21)$$

Discretizing the above process by Euler scheme on $0 \leq t \leq T$ for a given discretization, $0 \leq t_1 < t_2 < \dots < t_n \leq T$ of final time interval $[0, T]$ is given as follows,

$$X_t = X_{t-1} + \mu X_{t-1} \Delta t + \sigma X_{t-1} \sqrt{\Delta t} n_{X,t-1}$$

Where,

$\Delta t = \tau_t - \tau_{t-1}$ is the length of the time discretization subinterval $[\tau_{t-1}, \tau_t]$ and $n_{X,t-1}$ is a standard normal random variable.

Euler-Maruyama scheme is a method used to approximate numerical solution of a continuous time processes. In practice, many stochastic differential equations are not explicitly solvable like CIR model therefore we can not get an analytical solution to a given continuous time process(Anqi Shao(2012)).

2.8 Milstein discretization scheme

Like Euler-Maruyama discretization scheme Milstein scheme also used to find the numerical solution of a given stochastic differential equations. For the above stochastic differential equation (2.21) we have,

$$X_t = X_{t-1} + \mu X_{t-1} \Delta t + \sigma X_{t-1} \sqrt{\Delta t} n_{X,t-1} + \frac{1}{2} \sigma^2 X_{t-1} (\Delta t^2 n_{X,t-1}^2 - \Delta t)$$

Where, Δt and $n_{X,t-1}$ are as defined above.

2.9 Strong Convergence Analysis

Milstein scheme has strong order of convergence one and Milstein will converges to the correct stochastic solution process faster than Euler-Maruyama as the step size Δt goes to zero(Akinbo B.J, Faniran T. and Ayoola E.O(2015)). Due to the property of it's strong convergency, Milstein discretization scheme give better approximation for a given continuous time processes. In our simulation result for reduced form valuation model we get the same result for Euler and Milstein discretization scheme. Since the diffusion coefficients for spot price process and CIR interest rate process are not constant we can easily distinguish the difference between the two schemes in contrast since the diffusion coefficient for convenience yield process is constant we may not get the real difference between Euler and Milstein scheme. In our Euler and Milstein simulation we take small interval since if the time interval is sufficiently small, the simulation almost does not distinguish with the truth.

2.10 Valuation model

Anh Ngoc Lai, Constantin Mellios(2015) extends the existing literature by pricing derivatives under incomplete information and derived simple closed-form solutions for their valuation model. Unlike our model they used Vasicek interest rate model for their three factor model. Eduardo S. Schwartz(1997) also gets analytical solution by first assuming the commodity spot price follows the stochastic process, and applying Itô's lemma and derives Ornstein-Uhlenbeck stochastic process. The conditional distribution of the logarithm of the spot price at time T under the equivalent martingale measure is log-normally distributed with some parameters. Kristian R. Miltersen and Eduardo S. Schwartz(1998) develops a model for value options on commodity futures in the presence of stochastic interest rates as well as stochastic convenience yields. Eduardo S. Schwartz(1998) presents the long-term model and a two-factor model and how to implement the long-term model using the parameters estimated for the two-factor model in Eduardo S. Schwartz(1997). Julio J. Lucia and Eduardo S. Schwartz(2001) spot electricity prices discussed briefly in this study. This study used the one and two factor Schwartz model to derive future, forward and spot prices of the commodity.

In Eduardo S. Schwartz(1997) article the proposed model has three parts. On the first place, valuation model is presented in one factor model, which is the spot price process. Here the instantaneous convenience yield and interest rate are both assumed to constant. Since the interest rate is constant in this model the futures price and forward price are equal. The model which presented here in one factor model is;

$$dS = \kappa(\mu - \ln S)Sdt + \sigma Sdz \quad (2.22)$$

By defining $X = \ln s$ and applying Ito's lemma, we will have

$$dX = \kappa(\alpha - X)dt + \sigma dz \quad (2.23)$$

Where, $\alpha = \mu - \frac{\sigma^2}{2\kappa}$ $\kappa > 0$ measures the degree of mean reversion to the long run mean log price, α . σ is the volatility of the process, dz is standard Brownian motion. Here we can easily observe that, since $X = \ln S$, the spot price of the commodity at time T is log-normally distributed under a constant interest rate. The futures price is then given by;

$$F(S, T) = \exp \left[e^{-\kappa T} \ln S + (1 - e^{-\kappa T})\alpha^* + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa T}) \right] \quad (2.24)$$

Or

$$\ln F(S, T) = e^{-\kappa T} S + (1 - e^{-\kappa T})\alpha^* + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa T}) \quad (2.25)$$

Where, $\alpha^* = \alpha - \lambda$ and λ is the market price of risk associated with spot price.

On the second place we found two factor model, spot price process and instantaneous convenience yield process. Here like one factor model the interest rate is also constant. The two factor model is given by;

$$dS = (\mu - \delta)Sdt + \sigma_1 Sdz_1 \quad (2.26)$$

$$d\delta = \kappa(\alpha - \delta)dt + \sigma_2 dz_2 \quad (2.27)$$

where, $dz_1 dz_2 = \rho dt$, ρ is correlation coefficient. By defining $X = \ln S$ and applying Ito's lemma, the process for the log price can be written as;

$$dX = (\mu - \delta - \frac{1}{2}\sigma_1^2)dt + \sigma_1 dz_1 \quad (2.28)$$

Under equivalent martingale measure the stochastic process for the two factor model can be expressed as;

$$dS = (r - \delta)Sdt + \sigma_1 S dz_1^* \quad (2.29)$$

$$d\delta = \left[\kappa(\alpha - \delta) - \lambda \right] dt + \sigma_2 dz_2^* \quad (2.30)$$

$dz_1^* dz_2^* = \rho dt$, λ is the market price risk associated with convenience yield. The futures price for the two factor model is given by;

$$F(S, \delta, T) = S \exp \left[-\delta \frac{1 - e^{-\kappa T}}{\kappa} + (r - \hat{\alpha} + \frac{1}{2} \frac{\sigma_2^2}{\kappa} - \frac{\sigma_1 \sigma_2 \rho}{\kappa}) T + \frac{1}{4} \frac{\sigma_2^2}{\kappa^3} \frac{1 - e^{-2\kappa T}}{\kappa^3} + (\hat{\alpha} \kappa + \sigma_1 \sigma_2 \rho - \frac{\sigma_2^2}{\kappa}) \frac{1 - e^{-\kappa T}}{\kappa^2} \right] \quad (2.31)$$

Where, $\hat{\alpha} = \alpha - \frac{\lambda}{\kappa}$

The last model proposed in Eduardo S. Schwartz(1997) is the three factor valuation model, spot price process, instantaneous convenience yield process and interest rate process.

$$\frac{dS(t)}{S(t)} = (r(t) - \delta(t))dt + \sigma_s dZ_s^*(t) \quad (2.32)$$

$$d\delta(t) = \kappa(\hat{\alpha} - \delta(t))dt + \sigma_\delta dZ_\delta^*(t) \quad (2.33)$$

$$dr(t) = a(m^* - r(t))dt + \sigma_r dZ_r^*(t) \quad (2.34)$$

With initial conditions $S(0) \equiv S_0$, $\delta(0) \equiv \delta_0$ and $r(0) \equiv r_0$.

Where, $\hat{\alpha} = \alpha - \frac{\lambda}{\kappa}$, three correlated standard Brownian motions, $dZ_s^* dZ_\delta^* = \rho_1 dt$, $dZ_\delta^* dz_r^* = \rho_2 dt$ and $dZ_s^* dz_r^* = \rho_3 dt$. a is the speed of adjustment, m^* the risk adjusted mean short rate of the interest rate and σ_r is the constant, strictly positive, instantaneous standard deviation of interest rate, $r(t)$.

From this three factor model we can see that unlike the reduced form of three factor valuation model, the interest rate is based on Vasicek interest rate process. Naturally, from Vasicek interest rate process we know that there is no guarantee that the process is always positive. In the reduced form of the three factor valuation model the interest rate is based on CIR process. It's known that CIR process is always positive. The futures price given in this three factor model is expressed as;

$$\begin{aligned}
F(S, \delta, r, T) = S \exp & \left[\frac{\delta(1-e^{-\kappa T})}{\kappa} + \frac{r(1-e^{-aT})}{a} + \frac{(\kappa\hat{\alpha} + \rho_1\sigma_s\sigma_\delta)((1-e^{-\kappa T}) - \kappa T)}{\kappa^2} - \frac{\sigma_\delta^2(4(1-e^{-\kappa T}) - (1-e^{-2\kappa T}) - 2\kappa T)}{4\kappa^3} \right. \\
& - \frac{(am^* + \rho_3\sigma_s\sigma_r)((1-e^{-aT}) - aT)}{a^2} - \frac{\sigma_r^2(4(1-e^{-aT}) - (1-e^{-2aT}) - 2aT)}{4a^3} + \rho_2\sigma_\delta\sigma_r \left(\frac{(1-e^{-\kappa T}) + (1-e^{-aT}) - (1-e^{-(\kappa+a)T})}{\kappa a(\kappa+a)} \right) \\
& \left. + \frac{\kappa^2(1-e^{-aT}) + a^2(1-e^{-\kappa T}) - \kappa a^2 T - a\kappa^2 T}{\kappa^2 a^2 (\kappa+a)} \right]
\end{aligned}$$

After having futures price's expression for each model, now it's straight forward to put the proposed models in state space formulation using Kalman filtering technique⁵. In our proposed reduced form of three factor commodity derivative valuation model, unlike Eduardo S. Schwartz(1997) one of the difficulties arises when there is no a closed form solution for proposed three factor valuation model. This is the only reason why we prefer finding numerical solution for reduced form of the three factor valuation model using the two basic known discretization schemes, i.e, Milstein discretization and Euler discretization schemes.

2.11 Financial market under incomplete information

Ann Ngoc Lai and Constantin Mellios(2015) also discussed on finding futures price of three factor valuation model under incomplete information.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space with a standard filtration $F = \{\mathcal{F}(t) : t \in [0, T]\}$, a finite time period $[0, T]$ and three correlated standard Brownian motion $Z_s^*(t)$, $Z_\delta^*(t)$ and $Z_r^*(t)$, defined on (Ω, \mathcal{F}) such that, $dZ_s^*(t)dZ_\delta^*(t) = \rho_{s\delta}dt$, $dZ_s^*(t)dZ_r^*(t) = \rho_{sr}dt$ and $dZ_r^*(t)dZ_\delta^*(t) = \rho_{r\delta}dt$ where,

ρ_{ij} represents correlation coefficient and $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$, for $i, j = S(t), \delta(t), r(t)$ with $i \neq j$ represents the covariances between $\frac{dS(t)}{S(t)}$, $d\delta(t)$ and $dr(t)$.

$F^{s,r}$ is the filtration generated by $S(t)$ and $r(t)$, and $F^{s,r} \equiv \sigma(S(s), r(s) : 0 \leq s \leq t)$, the σ -algebra, represents the information available by observing the processes $S(t)$ and $r(t)$ up to time t , such that $F^{s,r}(t) \subset \mathcal{F}(t)$.

The spot price of the commodity, $S(t)$, satisfies the stochastic differential equation(SDE),

$$\frac{dS(t)}{S(t)} = (\mu_s(t) - \delta(t))dt + \sigma_s dZ_s^*(t) \tag{2.35}$$

with initial condition $S(0) \equiv S$. Here, $\mu_s(t) = r(t) + \lambda_s\sigma_s$ is the instantaneous expected rate of return of the spot price, σ_s is the constant strictly positive, instantaneous standard deviation of the spot price and λ_s is the constant market price of risk associated with the spot price process. $r(t)$ is the instantaneous riskless interest rate and $\delta(t)$ is the instantaneous convenience yield.

⁵see Appendix 4

Interest rates have an impact on spot commodity prices and on convenience yields. The SDE of the short rate follows a mean-reverting process;

$$dr(t) = \alpha(\hat{\beta} - r(t))dt + \sigma_r dZ_r^*(t) \quad (2.36)$$

with initial condition $r(0) \equiv r$. Here, α and $\hat{\beta}$ are constants, and σ_r is the constant, strictly positive, instantaneous standard deviation of $r(t)$. The short rate has a tendency to revert to a constant long-run interest rate level, $\hat{\beta}$, with a speed of mean reversion α .

SDE for convenience yield is given by as follows,

$$d\delta(t) = k(\hat{\delta} - \delta(t))dt + \sigma_\delta dZ_\delta^*(t) \quad (2.37)$$

with initial condition $\delta(0) \equiv \delta$. Here, k and $\hat{\delta}$ are constant positive scalars, σ_δ is the constant, strictly positive, instantaneous standard deviation of $\delta(t)$.

If the correlation coefficient between the spot commodity price and the convenience yield is positive, then a weak mean-reverting effect is induced by the stochastic behavior of the convenience yield (Anh Ngoc Lai and Constantin Mellios(2015)).

2.12 Valuation of Commodity derivatives

The investor does not observe the convenience yield but draws inferences about it from her (his) observations of the spot price of the commodity and the interest rate (Anh Ngoc Lai and Constantin Mellios(2015)). Here, it is clear that the investor views the prior distribution of $\delta(0)$ as Gaussian distribution with given mean $m(0)$ and variance $\gamma(0)$. The conditional mean and the estimation error are defined as, $m(t) = E[\delta(t)|F^{s,r}(t)]$ and $\gamma(t) = E[(\delta(t) - m(t))^2|F^{s,r}(t)]$ respectively.

SDE for conditional mean, $m(t)$, and estimation error, $\gamma(t)$, derived as follows by using Kalman-Bucy filtering technique.

$$dm(t) = k \left(\hat{\delta} - m(t) \right) dt + \left(\sigma_\delta \rho_{s_1} - \frac{\gamma(t)}{\sigma_s(1 - \rho_{sr}^2)} \right) dz_s(t) + \left(\sigma_\delta \rho_r + \frac{\rho_{sr} \gamma(t)}{\sigma_s(1 - \rho_{sr}^2)} \right) dZ_r(t) \quad (2.38)$$

$$d\gamma(t) = \left[\sigma_\delta^2(1 - \rho_{r\delta}^2) - \sigma_\delta^2 \rho_{s_1}^2(1 - \rho_{sr}^2) - 2 \left(k - \frac{\sigma_\delta \rho_{s_1}}{\sigma_s} \right) \gamma(t) - \frac{\gamma(t)^2}{\sigma_s^2(1 - \rho_{sr}^2)} \right] dt$$

$$\gamma(t) = \sigma_s^2(1 - \rho_{sr}^2) \left[\sqrt{\Delta} - \theta + 2\sqrt{\Delta} \frac{\Gamma e^{-2\sqrt{\Delta}t}}{1 - \Gamma e^{-2\sqrt{\Delta}t}} \right] \quad (2.39)$$

With initial conditions $m(0) \equiv m$ and $\gamma(0) \equiv \gamma$. Where,

$$\rho_{s_1} = \frac{(\rho_{s\delta} - \rho_{sr}\rho_{r\delta})}{1 - \rho_{sr}^2}, \quad \rho_r = \frac{(\rho_{r\delta} - \rho_{sr}\rho_{s\delta})}{1 - \rho_{sr}^2}, \quad \theta = k - \frac{\sigma_\delta \rho_{s_1}}{\sigma_s}, \quad \Delta = k^2 + \frac{\sigma_\delta^2(1 - \rho_{r\delta}^2)}{\sigma_s^2(1 - \rho_{sr}^2)} - 2k \frac{\sigma_\delta \rho_{s_1}}{\sigma_s},$$

$$\Gamma = \frac{\gamma - \gamma_\infty}{\gamma + \sigma_s^2(1 - \rho_{sr}^2)(\sqrt{\Delta} + \theta)}.$$

$$\gamma_\infty = \sigma_s^2(1 - \rho_{sr}^2)(\sqrt{\Delta} - \theta), \quad dZ_s(t) = \frac{1}{\sigma_s} \left[\frac{dS}{S} - (\mu_s - m(t))dt \right] = dZ_s^* + \frac{m(t) - \delta(t)}{\sigma_s} dt \text{ and}$$

$$dZ_r(t) = \frac{1}{\sigma_r} (dr(t) - \alpha(\beta - r(t))dt) = dZ_r^*(t).$$

Here, $z_s(t)$ and $Z_r(t)$ are Wiener processes relative to the filtration $F^{s,r}$.

Here, as time passes, the estimation error converges to a stable steady state. i.e,

$\gamma(\infty) \equiv \gamma_\infty = \sigma_s^2(1 - \rho_{sr}^2)(\sqrt{\Delta} - \theta)$, as $t \rightarrow \infty$. We can observe that, even for a long period, the investor can not accurately estimate the convenience yield. When the convenience yield is deterministic, i.e. $\sigma_\delta = 0$, then $\gamma_\infty = 0$.

2.13 Commodity Futures Prices

By using the general framework above, we can price commodity futures and options on commodity futures contracts. Since the futures contract is assumed to be marked to market continuously and then to have always zero-value, the futures price, under the risk-neutral probability measures, \mathbf{Q} , is a martingale. At maturity date T , the futures price is equal to the underlying spot price.

In arbitrage free, the price of a futures contract at date 0, $H(S, r, m, 0, T) \equiv H(T)$, of maturity date T written on a commodity is equal to: $H(T) = E^Q[S(T)]$, whose solution is given by,

$$\begin{aligned} H(T) = & \text{Sexp} \left\{ \left[\beta - \bar{\delta} + \frac{\sigma_r^2}{2\alpha^2} + \frac{\sigma_{sr}}{\alpha} - \frac{\sigma_{r\delta}}{\alpha K} + \frac{\sigma_\delta^2}{2K^2} - \frac{\sigma_{s\delta}}{K} \right] T \right. \\ & - \left[\beta - r + \frac{\sigma_r^2}{2\alpha^2} + \frac{\sigma_{sr}}{\alpha} - \frac{\sigma_{r\delta}}{\alpha(\alpha + K)} \right] D_\alpha(T) - \left[m - \bar{\delta} + \frac{\sigma_\delta^2}{2K^2} - \frac{\sigma_{s\delta}}{K} - \frac{\sigma_{r\delta}}{K(\alpha + K)} \right] D_k(T) \\ & - \frac{\sigma_{r\delta}}{\alpha + k} D_\alpha(T) D_k(T) - \frac{\sigma_r^2}{4\alpha} D_\alpha^2(T) - \frac{1}{2} \left[\frac{\sigma_\delta^2}{2K} - \gamma \right] D_k^2(T) \\ & \left. - \frac{\sigma_s(\lambda_s - \rho_{sr}\lambda_r)}{K} \left[\ln \left(\frac{1 - \Gamma e^{-2\sqrt{\Delta}T}}{1 - \Gamma} \right) - \frac{2\sqrt{\Delta}}{K} e^{KT} \sum_{n=1}^{\infty} \Gamma^n D_{2n\sqrt{\Delta}-K}(T) \right] \right\}, \end{aligned} \quad (2.40)$$

Where, $\beta = \bar{\beta} - \frac{\lambda_r\sigma_r}{\alpha}$

The system of stochastic differential equations(SDEs) (2.35), (2.36) and (2.37), in partially observable economy, is equivalent, in the fully observable economy, to the following equations:

$$\frac{dS(t)}{S(t)} = (r(t) + \lambda_s\sigma_s - m(t))dt + \sigma_s dZ_s(t), \quad (2.41)$$

$$dr(t) = \alpha(\bar{\beta} - r(t))dt + \sigma_r dZ_r(t), \quad (2.42)$$

$$dm(t) = k \left(\bar{\delta} - m(t) \right) dt + \left(\sigma_\delta \rho_{s_1} - \frac{\gamma(t)}{\sigma_s(1 - \rho_{sr}^2)} \right) dZ_s(t) + \left(\sigma_\delta \rho_r + \frac{\rho_{sr}\gamma(t)}{\sigma_s(1 - \rho_{sr}^2)} \right) dZ_r(t). \quad (2.43)$$

With initial conditions, $S(0) = S$, $r(0) = r$ and $m(0) = m$ respectively. By applying Ito's lemma to equation (2.41) gives:

$$S(T) = \text{Sexp} \left\{ \int_0^T (r(u) - m(u)) du + \left[\lambda_s\sigma_s - \frac{1}{2}\sigma_s^2 \right] T + \sigma_s Z_s(T) \right\} \quad (2.44)$$

With initial condition $S(0) = S$. From equation (2.42) above we have, $r(t) = re^{-\alpha t} + \alpha\bar{\beta}D_\alpha(t) + \sigma_r e^{-\alpha t} \int_0^t e^{\alpha u} dZ_r(u)$, with initial condition $r(0) = r$.

$$\int_0^T r(v)dv = \bar{\beta}T + (r - \bar{\beta})D_\alpha(T) + \sigma_r \int_0^T \int_0^V e^{-\alpha(v-u)} dZ_r(u)dv.$$

By applying Fubini's theorem⁶, to above equation we have,

$$\int_0^T r(v)dv = \bar{\beta}T + (r - \bar{\beta})D_\alpha(T) + \sigma_r \int_0^T D_\alpha(u, T) dZ_r(u) \quad (2.45)$$

By the same procedure, applying Ito's lemma to equation (2.43) yields,

$$m(t) = me^{-kt} + k\bar{\delta}D_k(t) + e^{-kt} \int_0^t e^{ku} \left(\sigma_\delta \rho_{s_1} - \frac{\gamma(u)}{\sigma_s(1-\rho_{sr}^2)} \right) dZ_s(u) + e^{-kt} \int_0^t e^{ku} \left(\sigma_\delta \rho_r + \frac{\rho_{sr}\gamma(u)}{\sigma_s(1-\rho_{sr}^2)} \right) dZ_r(u),$$

with initial condition $m(0) = m$.

$$\int_0^T m(v)dv = \bar{\delta}T + (m - \bar{\delta})D_k(T) + \int_0^T \int_0^v e^{-k(v-u)} \left(\sigma_\delta \rho_{s_1} - \frac{\gamma(u)}{\sigma_s(1-\rho_{sr}^2)} \right) dZ_s(u)dv$$

$$+ \int_0^T \int_0^v e^{-k(v-u)} \left(\sigma_\delta \rho_r + \frac{\rho_{sr}\gamma(u)}{\sigma_s(1-\rho_{sr}^2)} \right) dZ_r(u)dv.$$

Applying Fubini's theorem gives,

$$\int_0^T m(v)dv = \bar{\delta}T + (m - \bar{\delta})D_k(T) + \int_0^T D_k(u, T) \left(\sigma_\delta \rho_{s_1} - \frac{\gamma(u)}{\sigma_s(1-\rho_{sr}^2)} \right) dZ_s(u)$$

$$+ \int_0^T D_k(u, T) \left(\sigma_\delta \rho_r + \frac{\rho_{sr}\gamma(u)}{\sigma_s(1-\rho_{sr}^2)} \right) dZ_r(u) \quad (2.46)$$

Inserting equations (2.45) and (2.46) into equation (2.44) implies

$$S(T) = S \exp \left\{ \left(\bar{\beta} - \bar{\delta} + \lambda_s \sigma_s - \frac{1}{2} \sigma_s^2 \right) T + (r - \bar{\beta})D_\alpha(T) - (m - \bar{\delta})D_k(T) \right.$$

$$\left. + \int_0^T \left(\sigma_s - D_k(u, T) \left(\sigma_\delta \rho_{s_1} - \frac{\gamma(u)}{\sigma_s(1-\rho_{sr}^2)} \right) \right) dZ_s(u) + \int_0^T \left(\sigma_r D_\alpha(u, T) - D_k(u, T) \left(\sigma_\delta \rho_r + \frac{\rho_{sr}\gamma(u)}{\sigma_s(1-\rho_{sr}^2)} \right) \right) dZ_r(u) \right\}$$

Under martingale, Q, by substituting equation (2.39) to the above equation, we have,

$$S(T) = S \exp \left\{ \left(\beta - \bar{\delta} - \frac{1}{2} \sigma_s^2 \right) T + (r - \beta)D_\alpha(T) - (m - \bar{\delta})D_k(T) - \frac{\sigma_s(\lambda_s - \lambda_r \rho_{sr})}{k} \left[\ln \left(\frac{1 - \Gamma e^{-2\sqrt{\Delta}T}}{1 - \Gamma} \right) \right. \right.$$

$$\left. - \frac{2\sqrt{\Delta}}{k} e^{-kT} \sum_{n=1}^{\infty} \Gamma^n D_{2n\sqrt{\Delta}-k}(T) \right] + \int_0^T \left(\sigma_s - D_k(u, T) \left(\sigma_\delta \rho_{s_1} - \frac{\gamma(u)}{\sigma_s(1-\rho_{sr}^2)} \right) \right) d\tilde{Z}_s(u)$$

$$\left. + \int_0^T \left(\sigma_r D_\alpha(u, T) - D_k(u, T) \left(\sigma_\delta \rho_r + \frac{\rho_{sr}\gamma(u)}{\sigma_s(1-\rho_{sr}^2)} \right) \right) d\tilde{Z}_r(u) \right\},$$

$$S(T) = S \exp X(T)$$

$$\text{where, } \bar{\delta} = \bar{\delta} + \frac{1}{k} \left[\sigma_s(\lambda_s - \rho_{sr}\lambda_r)(\sqrt{\Delta} - k) - \lambda_r \sigma_\delta \rho_r \bar{\delta} \right] \text{ and } \beta = \bar{\beta} - \frac{\lambda_r \sigma_r}{\alpha}$$

The future price in free of arbitrage is given by: $H(T) = E^Q[S(T)]$. Since $X(T)$ is normally distributed, then:

$$H(T) = E^Q[S(T)] = \exp \left\{ SE^Q[X(T)] + \frac{1}{2} \text{Var}[X(T)] \right\}$$

where, $E^Q[X(T)]$ is the expectation of the stochastic variable $X(T)$ under Q and $\text{Var}[X(T)]$ is the variance of the stochastic variable $X(T)$ under Q.

⁶Fubini's theorem: $\int_0^b \int_0^y f(x, y) dx dy = \int_0^b \int_x^b f(x, y) dy dx$

$$E^Q[X(T)] = \left(\beta - \bar{\delta} - \frac{1}{2}\sigma_s^2 \right) T + (r - \beta)D_\alpha(T) - (m - \bar{\delta})D_k(T) - \frac{\sigma_s(\lambda_s - \lambda_r \rho_{sr})}{k} \left[\ln \left(\frac{1 - \Gamma e^{-2\sqrt{\Delta}T}}{1 - \Gamma} \right) - \frac{2\sqrt{\Delta}}{k} e^{-kT} \sum_{n=1}^{\infty} \Gamma^n D_{2n\sqrt{\Delta}-k}(T) \right] \quad (2.47)$$

$$\begin{aligned} Var[X(T)] &= \int_0^T \left[\left(\sigma_s - D_k(u, T) \left(\sigma_\delta \rho_{s_1} - \frac{\gamma(u)}{\sigma_s(1-\rho_{sr}^2)} \right) \right)^2 \right. \\ &+ \left(\sigma_r D_\alpha(u, T) - D_k(u, T) \left(\sigma_\delta \rho_r + \frac{\rho_{sr}\gamma(u)}{\sigma_s(1-\rho_{sr}^2)} \right) \right)^2 \\ &+ \left. 2\rho_{sr} \left(\sigma_s - D_k(u, T) \left(\sigma_\delta \rho_{s_1} - \frac{\gamma(u)}{\sigma_s(1-\rho_{sr}^2)} \right) \right) \left(\sigma_r D_\alpha(u, T) - D_k(u, T) \left(\sigma_\delta \rho_r + \frac{\rho_{sr}\gamma(u)}{\sigma_s(1-\rho_{sr}^2)} \right) \right) \right] du \\ &= \int_0^T \left[\sigma_s^2 + \sigma_r^2 D_\alpha(u, T)^2 + \sigma_\delta^2 \left(\rho_{s_1}^2 + \rho_r^2 + 2\rho_{sr}\rho_{s_1}\rho_r \right) D_k(u, T)^2 - 2\sigma_s\sigma_\delta \left(\rho_{s_1} + \rho_{sr}\rho_r \right) D_k(u, T) \right. \\ &+ \left. 2\sigma_{sr} D_\alpha(u, T) - 2\sigma_\delta\sigma_r(\rho_{sr}\rho_{s_1} + \rho_r) D_\alpha(u, T) D_k(u, T) + 2\gamma(u) D_k(u, T) - \frac{2\sigma_\delta\rho_{s_1}\gamma(u)}{\sigma_s} D_k(u, T)^2 \right. \\ &+ \left. \frac{\gamma(u)^2}{\sigma_s^2(1-\rho_{sr}^2)} D_k(u, T)^2 \right] du \\ &= \int_0^T \left[\sigma_s^2 + \sigma_r^2 D_\alpha(u, T)^2 + \sigma_\delta^2 (\rho_{s_1}^2 + \rho_r^2 + 2\rho_{sr}\rho_{s_1}\rho_r) D_k(u, T)^2 - 2\sigma_s\sigma_\delta (\rho_{s_1} + \rho_{sr}\rho_r) D_k(u, T) \right. \\ &+ \left. 2\sigma_{sr} D_\alpha(u, T) - 2\sigma_\delta\sigma_r(\rho_{sr}\rho_{s_1} + \rho_r) D_\alpha(u, T) D_k(u, T) + 2\gamma(u) D_k(u, T) \right. \\ &- \left. \frac{2\sigma_\delta\rho_{s_1}\gamma(u)}{\sigma_s} D_k(u, T)^2 + \frac{\gamma(u)^2}{\sigma_s^2(1-\rho_{sr}^2)} D_k(u, T)^2 \right] du \end{aligned}$$

Here, the following expressions can be established

$$\rho_{s_1}^2 + \rho_r^2 + 2\rho_{sr}\rho_{s_1}\rho_r = \rho_{r\delta}^2 + \rho_{s_1}^2(1 - \rho_{sr}^2), \quad \sigma_s\sigma_\delta(\rho_{s_1} + \rho_{sr}\rho_r) = \sigma_{s\sigma} \text{ and } \sigma_\delta\sigma_r(\rho_r + \rho_{sr}\rho_{s_1}) = \sigma_{r\delta}.$$

$$\begin{aligned} Var[X(T)] &= \int_0^T \left[\sigma_s^2 + \sigma_r^2 D_\alpha(u, T)^2 + 2\sigma_{sr} D_\alpha(u, T) - 2\sigma_{s\delta} D_k(u, T) + \sigma_\delta^2 (\rho_{r\delta}^2 + \rho_{s_1}^2 (1 - \rho_{sr}^2)) D_k(u, T)^2 \right. \\ &- \left. 2\sigma_{r\delta} D_\alpha(u, T) D_k(u, T) + 2\gamma(u) D_k(u, T) - \frac{2\sigma_\delta\rho_{s_1}\gamma(u)}{\sigma_s} D_k(u, T)^2 + \frac{\gamma(u)^2}{\sigma_s^2(1-\rho_{sr}^2)} D_k(u, T)^2 \right] du \end{aligned}$$

By substituting equation (2.39) into above equation, we have

$$\begin{aligned} Var[X(T)] &= \int_0^T \left[\sigma_s^2 + \sigma_r^2 D_\alpha(u, T)^2 + 2\sigma_{sr} D_\alpha(u, T) - 2\sigma_{s\delta} D_k(u, T) - 2\sigma_{r\delta} D_\alpha(u, T) D_k(u, T) \right. \\ &+ \left. \sigma_\delta^2 (\rho_{r\delta}^2 + \rho_{s_1}^2 (1 - \rho_{sr}^2)) D_k(u, T)^2 \right] du + \sigma_s^2 (1 - \rho_{sr}^2) \int_0^T \left[(\Delta - \Theta^2) D_k(u, T)^2 \right. \\ &+ \left. 2(\sqrt{\Delta} - \Theta) e^{-k(T-u)} D_k(u, T) \right] du - \frac{4\sigma_s^2(1-\rho_{sr}^2)\Delta}{k^2} \int_0^T \left[\frac{\Gamma e^{-2\sqrt{\Delta}u}}{(1-\Gamma e^{-2\sqrt{\Delta}u})^2} \right] du \\ &- \frac{4\sigma_s^2(1-\rho_{sr}^2)\sqrt{\Delta}}{k^2} \int_0^T \left[(2\sqrt{\Delta} - k) \frac{\Gamma e^{-2\sqrt{\Delta}u}}{1-\Gamma e^{-2\sqrt{\Delta}u}} e^{-k(T-u)} - (\sqrt{\Delta} - k) \frac{\Gamma e^{-2\sqrt{\Delta}u}}{1-\Gamma e^{-2\sqrt{\Delta}u}} e^{-2k(T-u)} \right. \\ &+ \left. 2\sqrt{\Delta} \left(\frac{\Gamma e^{-2\sqrt{\Delta}u}}{1-\Gamma e^{-2\sqrt{\Delta}u}} \right)^2 e^{-k(T-u)} - \sqrt{\Delta} \left(\frac{\Gamma e^{-2\sqrt{\Delta}u}}{1-\Gamma e^{-2\sqrt{\Delta}u}} \right)^2 e^{-2k(T-u)} \right] du \end{aligned}$$

The computation of the integrals in the above equation, involving hypergeometric functions,⁷ gives:

⁷Has a series expansion: $H(a, b, c, d) = 1 + \frac{ab}{1!c} d + \frac{a(a+1)b(b+1)}{2!(c+1)} d^2 + \dots$

$$\begin{aligned}
\int_0^T \frac{\Gamma e^{-2\sqrt{\Delta}u}}{1-\Gamma e^{-2\sqrt{\Delta}u}} e^{-k(T-u)} du &= \frac{1}{k} \left(e^{-kT} H\left(1, \frac{k}{2\sqrt{\Delta}}, \frac{k+2\sqrt{\Delta}}{2\sqrt{\Delta}}, \frac{1}{\Gamma}\right) - H\left(1, \frac{k}{2\sqrt{\Delta}}, \frac{k+2\sqrt{\Delta}}{2\sqrt{\Delta}}, \frac{e^{2\sqrt{\Delta}T}}{\Gamma}\right) \right), \\
\int_0^T \frac{\Gamma e^{-2\sqrt{\Delta}u}}{1-\Gamma e^{-2\sqrt{\Delta}u}} e^{-2k(T-u)} du &= \frac{1}{2\sqrt{\Delta}} \left(e^{-2kT} H\left(1, \frac{k}{\sqrt{\Delta}}, \frac{k+\sqrt{\Delta}}{\sqrt{\Delta}}, \frac{1}{\Gamma}\right) - H\left(1, \frac{k}{\sqrt{\Delta}}, \frac{k+\sqrt{\Delta}}{\sqrt{\Delta}}, \frac{e^{2\sqrt{\Delta}T}}{\Gamma}\right) \right), \\
\int_0^T \left(\frac{\Gamma e^{-2\sqrt{\Delta}u}}{1-\Gamma e^{-2\sqrt{\Delta}u}} \right)^2 e^{-k(T-u)} du &= \frac{1}{2\sqrt{\Delta}} \left(\frac{\Gamma e^{-kT}}{1-\Gamma} - \frac{\Gamma e^{-2\sqrt{\Delta}T}}{1-\Gamma e^{-2\sqrt{\Delta}T}} \right) - \frac{(2\sqrt{\Delta}-k)}{2k\sqrt{\Delta}} \left(e^{-kT} H\left(1, \frac{k}{2\sqrt{\Delta}}, \frac{k+2\sqrt{\Delta}}{2\sqrt{\Delta}}, \frac{1}{\Gamma}\right) \right. \\
&\quad \left. - H\left(1, \frac{k}{2\sqrt{\Delta}}, \frac{k+2\sqrt{\Delta}}{2\sqrt{\Delta}}, \frac{e^{2\sqrt{\Delta}T}}{\Gamma}\right) \right), \\
\int_0^T \left(\frac{\Gamma e^{-2\sqrt{\Delta}u}}{1-\Gamma e^{-2\sqrt{\Delta}u}} \right)^2 e^{-2k(T-u)} du &= \frac{1}{2\sqrt{\Delta}} \left(\frac{\Gamma e^{-2kT}}{1-\Gamma} - \frac{\Gamma e^{-2\sqrt{\Delta}T}}{1-\Gamma e^{-2\sqrt{\Delta}T}} \right) - \frac{(\sqrt{\Delta}-k)}{2k\sqrt{\Delta}} \left(e^{-2kT} H\left(1, \frac{k}{\sqrt{\Delta}}, \frac{k+\sqrt{\Delta}}{\sqrt{\Delta}}, \frac{1}{\Gamma}\right) \right. \\
&\quad \left. - H\left(1, \frac{k}{\sqrt{\Delta}}, \frac{k+\sqrt{\Delta}}{\sqrt{\Delta}}, \frac{e^{2\sqrt{\Delta}T}}{\Gamma}\right) \right),
\end{aligned}$$

Moreover:

$$\frac{4\sigma_s^2(1-\rho_{sr}^2)\Delta}{k^2} \int_0^T \left[\frac{\Gamma e^{-2\sqrt{\Delta}u}}{(1-\Gamma e^{-2\sqrt{\Delta}u})^2} \right] du = \frac{2\sigma_s^2(1-\rho_{sr}^2)\sqrt{\Delta}}{k^2} \left(\frac{\Gamma}{1-\Gamma} - \frac{\Gamma e^{-2\sqrt{\Delta}T}}{1-\Gamma e^{-2\sqrt{\Delta}T}} \right).$$

By inserting the above hypergeometric functions and the last integral above into $Var[X(T)]$, then we have:

$$\begin{aligned}
Var[X(T)] &= \left[\sigma_s^2 + \frac{\sigma_r^2}{\alpha^2} + \frac{2\sigma_{sr}}{\alpha} - \frac{2\sigma_{r\delta}}{\alpha k} + \frac{\sigma_\delta^2}{k^2} - \frac{2\sigma_{s\delta}}{k} \right] T \\
&\quad - \left[\frac{\sigma_r^2}{\alpha^2} + \frac{2\sigma_{sr}}{\alpha} - \frac{2\sigma_{r\delta}}{\alpha(\alpha+k)} \right] D_\alpha(T) - \left[\frac{\sigma_\delta^2}{k^2} - \frac{2\sigma_{s\delta}}{k} - \frac{2\sigma_{r\delta}}{k(\alpha+k)} \right] D_k(T) \\
&\quad + \frac{2\sigma_{r\delta}}{\alpha+k} D_\alpha(T) D_k(T) - \frac{\sigma_r^2}{2\alpha} D_\alpha^2(T) - \left[\frac{\sigma_\delta^2}{2k} - \gamma \right] D_k^2(T)
\end{aligned} \tag{2.48}$$

But we have:

$$H(T) = E^Q[S(T)] = exp\left\{ SE^Q[X(T)] + \frac{1}{2} Var[X(T)] \right\} \tag{2.49}$$

where, $E^Q[X(T)]$ is the expectation of the stochastic variable $X(T)$ under Q .

By inserting (2.47) and (2.48) into (2.49), we have:

$$\begin{aligned}
H(T) &= S exp\left\{ \left[\beta - \bar{\delta} + \frac{\sigma_r^2}{2\alpha^2} + \frac{\sigma_{sr}}{\alpha} - \frac{\sigma_{r\delta}}{\alpha K} + \frac{\sigma_\delta^2}{2K^2} - \frac{\sigma_{s\delta}}{K} \right] T \right. \\
&\quad - \left[\beta - r + \frac{\sigma_r^2}{2\alpha^2} + \frac{\sigma_{sr}}{\alpha} - \frac{\sigma_{r\delta}}{\alpha(\alpha+K)} \right] D_\alpha(T) - \left[m - \bar{\delta} + \frac{\sigma_\delta^2}{2K^2} - \frac{\sigma_{s\delta}}{K} - \frac{\sigma_{r\delta}}{K(\alpha+K)} \right] D_k(T) \\
&\quad - \frac{\sigma_{r\delta}}{\alpha+k} D_\alpha(T) D_k(T) - \frac{\sigma_r^2}{4\alpha} D_\alpha^2(T) - \frac{1}{2} \left[\frac{\sigma_\delta^2}{2K} - \gamma \right] D_k^2(T) \\
&\quad \left. - \frac{\sigma_s(\lambda_s - \rho_{sr}\lambda_r)}{K} \left[\ln\left(\frac{1-\Gamma e^{-2\sqrt{\Delta}T}}{1-\Gamma} \right) - \frac{2\sqrt{\Delta}}{K} e^{KT} \sum_{n=1}^{\infty} \Gamma^n D_{2n\sqrt{\Delta}-K}(T) \right] \right\}
\end{aligned}$$

Where, $\beta = \beta - \frac{\lambda_r\sigma_r}{\alpha}$

The model, i.e incomplete information model, provides an analytical solution for commodity futures prices in a partially observable economy. Under incomplete information, the futures prices are functions of the price of risk associated with the spot price and the short rate, λ_s and λ_r respectively. The expectation of $S(T)$ depends on the initial estimate of the convenience yield. The uncertainty, γ , about the initial value of the convenience yield influences both the expectation and the variance of $S(T)$. The expectation of $S(T)$ may rise or fall as a consequence of the effect of γ and λ_s . We can easily observe from equation (2.40), γ influences the variance of $S(T)$ through the term, $-\left[\frac{\sigma_\delta^2}{2k} - \gamma\right]$. Indeed, from expression $-\left[\frac{\sigma_\delta^2}{2k} - \gamma\right]$ we can easily conclude that for reasonable and sufficiently low value of γ , the market sector (investor) is more confident about his/her initial estimator which implies that the difference $-\left[\frac{\sigma_\delta^2}{2k} - \gamma\right]$ is negative, and vice versa. If $\lambda_s > 0$, γ has an opposite impact on the expectation (positive covariance) and on the variance terms.

Chapter 3

MODEL DISCRETIZATION

3.1 Introduction

In this study, our contributions to the existing literature are three folds: First, we extend the Eduardo S. Schwartz(1997) three factor model by adding new feature, which is Vasicek interest rate process in Eduardo S. Schwartz(1997) model is replaced by mean reverting Cox-Ingersoll-Ross(CIR) process as described by Cox et al.(1985). This new feature prevents negative interest rate. Second, we provide numerical solution for reduced form three factor commodity derivative valuation model by using two known discretization techniques, i.e, Euler-Maruyama and Milstein discretization techniques. Third, we study the strong convergence between Euler-Maruyama and Milstein discretization methods.

3.2 Financial Market

Assume we have (Ω, F, P) complete probability space with a standard filtration $F = \{\mathcal{F}(t) : t \in [0, T]\}$, a finite time period $[0, T]$. Assume we have three stochastic processes i.e, the spot price process of the underlying commodity, S , the instantaneous convenience yield process, δ , and the instantaneous interest rate process, r as presented in Eduardo S. Schwartz(1997). First we discuss joint stochastic process for the two state variables i.e, the spot price process and the instantaneous convenience yield under the equivalent martingale measure can be expressed as:¹

$$\frac{dS(t)}{S(t)} = (\mu - \delta(t))dt + \sigma_s dZ_s(t) \quad (3.1)$$

$$d\delta(t) = \kappa(\alpha - \delta(t))dt + \sigma_\delta dZ_\delta(t) \quad (3.2)$$

with initial conditions $S(0) \equiv S_0$ and $\delta(0) \equiv \delta_0$. Two correlated standard Brownian motions Z_s and Z_δ such that, $dZ_s dZ_\delta = \rho dt$, here ρ^2 stands for correlation coefficient between the two Brownian motions. $\kappa > 0$ is the magnitude of the speed of adjustment of the long run

¹see Eduardo S. Schwartz(1997)

²Positive correlation between spot price and convenience yield is induced by the level of commodities: when inventories of the commodity decreases the spot price should increase since the commodity is scarce and the convenience yield should also increase since futures prices will not increase as much as the spot price, and vice versa(Carmona and Ludkovski(1991))

mean α , σ_s and σ_δ represents, respectively, constant, strictly positive, instantaneous standard deviation of the spot price and convenience yield.

Letting $X = \ln S^3$ and applying Ito's lemma to equation (3.1) gives:

$$dX = (\mu - \delta - \frac{1}{2}\sigma_s^2)dt + \sigma_s dZ_s(t) \quad (3.3)$$

The stochastic differential equations for the state variables spot price and convenience yield under equivalent martingale measure can be expressed as:⁴

$$\frac{dS(t)}{S(t)} = (r(t) - \delta(t))dt + \sigma_s dZ_s^*(t) \quad (3.4)$$

$$d\delta(t) = \left[\kappa(\alpha - \delta(t)) - \lambda \right] dt + \sigma_\delta dZ_\delta^*(t) \quad (3.5)$$

$dZ_s^* dZ_\delta^* = \rho dt$, λ is constant market price risk associated with convenience yield.

Interest rates have an impact on spot commodity prices and on convenience yields.⁵ The reduced form three factor valuation model can be expressed as follows by using equation (3.4), equation (3.5) and the CIR interest rate process:

$$\frac{dS(t)}{S(t)} = (r(t) - \delta(t))dt + \sigma_s dZ_s^*(t) \quad (3.6)$$

$$d\delta(t) = \kappa(\hat{\alpha} - \delta(t))dt + \sigma_\delta dZ_\delta^*(t) \quad (3.7)$$

$$dr(t) = a(m^* - r(t))dt + \sigma_r \sqrt{r(t)} dZ_r^*(t) \quad (3.8)$$

With initial conditions $S(0) \equiv S_0$, $\delta(0) \equiv \delta_0$ and $r(0) \equiv r_0$.

Where, $\hat{\alpha} = \alpha - \frac{\lambda}{\kappa}$, three correlated standard Brownian motions, $dZ_s^* dZ_\delta^* = \rho_1 dt$, $dZ_\delta^* dz_r^* = \rho_2 dt$ and $dZ_s^* dz_r^* = \rho_3 dt$. a is the speed of adjustment, m^* the risk adjusted mean short rate of the interest rate and σ_r is the constant, strictly positive, instantaneous standard deviation of interest rate, $r(t)$.

The SDE of the short rate follows a mean-reverting process as Cox-Ingersoll-Ross(CIR).⁶ If $2am^* > \sigma_r^2$, the CIR process is strictly positive, otherwise non-negative. Hence, the CIR interest rate model depicts the actual condition of the market where interest rate is non-negative unlike Vasicek interest rate model. The CIR model is mean reverting in nature. If the process deviates from the stationary mean level m^* , it is brought back to m^* at the rate of a .

Let $X_t = \ln S_t$, then from Ito's Lemma, we obtain the dynamics of the process X as follows:

$$dX_t = (r_t - \delta_t - \frac{1}{2}\sigma_s^2)dt + \sigma_s dZ_s^* \quad (3.9)$$

³It is convenient to introduce a special notation for logarithm of the spot price $X = \ln S$. Indeed, the model (3.1) is linear in the state vector $[X, \delta]$ (Carmona and Ludkovski(1991))

⁴ Eduardo S. Schwartz(1997) used this expression.

⁵Equilibrium models of commodity contingent claims assume that interest rates are zero or constant and do not study the relation between convenience yields and interest rates(Anh Ngoc Lai and Constantin Melios(2015))

⁶see John C. Cox, Jonathan E. Ingersoll, Jr. and Stephen A. Ross(1985)

Let the futures price at time t with maturity T is $F(t, T)$. Then we have:

$$F(t, T) = E[S_T | S_t] = E[e^{X_T} | X_t]$$

Where, E is the expectation take with respect to the risk neutral process.

To compute the expectation, the natural way is to find the transition density, $\mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)$, and applying Kolmogorov backward equation gives the expression for transition density as:

$$\begin{aligned} & \frac{\partial \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial t} + \left\{ r - \delta - \frac{1}{2} \sigma_s^2 \right\} \frac{\partial \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial X} \\ & \quad + \kappa(\hat{\alpha} - \delta) \frac{\partial \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial \delta} \\ & + a(m^* - r) \frac{\partial \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial r} + \frac{1}{2} \sigma_s^2 \frac{\partial^2 \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial X^2} \\ & \quad + \frac{1}{2} \sigma_\delta^2 \frac{\partial^2 \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial \delta^2} + \frac{1}{2} \sigma_r^2 r \frac{\partial^2 \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial r^2} \\ & + \rho_1 \sigma_s \sigma_\delta \frac{\partial^2 \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial X \partial \delta} + \rho_2 \sigma_\delta \sigma_r \sqrt{r} \frac{\partial^2 \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial \delta \partial r} \\ & \quad + \rho_3 \sigma_s \sigma_r \sqrt{r} \frac{\partial^2 \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial X \partial r} = 0 \end{aligned} \quad (3.10)$$

With terminal boundary condition:

$$\mathcal{P}(X_T, \delta_T, T | X_t, \delta_t, t = T) = \hat{\delta}(X_T - X_t, \delta_T - \delta_t, 0) \quad (3.11)$$

To get futures price's expression⁷, we multiply both sides by e^{X_t} and integrating with respect to X_t gives:

$$\begin{aligned} & \frac{\partial F(t, T)}{\partial t} + \left\{ r - \delta - \frac{1}{2} \sigma_s^2 \right\} \frac{\partial F(t, T)}{\partial X} + \kappa(\hat{\alpha} - \delta) \frac{\partial F(t, T)}{\partial \delta} + a(m^* - r) \frac{\partial F(t, T)}{\partial r} + \frac{1}{2} \sigma_s^2 \frac{\partial^2 F(t, T)}{\partial X^2} \\ & \quad + \frac{1}{2} \sigma_\delta^2 \frac{\partial^2 F(t, T)}{\partial \delta^2} + \frac{1}{2} \sigma_r^2 r \frac{\partial^2 F(t, T)}{\partial r^2} + \rho_1 \sigma_s \sigma_\delta \frac{\partial^2 F(t, T)}{\partial X \partial \delta} \\ & \quad + \rho_2 \sigma_\delta \sigma_r \sqrt{r} \frac{\partial^2 F(t, T)}{\partial \delta \partial r} + \rho_3 \sigma_s \sigma_r \sqrt{r} \frac{\partial^2 F(t, T)}{\partial X \partial r} = 0 \end{aligned} \quad (3.12)$$

Subject to boundary condition $F(t = T, T) = e^{X_T}$

Assuming the solution of the above Kolmogorov backward equation has an exponential affine form:

$$F(t, T) = e^{A_0(t) + A_1(t)X_t + A_2(t)\delta_t + A_3(t)r_t} \quad (3.13)$$

Since $F(t = T, T) = e^{X_T}$, then we have:

$$A_0(T) = 0, \quad A_1(T) = 1, \quad A_2(T) = 0 \text{ and } A_3(T) = 0$$

⁷see appendix 3

Differentiating equation (5.22) with respect to t , X_t , δ_t and r_t gives:

$$\frac{\partial F(t,T)}{\partial t} = \left(\frac{dA_0(t)}{dt} + X_t \frac{dA_1(t)}{dt} + \delta_t \frac{dA_2(t)}{dt} + r_t \frac{dA_3(t)}{dt} \right) F(t,T)$$

$$\frac{\partial F(t,T)}{\partial X} = A_1(t)F(t,T)$$

$$\frac{\partial^2 F(t,T)}{\partial X^2} = (A_1(t))^2 F(t,T)$$

$$\frac{\partial F(t,T)}{\partial \delta} = A_2(t)F(t,T)$$

$$\frac{\partial^2 F(t,T)}{\partial \delta^2} = (A_2(t))^2 F(t,T)$$

$$\frac{\partial F(t,T)}{\partial r} = A_3(t)F(t,T)$$

$$\frac{\partial^2 F(t,T)}{\partial r^2} = (A_3(t))^2 F(t,T)$$

$$\frac{\partial^2 F(t,T)}{\partial X \partial \delta} = A_1(t)A_2(t)F(t,T)$$

$$\frac{\partial^2 F(t,T)}{\partial X \partial r} = A_1(t)A_3(t)F(t,T)$$

$$\frac{\partial^2 F(t,T)}{\partial \delta \partial r} = A_2(t)A_3(t)F(t,T)$$

Equation (5.21) becomes:

$$\begin{aligned} & \left(\frac{dA_0(t)}{dt} + X_t \frac{dA_1(t)}{dt} + \delta_t \frac{dA_2(t)}{dt} + r_t \frac{dA_3(t)}{dt} \right) F(t,T) + \left\{ r - \delta - \frac{1}{2}\sigma_s^2 \right\} A_1(t)F(t,T) \\ & + \kappa(\hat{\alpha} - \delta)A_2(t)F(t,T) + a(m^* - r)A_3(t)F(t,T) + \frac{1}{2}\sigma_s^2(A_1(t))^2 F(t,T) \\ & + \frac{1}{2}\sigma_\delta^2(A_2(t))^2 F(t,T) + \frac{1}{2}\sigma_r^2 r(A_3(t))^2 F(t,T) \\ & + \rho_1\sigma_s\sigma_\delta A_1(t)A_2(t)F(t,T) + \rho_2\sigma_\delta\sigma_r\sqrt{r}A_2(t)A_3(t)F(t,T) + \rho_3\sigma_s\sigma_r\sqrt{r}A_1(t)A_3(t)F(t,T) = 0 \end{aligned}$$

Dividing both sides by $F(t,T)$ gives:

$$\begin{aligned} & \frac{dA_0(t)}{dt} + X_t \frac{dA_1(t)}{dt} + \delta_t \frac{dA_2(t)}{dt} + r_t \frac{dA_3(t)}{dt} + \left\{ r - \delta - \frac{1}{2}\sigma_s^2 \right\} A_1(t) + \kappa(\hat{\alpha} - \delta)A_2(t) + a(m^* - r)A_3(t) \\ & + \frac{1}{2}\sigma_s^2(A_1(t))^2 + \frac{1}{2}\sigma_\delta^2(A_2(t))^2 + \frac{1}{2}\sigma_r^2 r(A_3(t))^2 \\ & + \rho_1\sigma_s\sigma_\delta A_1(t)A_2(t) + \rho_2\sigma_\delta\sigma_r\sqrt{r}A_2(t)A_3(t) + \rho_3\sigma_s\sigma_r\sqrt{r}A_1(t)A_3(t) = 0 \\ & \implies \begin{cases} \frac{dA_1(t)}{dt} = 0 \\ \frac{dA_2(t)}{dt} - A_1(t) - \kappa A_2(t) = 0 \\ \frac{dA_3(t)}{dt} + A_1(t) - aA_3(t) = 0 \end{cases} \\ & \frac{dA_0(t)}{dt} - \frac{1}{2}\sigma_s^2 A_1(t) + \kappa\hat{\alpha}A_2(t) + am^*A_3(t) + \frac{1}{2}\sigma_s^2(A_1(t))^2 + \frac{1}{2}\sigma_\delta^2(A_2(t))^2 + \frac{1}{2}\sigma_r^2 r(A_3(t))^2 \\ & + \rho_1\sigma_s\sigma_\delta A_1(t)A_2(t) + \rho_2\sigma_\delta\sigma_r\sqrt{r}A_2(t)A_3(t) + \rho_3\sigma_s\sigma_r\sqrt{r}A_1(t)A_3(t) = 0 \end{aligned}$$

Since $A_1(T) = 1$ then we have $A_1(t) = 1$

$$\begin{aligned} & \Rightarrow \begin{cases} A_1(t) = 1 \\ \frac{dA_2(t)}{dt} - 1 - \kappa A_2(t) = 0 \\ \frac{dA_3(t)}{dt} + 1 - aA_3(t) = 0 \end{cases} \\ & \frac{dA_0(t)}{dt} - \frac{1}{2}\sigma_s^2 + \kappa\hat{\alpha}A_2(t) + am^*A_3(t) + \frac{1}{2}\sigma_s^2 + \frac{1}{2}\sigma_\delta^2(A_2(t))^2 + \frac{1}{2}\sigma_r^2r(A_3(t))^2 \\ & + \rho_1\sigma_s\sigma_\delta A_2(t) + \rho_2\sigma_\delta\sigma_r\sqrt{r}A_2(t)A_3(t) + \rho_3\sigma_s\sigma_r\sqrt{r}A_3(t) = 0 \\ & \Rightarrow \begin{cases} A_1(t) = 1 \\ \frac{dA_2(t)}{dt} = 1 + \kappa A_2(t) \\ \frac{dA_3(t)}{dt} = -1 + aA_3(t) \end{cases} \\ & \frac{dA_0(t)}{dt} = \frac{1}{2}\sigma_s^2 - \kappa\hat{\alpha}A_2(t) - am^*A_3(t) - \frac{1}{2}\sigma_s^2 - \frac{1}{2}\sigma_\delta^2(A_2(t))^2 - \frac{1}{2}\sigma_r^2r(A_3(t))^2 \\ & - \rho_1\sigma_s\sigma_\delta A_2(t) - \rho_2\sigma_\delta\sigma_r\sqrt{r}A_2(t)A_3(t) - \rho_3\sigma_s\sigma_r\sqrt{r}A_3(t) \\ & \Rightarrow \begin{cases} A_1(t) = 1 \\ A_2(t) = \frac{e^{\kappa(t-T)} - 1}{\kappa} \\ A_3(t) = \frac{1 - e^{a(t-T)}}{a} \end{cases} \end{aligned}$$

After substituting and straightforward calculations we have:

$$\begin{aligned} & \frac{dA_0(t)}{dt} = \frac{\rho_2\sigma_\delta\sigma_r\sqrt{r}}{\kappa a} e^{(\kappa+a)(t-T)} - \frac{\sigma_\delta^2}{2\kappa^2} e^{2\kappa(t-T)} - \frac{\sigma_r^2 r}{2a^2} e^{2a(t-T)} + \left(\frac{\sigma_\delta^2}{\kappa^2} - \hat{\alpha} - \frac{\rho_1\sigma_s\sigma_\delta}{\kappa} - \frac{\rho_2\sigma_\delta\sigma_r\sqrt{r}}{\kappa a} \right) e^{\kappa(t-T)} \\ & + \left(m^* + \frac{\sigma_r^2 r}{a^2} - \frac{\rho_2\sigma_\delta\sigma_r\sqrt{r}}{\kappa a} - \frac{\rho_3\sigma_s\sigma_r\sqrt{r}}{a} \right) e^{a(t-T)} + \hat{\alpha} - m^* - \frac{\sigma_\delta^2}{2\kappa^2} - \frac{\sigma_\delta^2}{2\kappa^2} - \frac{\sigma_r^2 r}{2a^2} + \frac{\rho_2\sigma_\delta\sigma_r\sqrt{r}}{\kappa a} - \frac{\rho_3\sigma_s\sigma_r\sqrt{r}}{a} \end{aligned}$$

From above expression of $\frac{dA_0(t)}{dt}$ we can simply observe that the expression is dependent of the process r hence we can say that unlike Eduardo S. Schwartz(1997) three factor model there is no closed form solution for reduced form three factor valuation model or the model is not explicitly solvable. This happens because unlike Eduardo S. Schwartz(1997) three factor model in our model the interest rate process is CIR process and in nature CIR process does not have closed form solution⁸ and the spot price process, S , in reduced form three factor valuation model, is related with explicitly unsolvable CIR interest rate, r , process. Due to this reason we use numerical solution methods for our model. By choosing a sufficiently small length of time interval, Δt , Milstein and Euler discretization scheme can be used to generate discrete observations of a continuous-time system.

⁸The CIR process is not explicitly solvable, hence the tractability of the CIR model is not as good as the Vasicek model in this regard, Anqi Shao(2012)

3.3 Discretization of reduced form of the three factor commodity derivative valuation model

3.3.1 Euler discretization scheme

Using an Euler discretization to simulate CIR process gives rise to the problem that while the process itself is guaranteed to be non negative, the discretization is not. General schemes, such as the Euler scheme or the Milstein scheme are in general not well defined because they can lead to negative values for which the square root is not defined (Anoi Shao(2012)). To tackle this problem in our simulation we use Diop's⁹ approach which is 'reflection scheme' taking the norm of the discretization¹⁰. Therefore the straightforward¹¹ Euler discretization scheme for valuation model is given by,

$$S_t = S_{t-1} + (\mu - \delta_{t-1})S_{t-1}\Delta t + \sigma_s S_{t-1}\sqrt{\Delta t}n_{S,t-1} \quad (3.14)$$

$$\delta_t = \delta_{t-1} + \kappa(\alpha - \delta_{t-1})\Delta t + \sigma_\delta \rho_1 \sqrt{\Delta t}n_{S,t-1} + \sigma_\delta \sqrt{1 - \rho_1^2} \sqrt{\Delta t}n_{\delta,t-1} \quad (3.15)$$

$$r_t = r_{t-1} + a(m - r_{t-1})\Delta t + \sigma_r \rho_3 \sqrt{r_{t-1}} \sqrt{\Delta t}n_{S,t-1} + \sigma_r \sqrt{r_{t-1}} \sqrt{1 - \rho_3^2} \sqrt{\Delta t}n_{r,t-1} \quad (3.16)$$

Where $n_{S,t-1}$, $n_{\delta,t-1}$ and $n_{r,t-1}$ are independent and identically distributed standard normal random variables.

3.3.2 Milstein discretization scheme

Milstein discretization¹² scheme is given by,

$$S_t = S_{t-1} + (\mu - \delta_{t-1})S_{t-1}\Delta t + \sigma_s S_{t-1}\sqrt{\Delta t}n_{S,t-1} + \frac{1}{2}\sigma_s^2 S_{t-1}(\Delta t^2 n_{S,t-1} - \Delta t) \quad (3.17)$$

$$\delta_t = \delta_{t-1} + \kappa(\alpha - \delta_{t-1})\Delta t + \sigma_\delta \rho_1 \sqrt{\Delta t}n_{S,t-1} + \sigma_\delta \sqrt{1 - \rho_1^2} \sqrt{\Delta t}n_{\delta,t-1} \quad (3.18)$$

$$r_t = r_{t-1} + a(m - r_{t-1})\Delta t + \sigma_r \rho_3 \sqrt{r_{t-1}} \sqrt{\Delta t}n_{S,t-1} + \sigma_r \sqrt{r_{t-1}} \sqrt{1 - \rho_3^2} \sqrt{\Delta t}n_{r,t-1} \quad (3.19)$$

Where $n_{S,t-1}$, $n_{\delta,t-1}$ and $n_{r,t-1}$ are as stated above.

⁹refer Berkaoui A., Bossy M. and Diop A.(2008)

¹⁰refer Aurélien Alfonsi(2005)

¹¹see Appendix 1

¹²refer Ola Elerian(1998)

3.4 Simulation of valuation model

Simulation of valuation model in the context of option pricing refers to a set of techniques to generate underlying values typically stock prices, convenience yield or interest rate over time. The dynamics of these stock price, convenience yield and interest rate are assumed to be driven by a continuous-time stochastic process. Simulation, however, is done at discrete time steps. Hence, the first step in any simulation scheme is to find a way to discretized a continuous-time process into a discrete time process.

To simulate reduced form of three factor valuation model we use Euler and Milstein discretization representations listed above of the model with different time intervals. For both discretization techniques, we use the same final time interval $[0,1]$. For each discretization schemes¹³, we choose $\Delta t = 10^{-3}$ and use 10^3 simulation paths and we choose $\Delta t = 10^{-1}$ and use 10^2 paths.

¹³The discretization alternatives with either the first-order Euler's approximation or the Milstein's approximation formats introduce discretization errors into the simulation and have higher computational cost because need small Δt .

Chapter 4

Empirical studies of the reduced form of the three factor commodity derivative valuation model

4.1 Simulation of valuation model

4.1.1 Milstein discretization simulation for T=1

We choose the following parameters to generate the trajectories in Milstein discretization scheme for valuation model.

Table 4.1: Parameters of valuation model for Milstein and Euler discretization scheme

σ_s	σ_δ	σ_r	ρ_1	ρ_2	ρ_3	κ	$\hat{\alpha}$	a	m^*
0.25	0.15	0.1	0.24	0.3	0.08	0.3	1	0.18	0.76

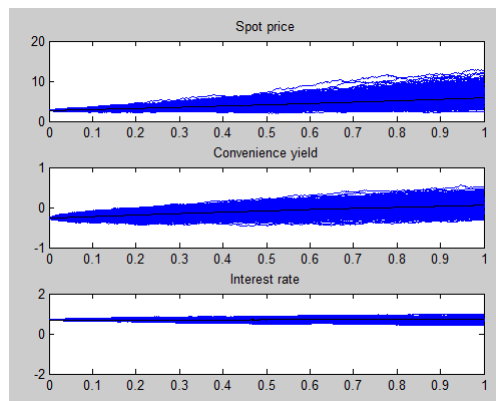


Figure 4.1: Milstein discretization simulation for reduced form three factor valuation model with $\Delta t = 10^{-3}$ and 10^3 simulation paths

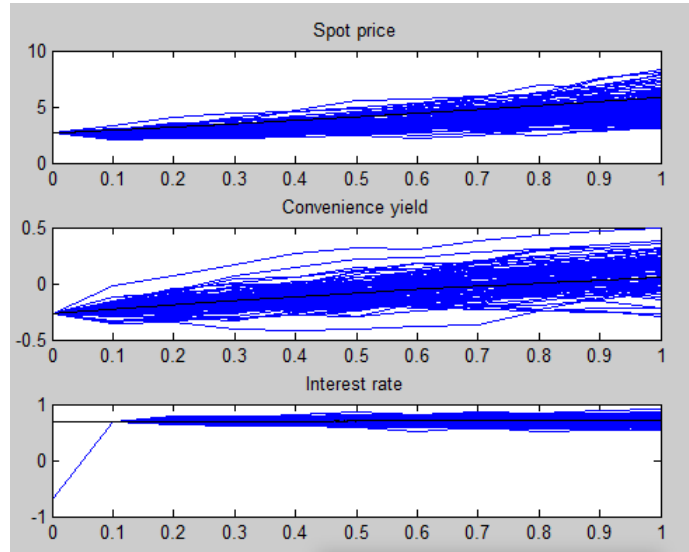


Figure 4.2: Milstein discretization simulation for reduced form three factor valuation model with $\Delta t = 10^{-1}$ and 10^2 simulation paths

4.1.2 Euler discretization simulation for $T=1$

We choose the same parameters as Milstein discretization scheme listed above.

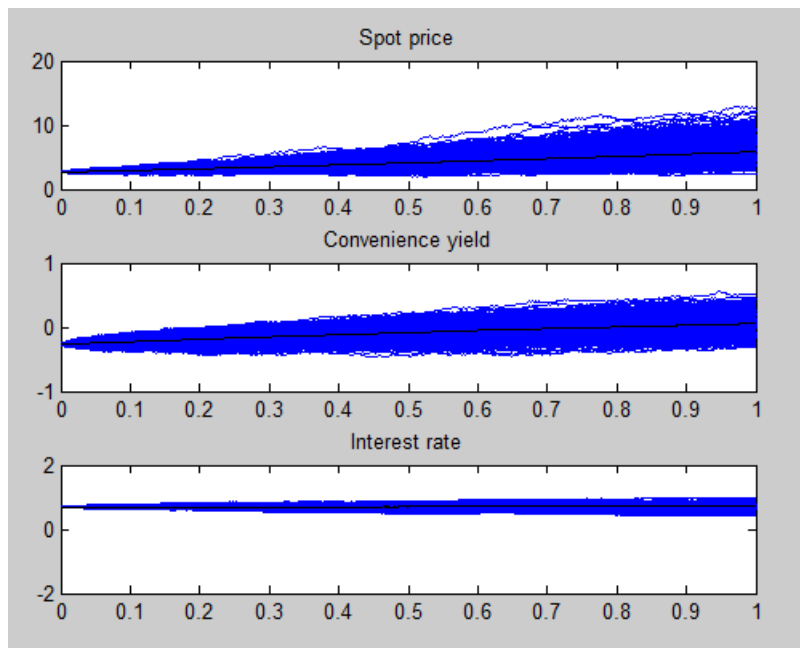


Figure 4.3: Euler discretization simulation for reduced form three factor valuation model with $\Delta t = 10^{-3}$ and 10^3 simulation paths

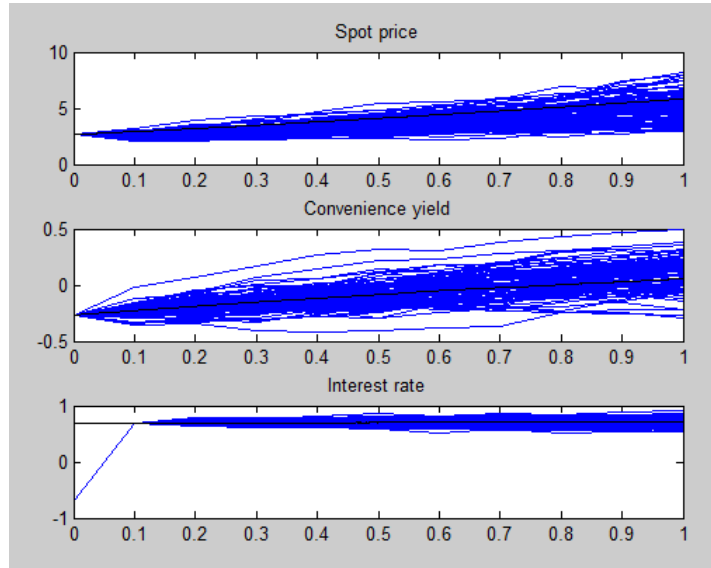


Figure 4.4: Euler discretization simulation for reduced form three factor valuation model with $\Delta t = 10^{-1}$ and 10^2 simulation paths

For above figures (4.1), (4.2), (4.3) and (4.4) lines indicated by blue represents the simulation paths and the line indicated by black represents the true mean of the factors spot price, S , convenience yield, δ and interest rate, r .

We can easily noted that the mean of 10^3 simulation paths using small $\Delta t = 0.001$ is appears to be closer to the true mean than the mean of 10^2 simulation paths using large $\Delta t = 0.01$ for Milstein and Euler discretization scheme. We can conclude that as the time interval(Δt) decreases and as we use more simulation paths the simulation result for both Milstein and Euler schemes achieves better approximation.

Since in our model the diffusion coefficients in the spot price process, S , and interest rate process, r , unlike Eduardo S. Schwartz(1997) three factor model, are not constant(they include process S and r respectively), the Milstein scheme and Euler scheme generates different results for spot price and interest rate so that we can easily distinguish Milstein and Euler scheme discretization.

4.1.3 Simulation results for both Milstein and Euler schemes as maturity expands

Euler discretization simulation for $T=5$

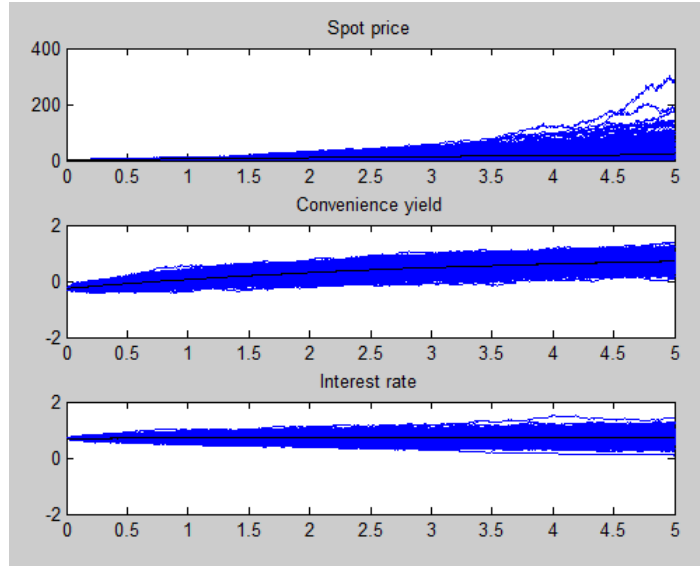


Figure 4.5: Euler discretization simulation for reduced form three factor valuation model with $\Delta t = 10^{-3}$ and 10^3 simulation paths

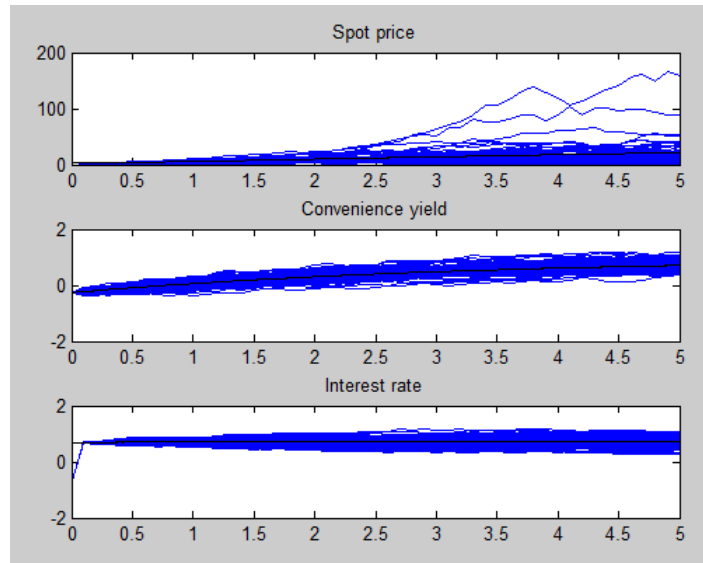


Figure 4.6: Euler discretization simulation for reduced form three factor valuation model with $\Delta t = 10^{-1}$ and 10^2 simulation paths

Milstein discretization simulation for $T=5$

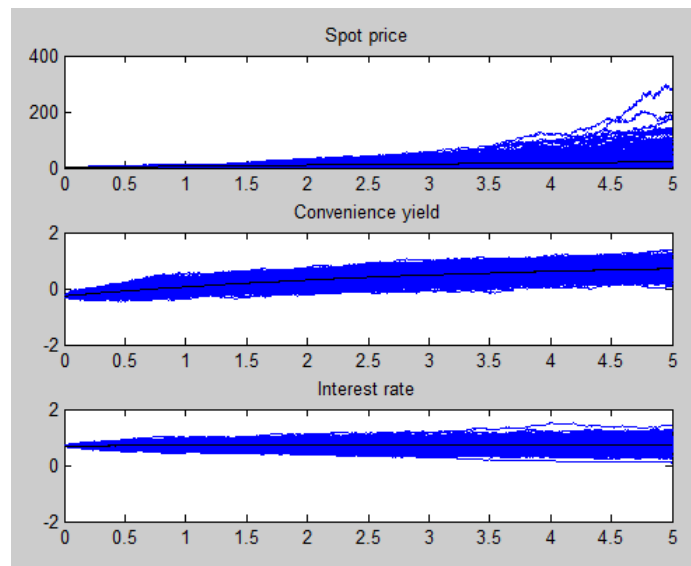


Figure 4.7: Milstein discretization simulation for reduced form three factor valuation model with $\Delta t = 10^{-3}$ and 10^3 simulation paths

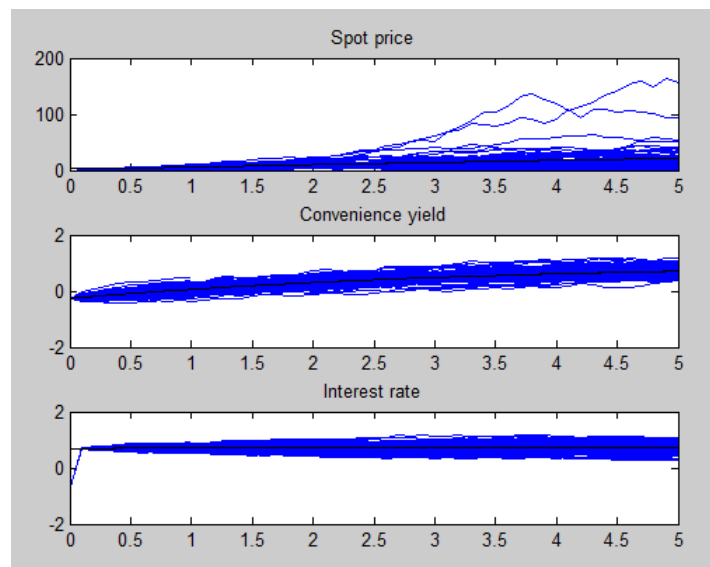


Figure 4.8: Milstein discretization simulation for reduced form three factor valuation model with $\Delta t = 10^{-1}$ and 10^2 simulation paths

Simulations for maturity, $T=25$

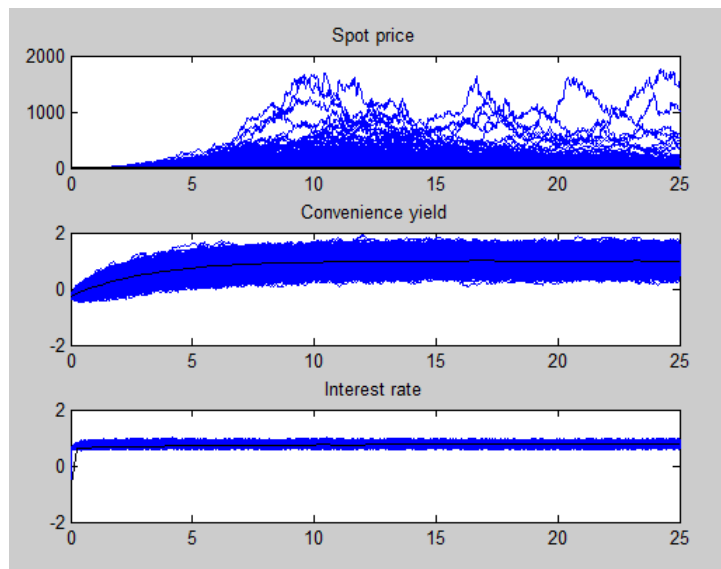


Figure 4.9: Simulation for reduced form three factor valuation model with $\Delta t = 0.025$ and 10^4 simulation paths

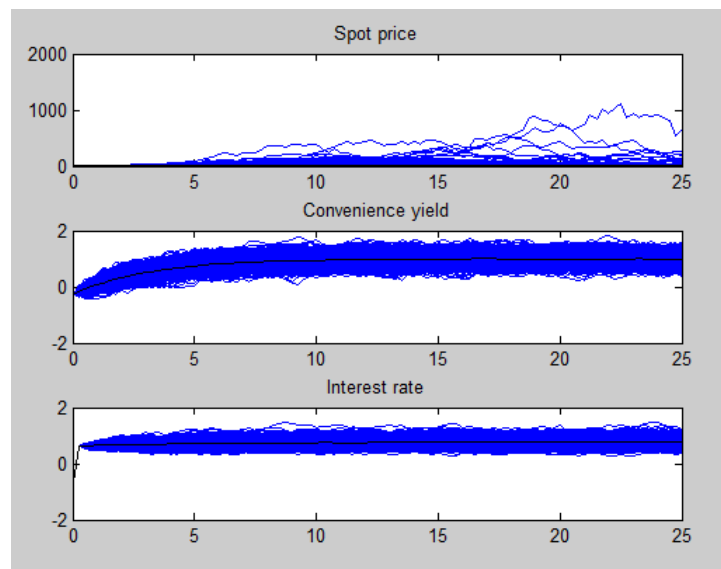


Figure 4.10: Simulation for reduced form three factor valuation model with $\Delta t = 0.25$ and 10^3 simulation paths

From above simulation results i.e from figure (4.5) to figure (4.10), we can observe that as maturity, T , increases, there appear more uncertainty in both Euler and Milstein schemes simulation results. These leads to a less accuracy in the simulations obtained by both Milstein and Euler schemes. So to get accurate results from both Milstein and Euler simulations one needs to have smaller maturity and also have smaller discrete time interval. When we observe above simulations for interest rate we can guarantee that the interest rate is always positive this happened due to unlike Eduardo S. Schwartz(1997) three factor model the reduced form of the three factor commodity derivative valuation model used CIR process. From numerical results listed in tables below we can conclude that Milstein scheme has better approximation than Euler scheme for reduced form of the three factor commodity derivative valuation model as discrete time interval and maturity get smaller.

The first ten numerical results of the simulation is listed in table below.

Table 4.2: Simulation results of Milstein scheme for the first ten simulation paths for $\Delta t = 10^{-5}$ and $T=1$

t	S	δ	r	True mean of S	True mean of δ	True mean of r	Ab. error in S	Ab. error in δ	Ab. error in r
0	2.658897758	-0.265611268	0.69674544	2.658897758	-0.265611268	0.69674544	0	0	0
1×10^{-5}	2.659807581	-0.265724802	0.696868195	2.658971699	-0.26563021	0.696763447	0.000835882	0.063735992530	0.000104748
2×10^{-5}	2.659391766	-0.265976129	0.697051102	2.659104144	-0.265618158	0.696735733	0.000287622	0.000357971	0.000315369
3×10^{-5}	2.659731848	-0.266181455	0.697104729	2.658997712	-0.265607887	0.696738081	0.000734135	0.000573568	0.000366649
4×10^{-5}	2.658860786	-0.266142717	0.697045966	2.658904728	-0.265602877	0.696747529	0.02960781932	0.00053984	0.000298438
5×10^{-5}	2.654995665	-0.266367026	0.697017383	2.658974969	-0.265628119	0.696749456	0.003979304	0.000738907	0.000267927
6×10^{-5}	2.652938128	-0.266872281	0.696629212	2.659107973	-0.265578497	0.696762256	0.006169845	0.001293784	0.000133044
7×10^{-5}	2.655396543	-0.266828903	0.697016206	2.659141004	-0.265584732	0.696760074	0.00374446	0.001244171	0.000256132
8×10^{-5}	2.653494379	-0.266616827	0.697019088	2.658894072	-0.265621916	0.696759831	0.005399692	0.000994912	0.000259256
9×10^{-5}	2.651565607	-0.266719849	0.697117397	2.658925909	-0.265581912	0.696783244	0.007360302	0.001137937	0.000334153

Table 4.3: Simulation results of Milstein scheme for the first ten simulation paths for $\Delta t = 0.1$ and $T=25$

t	S	δ	r	True mean of S	True mean of δ	True mean of r	Ab. error in S	Ab. error in δ	Ab. error in r
0	2.658897758	-0.265611268	0.69674544	2.658897758	-0.265611268	0.69674544	0	9.99201×10^{-16}	3.9968×10^{-15}
0.1	2.8764687	-0.209982172	0.694430604	2.91942026	-0.229401124	0.696887722	0.04295156	0.019418952	0.002457119
0.2	3.032216596	-0.226570142	0.711084713	3.181260525	-0.196434354	0.698539136	0.149043929	0.030135788	0.012545576
0.3	3.330546662	-0.279468072	0.677933193	3.480092229	-0.158283892	0.701295389	0.149545567	0.12118418	0.023362196
0.4	3.562232922	-0.189911653	0.68833876	3.763128822	-0.124788895	0.702809739	0.2008959	0.065122758	0.014470979
0.5	4.647714807	-0.1607325	0.730843446	4.079599242	-0.090808489	0.702431296	0.568115565	0.069924012	0.02841215
0.6	5.686869322	-0.029122165	0.759842598	4.4167477	-0.058013842	0.703391898	1.270121622	0.028891676	0.0564507
0.7	5.926514706	0.032340895	0.703273418	4.80783989	-0.026132385	0.702349253	1.118674817	0.05847328	0.000924165
0.8	6.504869172	0.133565821	0.720889024	5.183422184	0.007771102	0.706396021	1.321446988	0.125794719	0.014493003
0.9	7.031248273	0.125421182	0.745278398	5.568159396	0.039803189	0.70564094	1.463088877	0.085617993	0.039637458

Table 4.4: Simulation results of Euler scheme for the first ten simulation paths for $\Delta t = 10^{-5}$ and $T=1$

t	S	δ	r	True mean of S	True mean of δ	True mean of r	Ab. error in S	Ab. error in δ	Ab. error in r
0	2.658897758	-0.265611268	0.69674544	2.658897758	-0.265611268	0.69674544	0	9.99201×10^{-16}	3.9968×10^{-15}
1×10^{-5}	2.659808265	-0.265724802	0.696868195	2.658971694	-0.26563021	0.696763447	0.000836571	9.45926×10^{-5}	0.000104748
2×10^{-5}	2.659393244	-0.265976129	0.697051102	2.659104076	-0.265618158	0.696735733	0.000289168	0.000357971	0.000315369
3×10^{-5}	2.659734138	-0.266181455	0.697104729	2.658997676	-0.265607887	0.696738081	0.000736462	0.000573568	0.000366649
4×10^{-5}	2.658863756	-0.266142717	0.697045966	2.658904687	-0.265602877	0.696747529	4.0931×10^{-5}	0.00053984	0.000298438
5×10^{-5}	2.654996613	-0.266367026	0.697017383	2.658974905	-0.265628119	0.696749456	0.003978293	0.000738907	0.000267927
6×10^{-5}	2.652939087	-0.266872281	0.696629212	2.659107939	-0.265578497	0.696762256	0.006168852	0.001293784	0.000133044
7×10^{-5}	2.655397217	-0.266828903	0.697016206	2.659141004	-0.265584732	0.696760074	0.003743787	0.001244171	0.000256132
8×10^{-5}	2.653495183	-0.266616827	0.697019088	2.658894077	-0.265621916	0.696759831	0.005398894	0.000994912	0.000259256
9×10^{-5}	2.651566519	-0.266719849	0.697117397	2.658925931	-0.265581912	0.696783244	0.007359411	0.001137937	0.000334153

Table 4.5: Simulation results of Euler scheme for the first ten simulation paths for $\Delta t = 0.1$ and $T=25$

t	S	δ	r	True mean of S	True mean of δ	True mean of r	Ab. error in S	Ab. error in δ	Ab. error in r
0	2.658897758	-0.265611268	0.69674544	2.658897758	-0.265611268	0.69674544	0	9.99201×10^{-16}	3.9968×10^{-15}
0.1	2.884606566	-0.209982172	0.694430604	2.919148458	-0.229401124	0.696887722	0.034541893	0.019418952	0.002457119
0.2	3.048167591	-0.226570142	0.711084713	3.181450973	-0.196434354	0.698539136	0.133283382	0.030135788	0.012545576
0.3	3.357501797	-0.279468072	0.677933193	3.479675777	-0.158283892	0.701295389	0.12217398	0.12118418	0.023362196
0.4	3.600642175	-0.189911653	0.68833876	3.763370411	-0.124788895	0.702809739	0.162728236	0.065122758	0.014470979
0.5	4.637055586	-0.1607325	0.730843446	4.079502817	-0.090808489	0.702431296	0.557552769	0.069924012	0.02841215
0.6	5.649612639	-0.029122165	0.759842598	4.417450614	-0.058013842	0.703391898	1.232162025	0.028891676	0.0564507
0.7	5.902035818	0.032340895	0.703273418	4.808412366	-0.026132385	0.702349253	1.093623452	0.05847328	0.000924165
0.8	6.49321752	0.133565821	0.720889024	5.184825199	0.007771102	0.706396021	1.308392321	0.125794719	0.014493003
0.9	7.036915783	0.125421182	0.745278398	5.569230982	0.039803189	0.70564094	1.467684801	0.085617993	0.039637458

To observe the difference between Milstein and Euler discretization schemes we can easily observe the equations given for both discretization schemes (i.e, equations from (3.14) to (3.19)). Milstein discretization adds additional terms in discretization of spot price process¹. Therefore, we can observe very slight difference in simulation results of both discretization schemes. To observe this difference it's enough to see above tables (table (4.2) to table (4.5)).

¹see equations (3.14) and (3.17)

4.1.4 Simulation results for spot price, convenience yield and interest rate

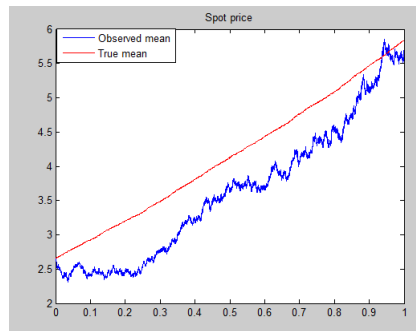


Figure 4.11: Simulation of spot price for $\Delta t = 10^{-5}$ and 10^3 simulation paths, $T=1$

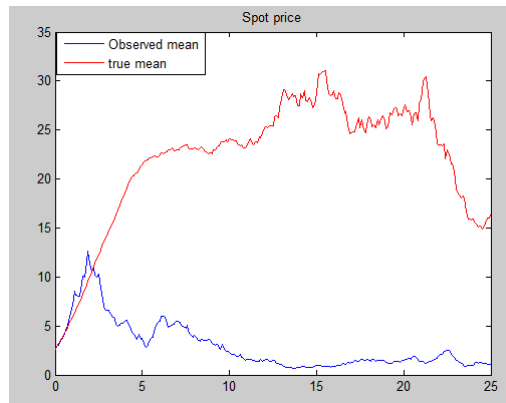


Figure 4.12: Simulation of spot price for $\Delta t = 0.1$ and 10^3 simulation paths, $T=25$

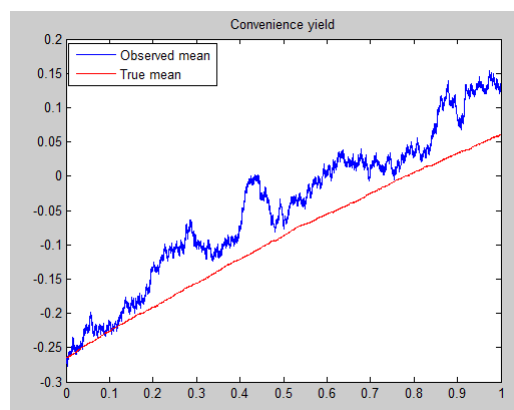


Figure 4.13: Simulation of convenience yield for $\Delta t = 10^{-5}$ and 10^3 simulation paths, $T=1$

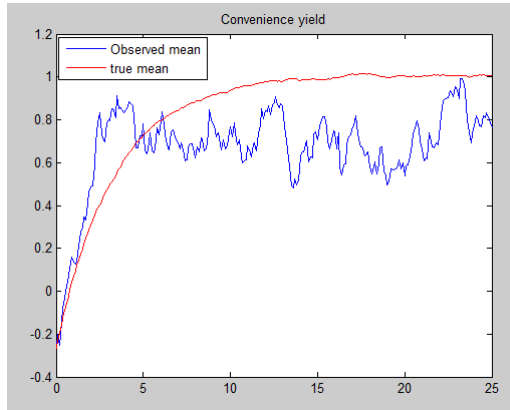


Figure 4.14: Simulation of convenience yield for $\Delta t = 0.1$ and 10^3 simulation paths, $T=25$

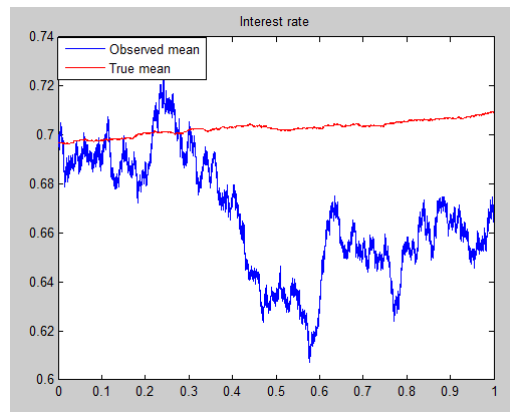


Figure 4.15: Simulation of interest rate for $\Delta t = 10^{-5}$ and 10^3 simulation paths, $T=1$

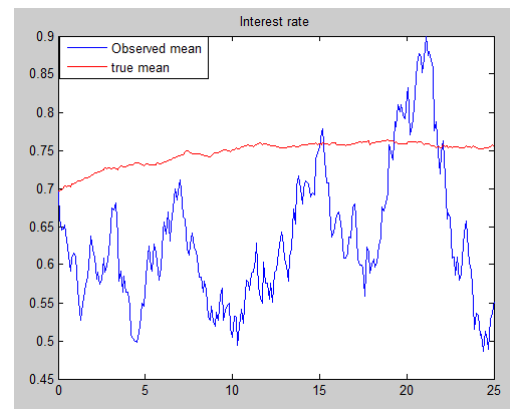


Figure 4.16: Simulation of interest rate for $\Delta t = 0.1$ and 10^3 simulation paths, $T=25$

In general, from table (4.2) to table (4.5) and from figure (4.11) to figure (4.16), for spot price, convenience yield and interest rate, we can easily observe that as time of maturity, T , expands and the time interval, Δt also increases, the results in simulation leads to uncertainty. Hence the accuracy of the simulation will be less. To get the best approximation for spot price, convenience yield and interest rate in reduced form three factor valuation model one can use less time of maturity, less time interval and more simulation paths for both Milstein and Euler dscretization schemes.

Chapter 5

Discussions, Conclusions and Recommendations

5.1 Discussions

From the beginning, this study was proposed to find analytical solution for the reduced form of three factor valuation model like Eduardo S.Schwartz(1997) three factor commodity derivative model. But, unlike Eduardo S. Schwartz(1997) three factor commodity derivative model we used CIR interest rate process as third factor and according to the nature of CIR interest rate process, CIR interest rate process has no closed form solution(see SDE (3.8)). Hence, due to the influence of CIR interest rate process to reduced form of the three factor commodity derivative valuation model, the proposed reduced form of three factor commodity derivative valuation model has no closed form solution. Due to this reason we focused on simulating the proposed model to find numerical solution for the joint stochastic differential equations¹ by using the two most known techniques, Milstein and Euler discretization techniques to find accurate approximation for reduced form of the three factor commodity derivative valuation model. According to our simulation results for both Euler and Milstein discretization schemes we used different maturity, $T(T=1, 5 \text{ and } 25)$ and different discrete time interval, $\Delta t(\Delta = 10^{-1}, 10^{-3}, 10^{-5} \text{ and } 0.25)$. From figure (4.1) to figure (4.4) we used the same maturity, $T=1$ but different discrete time interval, $\Delta t(\Delta t = 10^{-1} \text{ and } 10^{-3})$ to observe the best approximation results while decreasing discrete time interval for both discretization schemes. From figure (4.5) to figure (4.8) we used the same maturity, $T=5$ and different discrete time interval, $\Delta t(\Delta t = 10^{-1} \text{ and } \Delta t = 10^{-3})$. Hence from those simulation results we find as maturity, T , and discrete time interval, Δt decreases the simulation result shows best approximation in both discretization schemes. As we can easily observe from table (4.2) to table (4.5) we find Milstein discretization technique in reduced form of the three factor commodity derivative valuation model has the best approximation than Euler discretization technique. As maturity, T , and discrete time interval, Δt , get bigger and bigger we find uncertainty in all simulation results.

The main objective of this study was to build a suitable reduced form of three factor valuation model like Eduardo S. Schwartz(1997) three factor model by adding a new feature , which is Vasicek interest rate process in Eduardo S. Schwartz(1997) model was replaced by CIR interest

¹see SDEs (3.6), (3.7) and (3.8)

rate process to keep the interest rate always positive. After developing reduced form of three factor valuation model we simulated the proposed model according to the two discretization schemes to find numerical solution(see from figure (4.1) to figure (4.4)).

The second objective was to study the performance of both Milstein and Euler discretization schemes in reduced form of three factor valuation model. We examined the Milstein and Euler schemes in terms of their performance to the true process. To understand the difference between the two discretization schemes, we might observe at the results given by each discretization schemes as time interval gets smaller. If we want to improve the accuracy of the simulation for both Milstein and Euler schemes, then we must reduce the time discretization step and use more simulation paths for the proposed three factor model. It's seen that from equations given in both discretization schemes, Milstein scheme is an extension of Euler by simply adding one term to spot price process, the Milstein scheme has more accurate approximation than Euler scheme, to understand this more it's better to observe the numerical results given in table (4.2) to table (4.5).

5.2 Conclusions

From the financial market point of view, it can be concluded that the proposed reduced form of three factor valuation model is better than the one proposed in Eduardo S. Schwartz(1997) model, Since Vasicek interest rate process in Eduardo S. Schwartz(1997) model has no guarantee from being negative unlike CIR interest rate process in valuation model. In addition to this, from simulation results of Milstein and Euler discretization schemes, we can conclude that Milstein scheme has better approximation than Euler scheme in reduced form of three factor valuation model.

5.3 Recommendations

From this study it would be recommended that to compare the two basic known discretization schemes i.e, Milstein and Euler, one can use different approaches like strong and weak convergence approaches or average mean square error approach. In average mean square error approach the standard deviation might be taken to judge the accuracy of the Milstein and Euler schemes using reduced form of three factor valuation model.

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Appendix 1

Euler discretization scheme

The simplest way to discretize continuous time process is the use of Euler discretization scheme(Akinbo B.J, Faniran T and Ayoola E.O, 2015).

Recall,

Assume we have stock price process S_t^2

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t \quad (5.1)$$

To simulate S_t over the final time interval, $[0, T]$ one can discretize a final time interval $[0, T]$ into N subintervals, i.e,

$$0 = t_1 < t_2 < \dots < t_n = T. \quad n = 0, 1, \dots, N$$

Setting the discrete time interval, $\Delta t = \frac{T}{N}$,

Integrating the above process from t to $t + dt$ gives,

$$S_{t+dt} = S_t + \int_t^{t+dt} \mu(S_u, u)du + \int_t^{t+dt} \sigma(S_u, u)dW_u \quad (5.2)$$

To find the value of S_{t+dt} at $t + dt$ knowing the value of S_t at t the Euler discretization scheme can be applied to equation (5.2) and gives general form of Euler scheme as follows,

$$S_{t+dt} = S_t + \mu(S_t, t)dt + \sigma(S_t, t)\sqrt{dt}Z \quad \text{where } Z \text{ is standard normal random variable.}$$

Now we can use Euler scheme above to discretize valuation model as follows,

Recasting valuation model with respect to independent wiener processes dz_s , dz_δ and dz_r like above gives,

$$dS_t = (\mu - \delta_t)S_t dt + \sigma_s S_t dz_s \quad (5.3)$$

$$d\delta_t = \kappa(\alpha - \delta_t)dt + \sigma_\delta(\rho_2 dz_s + \sqrt{1 - \rho_1^2} dz_\delta) \quad (5.4)$$

$$dr_t = a(m - r_t)dt + \sigma_r \sqrt{r_t}(\rho_3 dz_s + \sqrt{1 - \rho_3^2} dz_r) \quad (5.5)$$

Then from above equation we can have Euler discretization for valuation model.

$$S_t = S_{t-1} + (\mu - \delta_{t-1})S_{t-1}\Delta t + \sigma_s S_{t-1}\sqrt{\Delta t}z_{nS, t-1} \quad (5.6)$$

²refer Akinbo B.J, Faniran T and Ayoola E.O, 2015

$$\delta_t = \delta_{t-1} + \kappa(\alpha - \delta_{t-1})\Delta t + \sigma_\delta \rho_1 \sqrt{\Delta t} n_{S,t-1} + \sigma_\delta \sqrt{1 - \rho_1^2} \sqrt{\Delta t} n_{\delta,t-1} \quad (5.7)$$

$$r_t = r_{t-1} + a(m - r_{t-1})\Delta t + \sigma_r \rho_3 \sqrt{r_{t-1}} \sqrt{\Delta t} n_{S,t-1} + \sigma_r \sqrt{r_{t-1}} \sqrt{1 - \rho_3^2} \sqrt{\Delta t} n_{r,t-1} \quad (5.8)$$

Where $n_{S,t-1}$, $n_{\delta,t-1}$ and $n_{r,t-1}$ are independent and identically distributed standard normal random variables.

Appendix 2

Milstein discretization scheme

Recasting three factor valuation model with respect to independent wiener processes dz_s , dz_δ and dz_r like above gives,

$$dS_t = (\mu - \delta_t)S_t dt + \sigma_s S_t dz_s \quad (5.9)$$

$$d\delta_t = \kappa(\alpha - \delta_t)dt + \sigma_\delta(\rho_2 dz_s + \sqrt{1 - \rho_1^2} dz_\delta) \quad (5.10)$$

$$dr_t = a(m - r_t)dt + \sigma_r \sqrt{r_t}(\rho_3 dz_s + \sqrt{1 - \rho_3^2} dz_r) \quad (5.11)$$

Then from above equation we can have Milstein discretization for valuation model.

$$S_t = S_{t-1} + (\mu - \delta_{t-1})S_{t-1}\Delta t + \sigma_s S_{t-1}\sqrt{\Delta t}n_{s,t-1} + \frac{1}{2}\sigma_s^2 S_{t-1}(\Delta t n_{s,t-1}^2 - \Delta t) \quad (5.12)$$

$$\delta_t = \delta_{t-1} + \kappa(\alpha - \delta_{t-1})\Delta t + \sigma_\delta \rho_1 \sqrt{\Delta t}n_{s,t-1} + \sigma_\delta \sqrt{1 - \rho_1^2} \sqrt{\Delta t}n_{\delta,t-1} \quad (5.13)$$

$$r_t = r_{t-1} + a(m - r_{t-1})\Delta t + \sigma_r \rho_3 \sqrt{r_{t-1}} \sqrt{\Delta t}n_{s,t-1} + \sigma_r \sqrt{r_{t-1}} \sqrt{1 - \rho_3^2} \sqrt{\Delta t}n_{r,t-1} \quad (5.14)$$

Where $n_{s,t-1}$, $n_{\delta,t-1}$ and $n_{r,t-1}$ are independent and identically distributed standard normal random variables.

Appendix 3

Schwartz(1997)³ three factor model is given by,

$$\frac{dS(t)}{S(t)} = (r(t) - \delta(t))dt + \sigma_s dZ_s^*(t) \quad (5.15)$$

$$d\delta(t) = \kappa(\hat{\alpha} - \delta(t))dt + \sigma_\delta dZ_\delta^*(t) \quad (5.16)$$

$$dr(t) = a(m^* - r(t))dt + \sigma_r dZ_r^*(t) \quad (5.17)$$

With initial conditions $S(0) \equiv S_0$, $\delta(0) \equiv \delta_0$ and $r(0) \equiv r_0$.

Where, $\hat{\alpha} = \alpha - \frac{\lambda}{\kappa}$, three correlated standard Brownian motions, $dZ_s^* dZ_\delta^* = \rho_1 dt$, $dZ_\delta^* dZ_r^* = \rho_2 dt$ and $dZ_s^* dZ_r^* = \rho_3 dt$. a is the speed of adjustment, m^* the risk adjusted mean short rate of the interest rate and σ_r is the constant, strictly positive, instantaneous standard deviation of interest rate, $r(t)$.

To find futures price of the three factor model, let $X_t = \ln S_t$, then from Ito's Lemma , we obtain the dynamics of the process X as follows:

$$dX_t = (r_t - \delta_t - \frac{1}{2}\sigma_s^2)dt + \sigma_s dZ_s^* \quad (5.18)$$

Let the futures price at time t with maturity T is $F(t, T)$. Then we have:

$$F(t, T) = E[S_T | S_t] = E[e^{X_T} | X_t]$$

Where, E is the expectation take with respect to the risk neutral process.

To compute the expectation, the natural way is to find the transition density, $\mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)$, and applying Kolmogorov backward equation gives the expression for transition density as:

³refer Schwartz(1997)

$$\begin{aligned}
& \frac{\partial \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial t} + \left\{ r - \delta - \frac{1}{2} \sigma_s^2 \right\} \frac{\partial \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial X} \\
& \quad + \kappa(\hat{\alpha} - \delta) \frac{\partial \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial \delta} \\
& + a(m^* - r) \frac{\partial \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial r} + \frac{1}{2} \sigma_s^2 \frac{\partial^2 \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial X^2} \\
& \quad + \frac{1}{2} \sigma_\delta^2 \frac{\partial^2 \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial \delta^2} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial r^2} \\
& + \rho_1 \sigma_s \sigma_\delta \frac{\partial^2 \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial X \partial \delta} + \rho_2 \sigma_\delta \sigma_r \frac{\partial^2 \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial \delta \partial r} \\
& \quad + \rho_3 \sigma_s \sigma_r \frac{\partial^2 \mathcal{P}(X_T, \delta_T, r_T, T | X_t, \delta_t, r_t, t)}{\partial X \partial r} = 0
\end{aligned} \tag{5.19}$$

With terminal boundary condition:

$$\mathcal{P}(X_T, \delta_T, T | X_t, \delta_t, t = T) = \hat{\delta}(X_T - X_t, \delta_T - \delta_t, 0) \tag{5.20}$$

To get futures price's expression, we multiply both sides by e^{X_t} and integrating with respect to X_t gives:

$$\begin{aligned}
& \frac{\partial F(t, T)}{\partial t} + \left\{ r - \delta - \frac{1}{2} \sigma_s^2 \right\} \frac{\partial F(t, T)}{\partial X} + \kappa(\hat{\alpha} - \delta) \frac{\partial F(t, T)}{\partial \delta} + a(m^* - r) \frac{\partial F(t, T)}{\partial r} + \frac{1}{2} \sigma_s^2 \frac{\partial^2 F(t, T)}{\partial X^2} \\
& \quad + \frac{1}{2} \sigma_\delta^2 \frac{\partial^2 F(t, T)}{\partial \delta^2} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 F(t, T)}{\partial r^2} + \rho_1 \sigma_s \sigma_\delta \frac{\partial^2 F(t, T)}{\partial X \partial \delta} \\
& \quad \quad + \rho_2 \sigma_\delta \sigma_r \frac{\partial^2 F(t, T)}{\partial \delta \partial r} + \rho_3 \sigma_s \sigma_r \frac{\partial^2 F(t, T)}{\partial X \partial r} = 0
\end{aligned} \tag{5.21}$$

Subject to boundary condition $F(t = T, T) = e^{X_T}$

Assuming the solution of the above Kolmogorov backward equation has an exponential affine form:

$$F(t, T) = e^{A_0(t) + A_1(t)X_t + A_2(t)\delta_t + A_3(t)r_t} \tag{5.22}$$

Since $F(t = T, T) = e^{X_T}$, then we have:

$$A_0(T) = 0, \quad A_1(T) = 1, \quad A_2(T) = 0 \text{ and } A_3(T) = 0$$

Differentiating equation (5.22) with respect to t , X_t , δ_t and r_t gives:

$$\frac{\partial F(t, T)}{\partial t} = \left(\frac{dA_0(t)}{dt} + X_t \frac{dA_1(t)}{dt} + \delta_t \frac{dA_2(t)}{dt} + r_t \frac{dA_3(t)}{dt} \right) F(t, T)$$

$$\frac{\partial F(t, T)}{\partial X} = A_1(t) F(t, T)$$

$$\frac{\partial^2 F(t, T)}{\partial X^2} = (A_1(t))^2 F(t, T)$$

$$\frac{\partial F(t,T)}{\partial \delta} = A_2(t)F(t,T)$$

$$\frac{\partial^2 F(t,T)}{\partial \delta^2} = (A_2(t))^2 F(t,T)$$

$$\frac{\partial F(t,T)}{\partial r} = A_3(t)F(t,T)$$

$$\frac{\partial^2 F(t,T)}{\partial r^2} = (A_3(t))^2 F(t,T)$$

$$\frac{\partial^2 F(t,T)}{\partial X \partial \delta} = A_1(t)A_2(t)F(t,T)$$

$$\frac{\partial^2 F(t,T)}{\partial X \partial r} = A_1(t)A_3(t)F(t,T)$$

$$\frac{\partial^2 F(t,T)}{\partial \delta \partial r} = A_2(t)A_3(t)F(t,T)$$

Equation (5.21) becomes:

$$\begin{aligned} & \left(\frac{dA_0(t)}{dt} + X_t \frac{dA_1(t)}{dt} + \delta_t \frac{dA_2(t)}{dt} + r_t \frac{dA_3(t)}{dt} \right) F(t,T) + \left\{ r - \delta - \frac{1}{2}\sigma_s^2 \right\} A_1(t)F(t,T) \\ & + \kappa(\hat{\alpha} - \delta)A_2(t)F(t,T) + a(m^* - r)A_3(t)F(t,T) + \frac{1}{2}\sigma_s^2(A_1(t))^2 F(t,T) \\ & + \frac{1}{2}\sigma_\delta^2(A_2(t))^2 F(t,T) + \frac{1}{2}\sigma_r^2(A_3(t))^2 F(t,T) \\ & + \rho_1\sigma_s\sigma_\delta A_1(t)A_2(t)F(t,T) + \rho_2\sigma_\delta\sigma_r A_2(t)A_3(t)F(t,T) + \rho_3\sigma_s\sigma_r A_1(t)A_3(t)F(t,T) = 0 \end{aligned}$$

Dividing both sides by $F(t,T)$ gives:

$$\begin{aligned} & \frac{dA_0(t)}{dt} + X_t \frac{dA_1(t)}{dt} + \delta_t \frac{dA_2(t)}{dt} + r_t \frac{dA_3(t)}{dt} + \left\{ r - \delta - \frac{1}{2}\sigma_s^2 \right\} A_1(t) + \kappa(\hat{\alpha} - \delta)A_2(t) + a(m^* - r)A_3(t) \\ & + \frac{1}{2}\sigma_s^2(A_1(t))^2 + \frac{1}{2}\sigma_\delta^2(A_2(t))^2 + \frac{1}{2}\sigma_r^2(A_3(t))^2 \\ & + \rho_1\sigma_s\sigma_\delta A_1(t)A_2(t) + \rho_2\sigma_\delta\sigma_r A_2(t)A_3(t) + \rho_3\sigma_s\sigma_r A_1(t)A_3(t) = 0 \end{aligned}$$

$$\implies \begin{cases} \frac{dA_1(t)}{dt} = 0 \\ \frac{dA_2(t)}{dt} - A_1(t) - \kappa A_2(t) = 0 \\ \frac{dA_3(t)}{dt} + A_1(t) - aA_3(t) = 0 \end{cases}$$

and

$$\begin{aligned} & \frac{dA_0(t)}{dt} - \frac{1}{2}\sigma_s^2 A_1(t) + \kappa\hat{\alpha}A_2(t) + am^*A_3(t) + \frac{1}{2}\sigma_s^2(A_1(t))^2 + \frac{1}{2}\sigma_\delta^2(A_2(t))^2 + \frac{1}{2}\sigma_r^2(A_3(t))^2 \\ & + \rho_1\sigma_s\sigma_\delta A_1(t)A_2(t) + \rho_2\sigma_\delta\sigma_r A_2(t)A_3(t) + \rho_3\sigma_s\sigma_r A_1(t)A_3(t) = 0 \end{aligned}$$

Since $A_1(T) = 1$ then we have $A_1(t) = 1$

$$\implies \begin{cases} A_1(t) = 1 \\ \frac{dA_2(t)}{dt} - 1 - \kappa A_2(t) = 0 \\ \frac{dA_3(t)}{dt} + 1 - aA_3(t) = 0 \end{cases}$$

$$\begin{aligned} & \frac{dA_0(t)}{dt} - \frac{1}{2}\sigma_s^2 + \kappa\hat{\alpha}A_2(t) + am^*A_3(t) + \frac{1}{2}\sigma_s^2 + \frac{1}{2}\sigma_\delta^2(A_2(t))^2 + \frac{1}{2}\sigma_r^2(A_3(t))^2 \\ & + \rho_1\sigma_s\sigma_\delta A_2(t) + \rho_2\sigma_\delta\sigma_r A_2(t)A_3(t) + \rho_3\sigma_s\sigma_r A_3(t) = 0 \end{aligned}$$

$$\begin{aligned} &\implies \begin{cases} A_1(t) = 1 \\ \frac{dA_2(t)}{dt} = 1 + \kappa A_2(t) \\ \frac{dA_3(t)}{dt} = -1 + aA_3(t) \end{cases} \\ \frac{dA_0(t)}{dt} &= \frac{1}{2}\sigma_s^2 - \kappa\hat{\alpha}A_2(t) - am^*A_3(t) - \frac{1}{2}\sigma_s^2 - \frac{1}{2}\sigma_\delta^2(A_2(t))^2 - \frac{1}{2}\sigma_r^2(A_3(t))^2 \\ &- \rho_1\sigma_s\sigma_\delta A_2(t) - \rho_2\sigma_\delta\sigma_r A_2(t)A_3(t) - \rho_3\sigma_s\sigma_r A_3(t) \\ &\implies \begin{cases} A_1(t) = 1 \\ A_2(t) = \frac{e^{\kappa(t-T)} - 1}{\kappa} \\ A_3(t) = \frac{1 - e^{a(t-T)}}{a} \end{cases} \end{aligned}$$

After some substitutions and straightforward calculations we will have the expression for futures prices for the above Schwartz(1997) three factor valuation model as:

$$\begin{aligned} F(S, \delta, r, T) &= S \exp \left[\frac{\delta(1-e^{-\kappa T})}{\kappa} + \frac{r(1-e^{-aT})}{a} + \frac{(\kappa\hat{\alpha} + \rho_1\sigma_s\sigma_\delta)((1-e^{-\kappa T}) - \kappa T)}{\kappa^2} - \frac{\sigma_\delta^2(4(1-e^{-\kappa T}) - (1-e^{-2\kappa T}) - 2\kappa T)}{4\kappa^3} \right. \\ &- \frac{(am^* + \rho_3\sigma_s\sigma_r)((1-e^{-aT}) - aT)}{a^2} - \frac{\sigma_r^2(4(1-e^{-aT}) - (1-e^{-2aT}) - 2aT)}{4a^3} + \rho_2\sigma_\delta\sigma_r \left(\frac{(1-e^{-\kappa T}) + (1-e^{-aT}) - (1-e^{-(\kappa+a)T})}{\kappa a(\kappa+a)} \right) \\ &\left. + \frac{\kappa^2(1-e^{-aT}) + a^2(1-e^{-\kappa T}) - \kappa a^2 T - a\kappa^2 T}{\kappa^2 a^2 (\kappa+a)} \right] \end{aligned}$$

Appendix 4

State space formulation of Schwartz(1997) one factor model

The measurement equation of one factor valuation model can be expressed using equation (2.25),

$$Z_t = W_t x_t + d_t + \varepsilon_t \tag{5.23}$$

$$t = 1, \dots, NT$$

Where,

$Z_t = [\ln F_{T_1}, \dots, \ln F_{T_N}]$ is a $N \times 1$ vector of observed(log) futures prices with maturities T_1, \dots, T_N ;

$$d_t = \left[(1 - e^{-\kappa T_1})a^* + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa T_1}), \dots, (1 - e^{-\kappa T_N})a^* + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa T_N}) \right] N \times 1 \text{ vector};$$

$W_t = [e^{-\kappa T_1}, \dots, e^{-\kappa T_N}]$ is a $N \times 1$ matrix; and ε_t is a $N \times 1$ vector of serially uncorrelated, normally distributed disturbances (this vector is introduced to account for actual errors in the data) with:

$$E[\varepsilon_t] = 0 \text{ and}$$

$$\text{cov}[\varepsilon_t] = H_t \text{ (} H_t \text{ here is the covariance matrix)}$$

The transition equation of Schwartz(1997) one factor model can be expressed using equation (2.23),

$$x_t = T_t x_{t-1} + c_t + R_t \eta_t \tag{5.24}$$

$$t = 1, \dots, NT$$

Where;

$x_t = [X_t]$ is a 1×1 vector of the state variable;

$T_t = [1 - \kappa \Delta t]$ is a 1×1 matrix;

$c_t = [\kappa \alpha \Delta t]$ is a 1×1 vector;

R_t is a 1×1 identity matrix;

η_t is a 1×1 vector of serially uncorrelated, normally distributed disturbances with;

$E[\eta_t] = 0$ and

$var[\eta_t] = Q_t = var[X_t] = [\sigma^2 \Delta t]$

Now, we can observe that the observations and state equation matrices, W_t , d_t , H_t , T_t , c_t and Q_t only depend on unknown parameters of the given model.

Kalman filter

One of the important aim of the Kalman filter implementation is to attain estimates for unknown parameters. This can be done by maximizing the quasi likelihood function with respect to the unknown parameters through an optimization activity. Once a model is put in a state space form which is specified by equations (5.23) and (5.24), the Kalman filter emerges as an efficient tool for computing the optimal estimator of the state vector at time t , based on the observations up to and including time t . When the disturbances and the initial state vector are normally distributed, the Kalman filter enables the likelihood function to be measured through the prediction error decomposition, and then this allows the estimation of any unknown parameters in the model.

Derivation of Kalman filter for valuation model

Let a_{t-1} denote the mean of x_{t-1} conditional on the observations up to and including time $t - 1$, i.e

$a_{t-1} = E[x_{t-1}|z_{1:t-1}]$. The initial state vector, x_0 , has a multivariate normal distribution with mean a_0 and covariance matrix P_0 . The disturbances ε_t and η_t also have multivariate normal distribution for $t = 1, \dots, T$ and are distributed independently of each other and of x_0 . The state vector at time $t = 1$ is given by;

$$x_1 = T_1 x_0 + c_1 + R_1 \eta_1 \quad (5.25)$$

Here, we can see from above expression, x is a linear combination of two vectors of random variables, x_0 and η_1 both with multivariate normal distributions, and a vector of constants, c_1 . Hence, x_1 is itself multivariate normal with a mean of;

$$a_1|_0 = T_1 a_0 + c_1 \quad (5.26)$$

and a covariance matrix

$$P_1|_0 = T_1 P_0 T_1' + R_1 Q_1 R_1' \quad (5.27)$$

$a_1|_0$ stands for the mean of the distribution of x_1 conditional on the information at time $t = 0$.

To obtain the distribution of x_1 conditional on z_1 ,

$$x_1 = a_1|_0 + (x_1 - a_1|_0) \quad (5.28)$$

$$z_1 = W_1 a_1|_0 + d_1 + W_1(x_1 - a_1|_0) + \varepsilon_t \quad (5.29)$$

We can easily see that the vector $[x_1' \ z_1']'$ has a multivariate normal distribution with a mean of;

$$\begin{bmatrix} a_1|_0 & (W_1 a_1|_0 + d_1)' \end{bmatrix}'$$

and a covariance matrix

$$cov[x_1' \ z_1']' = \begin{bmatrix} P_1|_0 & P_1|_0 W_1' \\ W_1 P_1|_0 & W_1 P_1|_0 W_1' + H_1 \end{bmatrix}$$

Now the distribution of x_1 , conditional on a specific value of z_1 , is multivariate normal with mean;

$$a_1 = a_1|_0 + P_1|_0 W_1' F_1^{-1} (z_1 - W_1 a_1|_0 - d_1) \quad (5.30)$$

and the covariance matrix;

$$P_1 = P_1|_0 - P_1|_0 W_1' F_1^{-1} W_1 P_1|_0 \quad (5.31)$$

Where; $F_1 = W_1 P_1|_0 W_1' + H_1$, we assume the inverse of F exists.

Repeating the procedure discussed above for $k = 2, \dots, T$, we will get the prediction and updating recursions for the Kalman filter.

For general form of Kalman filter, we consider two recursions of Kalman filter, namely, prediction and updating.

The two prediction equations are;

$$a_t|_{t-1} = T_t a_{t-1} + c_t \quad (5.32)$$

$$P_t|_{t-1} = T_t P_{t-1} T_t' + R_t Q_t R_t' \quad (5.33)$$

Where; $a_t|_{t-1}$ is the mean of the distribution of x_t conditional on the information up to time $t - 1$, and $P_t|_{t-1}$ is the covariance matrix of the estimation error.

Whenever the new observation, z_t , is available, the estimator of x_t which is $a_t|_{t-1}$, can be updated.

Hence the two updating equations are therefore;

$$a_t = a_t|_{t-1} + P_t|_{t-1} W_t' F_t^{-1} (z_t - W_t a_t|_{t-1} - d_t) \quad (5.34)$$

$$P_t = P_t|_{t-1} - P_t|_{t-1} W_t' F_t^{-1} W_t P_t|_{t-1} \quad (5.35)$$

Where $F_t = W_t P_t|_{t-1} W_t' + H_t$

Taking equations (5.32), (5.33), (5.34) and (5.35) together forms the Kalman filter.

The initial values for the Kalman filter may be specified in terms of a_0 and P_0 , or $a_1|_0$ and $P_1|_0$. Having these initial conditions, the Kalman filter then gives the optimal estimator of the state vector once the new observation becomes available. Peculiarly, when all T observations have processed, the Kalman filter then generates the optimal estimator of the current state vector, based on the full information we have. This estimator contains all information required to make optimal predictions of future values of both the state and the observations.