MODELLING THE IMPACT OF INTEREST RATE FINANCIAL CRISIS

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DECLARATION

This Research Thesis is my original work and has not been presented for a degree in any other University.

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DEDICATION

This work is deidcated to the Almighty God. Secondly, to my lovely parents Mr. and Mrs Mwithalii for their financial support and moral encouragement they have given me to pursue my career. They taught me the value of education and hard work.

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ABRREVIATIONS

CDS-Credit Default Swap

CIR-Cox–Ingersoll–Ross

EMU-European Monetary Union.

EONIA - Euro Over Night Index Average

EU-European Union

EURIBOR -Euro Interbank Offered Rate

FRA -Forward Rate Agreement

FX - Foreign Exchange

HJM-Heath–Jarrow–Morton

IRS-Interest rate swap

ISDA-International Swaps and Derivatives Association

LIBOR - London Interbank Offered Rate

OIS - Overnight Indexed Swap

OTC - Over the Counter

SDE - Stochastic Differential Equation

ABSTRACT

Financial crisis of 2007-2008 changed the assumptions underlying market models theoretically and in practice. Therefore, this research focuses on modelling the impact of interest rate financial crisis. The aim was to describe pricing of financial derivatives mainly interest rate swap pricing method used under both pre and post crisis period. To model riskless rate/short rate and credit risk factors in the interbank market as well as price a defaultable zero coupon bond with credit adjustment. Riskless rate was modelled by use of a no arbitrage model CIR++ model while default intensity simulation was modelled following a Monte-Carlo method using stochastic processes (CIR) for parameterization. In simulating short rate exact method of simulation was used by assuming that CIR++ model increments follow a non- central chi-square distribution and model parameters estimated by use of maximum likelihood method. Both pre and post crisis period parameters were estimated and compared to analyse the impact of the crisis.In modelling credit risk reduced form modelling approach was used and CIR model used to model default intensity. For default intensity simulation Euler scheme method was used for discretization.

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CHAPTER 1

INTRODUCTION

1.1 Background Information

The pre- crisis interest rate models are no longer adequate in modern context .The problem of developing new interest rate models has attracted significant attention from researchers working in financial institutions and also in academia following the financial crisis that begun in the second half of 2007 . Researchers are focusing on new techniques needed for valuation of interest rate derivatives. The importance of up to date models can be well explained when viewing the fixed income market as a part of the global derivatives market.

The term fixed-income market describes a sector of the global financial market on which various interest rate-sensitive instruments, such as bonds, forward rate agreements, swaps, swaptions, caps /floors are traded. Zero coupon bonds are the simplest fixed-income products, which deliver a constant payment (often set to one unit of cash for simplicity) at a pre-specified future time referred to as maturity(Grbac and Runggaldier, 2015). Nevertheless, their value at any time before maturity depends on the stochastic fluctuation of interest rates which is also true for other fixed-income derivatives. Fixed-income instruments signify the largest portion of the global financial market larger than even equities. Therefore developing realistic and analytically tractable models for the dynamics of the term structure of interest rates is of great benefit for the financial industry(Grbac and Runggaldier, 2015).

According to the yearly statistics provided by the bank for International Settlements, the notional amounts outstanding each year for over-the-counter (OTC) interest rate derivatives add up to 80 % of the total trade volume in OTC derivatives that is ($505 trillion out of the total volume of $630 trillion corresponded to interest rate derivatives in 2014 (Grbac and Runggaldier, 2015). The financial crisis affected all fixed-income markets hence prompting researchers to come up with better models that will be applicable in both pre and post crisis times. The crisis triggered a profound evolution of classical frameworks that were adopted for trading derivative assets.

Precisely the credit and liquidity issues were assumed to have minute impacts on the prices of financial instruments, both plain vanillas and exotics. Today, whether there is crisis or not, the market has learnt the lesson and persistently shows such effects. These are clearly visible in the market quotes of plain vanilla interest rate derivatives, such as Deposits, forward rate agreements (FRA), Swaps (IRS) and options (Caps, Floors and Swaptions) (Bianchetti and Carlicchi, 2012).

Since August 2007 the primary interest rates of the interbank market, for example Libor , Euribor, Eonia, and Federal Funds rate show huge basis spreads that have raised up to 200 basis points. The crisis led to market divergence between Libor and OIS rates, FRA and forward rates, the explosion of basis swaps spreads, as well as the diffusion of collateral agreements in terms of credit and liquidity effects. The implied market frictions has prompted a market segmentation of the interest rate market into sub-areas, corresponding to instruments with risky underlying Libor rates distinct by tenors, and risk free overnight rates, and those that are characterized, in principle, by different internal dynamics, liquidity and credit risk premia reflecting the different views and preferences of the market players(Bianchetti and Carlicchi, 2012). Comparable divergences are also found between FRA rates and the forward rates implied by two consecutive Deposits, and among swap rates with different floating legs. Lately, the market has also included the effect of collateral agreements widely diffused among derivatives counterparties in the interbank market(Bianchetti and Carlicchi, 2012).

The standard no arbitrage frameworks adopted to price derivatives that were developed over forty years following the Copernican Revolution of the Black and Scholes (1973) and Merton (1973a)have become obsolete after the market evolution. The ultimate idea of the construction of a single risk free yield curve, reflecting at the same time the present cost of funding of future cash flows and the level of forward rates, has been ruled out. Financial experts have therefore been forced to start the development of a new theoretical framework, and to review from scratch the no-arbitrage models used on the market for derivatives' pricing and risk analysis. This is because the initial approach did not take into account market information displayed by the basis swap spreads which became huge during the crisis period and could no longer be ignored. It also did not take into account that interest rate market had become segmented into sub areas according to different underlying tenors, that were characterized by different dynamics for example short rate process(Morini, 2009) .

Old frameworks are referred to as Classical while new frameworks are referred to as modern to mark the shift of paradigm induced by the crisis(Bianchetti and Carlicchi, 2012). The classical models did not account for credit and liquidity risks hence can no longer be applied in post crisis period.

1.2 Problem Statement

The credit crisis of 2007–2008 and the Eurozone sovereign debt crisis in 2009– 2012 have had a great impact to all financial markets and have permanently changed the way in which the market functioned in practice. This has also changed the way in which their theoretical models were developed, therefore making it easier for one to distinguish between a pre-crisis and a post-crisis setting. There are some key features that were put forward by the crisis which include counterparty risk, which is the risk of a counterparty failing to fulfil its obligations in a financial contract, and liquidity or funding risk, which is the risk of excessive costs of funding which occurs in a financial contract due to lack of liquidity in the market. These issues have raised a concern in the fixed income market.

The reason being that, the underlying interest rates for most fixed-income instruments are the market rates such as Libor or Euribor rates and the manner in which the market quotes for these rates are constructed reflects both the counterparty and the liquidity risk of the interbank market. Review of the quoted prices for related instruments reveals that the relationships between Libor rates of different maturities that were previously considered standard, and held reasonably well before the crisis, have broken down and presently each of the rates needs to be modelled as a separate thing. Consequently, major spreads are also observed between Libor/Euribor rates and the swap rates based on the overnight indexed swaps (OIS), which were following each other closely before the crisis.

Therefore by carrying out this research, we hope to show the term structure of short rate, how to price risk factors that were put foward by the crisis in the pricing framework that will not only be consistent with interest rate market practices, but also applicable to both pre and post crisis periods. We ought to incorporate credit risk factor which was considered negligible in the pre- crisis period and how it is priced in the interbank market as a premia for taking risk. The multi curve modelling practice is considered as the best interest rate modelling approach that factors in the market aspects that were not reflected in the pre- crisis period approaches.

1.3 Justification of the study

Interest rate modelling is one of the fundamental areas in finance. This is because of its great benefits in financial institutions specifically in risk management, financial reporting and its importance in forecasting the future investment returns. Therefore, developing better frameworks for risk modelling is of huge benefit in financial industry because it will minimize the risk of financial market crash.

The notable consequences of the credit crunch crisis started in August 2007. This turmoil in financial markets crisis forced market participants to re-evaluate financial instruments market risks. The emergence of the re-evaluation of market risks is mainly represented by the presence of “tenor and FX basis risks”. Tenor dependence refers to re pricing periods associated with the floating instruments, especially the floating side of interest rate swaps, while the FX basis refers to the relation in different currencies. Statistical evidence of financial market data reveals the persisting relevance of tenor and FX basis risks. It is highly unlikely that markets will return to previous conditions. Accordingly financial institutions are exposed to increased risk of higher volatility in profit

& loss of their financial statements resulting from derivatives and other financial instruments (Schubert, 2012).

These facts from financial markets push financial institutions to adopt to new valuation methodologies for the pricing and risk assessment of financial instruments, which is commonly summarized by the term "multi-curve valuation models" (Schubert, 2012) .Thus our approach to multiple curves modelling and credit pricing will be of importance for the purpose of pricing interest rate derivatives, hedging and/or risk management in financial institutions. This will also contribute to the on-going efforts in financial institutions to extend the new valuation models to financial accounting as well as aligning the financial reporting with economic risk assessment of hedging activities. Multi curve models takes into account the collateralization of over-the-counter (OTC) derivatives as well as additional risk factor for example tenor basis spreads as a result of different crediot levels attached to different tenors in order to derive market consistent prices for derivatives.

Before the crisis differences in pricings according to the tenor were considered negligible. While during the financial crisis the differences in pricing according to the tenor became more and more significant and have to be factored into pricing models accordingly (Schubert, 2012). As a result of crisis different tenors bear different credit levels and therefore, the study will be of benefit to the financial world for example banks when pricing derivatives with different tenors according to the current market trends and practices. Pricing of credit risk and liquidity risk is as well a considerable aspect when pricing interest rate derivatives.

1.4 Objective of the study

1.4.1 General Objective

The main objective of this study is to model the impact of interest rate crisis.

1.4.2 Specific Objectives

1. To describe pricing of interest rate derivatives under single and multiple curve frameworks and show various basis spreads that characterize post crisis fixed income markets.

2. To model short rate/risk free interest rate .

3. To discretize CIR model for credit risk modelling in the interbank market.

4. To price a defaultable zero coupon bond with credit risk adjustment prior to default.

1.5 Scope

The study was limited only to modern or post crisis modelling and specifically XIBOR model that complements multi-curve modelling framework. Illustrations, practical examples and comparison were used where necessary for the purpose of better understanding. However, it was not possible to compare all the models available for pre and post crisis interest rate modelling but XIBOR model was used as a reference model.

CHAPTER 2

LITERATURE REVIEW

One curve world(single curve) is based on the general idea that all interest rate derivatives depend on only one curve, which is supposed to be at the same time the risk-free curve and the curve relevant for Ibor which also discounts the fixed future cash-flows. The discount factor at time t for a maturity u is denoted $P(t,u)$. The theoretical deposits underlying the Ibor indexes are priced using the same curve. At any fixing date, the deposit that pays the notional at the settlement date and receives the notional plus the Ibor interest at maturity is supposed to be a fair deposit; this is a deposit for which the total present value is zero. The total value includes the initial settlement of notional and the final payment of notional plus interest. For the purpose of computing the present value, the unique curve is used(Henrard, 2014).

In single-curve framework, a single curve is used for both discounting and forwarding, which implies that the same instrument is used to derive both curves(Backas and Höijer, 2012). Single curve was based on Euribor which was assumed to be risk free, but since the 2007 crisis Euribor rates with different underlying tenors exhibit different credit and liquidity risks hence it's no longer considered a reliable risk-free proxy. As a consequence the single-curve pricing approach using Euribor for discounting has been abandoned. Overnight index Rates (OIS) Eonia rate in the Euro market have turned to be the best risk -free proxy in the post-crisis environment hence discounting should be done using the OIS discount curve(Backas and Höijer, 2012).

In researchBackas and Höijer (2012) found out that the pre-crisis single-curve framework under-estimates the price of the swap and the delta risk that the swap position is exposed to hence not fit for determining the term structure of interest rate and pricing interest rate derivative.

Interbank cash market rates are perceived as the foundation for the settlement of a wide range of interest rate contracts for example swaps and forward rate agreements. The premia for credit and liquidity risk in interbank markets are mirrored in derivative instruments in the form of basis spreads. Occurrence of such spreads (for example tenor basis spread, Euribor- Eonia Ois spread/Libor- OIS spread, FRA -implied foward rate spreads and basis swap spreads) consequently nullifies the traditional fixed income valuation framework of classical short rate, HJM and Libor market models(Gallitschke *et al.*, 2014).

The financial Market participants and academic researchers have adopted multi-curve models to address post-crisis market realities. Multi-curve models were first proposed by (Fruchard, 1995) in the context of currency swaps .The multi-curve approach has been applied byKijima *et al.* (2009) to rates of varying qualitiesMorini (2009) and Bianchetti (2010)also adopted this model to multiple-tenor curves, hence multi-curve modelling has become the preferred approach in post-crisis fixed income pricing. Like in classical curve construction, curves in a multi curve framework are of three forms; Discount, spot and foward curve. Multi curve approach has different curves depending on the tenor of the underlying bootstrapping instrument instead of one single curve as is in the case of pre-crisis framework. For example for tenors 1M, 3M, 6M and 12M, four different spot and foward curves can be constructed. However the approach requires the construction of a separate discount curve with different bootstrapping instrument from the one used in constructing foward curves.

Prior to crisis discount curve was derived from the Euribor rate which was assumed to be the risk free rate in the interbank market. As a consequence of credit risk the market practice has changed hence Euribor is no longer considered a risk free proxy. Market participants have now adopted Eonia as the risk free rate hence a discount curve is constructed from Eonia overnight index swap rates.

In pre -crisis times Libor, risk-free and funding were in some way considered equivalent (Henrard, 2014). 9 August 2007 is probably when multi curve modelling became a concept/subject of concern to financial markets, before 2007, the standard textbook formulas, using a unique multipurpose curve in each currency were generally used in banks and software packages. In literature the multi curve framework has received many names for example: two curves, multi-curve framework, derivative tenor curves, funding- Ibor, discounting-estimation, discounting-forecast, discounting -forward and multi-curve market. Multi curve has been recommended as the best approach in post crisis period for example(Henrard, 2014).

Classical short rate models are formulated and constructed based on the non-arbitrage relationships, which allow to hedge forward-rate agreements in terms of zero-coupon bonds. As a result models predict that forward rates of different tenors are related to each other by sharp constraints, consequently, due to market evolution the non-arbitrage relationships might not hold in practice (let us consider for example basis-swap spreads which, from a theoretical point of view should be equal to zero, but they are actually traded in the market at quotes larger than zero)(Pallavicini and Tarenghi, 2010).

The sharp constraints market in practice virtually presents situations of possible arbitrage violations. This may be traced from what happened starting from summer 2007, with the rising of the credit crunch, where market quotes of forward rates and zero-coupon bonds began to violate the usual non-arbitrage relationships in a deep way, both under the pressure of a liquidity crisis, which reduced the credit lines needed to hedge unfunded products using zero coupon bonds, and the possibility of a systemic break-down suggesting that counterparty risk cannot be considered negligible any more(Pallavicini and Tarenghi, 2010).

Before summer 2007 Overnight Indexed Swaps (OIS) and standard Interest Rate Swaps (IRS) depicted a very low spread and constant, there by indicating low liquidity and counterparty risk, and basis-swap spreads were nearly zero, consistently with the usual interest-rate models predictions. However with the beginning of the crisis, this situation changed abruptly: spreads between OIS and IRS widened, and traded basis-swap spreads are now significantly different from zero(Pallavicini and Tarenghi, 2010).

Because of the high volatility of interest rates as depicted in time of crisis, stochastic interest rate models are considered as fit to model interest rate movements (Dagıstan, 2010). Following market evolutions several models have been introduced in past to capture the market changes. Merton in 1973 Merton (1973b) proposed the first stochastic interest rate model. Vasicek in 1977 Vasicek (1977) model followed Merton's model and other models such as; Dothan in 1978 Dothan (1978), Cox-Ingersoll and Ross in 1985 Cox *et al.* (1985), Ho-Lee in 1986 Ho and LEE (1986), Hull-White Extended Vasicek in 1990 Hull and White (1990), Black-Derman-Toy in 1990 Black *et al.* (1990) and CIR++ in 2001 which is a preferred interest rate model because the process is non negative. The other models include the Heath-Jarrow-Morton (HJM) framework which was first published by(Heath *et al.*, 1992a). HJM is a general setting to model the evolution of the forward rate curve.(Heath *et al.*, 1992a) describes this model as a model that can be used to price and hedge interest rate derivatives. However, a well-established approach has been elaborated in various textbooks such as Brigo and Mercurio (2007), Hull (2005), and (Andersen and Piterbarg, 2010). However, the major difficulty in implementing the HJM model for pricing as explained byBeyna (2013), is that the model dynamics are non-Markovian in general. A restriction to the Markovian form simplifies the accessibility hence increases the usage in practice. The transformation to Markovian dynamics is done by assuming, that the forward rate volatility is not path dependent and does not depend on path dependent quantities. Therefore this subclass incorporates many earlier developed/well-known models, such as Ho and LEE (1986), (Vasicek, 1977) andHull and White (1990), (Beyna, 2013).

Cheyette (1994), Introduced a class of markovian HJM models by imposing a structure on the volatility function instead of fixing the volatility function.Cheyette (1994) developed an approach where the resulting system of Stochastic Differential Equations (SDE) describing the yield curve dynamics, breaks down from a high-dimensional process into a low dimensional structure of Markovian processes. Jamshidian (1991) names this type of volatility structure quasi-Gaussian. As a consequence of the popularity of the LIBOR market model starting 1997, the approach ofCheyette (1994) was abandoned. However, it was rediscovered and developed further by Andersen and Piterbarg (2010)and (Kohl-Landgraf, 2007).Heath *et al.* (1992b) standardized interest rate derivatives valuation approach on the basis of mainly two assumptions: the first one suggests, that it is not possible to gain riskless profit (No-arbitrage condition), and the second one assumes the completeness of the financial market.

The general setting of HJM mainly suffers from two disadvantages: first disadvantage being the difficulty to apply the model in market practice and the extensive computational complexity caused by the high-dimensional stochastic process of the underlying. The development of the LIBOR market model which combines the general risk-neutral yield curve model with market standards improved the first disadvantage. Restricting the general HJM model to a subset of models with a specific parameterization of the volatility function can solve the second disadvantage(Bianchetti and Carlicchi, 2012). The problem in the HJM framework is the path dependency of the spot interest rate since it is non-Markovian in general. The non-Markovian structure of the dynamics makes it difficult to apply standard economic methods like Monte Carlo simulation or valuation via partial differential equations, because the entire history has to be carried, which increases the computational complexity and effort(Beyna, 2013).

Cash market spreads is as a result of credit riskTaylor and Williams (2009), however,Trolle (2013), relate it to liquidity risk. Models merely based on credit risk may fail to generate realistic tenor basis spreads, simply because the tenor basis mainly signifies liquidity risk, therefore it is difficult to separate credit and liquidity risks(Michaud and Upper, 2008). The structure of interbank reference rates can be decomposed into term, credit and liquidity premia.Taylor and Williams (2009)) view cash market spreads as a result of credit risk while Fukuda (2012), attribute spreads to liquidity risk. The sudden divergence between various rates for example Euribor-OIS spread can be attributed to monetary policy decisions adopted by international authoritites in response to the financial crisis and the impact of credit crunch on the credit and liquidity market view as well as different financial meaning and dynamics of those rates(Bianchetti and Carlicchi, 2012).Credit has become a major concern to financial intermediaries (Mishkin,2010).

Therefore, XIBOR model is viewed as the first consistent model that clearly includes credit and liquidity risks aspect in the interbank cash transactions and it was first proposed by(Gallitschke *et al.*, 2014). The XIBOR model mainly focuses on the three fundamental interbank risks namely,interest rate risk,credit and liquidity risk which were the major risks that the financial market faced in 2007. The model complements multi culve approach in interest rate market pricing. This model was first proposed by Gallitschke *et al.* (2014) but in modelling credit risk and short rate they used log normal Black and karanski model byBlack and Karasinski (1991). In this thesis CIR++ model in the context of Brigo and Mercurio (2001a)has been used because of the analytical tractability of the model and its ability to produce a term structure that reflects the true market situation. Also under the CIR++ model rates can never be negative because of the shift extension(Brigo and Mercurio, 2001a).

Different credit risk models are useful depending on how default event is defined, several modelling techniques of credit risk modelling have been deviced to model default event as well as default probability. The two major approaches is by use of structural form models and reduced form models.In recent years researchers have proposed a third approach which is by use of incomplete information models. Structural models are based on the fact that default event occurs based on the balance sheet. That is when liabilities are more than the assets of an organization. The reference model under structural models is the Black-Scholes-Merton model (1973) which has found several extensions afterwards. This model assumes that default event takes place only at maturity date and there is complete observation of information regarding firm's asset value by the model. This is a strong assumption underlying the model that can not be confirmed in reality (Pereira, 2013).

TheMerton (1973b) has been extended by several researchers to nullify the assumptions that are too restrictive. In this regard,Leland and Toft (1996) developed an optimal leverage model and risky corporate bond prices for arbitrary debt maturity in which maturity date of the debt is considered as a function of either tax advantages, bankruptcy costs and agency costs. This model is based on the assumption that interest rate is constant. The other extension of Merton model is letting default event take place before debt maturity which gives the model a better market realistic model. Following this extension the models were now referred to as second structural form-models and an example is the(Black and Cox, 1976).Longstaff and Schwartz (1995) developed a better feature by allowing interest rates to be stochastic by use of Vasicek process 2.Hull and White (1995) developed another model that allows default time to be random and occuring at a fixed length of time assuming default first takes place when a firm's value reach default boundary. Kim *et al.* (1993)introduced another model allowing the value of a firm's asset to follow a stochastic diffusion process where the interest rate is a Cox-Ingersoll-Ross (CIR) model by (Cox *et al.*, 1985). Thus, in this research the stochastic model used to model credit risk is Cox-Ingersoll-Ross (CIR) model to parameterize default intensity.

In his research Pereira (2013) recommends use of reduced form models to model credit risk because the basis for modelling credit is the fact that investors don't have complete information about default of any party in the contract. Reduced form models are based on the assumption that default event is a stochastic process and limited information is available about the event of default by any of the borrowers.These models assume that default is not directly related to the variability in a firm´s asset value like the case of structural models,rather there is an underlying relation that exists, like the assumption of Lando (1998) that implies that intensity function depends on other different state variables. Default intensity in reduced form model follow a non negative process and this justifies why we use CIR to model default intensity because CIR process always gives positive values. The first reduced model was by Jarrow and Turnbull (1995) whose assumption is that default default time is the stopping time at the first jump of an independent poisson process N_t and the intensity process λ_t .

A bond is a debt instrument obliging the issuer to pay back to the lender initial amount borrowed and interest over a certain period of time called maturiy. According to "Bank for International Settlements, in 2009 the bond market size was estimated at $82.2 trillion which USA bond market formed $31.2 trillion. Bonds are considered as safe investments because their prices are less volatile as compared to stock prices. However, they are also subject to some market risks such as default risk and the interest rate risk (Dagıstan, 2010). In this thesis we will focus on default risk or credit risk on bonds. Interest rates and bond prices are inversely related when interest rates go up then the price of a bond reduces (Mishkin *et al.*, 2010).

2.1 Interbank Market Evolution After the Credit Crisis

As a result of the credit crisis there emerged various spreads as observed in the market quotation of plain vanilla interest rate linear instruments such as deposits, FRA, swaps, basis swaps and OIS. A discussion of the basis spreads between Euribor and Eonia rates/ Libor and Fed Fund rates with different tenors which affects market quotes of various financial instruments such as (deposits,FRAs, Swaps, basis swaps and OIS), foward rate basis spreads and tenor basis spread is provided below.

2.1.1 Euribor - OIS Spread

The Euribor - OIS basis is as a result of different credit and liquidity risks reflected by Euribor and Eonia rates. The divergence is not as a result of counterparty risk carried by the financial contracts that are exchanged in the interbank market by the risky counterparties, but it is dependent on the different fixing levels of the underlying rates that is Euribor and Eonia rates. Clearly it is difficult to separate credit and liquidity risks because they refer to a money market rates with bilateral credit risk rather than the default of a one counterparty in a single derivative deal(Morini, 2009).

2.1.2 FRAs Versus Foward Rates

There emerged a sudden divergence between implied foward rates and the quoted FRAs rates in August 2007. Before 2007 FRAs could be replicated by buying one deposit with the same maturity as the FRA and selling another deposit with maturity identical to the FRAs start date(Mercurio, 2009). This is the reason exchanges of payments in interest rate swap could be expressed interms of FRAs. This feature has changed hence the implied foward rates between two consecutive deposits now difffer from the quoted FRA rates hence as a consequence it has become unsuitable to replicate them. Though(Mercurio, 2009),explains that the two rates are actually allowed to differ ,their huge divergence can be explained by future credit and liquidity issues. According to classical no arbitrage formula foward rates between time t and T_2can be derived from two consecutive deposits, that is $P(t,T_2) = P(t,T_1)P(T_1,T_2)$(3.2.7).

2.1.3 Basis Swaps

A basis swap is defined as an interest rate swap which involves exchange of two floating interest payments.The floating rate is indexed to different tenors for example 3 month Euribor and 6 month Euribor or different interest rate basis for example one leg tied to Euribor and the other leg to T-bill rate(Backas and Höijer, 2012). Interest rate risk may arise as a result of having liabilities and assets indexed to different floating interest rate or interest rate maturities i.e tenors that have different credit levels ,thus the reason of going into basis swap is to hedge the risk. Basis risk is the risk that the normal relationship between two floating rates might change. In classical modelling basis swaps were valued using two plain interest rate vanilla swap where the fixed legs were similar and the the floating leg of each swap was indexed to one of the tenors in a basis swap. A basis swap could be thought of a floating for floating swap with basis swap spread attached to the floating leg with the shortest tenor(Morini, 2009). The present value of a basis swap can be expressed as(Bianchetti and Carlicchi, 2011).

$$PV_{BS} = N\left(\Sigma_{j=1}^{m} P\left(t,T_j\right)\left(F_x\left(t,T_j\right)+R_{BS}\right)\delta\left(t,T_j\right) - \Sigma_{i=1}^{n} P\left(t,T_j\right)F_y\left(t,T_j\right)\delta\left(t,T_i\right)\right) \tag{2.1.1}$$

Where $F_x\left(t,T_j\right)$ and $F_y\left(t,T_j\right)$are foward rates with tenors x and y , R_{BS} is the basis swap spread and i and j denotes the two legs in a swap. The initial value of the basis swap should be equivalent to zero and therefore R_{BS} can be expressed as

$$R_{BS} = \frac{\Sigma_{i=1}^{n} P(t,T_i)\left(F_y(t,T_i)\right)\delta(t,T_i) - \Sigma_{j=1}^{m} P\left(t,T_j\right)\left(F_x\left(t,T_j\right)\right)\delta\left(t,T_j\right)}{\Sigma_{j=1}^{m} P\left(t,T_j\right)\left(\delta\left(t,T_j\right)\right)} \tag{2.1.2}$$

Assuming that the two payments take place on the same date that is $i = j$ and that the payment frequency of the two legs is the same then equation (2.1.2) can be expressed as

$R_{BS} = F_y\left(t,T_j\right) - F_x\left(t,T_j\right)$ which indicates that the basis spread is the difference between foward rates of two similar interest rates with different tenors. In pre- crisis the difference between interest rates with different tenors was considered negligible hence was assumed to have no effect on the foward rates, the basis spread was assumed to be zero. In 2007 the financial market changed and the basis spreads increased considerably hence becoming more inconsistent and this is attributed to credit and liquidity risk in the market. This leads to pricing of credit in the interbank markets to compensate for risks taken by investors over a period of time.

2.1.4 Collateralization and OIS discounting

Counter party risk is the risk that the counterparty will default and eventually will not be able to honour its contractual obligations.The major driver of 2007 financial crisis is the counterparty credit risk, this prompted interbank market to introduce collateral agreements on all OTC derivatives to reduce the counterparty risk. More than 70% of OTC derivatives were collateralized by 2010 (ISDA,2010). As a consequence derivatives prices quoted in the interbank markets can be viewed to be counterparty free OTC transactions. This also means the risk free overnight rate can be used as a discounting rate and can be used for discounting curve construction.

2.1.5 Credit and Liquidity Risk

Credit risk is the risk of failure of a counterparty to meet its obligations. Credit risk is essential in the lending market because the price of a loan is determined by the term of the loan and the credit quality of the debt issuer.There is no collateral received by the lending bank as protection against default by the borrowing bank for loans underlying Euribor/Libor, hence these rates carry compensation for solvency issue referred to as credit risk. Credit and liquidity risks were considered negligible in the interbank market before crisis, however liquidity needs had an effect on spreads of various rates for example Euribor - Eonia OIS spread .

The other evidence of credit and liquidity risk is the difference in Euribor rates with different tenors. The difference between similar rates with different tenors has called for pricing of each tenor as different market but ensuring that there is no arbitrage opportunity. If banks fear that occurence of some events could threaten their access to funding they may decide to hoard liquidity in times of financial turmoil, liquidity risk affects the rate at which the banks are willing to lend at. Credit default swap spreads (CDS) of Euribor panel banks increased, that is the price of insuring against a default increased as a result of fear of possible default by interbank counterparties. This coincides with the explosion of the Euribor - Eonia OIS spread, the FRA - implied foward rate spreads and the basis swap spreads(Backas and Höijer, 2012).

CHAPTER 3

METHODOLOGY

The aim of this study is to provide a description of the pre and post crisis pricing frameworks (single curve and multiple curve models), focus on interbank market model (XIBOR), provide a detailed discussion of the models used to model riskless rate and credit risk as fundamental risk factors in XIBOR model as well as price a defaultable zero coupon bond.

3.1 Interbank market Modelling

Various basis spreads feature the post crisis fixed income market, such spreads include; XIBOR-OIS spreads; tenor basis spreads as well as the forward basis. There existed a historically stable relationships between cash market rates, swap rates, and treasury rates that collapsed as a result of the financial crisis of 2007. This resulted to emergency of various basis spreads that ruled out the classical fixed income pricing models, assumptions and practices.Gallitschke *et al.* (2014) Explains that XIBOR model calibrates well to pre and post crisis swap market data.

Financial experts have since developed a class of reduced form multi-curve models to deal with the new market realities and practices. In classical modelling credit and liquidity risks were assumed to be negligible which turned to be the major cause of 2007 financial turmoil which was referred to as a credit crunch. Liquidity is defined as the risk of money market freeze(Gallitschke *et al.*, 2014). Interbank cash transactions involve pricing of Credit and liquidity risk hence it generates XIBOR-OIS spreads, tenor basis spreads and the forward basis.

3.1.1 Interbank Market.

Interbank market is an over-the-counter market (OTC) where banks negotiate interbank loans with different maturities that range from one day to 12 months (Roussellet,2013). Companies, entrepreneurs and consumers rely on the banking system for credit and loans, thus the interbank cash market provides financial institutions with the short-term unsecured funding they require to meet their day-to-day obligations. As a result of tension in the banking market other financial markets are affected leading to widening of the spreads, increased volatility, disruptions of liquidity and also can lead to financial stress. XIBOR include LIBOR, EURIBOR and other relevant benchmark interbank market rates. These rates characterize the conditions for unsecured funding in the financial sector, highlighting particularly the prevailing interbank credit and liquidity risk environment.

During pre -crisis period major banks were viewed to be default-free and liquidity was taken for granted. The collapse of investment banks Bear Stearns, Lehman Brothers, and several other financial institutions as well as the ensuing collapse of liquidity in the cash market disqualified this assumption, leading to a paradigm shift in interbank money markets. Therefore, in the post crisis world both credit and liquidity risk are priced into cash market transactions(Brunnermeier, 2009).

3.2 Single curve approach

To illustrate how a single curve worked focus on interest rate derivatives for example; IRS. Interest rate swap was initiated in early 1980s and since then the pre- crisis approach had been applied for close to three decades.Pricing an interest rate swap was considered to be straight foward in pre-crisis pricing methodologies. The framework was based on building of a single spot curve to calculate forward rates and discount factors. The focus is on how to construct a spot curve and computations of forward rates and discount factors for pricing of an interest rate swap as an interest rate financial derivative.

3.2.1 Discount Curve

The price of any financial instrument including interest rate swaps, is equal to the sum of all future discounted cash flows or the present value of the financial instrument's expected cash flows under the risk-neutral measure(Björk, 2009). Therefore a discount factor between time t and T, is the amount at time t that is "equivalent" to one unit of currency payable at time T" (Brigo and Mercurio, 2007). The price of a zero coupon bond is used as a discount factor in swap pricing. A zero coupon bond is a financial instrument that guarantees a return of one unit of currency at maturity T, no coupon payments are made. The price of a zero coupon bond at time T is denoted by $P(t,T)$where

$$P(t,T) = \mathbb{E}\left[e^{-\left(\int_t^T r_s ds\right)} / \mathcal{F}_t\right] \tag{3.2.1}$$

r$_s$denotes the instantaneous rate/instantaneous rate at which the bank accrues, $\mathcal{F}_t$ denotes the available market information at time t and $\mathbb{E}$denotes the expectation under the risk-neutral probability measure. The instantaneous rate is also known as the short rate and it is often modelled as a stochastic process. All bond prices and derivative instrument prices depend on the short rate process in a risk-neutral world.

3.2.2 Spot Curve

Spot curve is a curve that gives yield to maturity of a zero coupon bond. When building a spot curve it is important to distinguish between simply and continously componded interest rates , a simply compounded investment of one unit of currency at time t will grow to the amount $\left(1+\frac{L(t,T)}{m}\right)^{m}$ at maturity time T,(Filipovic, 2009). From this formula $L(t,T)$is the simply compounded interest rate at time t with maturity T, and m is the number of compounding periods per year. As m increases that is tends to infinity, $m \to \infty$, $\left(1+\frac{L(t,T)}{m}\right)$converges to a constant. From this perspective we can define continously compounded interest rate $R(t,T)$ as the constant rate at which the investment of $P(t,T)$ units of currency yields a unit amount of currency at maturity as

$$\lim_{m\to\infty}\left(1+\frac{L(t,T)}{m}\right)^{m} = e^{(R(t,T)\delta(t,T))} \tag{3.2.2}$$

where $\delta(t,T)$represents the year fraction between t and T. The year fraction defination depends on the day count convention used, for example the commonly used day count convention is Actual/360, here one year is considered to be 360 days long.

$$\delta(t,T) = \frac{D_2 - D_1}{360} \tag{3.2.3}$$

$D_2 - D_1$ represents the actual number of days between the dates $D_1 = [y_1, m_{1,} d_1]$and $D_2 = [y_{2,} m_2, d_2]$. y, m and d stands for year,month and day respectively. From the defination of spot rate, $L(t,T)$can be determined by use of the following equation(Brigo and Mercurio, 2007).

$$P(t,T)\left(1+\left(R(t,T)\,\delta(t,T)\right)\right) = 1 \tag{3.2.4}$$

Solving we get

$$L(t,T) = \frac{1-P(t,T)}{\left(R(t,T)\,\delta(t,T)\right)} \tag{3.2.5}$$

Where $P(t,T)$is the price of a T -year zero coupon bond also known as the discount factor.Using equation (3.2.2) and the expression $P(t,T)\,e^{R(t,T)\delta(t,T)} = 1$ we can transform simply compounded spot rate $L(t,T)$to a continously compounded spot rate $R(t,T)$,that is

$$R(t,T) = -\frac{\ln P(t,T)}{\delta(t,T)} \tag{3.2.6}$$

We derive simple and continuous spot rates because the market interest rates are usually simply compounded rates whereas when pricing interest rate derivatives continuously compounded rates are used(Hull, 2009). Therefore when building a pricing curve there is a need to transform market rates to continously compounding rates.

The yield curve/spot curve/term structure of interest rates can take either of the four shapes, normal, upward sloping, downward sloping and humped.

Example of a simulated zero coupon bond yield curve with continous compounding (upward sloping) is as shown in figure 3.2.1. The yield increases as the term increases and it slops upwards indicating the investors expectation of future increase in interest rates.

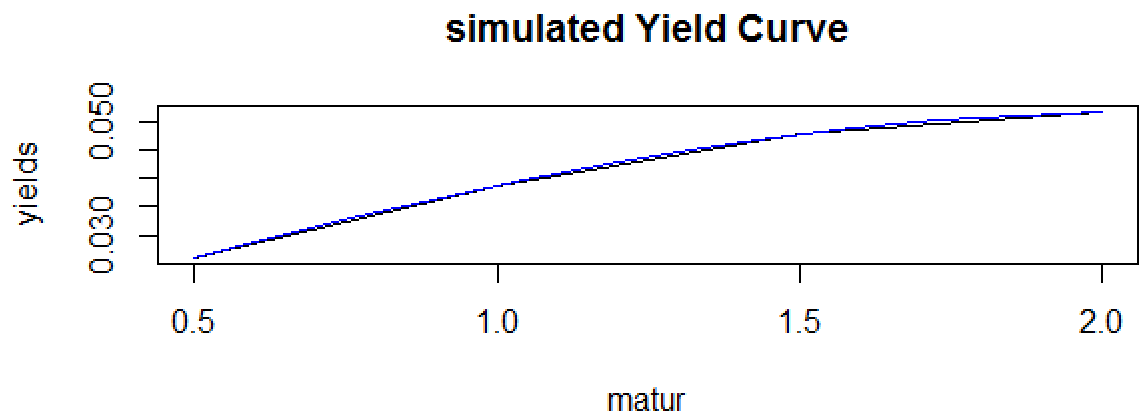

Figure 3.2.1: upward sloping zero coupon bond yield curve

3.2.3 Foward Curve

From classical, no-arbitrage interest rate theory, the implied forward rate over the time interval $[t,T]$ can be derived from two consecutive zero coupon bonds by use of the following equality (Filipovic, 2009).

$$P(t,T_2) = P(t,T_1)\,P(T_1,T_2) \tag{3.2.7}$$

From the equation (3.2.7) any cash flow that is discounted from $T_2$to t should have the same value regardless of whether it was discounted straight from $T_2$to t or discounted in two steps that is from $T_2$to $T_1$and $T_1$to t (Bianchetti, 2008). Conseqently the discount factor $P(T_1,T_2)$is referred to as the foward discount factor and it is the price of a zero coupon bond at time $T_1$with maturity time T_2. Considering the simply compounded equation (3.2.5), then by equation (3.2.7) the simply compounded foward rate can be expressed as

$$F_s(t,T_1,T_2) = \frac{1}{\delta(T_1,T_2)}\left(\frac{1}{P(T_1,T_2)}-1\right) = \frac{1}{\delta(T_1,T_2)}\left(\frac{P(t,T_1)}{P(t,T_2)}-1\right) \tag{3.2.8}$$

The continously compounded foward rates is implied by the continously componded spot rate in (3.2.6) and therefore can be expressed as

$$F_c(t,T_1,T_2) = \text{-}\tfrac{\ln P(t,T_2) - \ln P(t,T_1)}{\delta(T_1,T_2)} \tag{3.2.9}$$

If we let $T_1 = T$ in equation (3.2.8) and consider the limit as maturity T_2approaches settlement date T_1 in the same equation / $T_2 - T_1 \downarrow 0$, then the instantaneous interest rate at time t which is $f(t,T)$ can be represented as(Brigo and Mercurio, 2007).

$$f(t,T) = \lim_{T_2 \to T^+} F_s(t,T,T_2)$$

$$= -\lim_{T_2 \to T^+} \tfrac{1}{P(t,T_2)} \left(\tfrac{P(t,T_2) - P(t,T)}{\delta(t,T_2)}\right)$$

$$= -\frac{1}{P(t,T)} \frac{\partial P(t,T)}{\partial T}$$

$$= -\frac{\partial \ln P(t,T)}{\partial T} \tag{3.2.10}$$

This defination gives the relationship between the zero coupon bond $P(t,T)$ and the instantaneous foward rate $f(t,T)$defined as

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right) \tag{3.2.11}$$

Equation (3.2.11) implies that all interest rates can be derived from the zero coupon bond price(Backas and Höijer, 2012).

For example with the following information we can build a smooth foward rate curve using cubic interpolation method to include all the maturities as follows

Table 3.2.1: Various terms and rates

Time	0.0	0.5	1.0	1.5	2.0
Rates	0.02	0.01	0.04	0.06	0.07

The value at any time period can be determined by cubic interpolation, for example the approximate value at time $1.25 = 0.0500$.

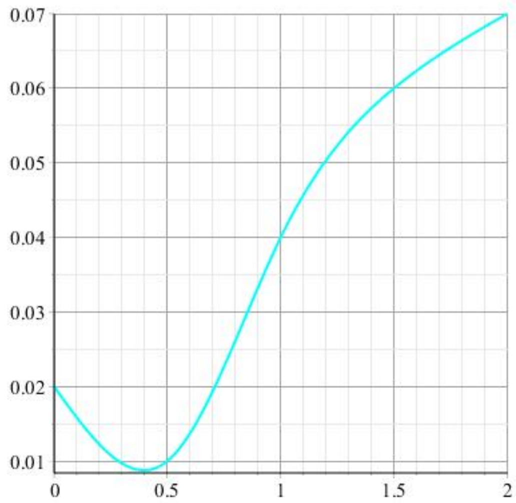

Figure 3.2.2: Foward rate curve (single curve framework)

One curve is interpolated to include all maturities under a single curve approach as displayed by figure 3.2.2. Instead of several curves for each maturity only a foward single curve is used.

3.3 Pricing an interest rate in a single curve framework

When pricing an interest rate swap the initial value should be equal to zero, therefore it is important to find the fixed rate K_{IRS} that will make the present value of the fixed leg equal to the present value of the floating leg, that is $PV_{IRS,Fixed} = PV_{IRS,Floating}$. An interest rate swap can be priced interms of a FRA or a bond but whichever the approach adopted will lead to the same value (Backas and Höijer, 2012).

3.3.1 Foward rate agreement

The foward rate agreement holder receives an interest rate payment for the contract period between T_1 and T_2 . However the contract is settled at time T_1 but the cash flows are exchanged at time T_2. This implies that the expected cash flows should be discounted from T_2 to T_1. The seller's FRA value at maturity time T_2 is

$$N\delta(T_1,T_2)(K_{FRA} - L(T_1,T_2)) \tag{3.3.1}$$

N is the notional amount in the contract. Refering to simply compounded spot rate in equation (3.2.5) and (3.2.7) which gives $P(t,T_1) = \frac{P(t,T_2)}{P(T_1,T_2)}$. Equation (3.3.1) can be rewritten as (Backas and Höijer, 2012)

$$N\delta(T_1,T_2)\left(K_{FRA} - \frac{1-P(T_1,T_2)}{\delta(T_1,T_2)P(T_1,T_2)}\right) = N\left(K_{FRA}\delta(T_1,T_2) - \frac{1}{P(T_1,T_2)} + 1\right) \tag{3.3.2}$$

Equation (3.3.2) represents the cash flows exchanged at time T_2 therefore to obtain the cash flows at time t the same equation should be discounted to time t which gives the value of the contract at time t as

$$NP(t,T_2)\left(K_{FRA}\delta(T_1,T_2)-\frac{1}{P(T_1,T_2)}+1\right)$$

$$=\mathrm{N}(K_{FRA}P(t,T_2)\,\delta(T_1,T_2)-P(t,T_1)+P(t,T_2)) \tag{3.3.3}$$

The contract value must be zero at time t for the FRA to be fair, consequently we set equation (3.3.3) to be zero. After solving for K_{FRA} then it is clear that the appropriate rate to use is the simply compounded foward rate defined in equation (3.2.8)(Backas and Höijer, 2012). Therefore

$$K_{FRA}=\frac{P(t,T_1)-P(t,T_2)}{P(t,T_2)\,\delta(T_1,T_2)}$$

$$=\frac{1}{\delta(T_1,T_2)}\left(\frac{P(t,T_1)}{P(t,T_2)}-1\right)$$

$$=F_s(t;T_1,T_2) \tag{3.3.4}$$

3.3.2 Pricing an interest rate swap interms of foward rate agreement

Pricing an interest rate refers to determining the value of the swap at initiation, thus the exchange of interest rate payment can be referred to as FRAs when pricing an interest rate swap (IRS) in terms of forward rate agreements (FRA). The first exchange of payments is referred as the inception of the contract while the remaining transactions must be computed. This can be done by assuming that forward rates are realized, that is future spot rates equals the forward rates(Backas and Höijer, 2012).

The fixed rate K_{IRS} should be determined ensuring that no side of the contract is given an arbitrage opportunity. To satisfy this constraint FRAs quotations can be used as representatives of the spot curve. The present value of the fixed leg can be represented as follows under the assumption that future fixed rate payments are known when entering into the contract at time t.

$$PV_{IRS,fixed}=K_{IRS}N\Sigma_{i=1}^{n}P(t,T_i)\,\delta(T_{i-1},T_i) \tag{3.3.5}$$

$P(t,T_i)$ represents discount factor at time t for maturity T_i, K_{IRS} fixed swap rate paid at times T_i.Thus the value of the fixed leg is the sum of all future discounted fixed cash flows.

The floating leg of the interest rate swap pays the rate with tenor $\left[S_{j-1}, S_j\right]$ at respective times S_j, Where j takes the values $j = 1, ., m$. As in the case of fixed leg we can also use FRAs quotation to derive the present value of the floating leg.

From equation (3.3.4) $K_{FRA} = F_s\left(t, S_{j-1}, S_j\right)$and therefore the present value of the floating leg equals

$$PV_{IRS,floating} = N\Sigma_{j=1}^{m} F_s\left(t, S_{j-1}, S_j\right) P\left(t, S_j\right) \delta\left(S_{j-1}, S_j\right) \tag{3.3.6}$$

If we consider an increasing date vector $T = \{T_0, ., T_n\}$and $S = \{S_0, ., S_m\}$ where $T_n = S_m > T_0 = S_0 > t_0$,then the fair swap rate $K_{IRS}(t, T, S)$can be derived by setting equation (3.3.5) equal to (3.3.6) and solve for K_{IRS} as follows;

$K_{IRS} N\Sigma_{i=1}^{n} P(t, T_i)\, \delta(T_{i-1}, T_i) = N\Sigma_{j=1}^{m} F_s\left(t, S_{j-1}, S_j\right) P\left(t, S_j\right) \delta\left(S_{j-1}, S_j\right)$

Solving the above equation gives K_{IRS} as defined by(Bianchetti and Carlicchi, 2011).

$$K_{IRS} = \frac{\Sigma_{j=1}^{m} F_s\left(t, S_{j-1}, S_j\right) P\left(t, S_j\right) \delta\left(S_{j-1}, S_j\right)}{\Sigma_{i=1}^{n} P\left(t, T_j\right) \delta\left(T_{j-1}, T_j\right)} \tag{3.3.7}$$

Using the simply compounded interest rate defination $F_s\left(t, S_{j-1}, S_j\right)$ and inserting it into equation (3.3.7), using the telescope summation and also the property $T_n = S_m \geq T_0 = S_0$, then the swap rate in (3.3.7) can be simplified into

$$K_{IRS} = \frac{\Sigma_{j=1}^{m} \left(\frac{P\left(t, S_{j-1}\right) - P\left(t, S_j\right)}{P\left(t, S_j\right) \delta\left(S_{j-1}, S_j\right)} \right) P\left(t, S_j\right) \delta\left(S_{j-1}, S_j\right)}{\Sigma_{i=1}^{n} P\left(t, T_j\right) \delta\left(T_{j-1}, T_j\right)}$$

$$= \frac{P(t, T_0) - P(t, T_n)}{\Sigma_{i=1}^{n} P\left(t, T_j\right) \delta\left(T_{j-1}, T_j\right)} \tag{3.3.8}$$

3.4 Multicurve approach

3.4.1 Pricing in a multi curve framework

In building a discount curve OIS rates which are considered risk free are used since Libor/ Euribor can no longer be considered risk free rates.Therefore the multi culve risk free discount factor $P_{OIS}(t, T)$ can be derived as (Backas and Höijer, 2012),

$$P_{OIS}(t,T) = \frac{1}{1 + K_{OIS}(t,T)\,\delta(t,T)} \tag{3.4.1}$$

Where K_{OIS} represents OIS rate i.e Eonia OIS rate at time t with maturity T and $\delta(t,T)$ the year fraction between time t and T. There is a need for construction of different fowarding curves depending on the maturity, also a second discount factor that is related to the foward curve is required . In construction of the second discount factor Libor/ Euribor rates are considered as risk free rates and therefore it is denoted as $P_L^x(T_1,T_2)$. The superscript x denotes the homogeneity requirements and takes the value of the underlying tenors for example $x = (1M, 3M, 6M, 12M)$. In multi curve framework the second discount factor can be represented as

$$P_L^x(T_1,T_2) = \frac{1}{1 + L^x(T_1,T_2)\,\delta^x(T_1,T_2)} \tag{3.4.2}$$

Where $L^x(T_1,T_2)$ is the risky spot libor/Euribor at time $T_1$for maturity T_2 (Backas and Höijer, 2012).

The foward rate agreement K_{FRA}^x can be calculated using the corresponding foward curve and applying the following formula;

$$K_{FRA}^x = F^x(t,T_1,T_2) = \frac{1}{\delta^x(T_1,T_2)}\left(\frac{P_L^x(t,T_1)}{P_L^x(T_1,T_2)} - 1\right) \tag{3.4.3}$$

Considering two increasing date vectors $T = \{T_0, \ldots.., T_n\}$ and $S = \{S_0, \ldots., S_m\}$ where $T_n = S_m > T_0 = S_0 > t_0$, then the fixed swap rate $K_{IRS}^x(t;T,S)$for the underlying tenor x is denoted as (Ametrano and Bianchetti, 2008).

$$K_{IRS}^x = \frac{\Sigma_{j=1}^m K_{FRA}^x P_{OIS}(t,S_j)\,\delta(S_{j-1},S_j)}{\Sigma_{i=1}^n P_{OIS}(t,S_j)\,\delta(S_{j-1},S_j)} \tag{3.4.4}$$

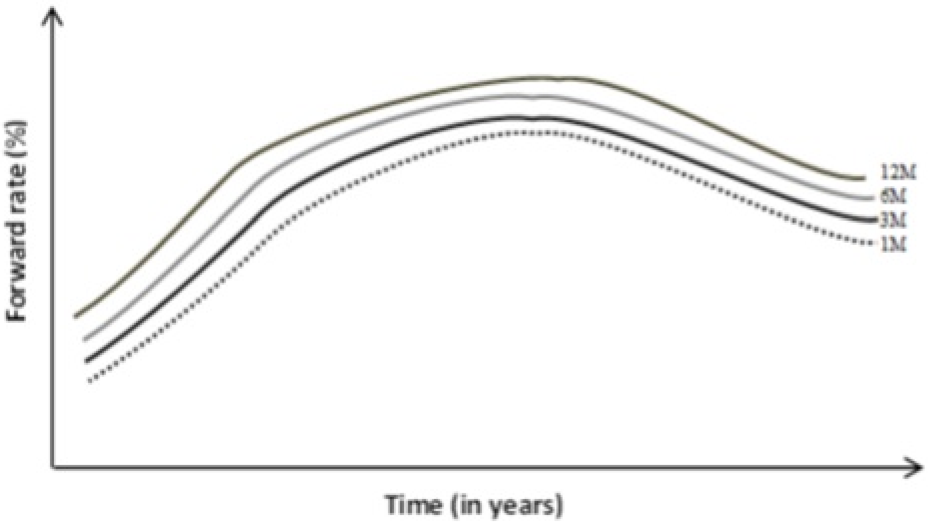

Figure 3.4.1: An example of foward curves under multi-curve pricing

There is a separate foward curve for each maturity under multiple curve approach as displayed by figure 3.4.1 unlike a single curve for all maturities shown in figure 3.2.2. Longer maturities have high interest rates as compared to shorter maturities as long maturities have a high probability of credit risk than shorter ones.

3.5 The General Xibor Model

This model supports and complements multi curve approach as a new pricing method. XIBOR model is a structural approach to post-crisis basis spreads in interbank market rates. Thus the XIBOR model provides the first consistent framework that is able to endogenously generate all relevant post-crisis basis spreads from essential risk factors such as interest rate risk, credit risk, and liquidity risk. The model offers arbitrage free explanation of why basis spreads emerge as well as emergency of multiple term structures. The model is based on three significant risk factors in the interbank market, that is; credit risk, interest rate risk and liquidity risk (Gallitschke *et al.*, 2014).

3.5.1 Interbank Interest Rates

XIBOR Rates refers to essential reference rates in interbank money markets; these include the London Interbank offer Rate (LIBOR), the Euro Interbank Offer Rate (EURIBOR) for term lending, Federal Funds Effective Rate (Fed Funds Rate) and the Euro Overnight Index Average (EONIA) for overnight transactions through the central bank. Overnight rates are accepted as the best representation for risk-free rates in the relevant currency in the cash market.The rates characterize the condition for unsecured lending in the interbank market taking into account credit and liquidity risk factors. Interbank rates are determined by a common panel-based submissions mechanism. For example Libor indicates the benchmark rate at which Libor contributer banks can obtain unsecured fund in a given currency in london interbank market. Euribor is the rate at which one panel bank in the Euro interbank within EMU offer deposits to the other at 11.00 am Brussels time(Eisenschmidt and Tapking, 2009). The rates for all maturities ranging from one day are submitted by the panel banks and the trimmed averaged interest rate for each maturity are published, that is the highest and the lowest 15% banks of each distribution tails are excluded. In 2010 the panel was composed of 42 EU banks with the highest business volume in the Euro zone money market and inaddition of some large international non EU banks with important Euro zone operations.

3.5.2 Basis Spreads

Basis spread refers to difference between various rates when pricing a financial instrument such as an interest rate swap. In an overnight indexed swap (OIS),the holder exchanges a fixed rate on an agreed notional for the geometric average of the relevant overnight rate(Fed Funds or Eonia rate) over a certain period (tenor). Overnight rates are considered to be risk-free, thus the OIS curve derived from OIS swap data is associated to the riskless discount curve(Hull and White, 2014). The XIBOR-OIS spread is the difference between the par swap rates of a XIBOR and an OIS swap with the same tenors and maturities and it is an important measure for funding conditions in the interbank money market(Thornton, 2009). Tenor basis spread that was negligible in the pre-crisis period has become significant in the post crisis period.

3.5.3 Forward Rate Agreements and the Forward Basis.

FRA rate is defined as the fixed rate $K = F(0;t;T)$at which the price of the forward rate agreement at inception equals zero. For a XIBOR-based forward rate agreement (FRA), where the payer pays the fixed rate K and receives the XIBOR rate $L(t;T)$and the receiver pays the XIBOR rate and receives the fixed rate K at a future date over the time interval $[t;T]$on a unit principal amount with simple compounding, the payer's cash flow at time T can be denoted by

$$FRA(T) = [L(t,T) - K](T - t) \tag{3.5.1}$$

There existed a connection between zero coupon bond, FRA rates and Libor rates before crisis. By definition a zero coupon bond is a financial contract which delivers one unit of cash at its maturity date $T > 0$. Its price at time $t < T$, denoted by $P(t,T)$, represents the expectation of the market concerning the future value of money and consequently the value at maturity, $P(T,T) = 1$. Historically in the classical approach interest rates are defined to be consistent with the zero coupon bond prices $P(t,T)$(Grbac and Runggaldier, 2015). Considering the discretely compounding forward rates and a notional amount of 1, we obtain the Post-Crisis Fixed-Income Markets setting expressed as:

$$F(t,T,S) = \frac{1}{S-T}\left(\frac{p(t,T)}{P(t,S)} - 1\right) \; t < T < S \tag{3.5.2}$$

Equation (3.5.2) represents the fair fixed rate at time t of a forward rate agreement (FRA), while the floating rate received at time S can be expressed as(Grbac and Runggaldier, 2015).

$$F(T,T,S) = \frac{1}{S-T}\left(\frac{1}{P(t,S)} - 1\right) \tag{3.5.3}$$

In pre - crisis framework, the Libor rate was assumed to be equal to the floating rate defined using zero coupon bond prices that is

$$L(T,T,S) = F(T,T,S) = \frac{1}{S-T}\left(\frac{1}{P(t,S)} - 1\right) \tag{3.5.4}$$

Where, $L(T,T,S)$ denotes the Libor rate at time T for the period $[T,S]$(Grbac and Runggaldier, 2015). This was based on the no arbitrage assumption since Libor panel, is refreshed regularly where banks of deteriorating credit quality are replaced with those of better credit quality, hence it contained no counterparty and liquidity risk making the assumption of risk freeness credible. The forward rate agreement $F(t,T,S)$ was also referred to as the forward Libor rate because it characterised the market expectation of the future value of the Libor rate which is often denoted by $L(t;T,S)$. The forward Libor rate was expressed either as a conditional expectation of the spot Libor rate under the forward martingale measure, or expressed using the quotient of the bond prices as shown below

$$F(t,T,S) = E^{\rho^s}\{L(T,T,S)/\vartheta_t\} = \frac{1}{S-T}\left(\frac{p(t,T)}{P(t,S)} - 1\right) = F(t,T,S) \tag{3.5.5}$$

As a result of the crisis and the assumption under which the spot Libor rate quotes are based, the assumption that the Libor rate is free of various interbank risks no longer hold. Therefore the expression $F(t,T,S) = E^{\rho^s}\{L(T,T,S)/\vartheta_t\} = \frac{1}{S-T}\left(\frac{p(t,T)}{P(t,S)} - 1\right)$ does not hold, that is

$$L(T,T,S) \neq \frac{1}{S-T}\left(\frac{1}{P(t,S)} - 1\right)$$

This has prompted researchers to develop new models and realistic market assumptions. The difference between the FRA rate and foward rate is referred to as the forward basisMorini (2009) as witnessed in the recent years. The appearance of the forward basis in XIBOR rates can be linked to liquidity and renewal effects(Gallitschke *et al.*, 2014).Liquidity risk explains the major proportion of the spread hence it is essential to adopt a model that factors the aspect of market freeze in.

3.6 Modelling under XIBOR model approach.

The construction of XIBOR model is based on three components that reflect fundamental risk factors in the interbank cash markets. These components are risk-free interest rates, credit and liquidity risk. The framework is suitable for modelling interbank risk and XIBOR rates and therefore the three risk factors are modelled. The assupmtions of the model are as follows:

(i) The model assumes that a bank is removed from the XIBOR panel only upon default and it is subsequently replaced by a bank drawn at random from the pool of surviving banks not yet included in the panel(Gallitschke *et al.*, 2014) .

(ii) No arbitrage opportunity

3.6.1 Riskless Term Structure (Interest rate risk)

The riskless rate $r^{O/N}$ is associated with the relevant overnight rate, hence since the riskless term structure is based on an overnight rate, it is important to use a short rate model for $r^{O/N}$ because it is the interest rate that applies for the shortest loan period. It is therefore assumed that the risk-neutral dynamics of $r^{O/N} = r^{O/N}(t)$ are specified by a suitable market-consistent short rate model. The associated discount curve (or OIS bond) is expressed as (Gallitschke *et al.*, 2014).

$$\delta(t;T) = P^{OIS}(t,T) = E_t\left[e^{-\int_t^T r^{O/N}(s)ds}\right] \tag{3.6.1}$$

The infinitesimal OIS forward rate is expressed as

$$f(t,T) = -\frac{\ln(\delta(t,T))}{\partial T} = -\frac{\ln\left(P^{OIS}(t,T)\right)}{\partial T} \tag{3.6.2}$$

It can therefore be noted that OIS bond is a theoritical financial instrument that is not traded in a real market world. The rate paid on a floating leg is the overnight rate rolled over the accrual period, that is the ex post compound interest rate realized by a rolling investment at the overnight rate over the time interval $[t,T]$. This is the rate paid in the floating leg of an OIS swap and it is denoted by;

$$L^{O/N}(t,T) = \frac{1}{T-t}\left(\exp - \int_t^T r^{O/N}(s)\,ds - 1\right) \tag{3.6.3}$$

3.6.2 Interbank Market Default Risk.

Credit risk is defined as the banks inability to meet its obligations or failure to repay payments owed to its counterparty in due time. Mathematically default can be modelled by means of default time τ. Default intensity can be modelled in various ways for example by use of structural models,incomplete models or reduced form intensity models. Here default risk in the interbank market is modelled through banks' credit default intensities. We denote all the participants in the interbank cash market by $i = 1.....1$. Therefore each index $i = 1.....1$ represent a fixed bank that is potentially relevant (now or in the future) for the determination of XIBOR rates, and τ^i denotes its default time. $\tau^i.....\tau^i$ can be modelled using doubly stochastic cox framework of(Lando, 1998) . Let $\xi = \{\xi(t)\}$ denote the full market filtration including jump events. For each bank i the risk-neutral default intensity or hazard rate is denoted by $\lambda^i = \{\lambda^i(t)\}$, which is the arrival rate of default at t , conditioning on all information available at time t(Duffie and Singleton, 2003) . The probability of default taking place in dt interval given that default has not occured so far is $\lambda^i(t)\,dt$ and can be expresed as;

$$\mathbb{Q}_t\left(\tau^i > t + dt/\tau^i > t\right) = \lambda^i(t)\,dt \tag{3.6.4}$$

The intensity process $\lambda^i(t)$ is said to varry hence it coincides with the foward default rate function inferring that survival is the only relevant information to default arriving over time(Pereira, 2013).

The risk neutral survival probability of bank i is

$$P^i_{OO}(t,T) = \mathbb{Q}_\mathrm{t}\left(\tau^\mathrm{i} > \mathrm{T}\right) = 1_{\left\{\tau^\mathrm{i} > \mathrm{T}\right\}}\mathbb{E}_\mathrm{t}\left[\mathrm{e}^{-\int_\mathrm{t}^\mathrm{T}\lambda^\mathrm{i}(\mathrm{s})\mathrm{ds}}\right]\cdot\lambda^\mathrm{i}(\mathrm{s}) : 0 \le \mathrm{s} \le \mathrm{T} \tag{3.6.5}$$

with $1_{\{\tau^i > T\}}$ a function that takes value 1 when $\tau^i > T$ and zero otherwise (Pereira, 2013).

To price a defaultable financial instrument such as bonds there is need to know its promised payment and payment in the event of default (recovery rate). Introducing two defaltable theoritical interbank bonds one with zero recovery and the other with recovery rate R_t both with no liquidity adjustment, then the price of a theoritical zero recovery interbank bond is given by

$$P^i_0(t,T) = E_t\left[e^{-\int_t^T\left(r^{O/N}(s) + \lambda^i(s)\right)ds}\right] \tag{3.6.6}$$

Which is a general pricing formula under the assumption that interest rate and default intensity are dependent. Assuming independence of interest rate and default intensity then we can split (3.6.6) as

$$P_0^i(t,T) = E_t\left[e^{-\int_t^T \left(r^{O/N}(s)+\lambda^i(s)\right)ds}\right] = E_t\left[e^{-\int_t^T r^{O/N}(s)ds}\right] E_t\left[e^{-\int_t^T \lambda^I(s)ds}\right] = \delta(t,T)\,\mathbb{Q}_t\left(\tau^i > T\right)$$

$$P_O^i(t,T) = E_t\left[e^{-\int_t^T r^{O/N}(s)ds}\right] 1_{\{\tau^i>T\}} = \delta(t,T)\,P_{OO}^i(t,T)$$

implying that under the risk neutral measure the price of a defaultable zero coupon bond is equivalent to the survival probability times the defaultable zero coupon bond price (Jorda, 2010) .

while the price of the interbank bond with recovery rate R can be given by

$$P^i(t,T) = \left(1 - \int_t^T f(t,s)\,P_O^i(t,s)\,ds\right) R_t + P_O^i(t,s)\,(1-R_t)\,,\ t \le \tau^i \tag{3.6.7}$$

where $R \in [0,1]$is the fractional recovery rate conditional on arrival of default event, $\mathbb{Q}_t$denotes the expectation under the risk neutral measure and E_t is the conditional expectation given information up to time-t. Interest rate r and default intensity λ^i are assumed to be independent under the pricing measure.

3.6.2.1 Cox Process/Model /Doubly stochastic process

The poisson process can be used to describe the default time τ^i of a bank where the default time τ^i is viewed as the first jump of a poisson process. The poisson process can be classified as either time homogeneous, time inhomogeneous or Cox process depending on the nature of the intensity function. The Cox process is different from the other two models in that the intensity $\lambda^i(t)$ is defined as time varrying and stochastic (Lan, 2011)

The process is said to be doubly stochastic because;

(i) The jump component is stochastic in nature

(ii) The probability of jumping or intensity is stochastic in nature

When predicting the likelihood of default the state variables are necessary, these variables include interest rate on a riskless debt and may include time, stock prices,credit ratings and other important variables relevant in predicting default(Lando, 1998).The default event varries randomly over time, at any time default can take place hence the probability of default is never null(Pereira, 2013). Given all available information default happens when a poisson random variable faces a discrete jump or shift (Sironi et al , 2004). Considering the default time

τ^i as the first jump of a poisson process with a randomly varrying mean arrival rate λ^ialso referred to as intensity and $\mathbb{Q}_{\mathrm{t}}$ the probability of an event under the risk neutral measure, then the cumulated intensity to time t can be expressed as(Lan, 2011).

$$\Lambda(t) = \int_t^T \lambda^i(t)\, ds \tag{3.6.8}$$

and the probability of a bank defaulting in the next dt interval is

$$\mathbb{Q}_{\mathrm{t}}\left\{\tau^{\mathrm{i}} \in [\mathrm{t, t+dt}] \mid \tau^{\mathrm{i}} \geq \mathrm{t,F_t}\right\} = \lambda^{\mathrm{i}}(\mathrm{t})dt \tag{3.6.9}$$

where F_t is a function that contains the default free market information up to time-t. The likelihood that the default time τ^i is greater than t is given as;

$$\mathbb{Q}_{\mathrm{t}}\left\{\tau^i \geq t\right\} = \mathbb{Q}_{\mathrm{t}}\left\{\Lambda\left(\tau^i\right) \geq \Lambda(t)\right\} = \mathbb{Q}_t\left\{\Lambda\left(\tau^i\right) \geq \int_0^t \lambda(s)\, ds\right\} = E_t\left[\mathbb{Q}_t\left\{\Lambda\left(\tau^i\right) \geq \int_0^t \lambda^i(s)\, ds\right\} \mid F_t\right] \tag{3.6.10}$$

Also the cumulated intensity at default time-t denoted as $\Lambda\left(\tau^i\right) = \xi$ is viewed as an exponential random variable that is independent of F_t , therefore

$$E_t\left[\mathbb{Q}_t\left\{\Lambda\left(\tau^i\right) \geq \int_0^t \lambda^i(s)\, ds\right\} \mid F_t\right] = E_t\left[\mathbb{Q}_t\left\{\xi \geq \int_0^t \lambda^i(s)\, ds\right\}\right] = E_t\left(e^{-\int_0^t \lambda^i(s)ds}\right) \tag{3.6.11}$$

3.6.3 Interbank Cash market liquidity

For banks to meet their day to day liquidity needs, they depend on interbank cash markets, however for longer periods the interbank money market cannot necessarily be relied upon to constantly provide funding.Therefore a liquidity shock is a significant risk in all cash transactions between financial institutions. Higher rates are charged by the refinancing bank in the event of a market freeze during the lifetime of an interbank market loan. The possibility of an interbank money market freeze at a random time σ is accounted for in the XIBOR model(Gallitschke *et al.*, 2014). Before the market freeze at time σ , the unsecured overnight borrowing is available at the overnight rate $r^{O/N}$. In the event of a market freeze (that is at time $t > \sigma$) banks can not access the interbank market at a risk free rate anymore but can obtain funding at a higher rate ;

$$r^{O/N}(t) + \rho_t > r^{O/N}(t) \tag{3.6.12}$$

If we denote the risk neutral intensity of σ by $\vartheta = \{\vartheta(s)\}$ then the liquidity factor can be espressed as

$$a_0(t,T) = 1_{\{\sigma>t\}} E_t\left[e^{-\int_t^T \vartheta(s)ds}\right] \tag{3.6.13}$$

If there is no liquidity risk then ρ_t is equivalent to zero.The random time σ is also modelled using the cox process. The interest rate, credit and liquidity components highlighted above are combined to determine interbank money market lending rates(Gallitschke *et al.*, 2014).When a bank (i) borrows money in the interbank market the lending bank (j) is said to have bought the interbank bond $\bar{P}^i$ of the borrowing bank (i). When determining the interest rate/price to be paid by the borrowing bank (i) the lender (j) takes into account the credit risk of the borrower and the funding risk of the transaction. The bond price $P^i(t,T)$ is adjusted for excess funding cost in case of liquidity freeze.This approach may be regarded as funding liquidity valuation adjustment. Bank $i's$ interbank bond price with liquidity adjustment can be given by

$$\bar{P}_l^i(t,T) = P^i(t,T) - E_t\left[\int_t^T e^{-\int_t^s r^{O/N}(u)du} E(s)ds\right] \tag{3.6.14}$$

where $E(s)$ is the excess funding cost at time s, even in the event of a liquidity freeze with the price adjustment continued funding is guaranteed. We can denote the notional amount to be refinanced at time s as

$$\overline{N}(s) = \bar{P}_l^i(t,T)\, e^{\int_t^s r^{O/N}(u)du} \tag{3.6.15}$$

In case of interbank market freeze or liquidity event $(\sigma \le s < \tau^i)$ the principal amount is refinanced at a higher rate $r^{O/N}(t) + \rho_t > r^{O/N}(t)$ hence the lender incurs excess cost of lending

$$E(s)\,ds = \rho_t \overline{N}(s)\,ds \tag{3.6.16}$$

Excess cost is applicable only when the borrower defaults hence

$$E_s = 1_{\{\sigma\le s\le \tau^i\}} \rho_t \overline{N}(s)\,ds = 1_{\{\sigma\le s\le \tau^i\}} \rho_t \bar{P}_l^i(t,T)\, e^{\int_t^s r^{O/N}(u)du} \tag{3.6.17}$$

3.6.3.1 Interbank Loan and funding cost

The price of bank $i's$ interbank loan/bond with liquidity adjustment is (Gallitschke *et al.*, 2014)

$$\bar{P}_l^i(t,T) = \frac{P^i(t,T)}{1+q_\rho^i(t,T)} \tag{3.6.18}$$

where

$$q_\rho^i(t,T) = \left[\int_t^T E_t\left[1_{\{\sigma \le s \le \tau^i\}}\rho_s ds\right]\right] \tag{3.6.19}$$

and bank $i's$ interbank unsecured funding costs in the interbank market are given by

$$\bar{L}^i(t,T) = \frac{1}{T-t}\left(\frac{1+q_\rho^i(t,T)}{P^i(t,T)} - 1\right) \tag{3.6.20}$$

$\bar{P}_l^i(t,T) = P^i(t,T)$ and $\rho = 0$ if there is no market liquidity freeze.

3.6.3.2 Determining XIBOR rates

XIBOR fixing are compiled by the calculation agent upon receipt of XIBOR quotes from XIBOR panel banks who provide the rates at which they could borrow were they to do so in the interbank market. Let I denote all the banks, and $P(t)$ the banks that are relevant as XIBOR contributors or panel banks with high credit quality, that is $P(t) \subseteq I$. Xibor rates are determined under the assumption that the submissions are informed by both current and previous interbank funding costs. Thus bank $i's$ Xibor estimate is given by (Gallitschke *et al.*, 2014).

$$L^i(t,T) = \omega L(t-1d, T-1d) + (1-\omega)\, quote^i\left[\bar{L}^j(t,T)\; j \in P(t)\right] \tag{3.6.21}$$

where $\omega \in [0,1]$ is defined as the weight factor and $quote^i[.]$ is a function that determines bank $i's$ current market view of all participating institutions in the interbank market. If $\omega = 0$ and $quote^i[L] = L^i$ then bank i will be compeletely transparent on its funding situation/decision. If $\omega > 0$ then it becomes possible to use the previous day fixing as an indicator of the overall cash market funding conditions. The formal defination of the XIBOR fixings is given by

$$L(t,T) = avg\left[L^i(t,T)\,, i \in P(t)\right] \tag{3.6.22}$$

where $avg[.]$represents the trimmed average. If the number of banks with high credit quality is not enough to determine the XIBOR rate then the XIBOR fixing is set as

$$L(t,T) = \frac{1}{T-t}\left(\exp\left(r^\infty (T-t)\right) - 1\right) \tag{3.6.23}$$

where $r^{\infty} > 0$ is a fixed stable rate.

3.7 Basis Spreads In the XIBOR Model

The assumpion when pricing derivatives under XIBOR framework is that they are fully collateralised, so credit and liquidity risks can be neglected but the risk should be priced in the cash market. Another assumption is that the interest rate applied to collateralised derivative instruments is the overnight rate $r^{O/N}(t)$.

3.7.1 Foward rate agreement in a XIBOR model

The fair price of a FRA at time zero is equivalent to (Gallitschke *et al.*, 2014)

$$\pi = E\left(e^{-\int_0^T r^{O/N}(s)ds}\left[L(t,T)-K\right](T-t)\right) \tag{3.7.1}$$

Therefore in a XIBOR model the par FRA rate is given by

$$F(0,t,T) = \frac{\ell(0,t,T)}{\delta(0,T)} \tag{3.7.2}$$

where for $t_0 \leq t \leq T$ we have

$$\ell(0,t,T) = E_{t_0}\left[e^{-\int_{t_0}^T r^{O/N}(s)ds} L(t,T)\right] \tag{3.7.3}$$

as the conditional expectation of the XIBOR rate at time t_0 over the interval $[t,T]$ discounted to time t_0 using the risk free overnight rate. In the pre crisis setting

$$\ell(0,t,T) = \frac{\delta(0,t)-\delta(0,T)}{T-t} \tag{3.7.4}$$

infering that the foward rate$F(0,t,T) = \frac{1}{T-t}\tilde{}\left(\frac{1+L(0,T)T}{1+L(0,T)T}-1\right)$ which no longer hold in the XIBOR model.

3.7.2 Overnight Index Rate (OIS Rate)

An OIS swap is an interest rate swap which involves exchange of a fixed rate $K > 0$ for a floating rate calculated from daily interest investment or ex post compounded overnight rate given by (Gallitschke *et al.*, 2014)

$$L^{O/N}(t,T) = \frac{1}{T-t}\left(e^{\int_t^T r^{O/N}(s)ds}-1\right) \tag{3.7.5}$$

At every time $t = m\triangle, m = 1,..,M$ the fixed leg of the OIS swap pays $K\triangle$ while the floating leg pays $L^{O/N}((m-1)\triangle, m\triangle)\triangle$, where $m\triangle$ is the maturity T of the swap. We will work with $\triangle = 1y$ as a standard market assumption. The fixed and floating fair price is given by

$$\pi^{fix} = K\triangle \sum_{m=1}^{M} \delta\left(0, m\triangle\right) \tag{3.7.6}$$

$$\pi^{OIS} = E\left[\sum e^{-\int_0^{m\triangle} r^{O/N}(s)ds} L^{O/N}\left(\left(m-1\right)\triangle, m\triangle\right)\triangle\right] = 1-\delta\left(t,T\right) \tag{3.7.7}$$

The associated pre crisis OIS swap rate is given by

$$swap^{OIS}\left(T\right) = \frac{1-\delta\left(t,T\right)}{\triangle\sum_{m=1}^{M}\delta\left(0,m\triangle\right)} \tag{3.7.8}$$

where $\triangle = 1y$

3.7.3 XIBOR - OIS Swaps

It is an interest rate swap that involves exchange of XIBOR based rates for an OIS rate with the same maturity with spread σ added on the OIS leg. Letting $T = m\triangle$ denote the swap maturity then, at time-t the OIS payer pays (Gallitschke *et al.*, 2014)

$$\left[L^{O/N}\left(\left(m-1\right)\triangle, m\triangle\right)+\sigma\right]\triangle \tag{3.7.9}$$

while the XIBOR payer pays

$$L\left(\left(m-1\right)\triangle, m\triangle\right)\triangle \tag{3.7.10}$$

The spread σ was considered negligible and approximately equal to zero ($\sigma = 0$) in pre - crisis framework . This made the correct value of each leg to be

$$1-\delta\left(0,T\right) \tag{3.7.11}$$

However, the fair prices of each leg in the post - crisis setting are;

$$\pi^{OIS} = E\left[\sum_{m=1}^{M} e^{-\int_0^{m\triangle} r^{O/N}(s)ds}\left[L^{O/N}\left(\left(m-1\right)\triangle, m\triangle\right)+\sigma\right]\triangle\right] \tag{3.7.12}$$

$$= 1-\delta\left(0,T\right)+\sigma\triangle\sum_{m=1}^{M}\delta\left(0,m\triangle\right)$$

$$\pi^{\triangle} = E\left[\sum_{m=1}^{M} e^{-\int_0^{m\triangle} r^{O/N}(s)ds}\left[L\left((m-1)\triangle, m\triangle\right)\right]\triangle\right] \tag{3.7.13}$$

$$= \triangle\sum_{m=1}^{M} \ell\left(0\left(m-1\right)\triangle, m\triangle\right)$$

By equating the two legs we obtain the $\triangle-$XIBOR-OIS par swap spread as follows

$$Spread_{OIS}^{\triangle}(T) = \frac{\sum_{m=1}^{M} \ell\left(0\left(m-1\right)\triangle, m\triangle\right) - \frac{1-\delta(0,T)}{\triangle}}{\sum_{m=1}^{M} \delta\left(0, m\triangle\right)} \tag{3.7.14}$$

where $\ell\left(t_0, t, T\right)$ is as defined in (3.7.3) and $T = M\triangle$ is the maturity of the swap.

In a XIBOR swap a fixed rate K (for $1y$) is exchanged for a floating rate attached to XIBOR rate. The payment for each leg is;

Fixed leg; K for$1y$ at $t = 1,2,3,,T$

Floating leg; $L\left((m-1)\triangle, m\triangle\right)\triangle$at time $t = m\triangle$, $m = 1,2,3.,M$.

The resulting par swap rate can be given by

$$Swap^{\triangle}(T) = \frac{\triangle\sum_{m=1}^{M} \ell\left(0\left(m-1\right)\triangle, m\triangle\right)}{\sum_{t=1}^{T} \delta\left(0, t\right)} \tag{3.7.15}$$

3.7.4 Tenor Basis Swap and Tenor basis Spread

Tenor basis swap is an interest rate swap that involves exchange of two similar rates (XIBOR rates) with different tenors. Consider a $\triangle/\alpha\triangle$ tenor basis swap where $\alpha = 2,3,4,.,$then the payment on both legs can be expressed as; (Gallitschke *et al.*, 2014)

payment on $\triangle - XIBOR$ leg is

$$\left[L\left((m-1)\triangle, m\triangle\right) + \sigma\right]\triangle$$

at $t = m\triangle$,for $m = 1,2,3,4,.,M$

and that of $\alpha\triangle - XIBOR$ leg is

$$\left[L\left((n-1)\alpha\triangle, \alpha n\triangle\right)\alpha\triangle\right]$$

at $t = n\alpha\triangle$ for $n = 1,2,.,N$, where maturity $T = M\triangle = N\alpha\triangle$.

For example if $\triangle = 0.25y$ and $\alpha = 2$, then we will be referring to a 3m/6m tenor basis swap. The tenor basis swap legs are valued as

$$\pi^{\triangle} = E\left[\sum_{m=1}^{M} e^{-\int_0^{m\triangle} r^{O/N}(s)ds}\left[L\left((m-1)\triangle, m\triangle\right)+\sigma\right]\triangle\right] \tag{3.7.16}$$

$$= \triangle\sum_{m=1}^{M}\ell\left(0,(m-1)\triangle, m\triangle\right)+\sigma\triangle\sum_{t=1}^{T}\delta\left(0,m\triangle\right)$$

$$\pi^{\alpha\triangle} = E\left[\sum_{n=1}^{N} e^{-\int_0^{n\alpha\triangle} r^{O/N}(s)ds}\left[L\left((n-1)\alpha\triangle, \alpha n\triangle\right)+\sigma\right]\alpha\triangle\right] \tag{3.7.17}$$

$$= \alpha\triangle\sum_{n=1}^{N}\ell\left(0,(n-1)\alpha\triangle, n\alpha\triangle\right)$$

Therefore as a result of the price difference between the tenors $\triangle$ and $\alpha\triangle$ with maturity $T = M\triangle = N\alpha\triangle$, we obtain tenor basis spread of a swap given by

$$Spread_{\triangle}^{\alpha\triangle}(T) = \frac{\alpha\sum_{n=1}^{N}\ell\left(0,(n-1)\alpha\triangle, n\alpha\triangle\right)-\sum_{m=1}^{M}\ell\left(0,(m-1)\triangle, m\triangle\right)}{\sum_{m=1}^{M}\delta\left(0,m\triangle\right)} \tag{3.7.18}$$

To compute the price of any XIBOR based interest rate linear product we need the discount curve $\delta(0,t)$and the function $\ell(0,t,T)$. We determine $\ell(0,t,T)$by use of monte carlo simulation since its not possible to compute it (Gallitschke *et al.*, 2014).

3.8 Dynamic XIBOR Model

The model is based on individual bank's dynamic stochastic risk factors; credit risk,interbank liquidity risk and interest rate risk/riskless term structure. The model explains the emergence of post crisis XIBOR spreads naturally from fundamentals in a unified, arbitrage free interbank cash market.The model is fully based on the significant risk factors and it generates basis spreads endogenously as risk premia associated to credit risk rather than taking basis spreads as input like in the case of static XIBOR model. It also does not assume the notion of hypothetical bank .The framework allows XIBOR fixings to be constructed from individual banks submissions through an explicit panel mechanism (Gallitschke *et al.*, 2014).

3.8.1 Riskless rate in a dynamic XIBOR model

The high volatility of interest rates in today's market call for application of stochastic interest rate models to model the interest rate movements. Interest rate models are also important in quantifying the interest rate risk of interest rate securities.The short rate is the limit of the interest rate of a bond as maturity tend to zero. Therefore, short rate can not be observed directly in the market althogh it can be thought of as a the overnight interest of reserve banks, because it is the interest rate with the shortest period in the market. Alternatively, yields of very short termed bonds (1 month or 3 months to maturity) can be used as approximate value of the short rate. We use the notation r_t to denote the short rate process because it changes over time hence it is important to have stochastic model that describes the dynamics of the short rate process r_t. Short rate defines all fundamental quantities (eg. yields, forward rates and bond prices) and also most stochastic interest rate models characterize the dynamics of the short rate and are therefore also called short rate models(Dagıstan, 2010).

The risk free rate $r^{O/N}$ is modelled by CIR ++ model under the assumption of (Brigo and Mercurio, 2001a)that the instantaneous short rate changes under the risk neutral measure given by the dynamics

$$r^{O/N}(t) = x^0(t) + \psi(t;\alpha) \tag{3.8.1}$$

where

$$dX^0(t) = k^0\left[\theta^0 - X^0(t)\right]dt + \sigma^0\sqrt{X^0(t)}dW^0(t) \tag{3.8.2}$$

subject to $x(0) = x_0$. k is the long term mean reversion coefficient, $\theta$is the long term mean of the process $x(t)$,σ is the volatility coefficient under the assumption that $k,\theta,\sigma > 0$ are positive constants(Dagıstan, 2010). $\psi(t;\alpha)$ is a deterministic shift extension that depends on the parameter vector α and time where $\alpha = (k,\theta,\sigma)$. $dW(t)$is a standard brownian motion under the risk-neutral measure. The model gives an exact fit to the term structure of interest rates, by letting $\psi(t;\alpha) = \psi^{CIR}(t;\alpha)$ where

$$\psi^{CIR}(t;\alpha) = f^M(0,t) - f^{CIR}(0,t;\alpha) \tag{3.8.3}$$

$$f^{CIR}(0,t;\alpha) = \frac{2k\theta\left(exp\{th\}-1\right)}{2h+(k+h)\left(exp\{th\}-1\right)} + x_0\frac{4h^2exp\{th\}}{\left[2h+(k+h)\left(exp\{th\}-1\right)\right]^2} \tag{3.8.4}$$

$h = \sqrt{k^2+2\sigma^2}$ and $f^M(0,t)$ is the market instantaneous foward rate.

3.8.2 Default intensity modelling

Default intensity is modelled by use of a reduced form model whose choice is based onDuffie and Singleton (2003) stochastic processes.Duffie and Singleton (2003)defined foward probability in the view of modelling credit risk.

3.8.2.1 Foward probability

All the available information up to time t is considered when modelling default probability of each borrower. Therefore any new information received in the market about the borrowers will have an impact on the default intensity(Pereira, 2013). If we let $p(t)$denote the probability that a potential borrower will not default in the next t years ,the probability of surviving to time s,given survival to t can be denoted by

$$p(s/t) = \frac{p(snt)}{p(t)} = \frac{p(s)}{p(t)} \tag{3.8.5}$$

meaning default probability to s given survival to t is $1 - p(s \mid t)$and this is what is reffered to as the foward probability. Also we let the distribution of T to be defined by $f(t)$under the assumption that the survival function at time t is given by $p(t) = p(T \geq t)$and the default intensity is defined by λ_t. Now assuming that the survival probability $p(t)$is differentiable function in t and is stricly positive, we can deduce that the distribution of T is also given by its hazard function simply because the survival function is also specified by hazard function. This leads to the expression

$$p(t) = e^{-\int_t^s f(u)du} \tag{3.8.6}$$

since s and t are two independent times and $s > t$ then we have $p(s \mid t) = e^{-\int_t^s f(u)du}$. Therefore the foward default rate is $f(t)$ and can be used to model the term structure of default risk (Pereira, 2013).

3.8.2.2 Doubly Stochastic Default Intensity

Let (Ω, F, P)be a probability space and $\mathcal{F}_t : t > 0$ a filtration, N an increasing counting process $\{T_0, ., T_n\}$of real random variables over the interval $[0, \infty]$, let also λ_t be a positive process such that for all t , $\int_0^t \lambda_s ds < \infty$ with almost certainity. In reduced form models default is defined as the first jump or arrival time τ of a poisson process with arrival intensity λ. There are properties of default intensity derived from poisson process which include;

The survival probability $p(t) = e^{-\lambda_t}$ implying that the time to default is exponentially distributed. The assumption under poisson process is that the intensity is constant, implying that the arrival risk over time in a default intensity model are independent. Therefore, default time is viewed as inaccessible meaning it is unpredictable. Hence λ_t can not be predicted on the basis of all available information up to time less than t. Therefore, the intensity can be allowed to varry deterministically(Duffie and Singleton, 2003). Thefore,Duffie and Singleton (2003), implied that if the intensity λ_t is allowed to varry deterministically then it coincides with the foward default intensity function f_x, meaning that survival is the only useful information to arrival of default risk over time.Duffie and Singleton (2003) conclude that if the intensity varries deterministically over time then,

$$p(t) = e^{-\int_0^t \lambda(t)dt} \tag{3.8.7}$$

where $\lambda(t)$ is the intensity at time t but incase of any additional time the intensity varries. This leads us to choose a model where intensity varries continously and this model is referred to as doubly stochastic default model by(Duffie and Singleton, 2003). The model is based on the information given by $\{\lambda(t) : t \geq 0\}$ which is a path of the default intensity where default is assumed to follow a poisson process with time varrying intensity. According to doubly stochastic property, for any$t$, if default time τ arrives after t, then the survival probability to a future time given $\mathcal{F}_t = \{\lambda(s)\} : 0 \leq s \leq t$ is given by

$$p(t) = E[P(\tau > s \mid F_t)] = E\left[e^{-\int_0^t \lambda(t)dt} \mid F_t\right] \tag{3.8.8}$$

which is the expected value of the interarrival times up to t. Since default intensity is time varrrying and default arrival conditional on available information follow a poisson process with random intensity then a doubly stochastic uncertainty arises(Pereira, 2013).The doubly stochastic implies that the filtration F_t contains enough information to help predict the intensity λ_t but the information is not enough to predict the counting process N. Therefore, the intensity model can be specified though it is difficult to compute default probability since we dont know the form that the intensity assumes over time. This implies that we need to know the parameterization of $\lambda(t)$ in order to use the doubly stochastic model to calculate default probability. Parameterization can be done by use of stochastic models such as CIR model that is we assume the intensity follow a CIR model in order to simulate it.

The individual bank's default intensity λ^i follow the dynamics

$$\lambda^i(t) = x^i(t) \tag{3.8.9}$$

Where $dX^i(t) = k^i\left[\theta^i - X^i(t)\right]dt + \sigma^i\sqrt{X^i(t)}dW^i(t)$ is a Feller diffusion model named after(Cox *et al.*,

1985) and $W^{I},..,W^{I}$ are correlated Wiener process with $dW^{i}(t)\,dW^{j}(t) = P^{ij}dt$ implying a non zero correlation between individual default risk process that helps capture likely spillover effects in the interbank cash market.

Dynamic XIBOR model assumes the following: (Gallitschke *et al.*, 2014)

(i) The model is based on fully dynamic and fully correlated model for interbank credit risk

(ii) The liquidity freeze intensity ϑ is assumed to be constant for simplicity

(iii) Interbank Bond Prices under Static XIBOR model

From the general representation of XIBOR model we have

$$\delta(t,T) = E_t\left[e^{-\int_t^T r^{O/N}(s)ds}\right] \tag{3.8.10}$$

While the risk neutral survival probability for each bank i is given by;

$$P_{00}^{i}(t,T) = 1_{\left\{\tau^{\mathrm{i}}>\mathrm{t}\right\}} E_{\mathrm{t}}\left[\mathrm{e}^{-\int_{\mathrm{t}}^{\mathrm{T}} \lambda^{\mathrm{i}}(\mathrm{s})\mathrm{ds}}\right] \tag{3.8.11}$$

3.8.3 XIBOR Quotes , panel and Fixings

Bank $i's$ XIBOR fixing can be expressed as (Gallitschke *et al.*, 2014)

$$L^{i}(t,T) = \omega L(t-1d, T-1d) + (1-\omega)\, quote^{i}\left[L^{j}(t,T)^{-} j \in P(t)\right] \tag{3.8.12}$$

where

$L(t-1d, T-1d)$ gives the relevant previous day fixing and (Gallitschke *et al.*, 2014)

$$L^{j}(t,T) = \frac{1}{T-t}\left(\frac{1+q_{\rho}^{j}(t,T)}{P^{j}(t,T)} - 1\right) \tag{3.8.13}$$

under the assumption of the best peer approach $quote^{i}[L] = min_{j\in p(t)} L^{j}$.

As explained in the general XIBOR framework in dynamic XIBOR model the bank reports a weighted average of the previous day fixing and the funding cost of its best peer in the panel. Also the initial panel P(0) is chosen according to the actual LIBOR panel at that date. A bank is removed from the panel upon default and it is immediately replaced by a bank with a better credit quality selected at random from a set of survivor banks not previously in the panel.

3.9 SIMULATION METHODS

There are several methods that can be used in simulation of CIR++ process among which include Exact simulation and approximate Euler simulation. Exact method was applied in simulating CIR++ and approximate method in simulating CIR model for default risk(credit risk modelling).

3.10 Exact Method

It is possible to simulate CIR++ process exactly by simulating CIR process and adding the deterministic shift extension. However, in CIR model the increments of the short-rate follows a non-central chi-square distribution. The transition density of $r(t)$ can be written as follows from the initial CIR SDE(Glasserman, 2003).

$$r(t) = \frac{\delta^2\left(1-e^{-k(t-u)}\right)}{4k}\chi_d^2\left(\frac{4k\left(e^{-k(t-u)}\right)}{\delta^2\left(1-e^{-k(t-u)}\right)}r(u)\right), t>u \tag{3.10.1}$$

where $d = \frac{4\mu k}{\delta^2}$

This indicates that, given $r(u), r(t)$ is distributed as $\frac{\delta^2\left(1-e^{-k(t-u)}\right)}{4k}$ times a non-central chi-square random variable with d degrees of freedom and non-centrality parameter

$$\lambda = \frac{4k\left(e^{-k(t-u)}\right)}{\delta^2\left(1-e^{-k(t-u)}\right)}r(u) \tag{3.10.2}$$

Therefore to simulate $r(t)$by use of the exact method at time $0 = t_0 < t_1 ..... < t_n$ we use the recurssion

$$r(t_{i+1}) = \frac{\delta^2\left(1-e^{-k(t-u)}\right)}{4k}\chi_d^2\left(\frac{4k\left(e^{-k(t-u)}\right)}{\delta^2\left(1-e^{-k(t-u)}\right)}r(t_i)\right) \tag{3.10.3}$$

This method is slow when the non centrality parameter is large and degrees of freedom is less than 1. The method is also slower than Euler approximate method (Shao, 2012).

3.11 Approximate Euler Scheme Simulation.

Euler scheme provides an approximate numerical solution of a stochastic differential equation. The Euler discretization scheme can be used to simulate the process $r(t)$ because the exact simulation method requires sampling from the non-central chi-square distribution. This method of simulation proposes simulating $r(t)$ at times $0 = t_0 < t_1 ..... < t_n$ by setting

$$r(t_{i+1}) = r(t_i) + k(\mu - r(t_i))[t_{i+1} - t_i] + \delta\sqrt{r(t_i)}\sqrt{(t_{i+1}) - t_i}Z_{i+1} \tag{3.11.1}$$

where $Z_1, Z_2, ..., Z_n$ are independent standard normal $N(0,1)$random variables.

Simulatin of CIR process using euler Euler scheme is fast and exact on a large number of time grids as compared to exact method. other schemes like Milstein scheme generally not well defined since they can result to negative values for which the square root is undefined. This method is applied in default modelling because it is faster and there is need to simulate CIR process along a time grid with many time points. One feature of derivatives is that they are path dependent that is, their price depends on many time points and not a single one. In order to price a path dependent derivative there is need to simulate an asset price process along many time points using monte carlo method (Shao, 2012).

3.12 Parameter Estimation

We use the maximum likelihood estimation to estimate the parameters of the CIR++ model, here the parameters of the CIR++ model were estimated by use of the 3 month maturity US daily yield data. The parameters of the short rate in the CIR model have a non-Gaussian distribution hence there are no closed form formulas for the maximum likelihood estimate of the parameters. This leads to use of numerical optimization to obtain the MLE's(Dagıstan, 2010). So we require the maximum likelihood of the CIR++ model. From the transition density of the short rate in 3.10.1 the cumulative distribution of $r(t)$can be given by

$$P(r(t) \leq y/r(u)) = F_{\chi^2_{d,\lambda}} \frac{4ky}{\delta^2\left(1 - e^{-k(t-u)}\right)} \tag{3.12.1}$$

with d and λ as defined above.

Therefore the pdf is

$$P_{r(t)}(y/r(u)) = CP_{\chi^2_{d,\lambda}}(Cy) \tag{3.12.2}$$

where $P_{\chi^2_{d,\lambda}}(Cy)$ is the non-central chi-square distribution density and $c = \frac{4k}{\delta^2\left(1-e^{-k(t-u)}\right)}$,Therefore, the likelihood and the log likelihood function can be expressed as

$$L(\mu, k, \delta; y) = \prod_{i=2}^{n} CP_{\chi^2_{d,\lambda}}(Cy_i/y_{i-1}) \tag{3.12.3}$$

$$l(\mu,k,\delta;y)=\sum_{1=2}^{n}log(c)+\sum_{i=2}^{n}log\left(P_{\chi^2_{d,\lambda}}(Cy_i/y_{i-1})\right) \tag{3.12.4}$$

,

where $y = r_1, r_2, \ldots.., r_n$. we will consider dt to be either 1 meaning monthly or daily. With the log likelihood numerical optimization can be used to find the MLE's. To test the accuracy of our MLE method we simulate the process using fixed parameters and estimate parameters of the test data.

3.13 Confidence interval (in simulating default risk)

The 95% confidence interval used has the form

$$\frac{1}{n}\sum_{1=1}^{n}X_i-1.96\frac{\sigma}{\sqrt{n}};\frac{1}{n}\sum_{1=1}^{n}X_i+1.96\frac{\sigma}{\sqrt{n}} \tag{3.13.1}$$

CHAPTER 4

SIMULATION AND ANALYSIS

4.1 Introduction

In this chapter the focus is on simulation and analysis, show various spreads in different market rates for example LIBOR rate, swap rates and overnight rates with different tenors with an aim of identifying the break point, simulate the short rate dynamics $r(t)$ by use of CIR++ model, estimate model parameters by use of maximum likelihood estimation method, simulate credit risk (default intensity) and show the effect of varrying model parameters and price a defaultable zero coupon bond before default.

4.1.1 Data Description

In the estimation of short rate model parameters daily LIBOR data was collected from January 2005 to December 2015 from federal reserve bank website. Daily swap market data was also collected from the same website to help identify the breakpoint as a result of the crisis.

4.2 Basis spreads that characterize post - crisis fixed income market

Before Financial crisis of 2007 similar rates for different tenors moved together, for example there was no significant difference between $3Month$ and $6Month$ LIBOR rates. As a result of the crisis the rates began to varry/ show a huge difference depending on the tenors hence calling for pricing of different tenors differently. Figure 4.1.1 shows a plot of $3M$ vs $6M$ LIBOR market data for the period between January 2005 to December 2015.

4.2.1 Pre and post crisis LIBOR rate plot

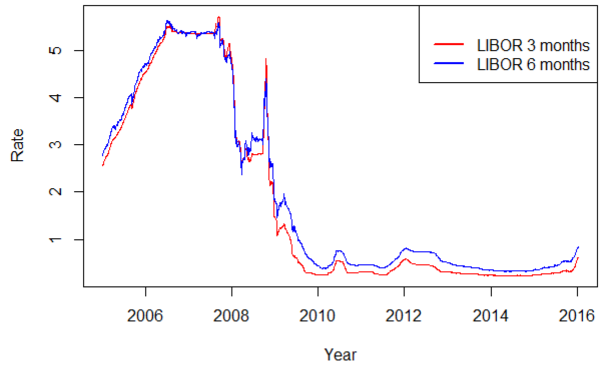

Figure 4.2.1: Spread between US 3M and 6M LIBOR daily rate market data

From figure 4.2.1 ,3 month and 6 month LIBOR rates were low and moved closely in 2005 and 2006 but went up between 2007 and 2008. From 2009 the rates went down and began to differ as displayed by the plot, 6 month LIBOR rate is greater than 3 month LIBOR rate and the difference can no longer be assumed negligible as before. This shows that there is a high perceived credit risk as the tenor increases and therefore it supports pricing of financial securities depending on the underlying tenor as well as pricing credit risk in the market. This is the key driver that motivates the research on new post crisis methods such using stochastic models to model short rate and pricing of credit risk in the interbank market.

4.2.2 Pre and post crisis swap rate market plot

Financial derivative rates also went up and began to differ depending on the underlying tenor, the spread between the rates indicates the level of credit and liquidity in the market. Figure 4.2.2 shows a plot of $5-year$ vs $10-year$ year maturity interest rate swap.

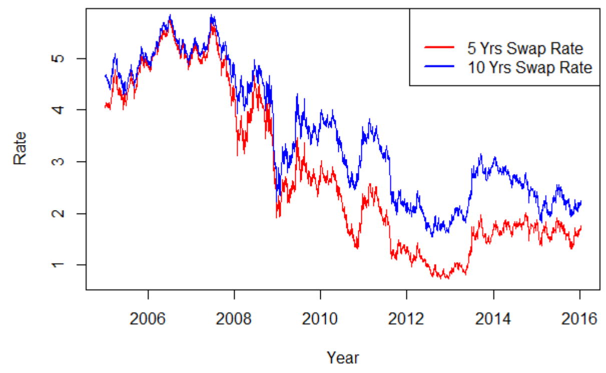

Figure 4.2.2: spread between 5 year and10 year swap market data

The difference between swap rates for different tenors was negligible before crisis for example 3 month and 6 month swap market rate as well as moved together before crisis, but after crisis there is a huge spread between them. Since financial crisis occurence the spread has no longer been negligible which signifies a major change in pricing of interest rate derivatives rendering the single curve pricing methods obsolete. The spread signifies high credit risk for long maturities as opposed to short maturities. Interbank rates are used as reference rates for pricing of interest rate derivatives and since as shown in figure 4.2.1 the rates went up and began to differ even the prices of IRS went up as shown in figure 4.2.2.

4.3 Instantaneous short rate Simulation (Risk - free rate Modelling)

In simulating the short rate path we have to define initial model parameters,therefore, we set initial parameters $k = 0.1, \theta = 0.06, \sigma = 0.06$. Figure 4.3.1 shows the plot of the exact simulation of CIR++ process for 5000 time steps. The short rate is stochastic that is it changes over time.The process approaches its long term mean slowly because of a small value of the mean reversion coefficient. The process also has low fluctuation because the volatility is small. The process starts at the initial value of 0.1 but it is pulled towards its long term mean of 0.06 which is the value of θ. The simulation results helps confirm the main advantage of CIR++ model in that the rates can never drop below 0%. This represents the real world situation where interest rates are not negative.

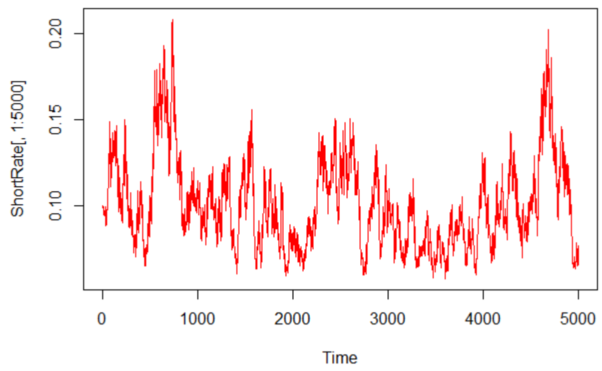

Figure 4.3.1: Simulated path of the short rate by CIR++ process

We also simulate the short rate path using parameters $k = 0.128, \theta = 0.052$ and $\sigma = 0.066$ to have a view of how the process behaves with variation in parameters.The process starts at the initial value of 0.1 but it is pulled towards its long term mean of 0.052 which is the value of $\theta$and the rates remain positive. As noted the process is mean reverting and the speed at which the mean reverts depends on the parameter k of the CIR++ model.

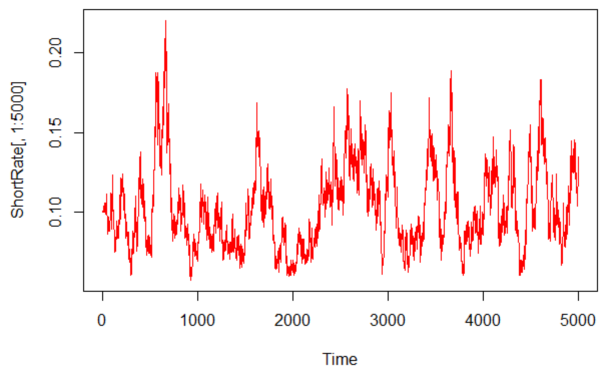

Figure 4.3.2: Short rate path simulation in CIR++ model

4.3.1 MLE results

To test the implementation of the MLE method, first test data was simulated and parameters of the test data estimated. For the case of $k = 0.1, \theta = 0.06, \sigma = 0.06$ the MLE of the test data was obtained as displayed in table 4.1. The parameters are close to the initial parameters used to generate test data hence the implementation of MLE can be considered accurate which helps us estimate model parameters by use of real data.

Table 4.3.1: MLE of the test data

Parameter	Value
$\hat{\theta}$	0.06123275
$\hat{k}$	0.09005961
$\hat{\sigma}$	0.06162169

Table 4.1 shows the maximum likelihood estimates of the CIR++ parameters. Standard values of CIR++ parameters θ,k and σ for US market were estimated using the historic 3 month period US daily LIBOR rate data(time series observation) for the period between January 2005 to July 2008 (pre crisis) and (post crisis) September 2009 to December 2013(Data source; Federal Reserve Economic data website). The reason for use of 3 month US LIBOR rate in estimating is because it is the shortest unsecured lending rate and we are modelling the short rate. The maximum likelihood estimates of the model parameters are as shown in table 4.2 below;

Table 4.3.2: MLE of the US 3 month LIBOR rate for pre and post crisis period

Parameter	Pre crisis period Value	Post crisis period Value
$\hat{\theta}$	1.00000000	0.22725025
$\hat{k}$	0.01000000	0.01000000
$\hat{\sigma}$	0.04679275	0.01736931

During pre-crisis period the long term mean θ is greater than the post crisis period as shown in table 4.2. The mean reversion coefficient is the same for the two periods meaning the rate of convergence to the long run mean is the same. The volatility during pre-crisis period is higher than post-crisis period indicating that the rates fluctuated more as a result of crisis and less after crisis. The analysis shows that interest rate risk in the market was higher during as result of the crisis as compared to after crisis period.

Table 4.3.3: A summary statistic of 3 month US Libor daily data (from Jan 2005 to Dec 2015)

Statistic	Mean	Max	Min	St.Dev	Ex. Kurt	Skewness
Value	1.7739	5.7250	0.1396	1.989089	-0.9553063	0.8659679

The standard deviation (1.989089) is high indicating high variability of the rates. The Excess kurtosis is (-0.9553063) which means Kurtosis is less than 3 and the skewness is greater than 0 indicating that the data is non-normal. Non normality is one of the stylized facts of the financial market data.

4.4 Default intensity simulation (Credit risk modelling)

In modelling credit risk reduced form modelling approach was used. The assumption was that the default intensity follow a CIR model (a diffusion model) proposed by(Cox *et al.*, 1985).The closed form of a CIR stochastic differential equation has been derived in(Duffie and Singleton, 2003). When simulating CIR process there is a need to specify the initial values of $\lambda(0)$,k,$\theta$and σ. Finding the exact values of these parameters might be a problem, therefore we can start by testing the different CIR process parameters (k,θ,σ) selected at random so as to appreciate how the default intensity process $\lambda(t)$ reacts.We first simulate the independently and identically distributed trajectories of the diffusion process and apply monte carlo simulation idea to assume that those trajectories are representatives of a random variable, then compute the expected value of the sum of those realizations (law of large number) as well as 95% confidence interval(Central limit theorem). This helps obtain a survival probability $P(s/t) = e^{-\int_t^s \lambda(u)du}$ which corresponds to the foward probability under deterministic variation of $\lambda(t)$assumption. Due to the velocity of the exponential function we compute the logarithm of the foward probability. Then we assume the variability of $\lambda(t)$ and compute the doubly stochastic survival probability

$$P(s/t) = E\left[e^{-\int_t^s \lambda(u)du}/F_t\right] \tag{4.4.1}$$

The logarithm of the functions of survival probability and doubly stochastic survival probability is calculated as well due to the velocity of the exponential function(Pereira, 2013).

We assume that default intensity is not constant but it is a function of other stochastic variables hence default time can be generated by poisson process referred to as doubly stochastic process or cox process. Here the intensity can be assumed to evolve as a square root diffusion process. The model can be expressed as 3.8.9

$$\lambda^i(t) = x^i(t) \tag{4.4.2}$$

CIR process $dX^i(t) = k^i\left[\theta^i - X^i(t)\right]dt + \sigma^i\sqrt{X^i(t)}dW^i(t)$

Survival probability $P(s/t) = e^{-\int_t^s \lambda(u)du}$

Expected values of survival probability given past information $P(s/t) = E\left[e^{-\int_s^t \lambda(u)du}/F_t\right]$

In simulating default intensity $\lambda^i(t)$ we test several values of the model parameters so as to appreciate the reaction of the default process $\lambda^i(t)$ to changes in model parameters.

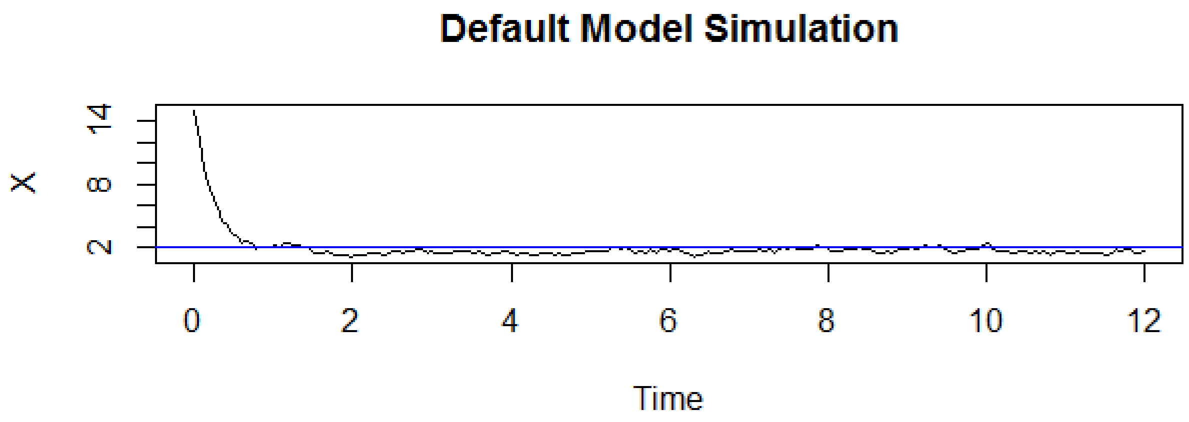

Figure 4.4.1: One simulation of CIR process

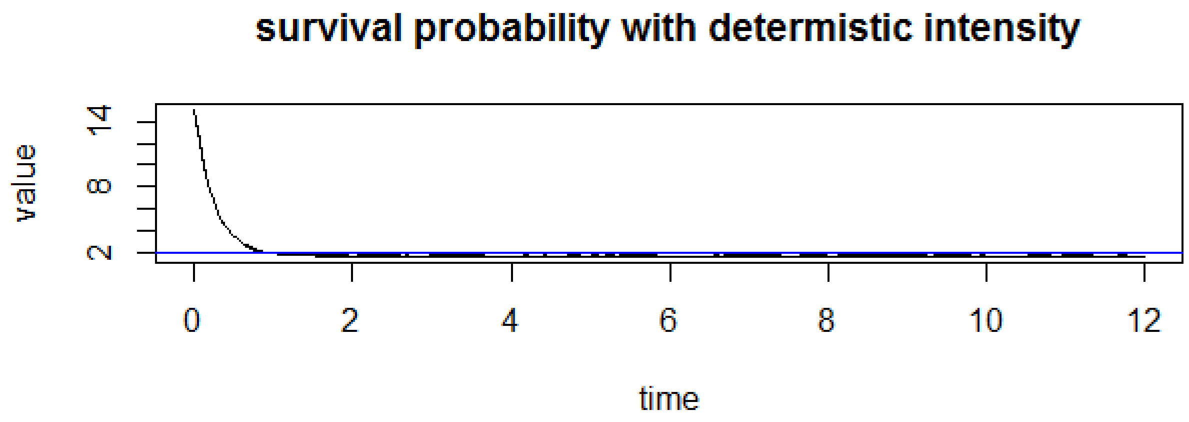

Figure 4.4.2: Survival probability to time s given survival to time t

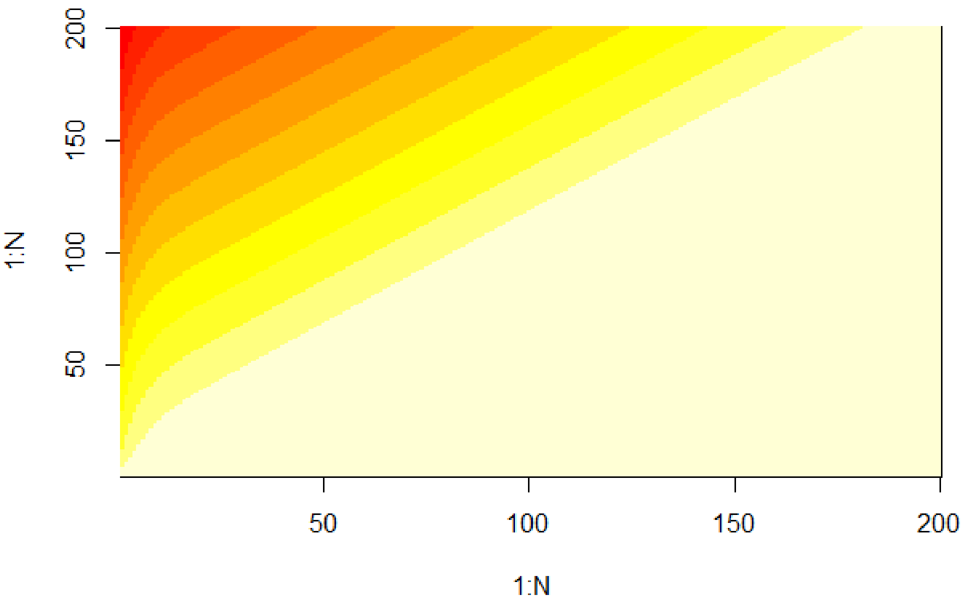

Figure 4.4.3: Log Matrix of the expected values of survival probability

From figure 4.4.1,the process behaves as expected because it beggins at the initial value of15 and at time 2 it reaches its long term mean of 2. We used the mean reversion coefficient (k=4) hence since it is a high value then the process converges to its long term mean quickly. The volatility $(\sigma = 0.4)$is low hence limiting the fluctuation of the process. From figure 4.4.2 the behaviour of the survival probability with deterministic intensity $\lambda(t)$is similar to that of a single realization of square root diffusion process. We simulated the survival probability with deterministic intensity $\lambda(t)$ and 1000 CIR process realizations.The survival probability curve is smooth and downward sloping reaching its long term mean at time 2. Figure 4.4.3 dispays a level curve of the log matrix of the doubly stochastic survival probability with a random/stochastic intensity $\lambda(t)$. Due to the fact that the effect of exponential has been removed the function decreases so fast to its final state.The function quickly decreases to its long run mean that it is not easy to notice the transition states. Also, even with the logarithm the survival probability function decreases rapidly to its final state. As the process approaches its final state there is an increase in various states and the dots of various states increase displaying the convexity of the doubly stochastic function(inequality of Jensen preserved).

4.4.1 Effect of reducing mean reversion coefficient on default model simulation

We simulate the model with smaller value of k to identify the effect of mean reversion coefficient (k) as well as the reaction of the default process $\lambda(t)$, here we repeat the simulation with $k = 0.3$ and obtain the following results.

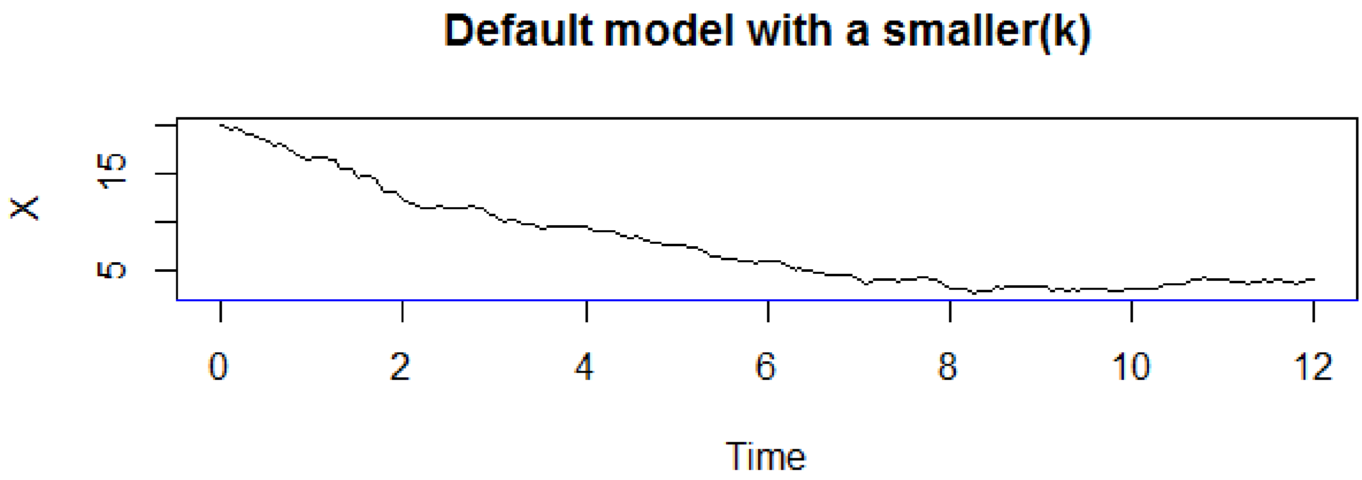

Figure 4.4.4: Effect of reducing mean reversion Coefficient on CIR process simulation

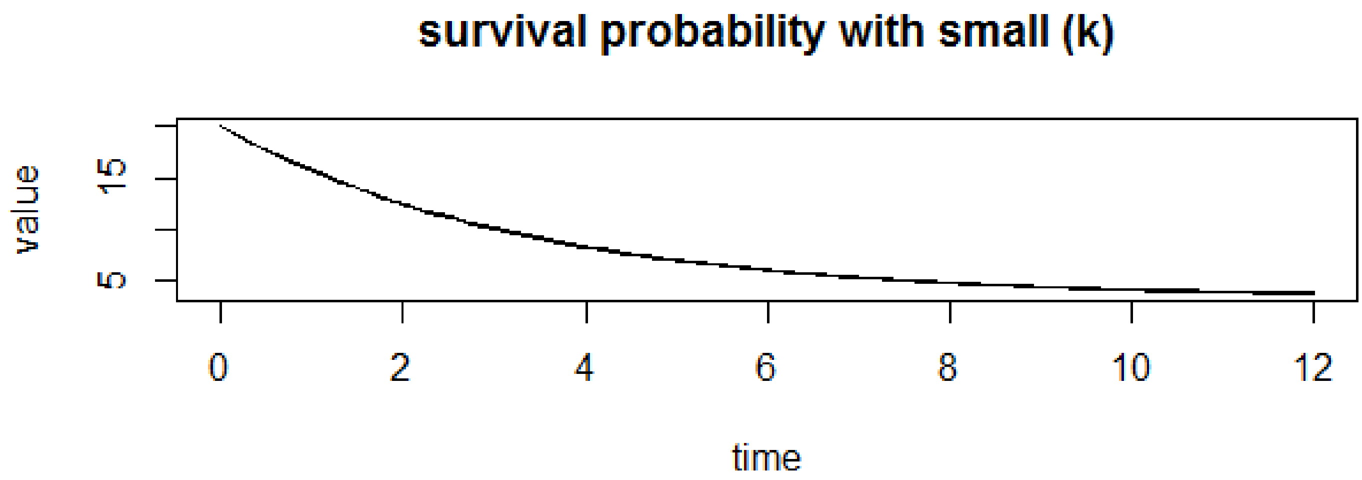

Figure 4.4.5: Effect of reducing mean reversion on Survival Probability Simulation

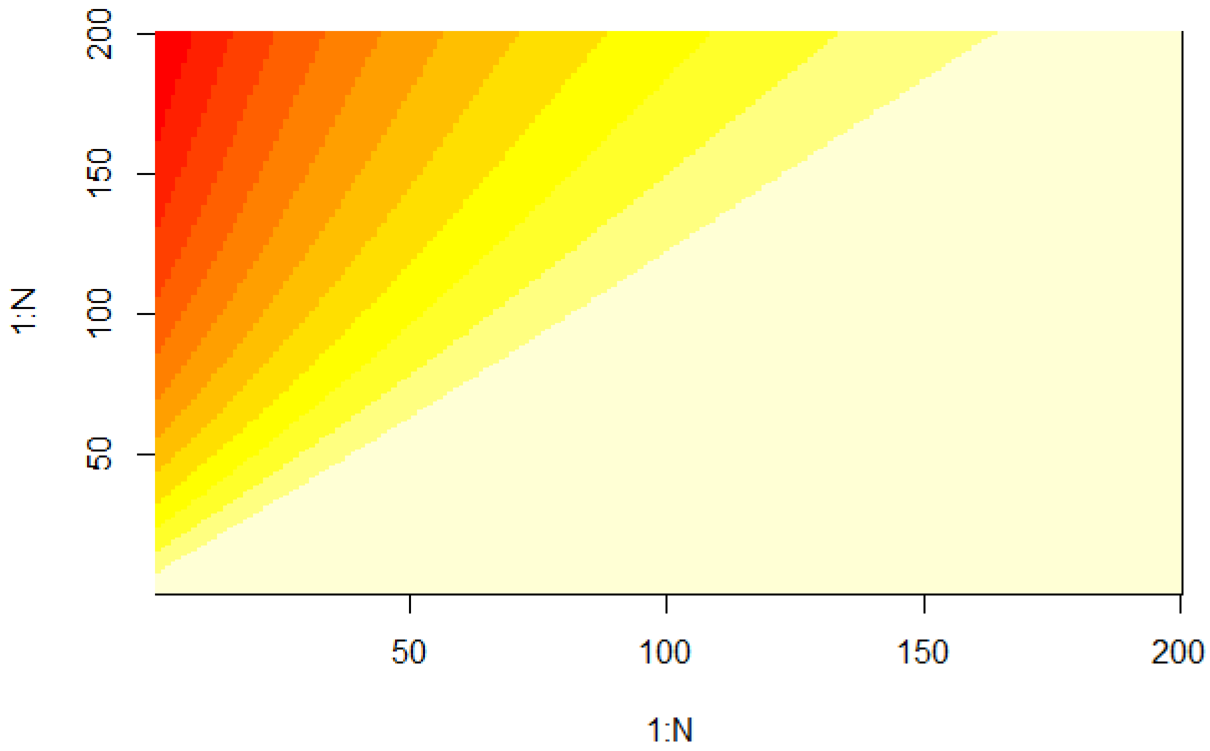

Figure 4.4.6: Effect of reducing mean reversion on Log matrix of the doubly stochastic survival probability

The process decreases slowly until it reaches its long term mean of 1, the process is slow and hence it takes more time to reach its long term mean as compared to the one with $k = 4$. It reaches its long term mean of $(\theta = 1)$ at time 8. The mean reversion coeffient is low the reason the process takes more time to reach its long term mean. Figure 4.4.6 displays the simulation of log matrix of the doubly stochastic survival probability with small mean reverting coefficient and also low volatility.Similarly the process seems to decrease slowly as compared to when the mean revertion is high. High values are transitioning to another state slowly as compared to small values.

4.4.2 Effect of reducing long term mean coefficient on default model simulation

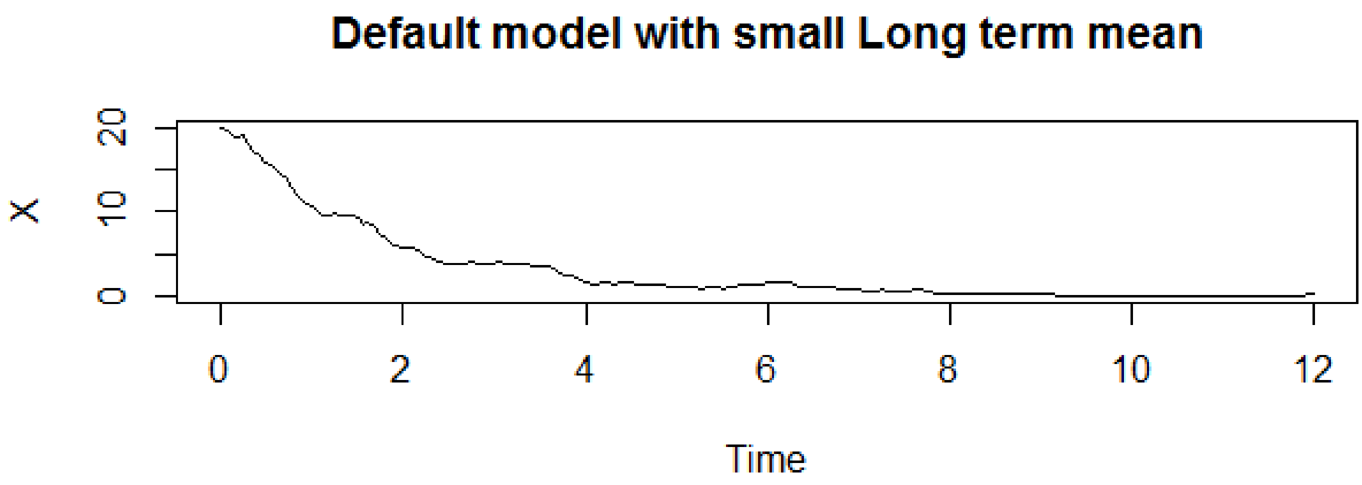

Figure 4.4.7: CIR process simulation with reduced long term mean

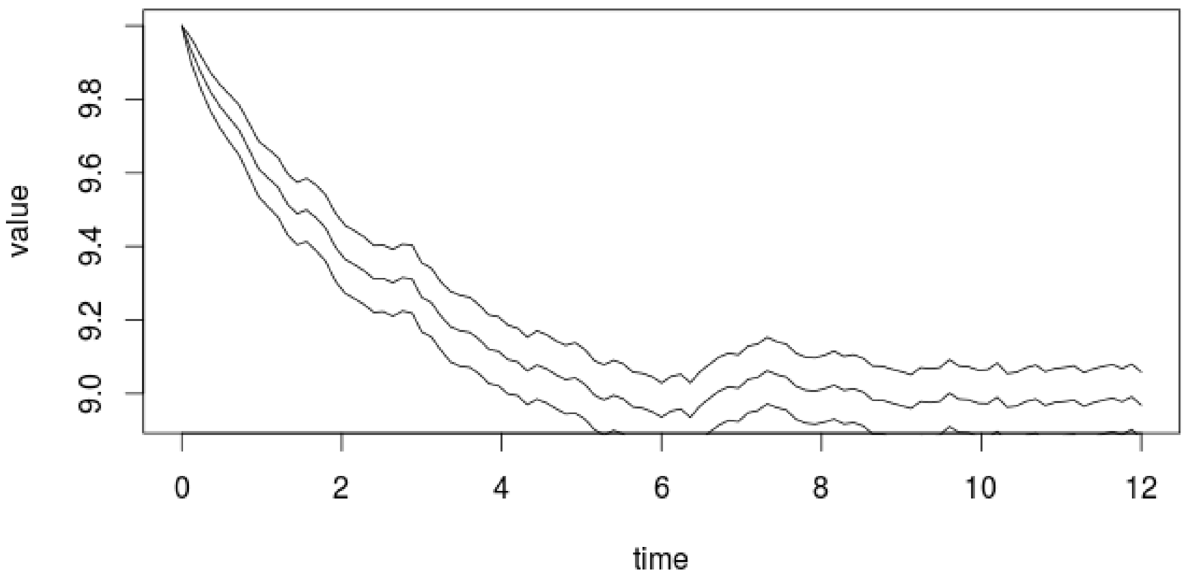

Figure 4.4.8: Survival probability Simulation with reduced long term mean

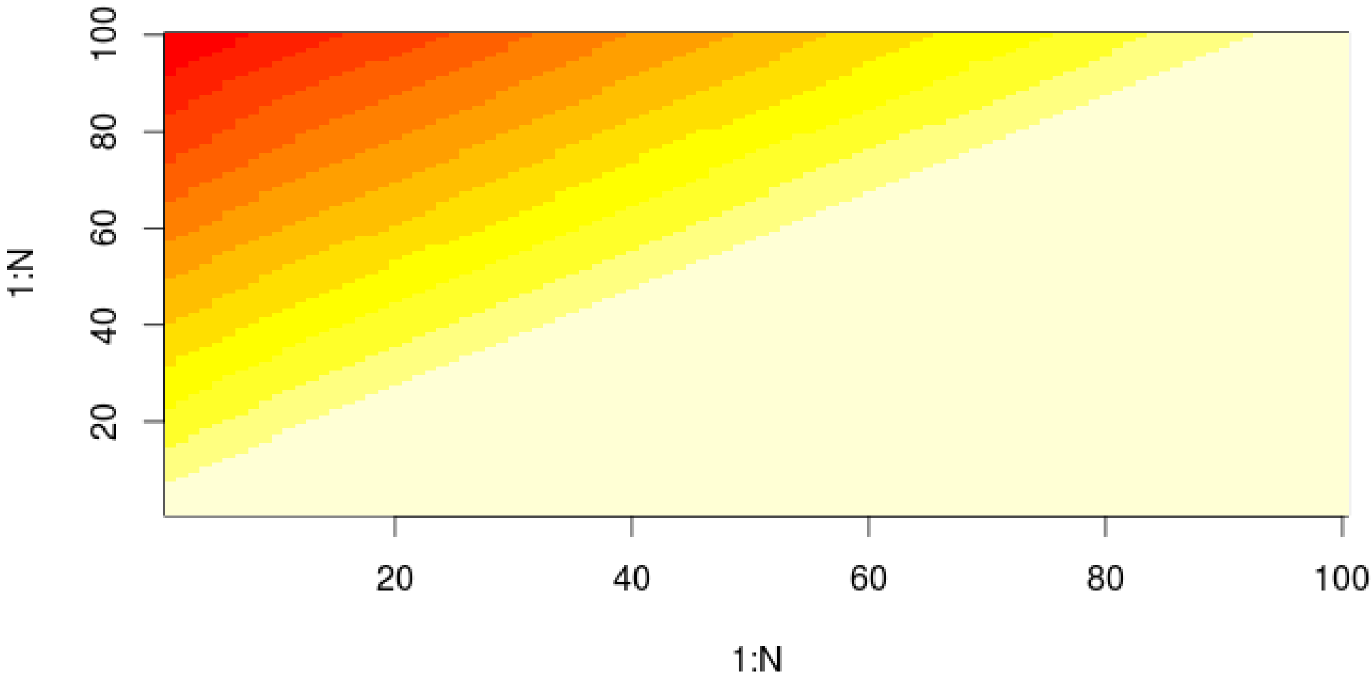

Figure 4.4.9: Expected survival probability simulation with reduced long term mean

Figure 4.4.7 shows the effect of reducing the longterm mean θ of the CIR process. We used $\theta = 0.05$.The process also decreases slowly to its long term mean because the mean reverting coefficient is low $k = 0.5$, and it does not highly fluctuate because of the low volatility coefficient $\sigma = 0.5$. The doubly stochastic survival probability (figure 4.4.9) takes more time to transit from one state to the other for higher values than smaller values that is the transitional states becomes greater as the process approaches its long term mean. Therefore the process takes more time for a final state to pass from one state to the other but it reaches the long run mean faster because of its convexity.

4.4.3 Behaviour of default model with large longterm mean

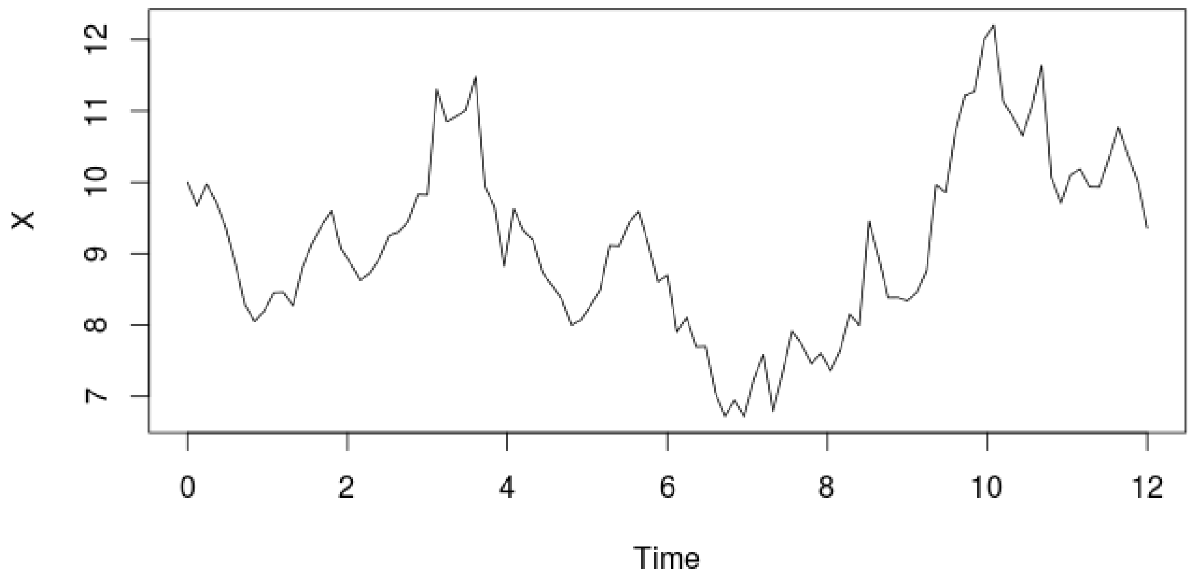

Figure 4.4.10: CIR process Simulation with large longterm mean(4.5)

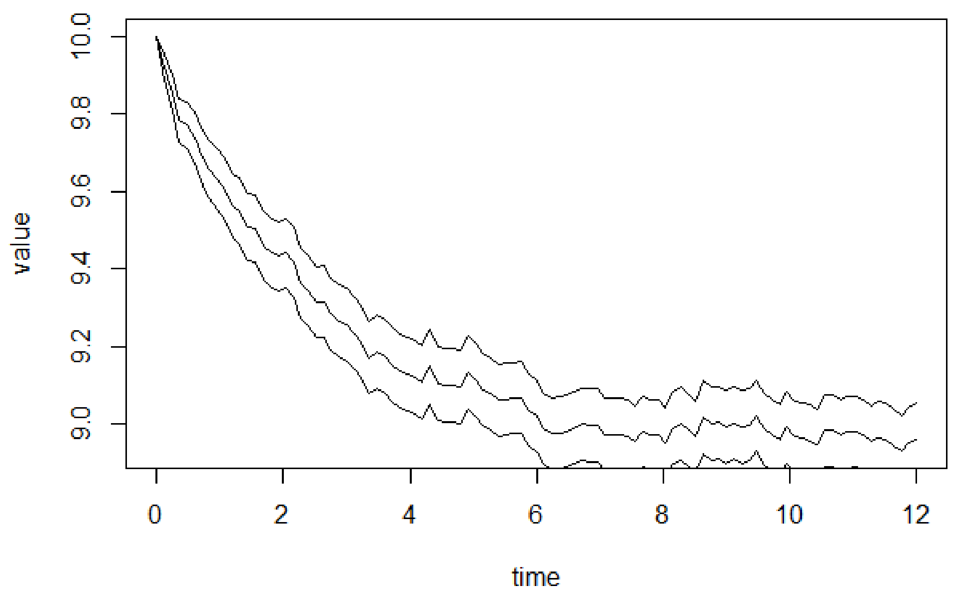

Figure 4.4.11: Survival probability with large longterm mean, $\theta = 4.5$

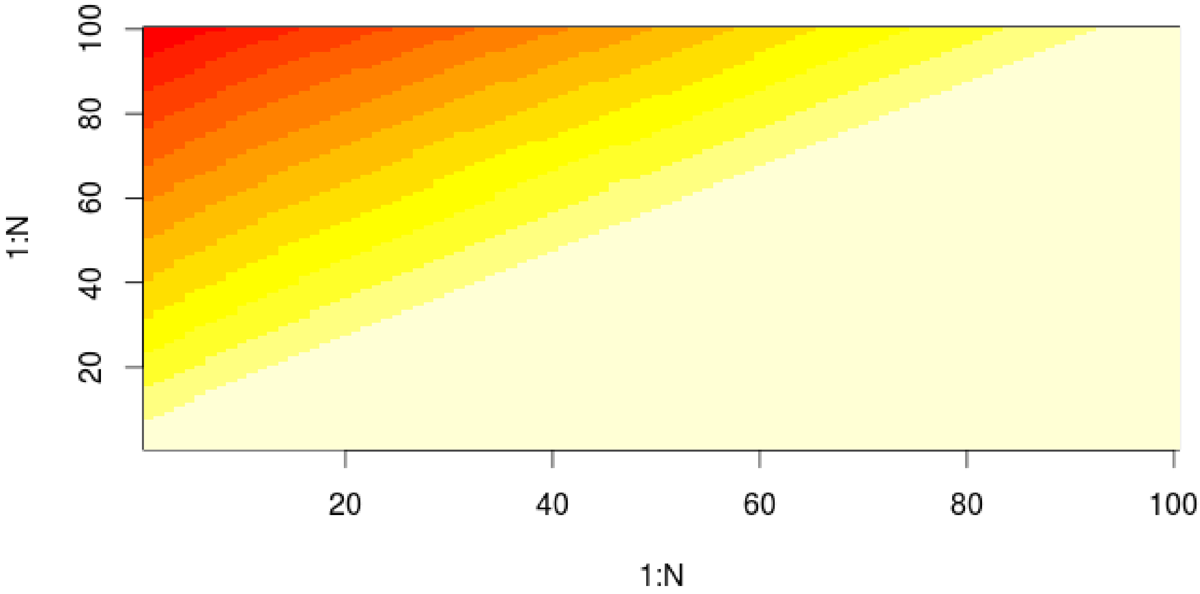

Figure 4.4.12: Expected value of the doubly stochastic survival probability with large longterm mean

Figure 4.4.10 shows the simulation of CIR process with increased long term mean $\theta = 4.5$ that fluctuates in turn of its long term mean with volatility $\sigma = 0.5$. Survival probability function (figure 4.4.11) is also decreasing fast and it reaches its long term mean in a short period. This could be because the longterm mean $\theta = 4.5$ is close to the initial value $\theta = 10$. The same happens to the log matrix of the doubly stochastic survival probability which decreases more rapidly to its final state but less rapidly than when the long term mean is distant from the initial value. These simulations resemble(Duffie and Singleton, 1999) simulations in literature.

4.4.4 Default model simulation usingDuffie and Singleton (1999) parameters

Duffie(1999), CIR parameters are $\theta = 0.559$,$k = 0.238$, $\sigma = 0.074$, we use these parameters to simulate default model.

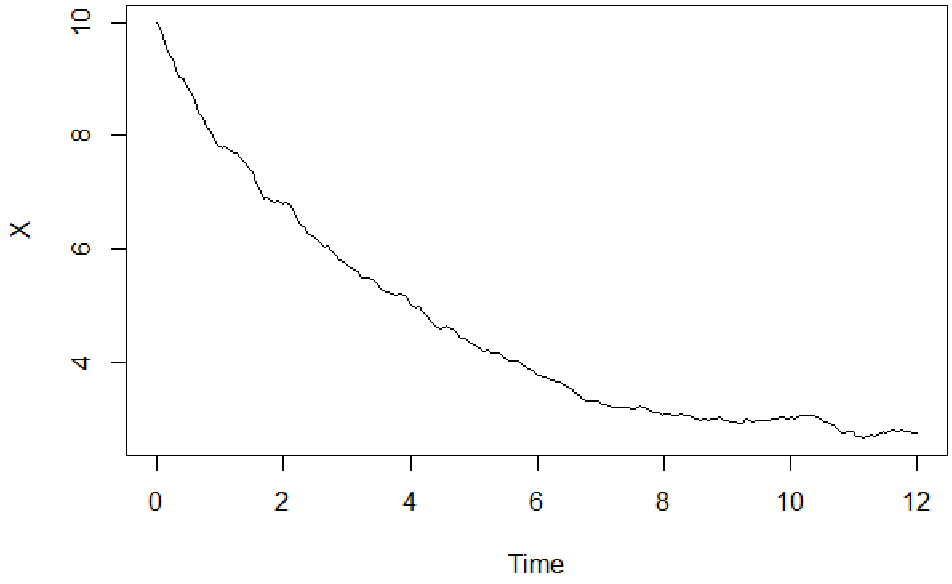

Figure 4.4.13: CIR process simulation usingDuffie and Singleton (1999) parameters

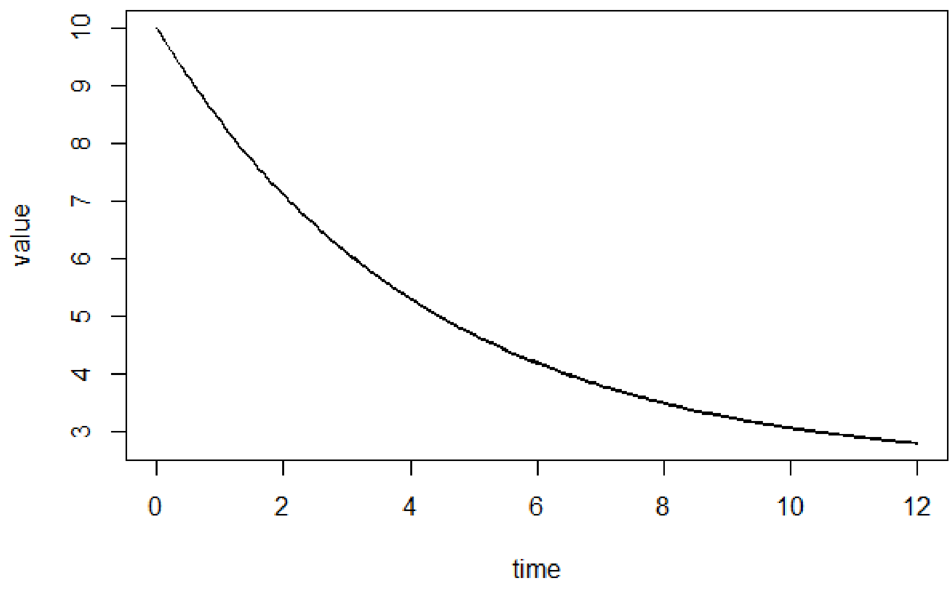

Figure 4.4.14: Survival simulation usingDuffie and Singleton (1999)parameters

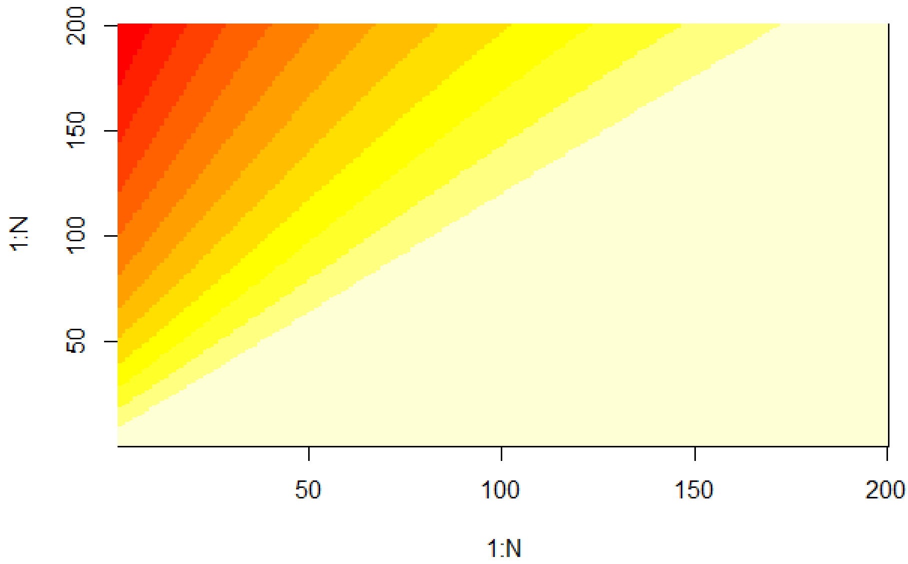

Figure 4.4.15: Expected value of the doubly stochastic survival probability usingDuffie and Singleton (1999) parameters

4.4.5 Simulation usingBrigo and Alfonsi (2005a) parameters

The parameter values are$\theta = 0.00125$,$k = 0.25$ and $\sigma = 0.1$ which we use to simulate default model

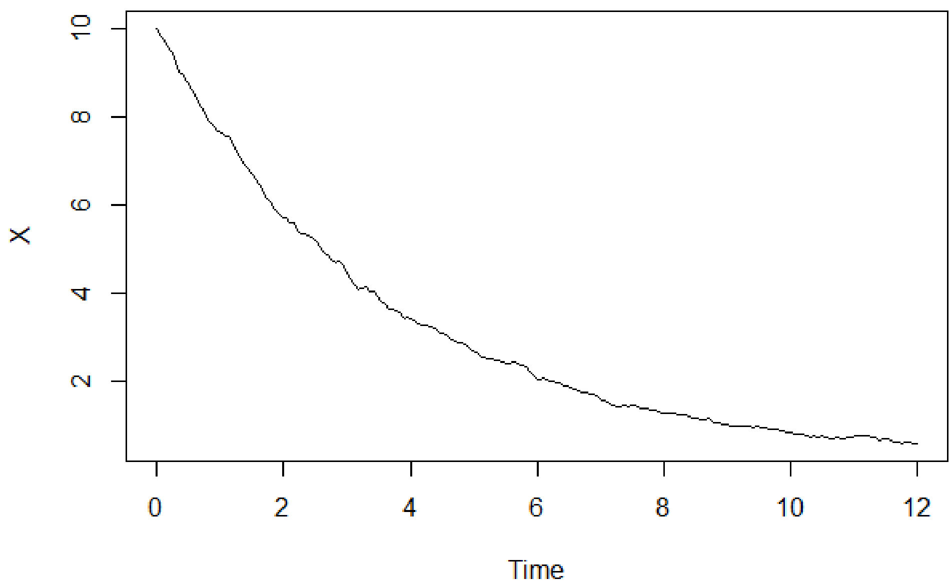

Figure 4.4.16: CIR process simulation using(Brigo and Alfonsi, 2005a) parameters

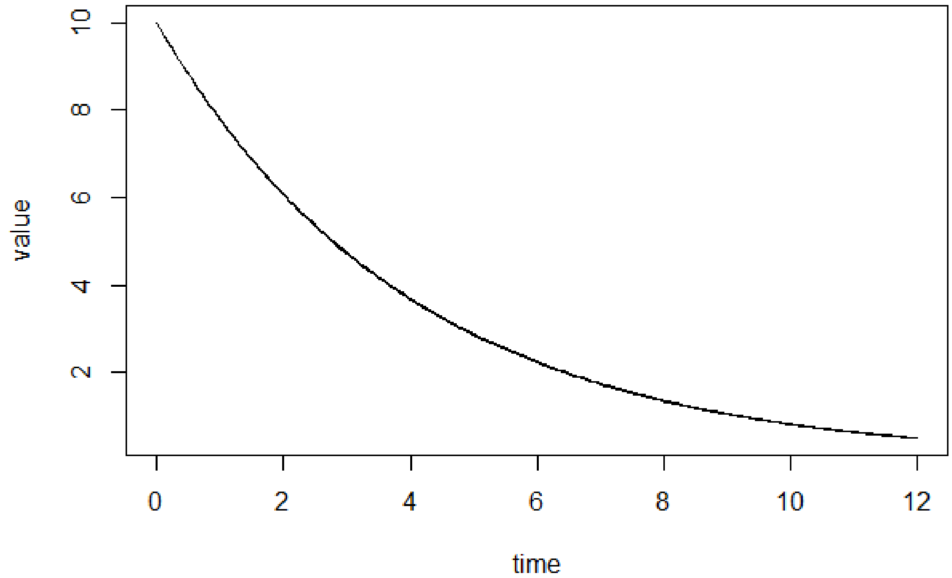

Figure 4.4.17: Survival Function Simulation using(Brigo and Alfonsi, 2005a)parameters

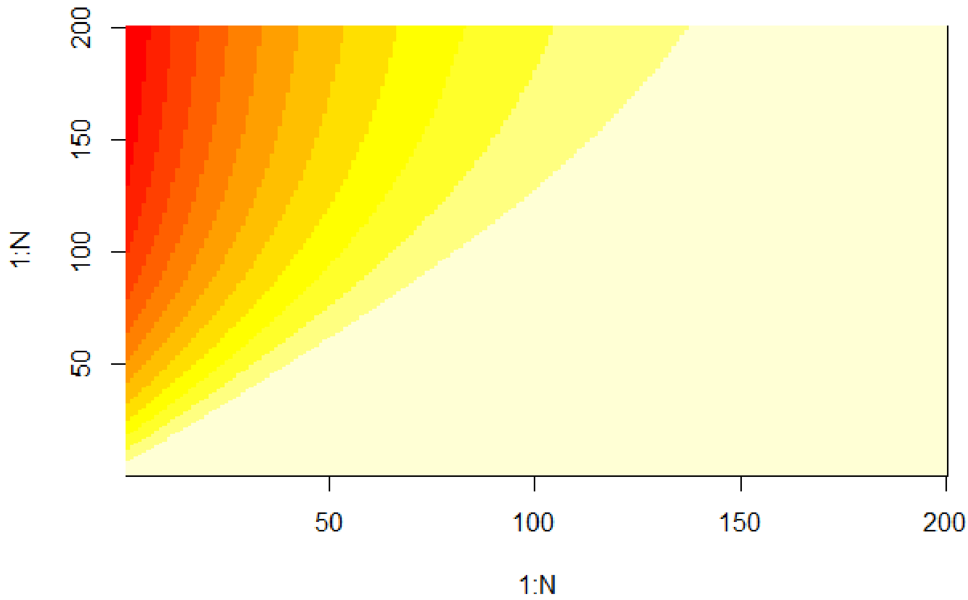

Figure 4.4.18: Expected Survival function usingBrigo and Alfonsi (2005a) parameters

Volatility parameters used for both cases are low and the mean reversion coefficients for both simulation are close implying a similar behaviour of the process. The difference occurs only between the doubly stochastic for the two simulations. The long run mean forDuffie and Singleton (1999) simulation is greater than that ofBrigo and Alfonsi (2005a), hence for simulation using(Duffie and Singleton, 1999) parameters the process is slower than that of(Brigo and Alfonsi, 2005a). This can be explained by the fact that if the distance between the initial value and the long run mean is significant then the process reaches its long run mean faster. The transitional states becomes greater as the process approaches its final state. Therefore the stochastic survival probability with a high long run mean reaches its final state rapidly than the one with low mean.

In conclusion the greater the mean reversion coefficient k ,the faster the process reaches its long term mean θ,and the smaller the transitional states and vise varsa is true. The closer the long term mean θ to the initial value of the default intensity $\lambda(0)$the quicker the process decreases to its long run mean θ and the smaller the transitional states. If the volatility is high then the CIR process becomes less stable and if the mean reversion function of the process is low with volatility the process also becomes less stable and the opposite is true. low mean reversion implies that the stochastic part of the CIR process is more dominant.

4.5 Price of a defaultable zero coupon bond prior to default

In this section we simulate the price of a zero coupon bond without recovery value and assuming that there is no correlation between risk free interest rate and default free intensity. The price of a zero coupon bond after default is equal to zero.The price of a defaultable zero coupon bond therefore can be expressed as 3.6.6

$$P_0^i(t,T) = E_t\left[e^{-\int_t^T\left(r^{O/N}(s)+\lambda^i(s)\right)ds}\right] = E_t\left[e^{-\int_t^T r^{O/N}(s)ds}\right]E_t\left[e^{-\int_t^T \lambda^i(s)ds}\right]$$

$$P_0^i(t,T) = \left\{E_t\left[e^{-\int_t^T\left(r^{O/N}(s)+\lambda^i(s)\right)ds}\right]\right\} \text{ for } t<T$$

0 if $t \geq T$

4.5.1 Discounted expected cash flow

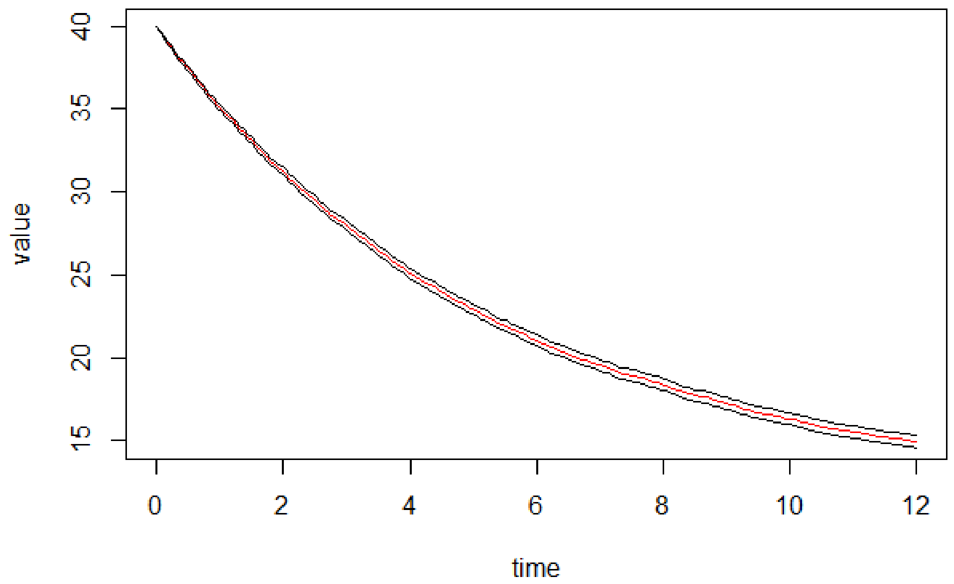

Figure:4.5.1. Discounted expected cash flow in logarithm form

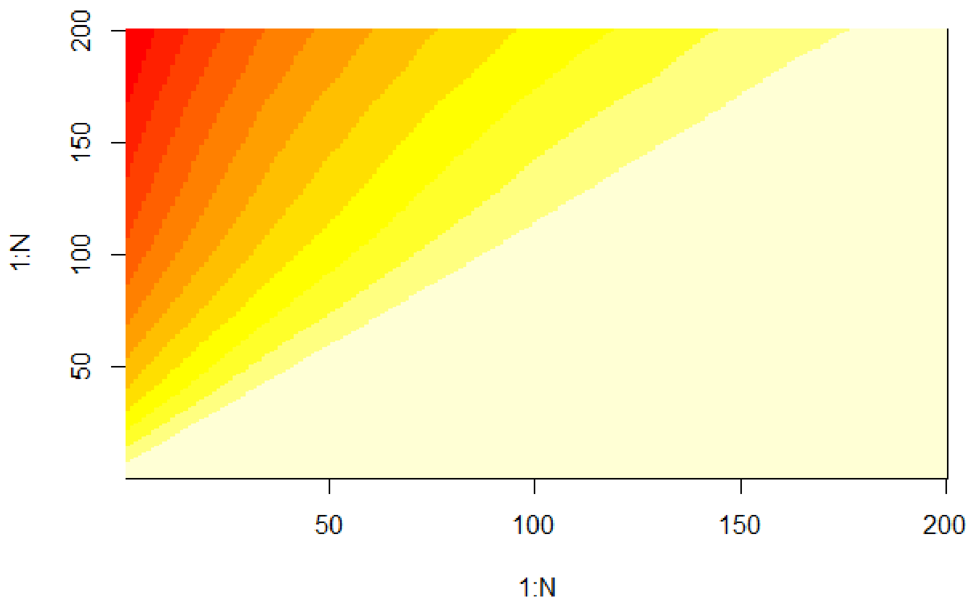

Figure:4.5.2. The price of a defaultable zero coupon bond

Figure 4.5.1 shows the result of simulation of a logarithm of the discounted expected cash flow. The discounted cash flow decrease with time because of the aspect of time value for money.The price of a zero coupon bond reduces with time due to the notion of time value for money. The risk free interest rate increases due to the default intensity that is viewed as a risk premium. The confidence interval is close to the estimated simulation mean hence the estimated mean is a true estimator of the mean.

Figure 4.5.2 shows a level curve of the price of a defaultable zero coupon bond ,after default has taken place the price becomes null.As we get nearer to the final state the states become greater due to the convexity of the curve. The level curve is non linear hence the states are not proportional. The process decreases until it reaches the stopping time τ where it becomes null. Bonds and interest rates have an inverse relationship that is, when one goes up the other goes down hence this result shows that because interest rates increases with increase in the term, then bond price decreases with increase in the term. The premia for credit risk as the term increases makes interest rates to go up hence bond prices decreases.

CHAPTER 5

CONCLUSION AND RECOMMENDATION

5.1 Conclusion

Pricing of interest rates has changed significantly as a result of 2007-2008 interest rate financial crisis that caused a change in the interest rate financial markets leading to adoption of multicurve framework as the best pricing method as compared to single curve pricing. It is of great benefit to correctly price interest rates because they are the basis for which derivative instruments are priced. Derivative markets specifically interest rate swap market has grown tremedously and they are used by financial institutions for risk management that is to hedge against fall or rise in interest rates. Borrowing and lending was considered to be done at a risk free rate LIBOR but as a result of the crisis overnight rate(OIS) is used as risk free interest rates in multiple curve pricing. Pricing is also done depending on the tenor of the underlying instrument because of different credit and liquidity risk associated to various tenors, for example the longer the maturity the higher the credit and liquidity risk is because financial market dynamics keep on changing.

Interest rate market is volatile and due to this feature stochastic models are suitable for modelling short rate rather than deterministic models.When modelling short rate there is need to understand some features for example, interest rates are mean reverting by nature and when we varry the mean reverting coefficient in the model different results are obtained as well as when other parameters are varried. The estimates of the model parameters for both periods using 3 month LIBOR US daily data shows the aspect of abnormality in the market for the crisis period because the parameters are large during pre-crisis than after crisis period. Volatility during crisis period is larger than post crisis period indicating that the rates fluctuated more. Fluctuation of interest rates support use of stochastic models because interest rates are never constant.

When modelling credit risk the most important element to consider is the market information one can perceive.The information can be completely available, partially available or not available at all. Information is what really makes up credit risk because investors have no clue of what the future holds hence the risk averse investors wants to be covered for taking any or extra risk. Reduced form modelling was applied with the assumption that the information is not fully available in the market. We were able to notice that the results of stochastic processes can be difficult to interpret so they have to be well calibrated. Default model behaves differently with change in model parameters but its important to specify well the parameters of a CIR model to have reliable results because a change in model parameters gives different results. For a CIR model to be meangiful it has to be well parameterized and the volatility controlled to limit the effect of the stochastic part of the process.

Interest rate increases through out time because of the risk premium that helps investors bear more risk as time increases. The price of a defaultable zero coupon bond reduces as time increases and the curve is downward sloping hence the simulation results are consistent to (Mishkin *et al.*, 2010) theory on bond price.

5.2 Recommendation

Interest rates are stochastic in nature and therefore stochastic models are the best models to use as compared to deterministic models. As a redult of 2007 crisis credit risk has to be priced in the interbank market in order to compensate for risk incase of default of one counterparty. In this research we estimated parameters using LIBOR data, other benchmark rates such as EURIBOR can also be used. Liquidity risk is also among the risks that should be modelled in order to accurately price derivatives, research can be done on how to price liquidity in the interbank market to avoid risk of market freeze.

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