ESTIMATION OF PARAMETERS OF A THREE-PARAMETER WEIBULL DISTRIBUTION BASED ON CENSORED DATA

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DECLARATION

This thesis is my original work and has not been submitted to any other University for examination.

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This thesis has been submitted for examination with our approval as University supervisors.

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DEDICATION

This thesis is dedicated to first my daughter Brenda Anova and my son Francisco Cauchy who missed my fatherly love and care at their tender ages for the sake of the course, secondly to my dear wife Teopilista Nabirye who had to bear the responsibility of a single mother scenario during the period of the study and finally to all friends, relatives and colleagues who might have missed my company during the course of the study. Most importantly I dedicate this thesis to the Almighty God who gave me strength and good health while doing this course.

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Though only my name appears on the cover of this thesis, many great people have contributed to its production. I owe my gratitude to all those people who have made this thesis possible and because of whom my graduate experience has been one that I will cherish forever.

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All views, errors and omissions of any kind are my own and should not be directed to any of the persons or organization mentioned above.

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LIST OF ABBREVIATIONS/ACRONYMS

CDF	Cumulative distribution function
CMLE	Corrected Maximum Likelihood Estimation
DFP	Davidon-Fletcher-Powell optimization
IDP	Internally Displaced Persons
LRA	Lord Resistance Army
MATLAB	Matrix Laboratory
MLE	Maximum Likelihood Estimation
MSE	Mean Square Error
pdf	Probability density function
RMSE	Root Mean Square Error
TD	Total deviation
WMLE	Weighted Maximum Likelihood Estimation

LIST OF NOMENCLETURES

L	Likelihood function
L_C	Corrected likelihood function
l	Log-likelihood function

ABSTRACT

In this thesis, parameters of a three-parameter Weibull distribution model are estimated based on a combination of both progressive and fixed type-I censoring using the techniques of maximum likelihood estimation, Corrected maximum likelihood estimation and Weighted maximum likelihood estimation. The objectives were to identify the best estimators for the three-parameter Weibull models' parameters based on fixed type I and progressively censored subjects by investigating their properties as well as analyzing the effect of both sample size and shape parameter to the resulting parameter estimates. Different samples sizes (20, 40, 60, 80, and 100) were considered and simulated with 25% of each being censored. The 25% were divided into progressive and fixed type-I censoring scheme. Davidon-Fletcher-Powell (DFP) optimization method in MATLAB program was used in estimation of the parameters from the parametric models by iteration with a one-step bias-correction as initial estimates for the required iterative procedure. Application was made to the Internally Displaced Persons dataset. Discussion was made on the parameter estimates of the three-parameter Weibull distribution from the three techniques based on a mixture progressive and fixed Type-I right censored sample. Different estimation procedures are studied and compared through a Monte Carlo simulation study. Based on the simulation results, the weighted maximum likelihood estimates was found to be superior in estimating the parameters of the Weibull distribution in terms of its bias, total deviation and Root Mean Square Error.

CHAPTER ONE:

INTRODUCTION

In this chapter, the background of the study, statement of the problem, objectives of study, significance of study and the basic concepts of the study that include censoring are discussed.

1.0. Background

The Weibull distribution is a very important life testing model and it provides the generalization of many other life testing distributions such as the two parameter exponential distribution (when the shape parameter is one), two parameter Weibull distribution (when the location/shift parameter is zero), one-parameter exponential distribution (when the shape parameter is one and the location parameter is zero) and a Rayleigh distribution (when the shape parameter is two). It is used in reliability studies, for example to study the breakdown of equipment or parts of equipment. It is also commonly used in other disciplines such as Medical research for clinical trials' study, and in cognitive psychology to study the time to complete a task. The Weibull distribution with non-zero shift has three parameters, denoted by: $\beta > 0$, the shape parameter responsible for the skew of the distribution; $\alpha > 0$, the scale parameter; finally γ, the shift parameter which is also a lower bound. The shift parameter also called the threshold parameter represents the time below which no failure occurs or the minimal survival time of all the items in a specific population. Hirose and Lai (1993) technical report states that the inclusion of the threshold parameter introduces many additional difficulties and makes valid inferences on the parameter or the functions thereof particularly difficult. However, the threshold parameter provides a very useful result and therefore better to battle with the difficulties associated with its estimation for better interpretation of the properties of a particular dataset.

A three-parameter Weibull distribution with a shift or location parameter has a probability density function (pdf), mathematically defined by:

$$f(t;\, \underline{\theta}) = \left(\frac{\beta}{\alpha}\right)\left(\frac{t-\gamma}{\alpha}\right)^{\beta-1} exp^{\left[-\left(\frac{t-\gamma}{\alpha}\right)^{\beta}\right]} \tag{1.1}$$

$t \geq \gamma \geq 0 \quad \alpha > 0\,, \qquad and\; \beta > 0$

and a cumulative distribution function (cdf)

$$F(t;\, \underline{\theta}) = 1 - exp^{\left[-\left(\frac{t-\gamma}{\alpha}\right)^{\beta}\right]},\, t \geq \gamma \tag{1.2}$$

Where $\underline{\theta}$ is the vector of $\alpha > 0$ the scale parameter which determines the spread of the distribution; $\beta > 0$ is the shape parameter also known as Weibull slope which determines the skewness of the distribution; and $\gamma > 0$ is the location parameter (Shift parameter) which determines the lower bound of the distribution. Therefore $\underline{\theta} = [\alpha, \beta, \gamma]$ for this case.

The three basic measures of central tendency (statistics) of this distribution, the mean, the variance and the Fisher skew are given by

$$
\left.\begin{aligned}
E(t) &= \gamma + \alpha\Gamma\left(1+\frac{1}{\beta}\right) \\
Var(t) &= \alpha^2\left(\Gamma\left(1+\frac{2}{\beta}\right)-\left(\Gamma\left(1+\frac{1}{\beta}\right)\right)^2\right) \\
Sk(t) &= \frac{\left(\Gamma\left(1+\frac{1}{\beta}\right)\right)^3 - 3\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(1+\frac{2}{\beta}\right)+\Gamma\left(1+\frac{3}{\beta}\right)}{\left(\Gamma\left(1+\frac{2}{\beta}\right)-\left(\Gamma\left(1+\frac{1}{\beta}\right)\right)^2\right)^{\frac{3}{2}}}
\end{aligned}\right\} \tag{1.3}
$$

in which Γ represents the Gamma function. Its shape can vary from a hyper-exponential (whenever $\beta < 1$) to a near symmetrical (when $\beta \approx 3.6$) to a negatively skewed distribution (as $\beta \rightarrow \infty$).

1.1. Weibull graphical properties

1.1.1. **Weibull Scale parameter, α**

A change in the scale parameter, α, has the same effect on the distribution as a change of the abscissa scale. Increasing the value of the scale parameter, α while holding values of the shape parameter, β and the location parameter, γ constant has the effect of stretching out the probability distribution function. Since the area under a probability density function curve is a constant value of one, the "peak" of the probability density function curve will also decrease with the increase of α, as indicated in the Fig.1.1. below. If α is increased, while β and γ are kept the same, the distribution gets stretched out to the right and its height decreases, while maintaining its shape and location. If α is decreased, while β and γ are kept the same, the distribution gets pushed in towards the left (*i.e.* towards its beginning or towards 0 or γ), and its height increases. α has the same unit as T, such as hours, miles, cycles, actuations, etc.

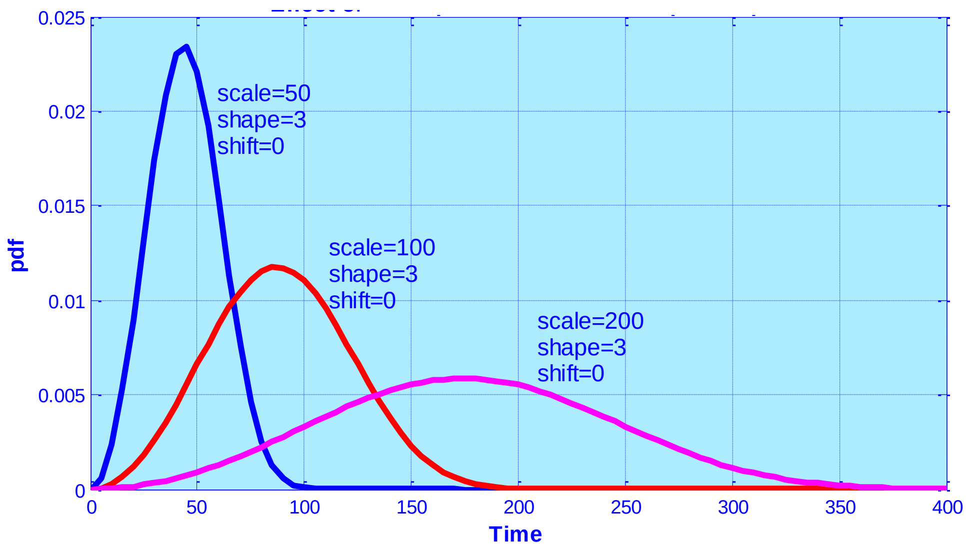

Fig. 1.1. *Illustration of the effect of the Scale parameter on the shape of the Weibull plot*

Scale parameter is always referred to as the characteristics life of the Weibull distribution and the term life comes from the common use of the Weibull distribution in modeling lifetime data (Scholz, 2008).

1.1.2. **Weibull Shape Parameter, β**

The Weibull shape parameter, β, is also known as the Weibull slope. This is because the value of β is equal to the slope of the line in a probability plot. Different values of the shape parameter can have marked effects on the behaviour of the distribution. In fact, some values of the shape parameter will cause the distribution equations to reduce to those of other distributions. For example, when $\beta = 1$, the probability density function of the three-parameter Weibull distribution reduces to that of the two-parameter exponential distribution. The parameter β is a pure number, *i.e.* it is dimensionless. The Fig.1.2 below shows the effect of different values of the shape parameter, β, on the shape of the *pdf* (while keeping γ and α constant). One can see that the shape of the probability density function curve can take on a variety of forms based on the value of β. When β=1 the resulting shape of the Weibull curve resembles that of an exponential distribution. For β=3 the resulting shape of the Weibull plot resembles the normal distribution plot while for β=0.5 the Weibull curve has a parabolic shape in the first quadrant. According to Scholz (2008) the scale parameter represents the 0.632 quantile of the Weibull distribution regardless of the value of the shape parameter but the spread of the distribution around the scale parameter gets small as the shape parameter increases.

The shape parameter, β is a very important parameter in the interpretation of the characteristics of a given dataset that has been assumed to follow a Weibull model (Scholz, 2008). When $\beta > 1$ the part or system for which the lifetime is being modeled by a Weibull distribution is subject to aging in the sense that an older system has a higher chance of failing during the next small time increment, Δt than the younger system. This situation is very common. For $\beta < 1$, the system has better chance of surviving the next small time increment, Δt as it get older. This is less common scenario in real life event but a good situation of this kind is the infants' mortality, where after initial early failures the survival gets better with age. Rate of failure therefore increases when $\beta > 1$, decreases when $\beta < 1$ and is constant when $\beta = 1$

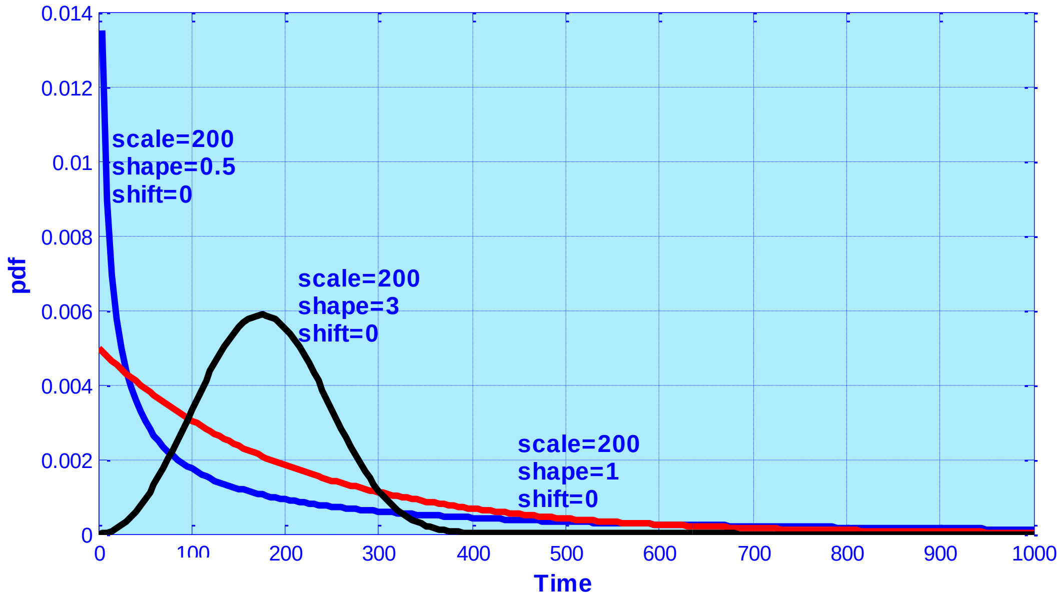

Fig. 1.2. *Illustration of the effect of the shape parameter on the shape of the Weibull plot*

1.1.3. **Weibull Location Parameter,** γ

The location parameter shifts the distribution to start from its value point. It is a point below which the hazard rate/rate of failure (motility rate) is zero. i.e. no subject experiences the event of interest before time= γ. Fig.1.3. below shows the Weibull pdf curves with varying location parameter and constant scale and shape parameters.

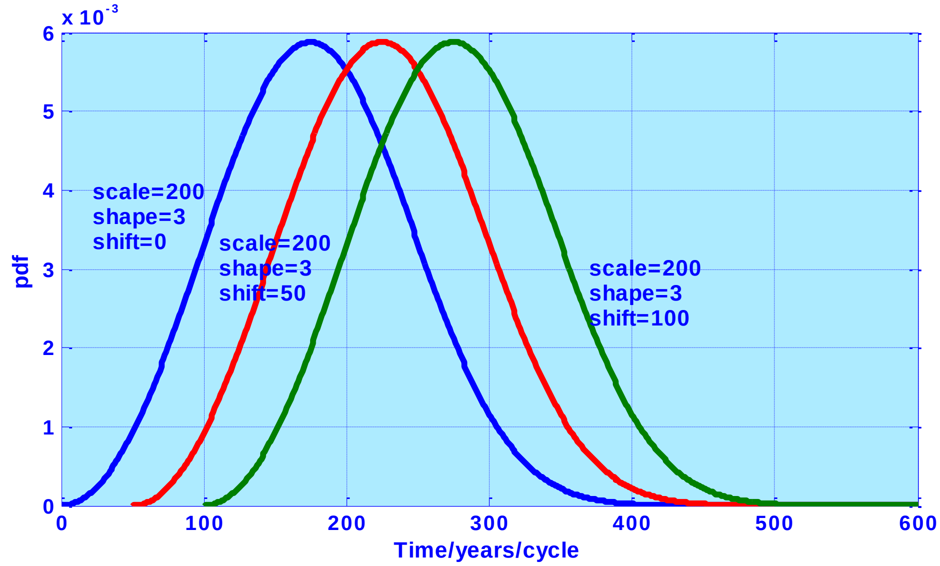

Fig. 1.3. *Illustration of the effect of the location parameter on the shape of the Weibull pdf plot*

1.2. Basic concepts under consideration

Under this subsection of the thesis, techniques used in analyzing time-to-event data are briefly mentioned, censoring and data consideration as well.

1.2.1. Time-to-event analysis techniques

In a typical life testing, n specimens are placed under observation and as each failure occurs, the cumulative life is recorded. This is referred to as 'complete sample' test or simply non-censored sample. In some other scenario, the study is stopped when a certain fixed number of subjects failed (type II censoring) or when a certain set time elapses (Type I censoring) this leads to incomplete sample. In this study progressive type I censoring was considered. Buis, (2006) states parametric, non-parametric and semi-parametric techniques as the three techniques used for analyzing the time-to-event data, each with its own limitation but parametric approach is thought to yield better results provided the assumptions made in the analysis are correct. With Parametric models, the outcome is assumed to follow a certain known distribution whose parameters need to be estimated in order to make inference about the statistics of the data. There are a number of texts that discuss comprehensively parametric time-to-event-models such as; Cox and Oakes (1984), Crowder, et al (1991), Gross and Clark (1995), Lawless (2003), Lee and Wang (2003), Nelson (1990), and Saleem, et al, (2012). Lee and Wang (2003) for instance suggest exponential, Weibull, lognormal and gamma distributions as the most commonly used parametric models in survival analysis. However most of them when discussing Weibull distribution they prefer two-parameter Weibull distribution to the three-parameter counterpart due to the complication they ought to experience when estimating the parameters under certain condition. In this study, the parameters of a three-parameter Weibull distribution are estimated using different techniques and application made in obtaining the parameter estimates using the IDP return time data which was assumed to follow a Weibull distribution with scale parameter α, shape parameter β and the location parameter γ

Singh, et al (2005) studied the point estimators of three-parameters for Exponentiated Weibull distribution under complete data and type II censored using various estimation methods such as maximum likelihood method, Bayes method and generalized maximum likelihood method. Numerical comparisons were obtained for the point estimators. In this study, consideration was done for maximum likelihood estimation (MLE), Corrected maximum likelihood estimation (CMLE), Weighted maximum likelihood estimation (WMLE) and maximum product of spacing estimation (MPS) in estimating the parameters of the three-parameter Weibull distribution. Parameter assumptions are also made resulting to a reduced parametric distribution whose parameters were also estimated.

1.2.2. Censoring

One important concept in survival analysis is how to handle censoring/missing observation that arise when the time when the event of interest happened to some subjects under study are unknown. In statistics, sociology, engineering, economics, and medical research, censoring occurs when the value of a measurement or observati on is only partially known. For example, suppose a study is conducted to determine the time at which a school going pupil's first teeth will be affected by caries after being subjected to a certain drug, in such a study a pupil may drop out of school by fifth year before developing caries infection in his/her teeth and as such it could be known that time to developing caries infection is at least five years when subjected to the drug (but may be more). Such a situation could occur if the individual withdrew from the study at a time when the event of interest has not happened yet or when the study came to an end before event of interest happen to a subject. In the above example the event of interest is caries' infection. Censoring also occurs when a value occurs outside the range of a measuring instrument. For example, a bathroom scale might only measure up to 150 kg. If a 175 kg individual is weighed using this bathroom scale, the observer would only know that the individual's mass is at least 150 kg.

The survival times of some individuals might not be fully observed due to different reasons. In life sciences, this might happen when the survival study (e.g., the clinical trial) stops before the full survival times of all individuals can be observed, or a person drops out of a study, or for long-term studies, when the patient is lost to follow up. In the industrial context, not all components might fail before the end of the reliability study. In such cases, the individual survives beyond the time of the study, and the exact survival time is unknown but is known to be more than a certain time.

During a survival study either the individual is observed to fail at time *T*, or the observation on that individual ceases at time *c*. Then the observation is min (*T*,*c*) and an indicator variable ∂_i shows if the individual is censored or not. The calculations for survivor functions (the probability that the time for happening of the event of interest is later than some specified time *t*. i.e. probability of a subject surviving pass a certain time t) and hazard function (the event rate at time t conditional on survival until time *t* or later (that is, $T \geq t$)) and must be adjusted to account for censoring. Statistics Toolbox functions such as *ecdf, ksdensity, coxphfit, mle* account for censoring.

The two main statistical quantities a time-to-event analyst is always interested at are:

Survival function which is the probability of a subject surviving beyond certain time t. one most important use of a survivor function is to predict quantiles of the survival time, and

Hazard function which is the probability of failure in an infinitesimally small time period between t and $t + \delta t$ given that the subject has survived up till time t. it is therefore a measure of risk. The greater the hazard between times t_1 and t_2, the greater the risk of failure in this time interval.

Several Types of censoring exist during the data collection of the time-to-event information such as:

Left censoring – this occurs when a data point is below a certain value but it is unknown by how much for example a dentist wants to determine how long it would take for a group of learners to develop caries but found out that some learners were already having the caries infection before the commencement of the study

Interval censoring – this occurs when a data point is somewhere on an interval between two values for example when a dentist investigates his subjects only during the school period but some learners developed caries infection during holiday of first term meaning the dentist knows that event of interest occurs between the first and the second term.

Right censoring – this occurs when a data point is above a certain value but it is unknown by how much for example when a subject withdraws from the study may be a learner has dropped out of school then the dentist only knows that the time to event is longer than the time when the learner dropped out**.**

There are several types of right censorship that have always been considered by researchers whose interests are in the time-to-event studies. These include;

(i) *Progressive Type I censoring* occurs if an experimenter has a set number of subjects or items and stops the experiment at a predetermined time, at which point any subjects remaining are right-censored. For instance a researcher may decide that he will stop the experiment at 36 months at which time m subjects out of the total N subject would have experienced the event of interest meaning the remaining n-m subjects will be right censored of type I

(ii) *Progressive Type II censoring* occurs if an experiment has a set number of subjects or items and stops the experiment when a predetermined number are observed to have failed; the remaining subjects are then right-censored. For example a researcher may consider stopping the study when 500 items failed from a cohort of 750 items and this means the remaining 250 will be right censored of type II

(iii) *Random* (or *non-informative*) *censoring* is when each subject has a censoring time that is statistically independent of their failure time. The observed value is the minimum of the censoring and failure times; subjects whose failure time is greater than their censoring time are right-censored.

Fig. 1.4 below illustrates censorship using 8 subjects in which some are left censored, others are right censored while others are uncensored.

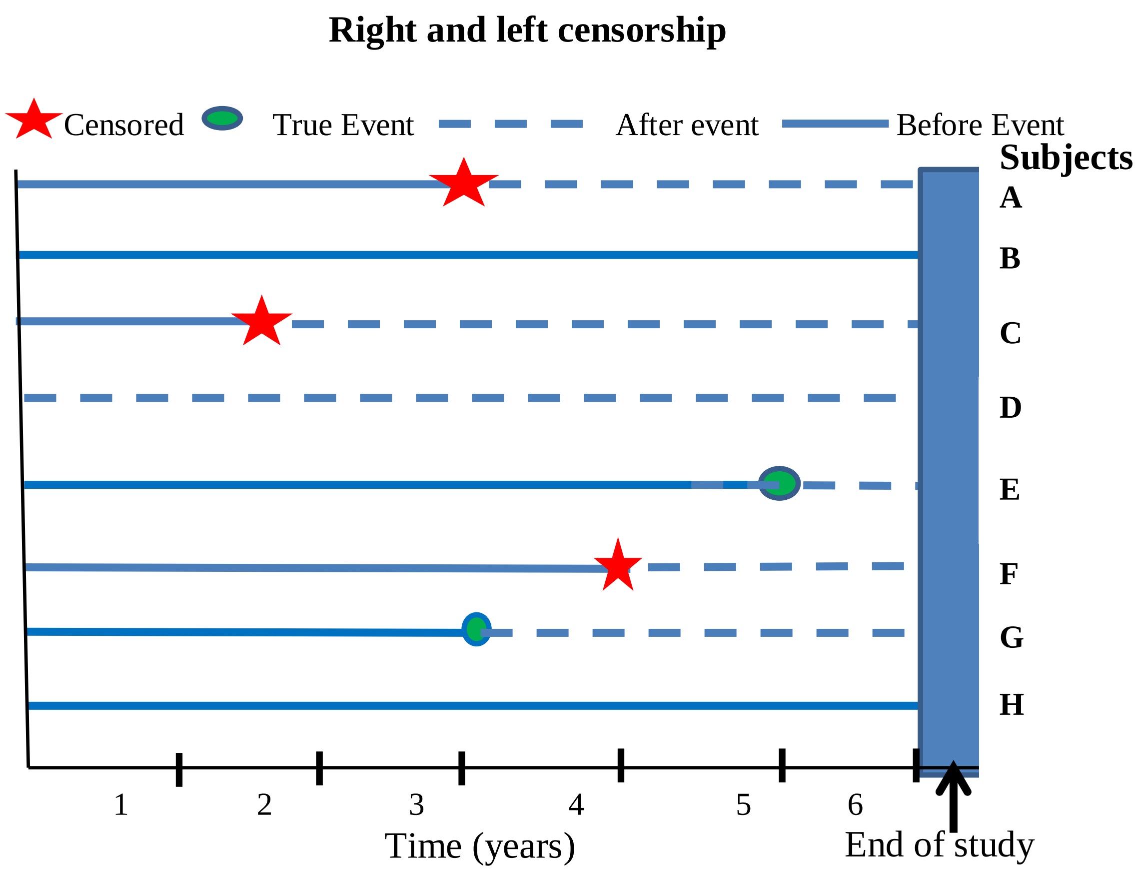

Fig.1.4. *Illustration of uncensored subjects and the right and left censorships.*

Explanation:

Subjects A, C, and F were censored in the 3rd, 2nd and 4th year respectively which means they were lost to study during these periods. The researcher therefore knows the occurrence of event of interest is longer than three years say, for subject A, longer than two years for subject C and longer than 4 years for subject F. B and H are censored at the end of the study which means the study ended before the occurrence of the event of interest on them. At the beginning of the study, event of interest had already happened to subject D and therefore this subject is left censored. Subjects E and G are uncensored since the event of interest are observed at the 5th and the 3rd periods respectively. In a nut shell therefore, subjects A, B, C, F and H are right censored, subjects E and G are uncensored (event of interest has happened to them and the time is known) and subject D is left censored (by the time the study was beginning, the event of interest had already occurred to it but the exact time nobody knows)

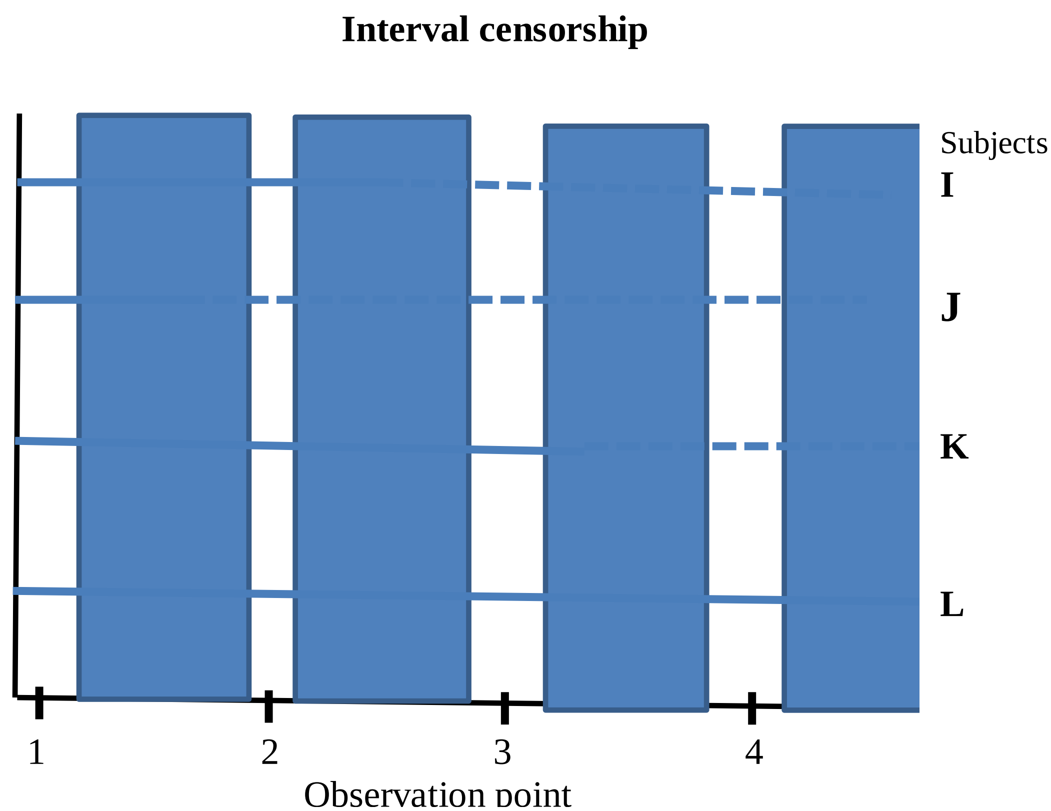

Fig. 1.5. *Illustration of the Interval Censored Mechanisms.*

Fig.1.5. above illustrates the interval censoring techniques in which for subject **J**, the researcher knows that the event of interest occurs between the first and the second time of observation, for instance using the previous example about a school dentist

studying the development of caries infection among the school going children, then he knows that caries infection developed on the teeth of the subject **J** between the first and the second period of observations. For subject **I**, the event of interest occurs between the second and the third observation point, subject **K** between the third and the fourth observation point and subject **L** event interval is not known. All the subjects **I, J**, and **K** event time are interval censored and the researcher only knows that the event of interest occurred within a certain interval but has no knowledge about the exact time.

1.2.3. Data consideration for the study

In life-testing experiments, it is common that complete information on failure times for all experimental units may not be available. Such unavailable information about the failure time of experiments is called censorship. Right censoring is one of the censoring techniques which is most often used in life-testing experiments. This saves the total time that would be spent for all the experimental units to fail and the cost associated with the entire study. The most common right censoring schemes that have been considered by many researchers are Type-I and Type-II censoring, but the conventional Type-I and Type-II censoring schemes do not have the flexibility of allowing removal of units at points other than at the terminal point of the experiment. For this reason, a more general censoring scheme called *progressive censoring scheme* is proposed in which when considering an experiment in which n units are placed on a life-test, then at the time of the first failure, ∂_1 units are randomly removed from the remaining $n - 1$ surviving units, at the second failure time, ∂_2

units are randomly removed from the remaining $n - 2 - \partial_1$ units. The test continues until kth time when the m^{th} subject fails, at which time all remaining $\partial_m = n - m - \partial_1 - \partial_2 - \ldots - \partial_{m-1}$ units are removed and termed fixed type I censored subjects. Some early works on progressive censoring can be found in Cohen (1963), and Thomas and Wilson (1972).

Survival or reliability or time-to-event data are often collected for analysis with the aim of determining the limits of safety. It may be believed that there is a threshold stress free level below which the material is safe but above which there is a non-negative probability of failure. There is therefore an interest in estimating the threshold level (if it exists) or some other quintile of the probability distribution of the failure of the subjects such as the survival or reliability function. The Weibull distribution is a very widely used parametric family which has been found in practice to be suitable for data on failure times. This is partly because it is one of the stable distributions of extreme value theory. It is therefore an important practical problem to be able to estimate the parameters of this distribution with the threshold (or endpoint) of the distribution of a particular interest (Smith and Naylor, 1987).

Survival data usually consist of the time until an event of interest occurs and the censoring information for each individual or component. Some features of lifetime data distinguish them from other types of data. First, the lifetimes are always positive values, usually representing time. Second, some lifetimes may not be observed exactly, so that they are known only to be larger than some value. Third, the distributions and analysis techniques that are commonly used are fairly specific to lifetime data. In the thesis, attempt is made to estimate the parameters of a three-parameter Weibull distribution under a combination of the progressive type I censoring scheme and fixed type I censoring scheme using the techniques of the Maximum likelihood estimation, corrected Maximum likelihood estimation and weighted Maximum likelihood estimation.

1.2.4. Application Data

In this thesis, application is made in estimating the parameters of a three-parameter Weibull distribution function based on the time that the internally displaced persons took to return to their ancestral homes after the Lord Resistance Army insurgency. The displacement of people into IDPs' Camps in Northern Uganda caused by Lord Resistance Army (LRA) war intensified in 2002. In 1987, Joseph Kony initiated the LRA rebel group in Uganda believed to have been rooted in the rebellion against President Yoweri Museveni's National Resistance Movement (NRM) government but later transformed into a brutally violent war in which civilians turned to be the main victims (Refugee law project report, Makerere University 2004). According to the same report, over 1.4 million people had been displaced and tens of thousands had been killed, raped or abducted by 2004. Over this period many people (12,000) have lost their lives, 20,000 abducted and 1.5 million lived in IDPs Camps (Nampindo et al, 2005). According to Ward (2001), the LRA is led by Joseph Kony, who proclaims himself a spirit medium and apparently wishes to establish a state based on his unique interpretation of the Ten Commandments of God. According to report on Juba talks (2006-2008), peace talks were held that resulted into an agreement between Uganda government and the LRA rebels' ceasefire, the two parties signed a truce on 26

August 2006 while Uganda government began a process of creating ‘satellite camps’ to decongest the main IDPs camps and by mid-2007, thousands of IDPs had moved into the decongested camps. However, the populace remained cautious about the prospect of a peace deal with many refusing to return to their ancestral homes before definite end to the insurgency. This motivates the application of the technique of time-to-event analysis to their return time.

The 590 subjects from the villages of; Atat, Ogor, Omwonylee, Abur, Adodo, Otudu and Arom villages, in Otuke District were studied and a retrospective dataset for the period 2007 to 2013 was generated and the parameters of a three-parameter Weibull distribution function estimated using the data.

1.3. Statement of the problem

Survival (time-to-event) data have a unique characteristic in the sense that their censoring nature leads to partial observation of the time of the occurrence of the events of interest by the researcher. These therefore lead to bias estimates when using parametric models such as the three-parameter Weibull distribution. There exist several approaches in estimating the parameters of a three-parameter Weibull distribution based on censored data and many types of censorship which need careful attention in handling them.

In this thesis, the parameters of a three-parameter Weibull distribution are estimated using the MLE, CMLE and WMLE based on the combination of progressive and fixed type I censorship with the intention of investigating the effect of these censoring mechanism on the parameter estimates and the best estimation techniques from among the three estimators that can reduce the estimates’ deviation from the true parameters. The effects of the sample sizes on the parameter estimates from the three estimators are also considered through simulation study.

1.4. Objectives of the Study

1.4.1. Main objective

The main objective of this study was to estimate the parameters of a three-parameter Weibull distribution based on censored data with application to the internally displaced persons return time to their ancestral homes in Northern Uganda after the Lord Resistance Army insurgency.

1.4.2. Specific objectives

In order to realize the main objective of the study, the specific objectives that have to be achieved are:

(i) To formulate the log-likelihood function for a three-parameter Weibull distribution function based on right censored data under the three estimation techniques of Maximum Likelihood Estimators (MLEs), corrected Maximum Likelihood Estimators (CMLE) and the weighted Maximum Likelihood Estimators (WMLE).

(ii) To derive the Maximum Likelihood Estimators (MLEs), corrected Maximum Likelihood Estimators (CMLE) and the weighted Maximum Likelihood Estimators (WMLE) of the parameters from (i) above

(iii) To investigate the properties of the estimators in (ii) above using the simulated data

(iv) To apply the estimators obtained from (ii) above to the application data

1.5. **Significance of the Study**

Much as the main aim of this study is at fulfilling the M.Sc. degree award of Pan African University, the results obtained in this thesis will be of paramount importance to academic world. Despite shading light on the estimation techniques while paying attention to the effect of censorship, sample sizes and parametric time-to-event analysis, the results obtained in this study will:

1. Help statisticians, Biostatisticians, engineers and practitioners interested in analysing the time-to-event data using parametric models and in particular using the three-parameter Weibull distribution to know under what circumstances one can use maximum likelihood estimation, weighted maximum likelihood estimations and corrected maximum likelihood method and how to determine the degree of accuracy using the idea of the total deviation.
2. Contribute to the existing literature in the estimation of the parameters of a three-parameter Weibull distribution based on right censored data;
3. Assist in generalizing what happens in estimating the parameters of a three-parameter Weibull distribution based on right censored data using the maximum likelihood estimation, weighted maximum likelihood estimations and corrected maximum likelihood estimation techniques.
4. Give insight to the return time of the internally displaced persons in northern Uganda after the Lord Resistance army insurgency.
5. Give insight into the effect of the different type I right censoring scheme into the parameter estimates of the parameters of a three-parameter Weibull distribution.

1.6. Overviews of the Thesis Chapters

This thesis is organized as follows;

chapter one gives brief background to the Weibull distribution models as one of the parametric models used in analyzing time-to-event data, common censoring techniques that forms the basis for the statement of the problems and the objectives of the study as well as the study justification which in all form the basis for this study.

Chapter two discusses related literature and other authors' contribution towards the study of a three-parameter Weibull models including the data consideration in the estimation of the parameter of the three-parameter Weibull distribution and the estimation techniques so far done by other researchers.

Chapter three gives the methodology used in achieving the study objectives. This chapter introduces different estimation techniques such as the Maximum likelihood estimation, the corrected Maximum likelihood estimation and the weighted Maximum likelihood estimation for the parameters and discusses ways of numerically analyzing the properties of these parameter estimators. Since none of the estimators has closed-form solution, this chapter gives an efficient and reliable way to obtain initial estimates for iterative procedures based on censored estimation which was in line with the one suggested by Harter and Moore (1966) and applied by Ng et al (2012) with a one-step bias correction for the location parameter. Then, there is review of the MLEs of model parameters, the corrected MLEs, and weighted MLEs estimators to progressively Type-I censoring, which are extensions of the results by Cheng and Iles

(1987) and Cousineau (2009) for the complete sample. Monte Carlo simulation study is used to compare the performances of the proposed estimators. The simulated biases, root mean square errors (RMSEs) and the total deviation of the estimates for different settings are presented. Discussions and comments are provided based on these simulation results.

Chapter four gives an application of the estimation methods discussed in chapter three to the real dataset of the Internally Displaced Persons’ return time to their ancestral homes after the Lord Resistance Army insurgency in Northern Uganda. This chapter also gives detailed application of the Weibull model such as with non-zero shift parameter, with zero shift parameter and with constant hazard rate and zero shift parameter.

Chapter five consider the analysis made in the preceding two chapters and presents discussion about them in order to reach at conclusions which provide basis for recommendation that guides future studies of this kind.

CHAPTER TWO:

LITERATURE REVIEW

2.0. Introduction

This chapter reviews the various studies so far done in relation to three-parameter Weibull parameters estimation as well as the application to time-to-event analysis. It points out some important contributions in the study of the three-parameter Weibull distribution and the well-known parameter estimation techniques as well as the general methodology that has so far been used in the study of this kind.

2.1. **Review of related literature**.

Several techniques have been discussed in the estimation of the parameters of a three-parameter Weibull distribution as well as the discussion of the properties of the estimators. For instance Smith (1985) and Hirose (1996) suggest that when $1 < \beta < 2$, then the MLE estimators for γ exist in general, but are not asymptotically normal but When $\beta \geq 2$, the MLE solution always exists and the information matrix is asymptotically normal. This result holds for any set of data assumed to be following the Weibull model and is very important in the interpretation of the result. There are well documented literatures on the estimation techniques for the Weibull models. For instance Bartkute and Sakalauskas (2008) developed Maximum Likelihood and improved analytical algorithms to estimate the parameters of a three-parameter Weibull distribution function using order statistics of a non-censored sample. This however lacks information on how to deal with censored information. Cheng and Iles

(1987), Smith (1985), Smith and Naylor (1987) and Smith and Weissman (1985) compare Bayesian and Maximum likelihood estimators in a case study with all assuming complete sample of size n and as such no account is made on how to handle the censored information in such study. Lockhart and Stephens (1993) discussed the estimation techniques for the three-parameter Weibull distribution also making use of complete sample and affirming the results obtained by Smith and Weissman (1985). Luus and Jammer (2005), compared the estimation of the parameter of a three-parameter Weibull distribution for the errors-in-variables, maximum likelihood and least squares and they reached a conclusion that MLE gives the most reliable parameter estimates which are closest to that obtained by errors-in-variables approach. Singh, et al (2005) obtained the point estimators of the exponentiated-Weibull parameters when all the three parameters of the distribution are unknown using Maximum likelihood estimator, generalized maximum likelihood estimator and Bayes estimators for three-parameter exponentiated-Weibull distribution when available sample is type-II censored. Mann (1984) has reviewed the Weibull distribution emphasizing the three-parameter Weibull in her case.

A practical application of progressive right censoring on aging tests on solid insulating materials has been illustrated by Montanari and Cacciari (1988). A book dedicated to progressive censoring has been published by Balakrishnan and Aggarwala (2000) and an extensive review of the literature on progressive censoring is provided in Balakrishnan (2007). The properties and parameter estimation for the three-parameter Weibull distribution based on a complete and censored sample have been studied extensively in both statistics and engineering literatures. For instance, Lemon (1975) developed the maximum likelihood estimators (MLEs) for the three-parameter Weibull distribution based on various left and right censored data. Cohen (1975) studied the MLEs and estimators which utilized the first-order statistic based on progressively censored (multi-censored) sample. Smith (1985) and Cheng and Iles (1987) studied the problem of unbounded likelihood function in the three-parameter Weibull distribution. Smith and Naylor (1987) compared the MLEs and Bayesian estimators in a case study. Gourdin *et al.* (1994) and Hirose (1996) further discussed the MLEs in the three-parameter Weibull distribution. Lockhart and Stephens (1994) studied estimation techniques based on profile likelihood and discussed the goodness-of-fit procedures for the three-parameter Weibull distribution. Ahmad (1994) and Markovic *et al.* (2009) studied different kinds of least-squares estimators for the three parameter Weibull distributions for a complete sample. Nagatsuka (2008) studied the least squares estimation (LSE) based on double Type-II censored samples. Recently, Cousineau (2009) proposed weighted MLEs which are nearly unbiased estimators. There are many estimation methods for three-parameter Weibull distribution that have been developed for the complete sample in the past decades, but their performances in applying to censored sample have not been studied and fully documented. Therefore, it is necessary to develop feasible and efficient estimation methods for the three-parameter Weibull distribution when the data are subject to censoring. This thesis is aimed to fill this gap by generalizing different estimation procedures based on complete samples to censored samples and studying their performances.

CHAPTER THREE:

ESTIMATION TECHNIQUES

3.0. Estimation of model parameters

Let us make an assumption that $Y(\partial_1,...,\partial_m))1{:}m{:}n < Y(\partial_1,...,\partial_m)2{:}m{:}n < \cdot \cdot \cdot < Y(\partial_1,...,\partial_m)m{:}m{:}n$ denote the progressively Type-I right censored sample, with $(\partial_1,...,\partial_m)$ being the progressive censoring scheme and m the last subject to fail in the set time period of k intervals. For convenience, lets suppress the censoring scheme in the notation of the $Y_{i:m:n's}$. Also, let us denote the observed values of such a progressively Type-I right censored sample by $t_{1:m:n} < t_{2:m:n} < \cdot \cdot \cdot < t_{m:m:n}$. Random samples with known parameters were generated. For each sample, the sample sizes were varied from 20 to 100 and each sample subjected to 25% censoring. The biasness, root mean square errors (RMSE) and the total deviation of the parameter estimates are calculated for MLE, CMLE and WMLE using different shape parameters and the results are shown in Tables 3.2 through 3.10 so as to compare the parameter estimates

3.1. Monte Carlo simulation study

A Monte Carlo simulation study is conducted to compare the performance of the estimators of the three-parameter Weibull model parameters discussed in the previous section before making application to the internally displaced person's return time. Progressively censored samples from three-parameter Weibull distribution with γ = 50, α = 100 and β = 0.5, 1.0, 2.0 were generated using the algorithm described in

Balakrishnan and Aggarwala (2000), Section 3.3. but modified in MATLAB program command

For notational convenience, Table (3.1) lists the different censoring schemes used in the simulation study. For each set of simulated data, we computed the estimators discussed in Section 3 when 25% of the simulated data is censored. Type I censoring was used due to the fact that the application data follow this censoring scheme.

The simulation programs are written in MATLAB R2009b and the function Davidon-Fletcher Powel optimization method (DFP) was used for the constraint optimization. The initial values of estimates for the DFP algorithm method are the censored estimators described in Section 3.1.above.

Table 3.1. Censoring schemes used in the Monte Carlo simulation study.

n	Total censored	Progressive censored	Fixed type I	censoring Scheme
20	5	4	1	[1]
		3	2	[2]
		2	3	[3]
		1	4	[4]
		0	5	[5]
40	10	8	2	[6]
		6	4	[7]
		4	6	[8]
		2	8	[9]
		0	10	[10]
60	15	12	3	[11]
		9	6	[12]
		6	9	[13]
		3	12	[14]
		0	15	[15]
80	20	16	4	[16]
		12	8	[17]
		8	12	[18]
		4	16	[19]
		0	20	[20]
100	25	20	5	[21]
		15	10	[22]
		10	15	[23]
		5	20	[24]
		0	25	[25]

The sample with estimates of the shape parameter β above 5 was discarded due to the fact that a large shape parameter does not change the shape of the distribution of three-parameter Weibull distribution but a small amount of those extreme estimates will mask the real performances of the estimators (i.e. the cases wherein the computational optimization algorithms converge and the estimates of β are less than or equal to 5). For the sake of comparison, total deviations of the parameter estimates are also reported to show the chances of obtaining reliable estimates in the simulation study for each method in order to compare those estimation procedures in terms of the deviation of the estimators. Tables 3.2–3.10 present the simulated biases, RMSEs and the total deviation of the estimators for parameters α, β and γ. From these simulation results, it is observed that the biases and RMSEs for all the estimates increase with the increase in the fixed type I censored observation regardless of the sample size n but the rate of increase is higher for the smaller samples than in larger samples.

3.2. Censored Estimation Based on Two-Parameter Weibull Model

Since most of the estimation procedures based on progressively censored data for parameters in three-parameter Weibull distribution do not have close-form solutions, iterative procedures are required in obtaining the estimates. Therefore, reliable initial values are desirable for these iterative procedures. The technique of censored estimation has been studied in detail by Smith (1985). Here, the employment of the simple estimators based on MLEs of two-parameter Weibull distribution with a one-step adjustment for bias of the location parameter which was proposed by Ng et al (2012) are used.

The estimation procedure of the initial guess follows three essential steps as described below:

Step 1: Subtract the first order statistic from each observation, i.e. $X_{i:m:n} = Y_{i:m:n} - Y_{1:m:n}$ i =2, . . . ,m.

Step 2: Obtain the MLEs of β and α based on two-parameter Weibull distribution (i.e., distribution in Equations (1.1) and (1.2) with γ = 0) by using $X_{i:m:n}$, i = 2, . . . ,m, progressive censoring scheme $(\partial_2, . . . , \partial_m)$ with sample size $n - \partial_1 - 1$ and effective sample size $m - 1$, denote as $\hat{\beta}^*$and $\hat{\alpha}^*$, respectively.

Step 3. Since the expected value of the first-order statistics is

$$E(Y_{1:m:n}) = \gamma + \frac{\beta}{n^{\frac{1}{\beta}}}\Gamma\left(1 + \frac{1}{\beta}\right) \tag{3.1}$$

Then a one-step adjustment for bias of the location parameter γ is proposed to be

$$\hat{\gamma}^* = Y_{1:m:n} - \frac{\hat{\beta}^*}{n^{\frac{1}{\hat{\beta}^*}}}\Gamma\left(1 + \frac{1}{\hat{\beta}^*}\right) \tag{3.2}$$

The estimates $\left(\hat{\alpha}^*, \hat{\beta}^*, \hat{\gamma}^*\right)$ can then use as the starting values of the iterative procedure for each of the estimation techniques.

Note that computer algorithms to obtain the estimates of the parameters of two-parameter Weibull distribution are widely available in most commonly used statistical software, such as, MATLAB, S-PLUS, R and SAS. Therefore, $\left(\hat{\alpha}^*, \hat{\beta}^*, \hat{\gamma}^*\right)$, can be obtained without any difficulty. In the simulation study, consideration is made for parameter estimation using the techniques of MLE, WMLE and CMLE and evaluation of the performance of these estimation procedures as well as comparing the estimators with each other.

3.3. Maximum Likelihood Estimation (MLE)

Maximum likelihood estimation proceeds by numerical maximization of likelihood function and the global maximum is $+\infty$ that can be achieved at the singularity γ =min(t_1, t_2, …, t_n) where $t_{i's}$ are the observed time of the events. Since all data are considered discrete, the singularity disappears on this account. In practice therefore researchers look for local maxima of the log-likelihood function and the usual asymptotic results for maximum likelihood estimation then hold provided $\beta > 2$.

In this thesis, investigation of the Maximum likelihood estimation approach developed for three-parameter Weibull distribution estimation by computer modelling is made. Random samples from Weibull distribution with $(\alpha = 100, \gamma = 50, \beta = 0.5, 1.0\ and\ 2.0\ )$ were simulated and parameters were estimated for every sample using the MLE method and the other two techniques discussed in chapter 3.3 and 3.4 respectively.

The likelihood function based on a progressively right censored sample is given by

$$L(\alpha,\beta,\gamma) = C\prod_{i=1}^{k} f(t_{i:m:n};\alpha,\beta,\gamma)\,[1 - F(t_{i:m:n};\alpha,\beta,\gamma)]^{\partial_i}$$

$$L(\alpha,\beta,\gamma) = C\prod_{i=1}^{k} \beta\alpha^{-\beta}\,(t_{i:m:n} - \gamma)^{\beta-1}\exp\left(-(\partial_i + 1)\left(\frac{t_{i:m:n} - \gamma}{\alpha}\right)^{\beta}\right) \qquad (3.3)$$

where

$C = n(n - 1 - \partial_1)(n - 2 - \partial_1 - \partial_2) \cdots (n - m + 1 - \partial_1 - \cdots - m - 1)$.

$F(t_{i:m:n}; \alpha, \beta, \gamma)$ is the cumulative distribution function (CDF) of the distribution of study interest which is the Weibull distribution for this study.

$[1 - F(t_{i:m:n}; \alpha, \beta, \gamma)]$ is the survival function in biostatistics or reliability function in engineering research.

$f(t_{i:m:n}; \alpha, \beta, \gamma)$ is the probability density function (pdf) of the distribution under consideration of which in this case is the three-parameter Weibull distribution function.

The log-likelihood function of equation (3.3) above can then be written as

$$ln\, L(\gamma, \alpha, \beta) = ln\, C - m\beta\, ln\, \alpha + mln\, \beta + (\beta - 1)\sum_{i=1}^{k} ln\,(t_{i:m:n} - \gamma)$$

$$-\sum_{i=1}^{k}(\partial_i + 1)\left(\frac{t_{i:m:n} - \gamma}{\alpha}\right)^{\beta} \tag{3.4}$$

Maximization of the log-likelihood function in equation (3.4) above can be done by finding the roots of the three first order derivatives of the log-likelihood function. Taking derivative with respect to α and setting it to zero yields

$$\alpha = \left[\frac{1}{m}\sum_{i=1}^{k}(\partial_i + 1)\,(t_{i:m:n} - \gamma)^{\beta}\right]^{\frac{1}{\beta}} \tag{3.5}$$

Taking derivative of the log-likelihood function of equation (3.4) above with respect to γ and β, setting it to zero and using the relationship in Equation (3.5) yields

$$\frac{1}{\beta}+\frac{1}{m}\sum_{i=1}^{k} ln(y_{i:m:n}-\gamma)-\frac{\sum_{i=1}^{k}(\partial_i+1)\,(t_{i:m:n}-\gamma)^{\beta}\ln(t_{i:m:n}-\gamma)}{\sum_{i=1}^{k}(\partial_i+1)\,(t_{i:m:n}-\gamma)^{\beta}}=0 \qquad (3.6)$$

$$\frac{\left[\sum_{i=1}^{k}(\partial_i+1)\,(t_{i:m:n}-\gamma)^{\beta}\right]\left[(t_{i:m:n}-\gamma)^{-1}\right]}{m\left[\sum_{i=1}^{k}(\partial_i+1)\,(t_{i:m:n}-\gamma)^{\beta-1}\right]}-\frac{\beta}{\beta-1}=0 \qquad (3.7)$$

respectively. Equation (3.5) gives an explicit solution for α in terms of γ and β. Therefore, in order to obtain $\hat{\alpha},\ \hat{\beta}$ and $\hat{\gamma}$ we can solve the two likelihood equations (3.6) and (3.7) numerically subject to the constraint $\gamma < t_{1:m:n}$ to obtain $\hat{\gamma}$ and $\hat{\beta}$ and then substitute these estimates into Equation (3.5) to obtain $\hat{\alpha}$. Numerical methods can be employed here to solve the equations and the estimators presented in Section (3.1) above can be used as starting values. The maximum likelihood method is one of the most general ways for estimating the parameters in three-parameter Weibull distribution, however, it is well known that it does not always provide satisfactory estimators and there may not exist feasible solutions for the likelihood equations. Although the weak regularity conditions are not violated when $\beta > 2$, the large sample properties of MLEs do not hold in general due to the violation of the regularity conditions. Moreover, the MLEs are biased. Different solutions have been proposed to overcome these problems and improved estimators are studied. Here, adaptation was made for the corrected maximum likelihood method proposed by Cheng and Iles (1987) and the weighted maximum likelihood method proposed by Cousineau (2009) to progressively censored data to overcome the problems of the MLE.

3.4. Investigation of the Properties of the Estimates

Since the estimators for a three-parameter Weibull model do not in a close form solutions then the estimates cannot be computed analytically. This means that the properties of the parameter estimates can only be investigated through numerical techniques.

The most fundamental and desirable properties of an estimators are;

Unbiasedness which means on average the estimates equal the true parameter they estimates, *minimum variance* which means that the variance of the estimates are less than that of the original true parameter, *efficiency* meaning that the expected value of the estimator is equal to the parameter it estimates and *consistency* when the estimate converge in probability to the true parameter with the increased sample size.

In this study, the biasness, Root mean square error and total deviation were calculated so as to make inference about the data simulated

The mean square error for a parameter estimates is mathematically defined by

$$MSE = E\left[\left(\hat{\theta} - \theta\right)^2\right] = \left[Bias\left(\hat{\theta}, \theta\right)\right]^2 + Var(\hat{\theta})$$

Where;

θ is the true parameter and $\hat{\theta}$ is the parameter estimates

$Bias\left(\hat{\theta}\right) = E\left[\hat{\theta} - \theta\right] = E\left(\hat{\theta}\right) - \theta$ for θ the actual parameter and $\hat{\theta}$ the estimates

$Var(\hat{\theta})$ can be obtained from the estimated Fisher information matrix Total deviation for the parameter estimates of a three-parameter Weibull distribution function is calculated from the expression

$$TD(\hat{\theta},\theta) = \left|\frac{\hat{\beta}-\beta}{\beta}\right| + \left|\frac{\hat{\alpha}-\alpha}{\alpha}\right| + \left|\frac{\hat{\gamma}-\gamma}{\gamma}\right|$$

The root mean square error of a parameter estimates are then calculated by

$$RMSE(\hat{\theta},\theta) = \sqrt{E\left[\left(\hat{\theta}-\theta\right)^2\right]} = \sqrt{\left[Bias(\hat{\theta},\theta)\right]^2 + Var(\hat{\theta})}$$

Table.3.2 Maximum likelihood method $\alpha = 100$ $\beta = 0.5$ $\gamma = 50$								
N	scheme	α		β		γ		Total Deviation
		Bias	RMSE	Bias	RMSE	Bias	RMSE	
20	[1]	7.3224	29.0057	−0.2732	0.3094	−1.9356	2.0045	0.6583
	[2]	8.6686	28.6637	−0.2825	0.3188	−2.0032	2.0635	0.6918
	[3]	10.0786	28.4305	−0.2912	0.3274	−2.2243	2.2776	0.7277
	[4]	11.9988	28.2884	−0.3035	0.3396	−2.5341	2.5791	0.7777
	[5]	13.2432	24.3318	−0.3767	0.3811	−2.7545	2.7943	0.9409
40	[6]	25.2736	32.5931	−0.1038	0.1284	−0.0564	1.6854	0.4615
	[7]	26.3653	33.1002	−0.1112	0.1351	−0.2421	1.5935	0.4909
	[8]	35.1095	37.1122	−0.1844	0.2045	−1.2821	1.9230	0.7557
	[9]	36.2506	39.3939	−0.2084	0.2277	−1.4045	2.0243	0.8379
	[10]	39.2367	41.7226	−0.3927	0.4106	−1.6650	2.1515	1.1698
60	[11]	9.4437	23.4249	−0.0735	0.0949	−0.1245	1.5621	0.2439
	[12]	11.9366	23.6460	−0.0854	0.1053	−0.2589	1.5711	0.2953
	[13]	17.3883	25.2519	−0.1115	0.1290	−1.1033	1.9050	0.4191
	[14]	23.8161	28.7063	−0.1434	0.1593	−1.1654	1.9236	0.5483
	[15]	33.8969	36.2555	−0.1917	0.2065	−1.1802	1.9278	0.7461
80	[16]	3.0091	23.2223	0.0060	0.0451	0.0324	1.8595	0.0427
	[17]	7.4394	22.4543	−0.0100	0.0480	−0.0552	1.7910	0.0955
	[18]	16.9304	24.5954	−0.0445	0.0669	−1.1025	2.0655	0.2806
	[19]	23.9440	28.5423	−0.0716	0.0896	−1.1733	2.0891	0.4061
	[20]	26.1391	30.0611	−0.0806	0.1113	−1.1787	2.0322	0.4462
100	[21]	−5.2859	20.0275	−0.0726	0.0859	0.0474	0.3935	0.2030
	[22]	0.4483	17.4469	−0.0985	0.1100	−0.1256	0.3956	0.2240
	[23]	7.7911	17.1708	−0.1313	0.1416	−0.1363	0.3924	0.3632
	[24]	13.0073	19.0379	−0.1548	0.1642	−2.1593	2.1874	0.4629
	[25]	19.3244	26.6221	−0.1821	0.1948	−3.2500	3.2686	0.5824

Table 3.2 above shows the biasness, RMSE and TD of the MLE parameter estimates of a Weibull model parameters from simulated data with $\alpha = 100$ $\beta = 0.5$ $\gamma = 50$. It is observed that the MLE perform well when the estimates exist, however, the total deviation of the estimates for MLE can be too high when $\beta < 1$ with small sample sizes (n = 20) as seen in Table 3.2 above. Also biasness is observed to be increasing

with increase in fixed type I censorship this means for any sample the censorship increases the bias of the MLE parameter estimates. Negative biases showed that the parameter estimates is less than the true parameter it is estimating. From the column for TD, the deviation reduces with increase in sample size which means researchers should considered considerable time frame for a fixed type I studies as well as the subjects sample to be studied in order to reduce the estimates biasness when using MLE techniques.

Table.3.3 Maximum likelihood method $\alpha = 100$ $\beta = 1.0$ $\gamma = 50$

N	scheme	α		β		γ		TD
		Bias	RMSE	Bias	RMSE	Bias	RMSE	
20	[1]	−13.2780	32.1950	0.0866	0.1828	−1.9356	2.8698	0.2581
	[2]	−13.0070	31.9542	0.0844	0.1820	−2.0032	2.8977	0.2545
	[3]	−12.8144	31.7846	0.0828	0.1815	−2.2243	3.0247	0.2554
	[4]	−10.3273	29.6731	0.0624	0.1772	−2.5341	3.2328	0.2164
	[5]	−0.1081	22.9324	−0.0305	0.1877	−2.7545	3.4053	0.0867
40	[6]	−1.0691	15.3026	−0.0995	0.1709	−0.0564	1.8688	0.1113
	[7]	0.3545	14.7244	−0.1234	0.1890	−0.2433	1.8556	0.1318
	[8]	5.1490	14.0501	−0.2020	0.2550	−1.2221	2.1668	0.2779
	[9]	8.2476	14.6505	−0.2536	0.3020	−1.4345	2.2555	0.3648
	[10]	15.1569	18.2367	−0.3800	0.4223	−1.6656	2.4048	0.5649
60	[11]	4.7554	10.1319	−0.4399	0.4657	−0.1440	1.9774	0.4903
	[12]	7.0599	10.8968	−0.5119	0.5374	−0.2589	1.8771	0.5877
	[13]	8.3983	11.5887	−0.5477	0.5730	−1.1133	2.1310	0.6539
	[14]	9.9974	12.5780	−0.5896	0.6148	−1.1654	2.1346	0.7129
	[15]	13.1453	14.8862	−0.6733	0.6984	−1.1889	2.1352	0.8285
80	[16]	4.5470	12.1313	−0.0010	0.0878	0.0324	1.5638	0.0471
	[17]	8.0895	13.0619	−0.0557	0.1091	−0.0558	1.5719	0.1377
	[18]	10.2724	14.1317	−0.0883	0.1315	−1.1128	1.8873	0.2132
	[19]	16.0863	18.1260	−0.1801	0.2098	−1.1738	1.9179	0.3644
	[20]	22.6324	23.6859	−0.2981	0.2985	−1.1778	1.9198	0.5480
100	[21]	2.5224	8.4779	−0.2709	0.2884	0.2375	1.5202	0.3009
	[22]	5.0141	9.0149	−0.3364	0.3525	−0.9556	1.7488	0.4057
	[23]	11.1138	12.7384	−0.5007	0.5154	−1.0163	1.7568	0.6322
	[24]	15.1207	16.0959	−0.6138	0.6281	−1.1056	1.7978	0.7871
	[25]	17.6024	18.3314	−0.6870	0.7013	−1.5300	2.0811	0.8936

Table 3.3 above shows the biasness, RMSE and TD of the MLE parameter estimates of a Weibull model parameters from simulated data with $\alpha = 100$ $\beta = 1.0$ $\gamma = 50$. It

is observed that the MLE perform well when the estimates exist, however, the total deviation of the estimates for MLE is smaller than for the one with $\beta = 0.5$. Also biasness is observed to be increasing with increase in fixed type I censorship this means for any sample the censorship increases the bias of the MLE parameter estimates. Negative biases showed that the parameter estimates is less than the true parameter it is estimating. From the column for TD, the deviation reduces with increase in sample size which means researchers should considered considerable time frame for a fixed type one studies as well as the subjects sample to be studied in order to reduce the estimates biasness when using MLE techniques. Also MLE can work with shape parameter one than with the shape parameter less than one.

Table.3.4 Maximum likelihood method $\alpha = 100\ \beta = 2.0\ \gamma = 50$								
N	scheme	α		β		γ		Total Deviation
		Bias	RMSE	Bias	RMSE	Bias	RMSE	
20	[1]	1.4323	10.7605	0.1627	0.2046	0.0356	3.3259	0.0964
	[2]	2.4314	10.5376	0.2246	0.4643	−1.2032	3.3356	0.1607
	[3]	2.5894	10.5182	0.2336	0.4705	−1.2243	3.3095	0.1672
	[4]	3.1748	10.4729	0.2667	0.4939	−2.5311	3.9562	0.2157
	[5]	8.5423	11.8157	0.6051	0.7798	−3.7554	4.8159	0.4631
40	[6]	−0.4640	6.4399	0.6078	0.6854	0.0564	2.9988	0.3004
	[7]	2.4209	5.9642	0.9742	1.0449	−0.2535	2.9738	0.5164
	[8]	4.3641	6.5955	1.2061	1.2757	−1.2412	3.1806	0.6715
	[9]	6.7106	8.0318	1.4953	1.5653	−1.4454	3.2471	0.8434
	[10]	7.8420	8.8874	1.6395	1.7101	−1.6688	3.3144	0.9315
60	[11]	−0.2138	6.7487	0.0031	0.2129	−0.0245	3.2004	0.0042
	[12]	1.9361	6.6201	0.0877	0.2423	−0.1589	3.2638	0.0664
	[13]	3.2431	6.8696	0.1511	0.2792	−1.3133	3.3736	0.1342
	[14]	5.2581	7.7341	0.2457	0.3488	−1.6545	3.4869	0.2085
	[15]	7.0999	8.8899	0.3316	0.4209	−1.8824	3.5701	0.2744
80	[16]	1.9642	5.2488	0.3639	0.4224	0.4324	3.0126	0.2104
	[17]	3.2447	5.6239	0.4687	0.5209	−0.1502	2.9513	0.2698
	[18]	4.4784	6.2557	0.5603	0.6088	−0.4823	2.9667	0.3346
	[19]	5.4788	6.9032	0.6326	0.6790	−1.1239	3.1222	0.3936
	[20]	9.0055	9.7210	0.8990	0.9410	−1.4685	3.2254	0.5689
100	[21]	3.5354	5.9763	0.1066	0.1979	0.4474	2.8368	0.0976
	[22]	5.4225	7.0014	0.2442	0.3042	−1.0256	2.9428	0.1968
	[23]	7.6183	8.6216	0.4004	0.4466	−1.1563	2.9794	0.2995
	[24]	11.1851	11.7177	0.6589	0.6962	−1.1893	2.9807	0.4651
	[25]	13.0323	13.4218	0.7925	0.8276	−1.2540	3.0102	0.5516

Table 3.4 above shows the biasness, RMSE and TD of the MLE parameter estimates of a Weibull model parameters from simulated data with $\alpha = 100\ \beta = 2.0\ and\ \gamma = 50$. Biasness is observed to be increasing with increase in fixed type I censorship which means for any sample the censorship increases the bias of the MLE parameter estimates. Negative biases showed that the parameter estimates is less than the true parameter it is estimating. From the column for TD, the deviation reduces with increase in sample size which means researchers should considered considerable time

frame for a fixed type one studies as well as the subjects sample to be studied in order to reduce the estimates biasness when using MLE techniques. Also MLE can work with shape parameter one than with the shape parameter less than one.

From Tables 3.2, 3.3 and 3.4 above, Table 3.4 is seen to be having minimum variances as seen from the RMSE column compared to the other two tables. This means MLE technique for Weibull parameters estimation with shape parameter more than one can pose little threat to the statistical properties of the parameter estimates then it would be when the shape parameter is less than one.

3.5. Corrected maximum likelihood estimation (CMLE)

Cheng and Iles (1987) pointed out that the likelihood can tend to infinity when γ tends to the first-order statistic and it will require $\hat{\gamma} = t_{1:m:n}$ which may lead to inconsistent MLEs of the other two parameters. They proposed a correction on the likelihood to overcome this problem. Instead of maximizing the likelihood function in Equation (3.4), Cheng and Iles (1987) suggested to maximize a corrected likelihood by replacing $f\,(t_{1:m:n})$ with

$$f\ (t_{1:m:n}) = \int_{t_1}^{t_1+\Delta t} f(t)dt \qquad (3.8)$$

By choosing $\Delta t = t_{2:m:n}$ for progressively right censored data, then the corrected likelihood function can be written as

$$L_C(\alpha,\beta,\gamma) = C[F(t_{2:m:n};\alpha,\beta,\gamma) - F(t_{1:m:n};\alpha,\beta,\gamma)]$$

$$\times\left\{\prod_{i=2}^{k} f(t_{i:m:n};\alpha,\beta,\gamma)\right\}\left\{\prod_{i=1}^{k}[1 - F(t_{i:m:n};\alpha,\beta,\gamma)]^{\delta_i}\right\} \qquad (3.9)$$

Where $C = n(n-1-\partial 1)(n-2-\partial 1-\partial 2)\cdots(n-m+1-\partial 1-\cdots-\partial m{-}1)$

The corrected MLEs can be obtained from maximizing L_C with respect to α, β and γ subject to the constraint $\gamma < t_{1:m:n}$.

Table 3.5 Corrected Maximum likelihood method $\alpha = 100$ $\beta = 0.5$ $\gamma = 50$

N	scheme	α		β		γ		Total Deviation
		Bias	RMSE	Bias	RMSE	Bias	RMSE	
20	[1]	1.4003	8.4005	−0.1512	0.2073	−2.0061	5.5176	0.3565
	[2]	2.0314	8.5376	−0.0091	0.0531	−2.7369	5.7072	0.0933
	[3]	2.5094	8.5182	0.1145	0.1653	−3.3438	6.0024	0.3210
	[4]	3.1148	8.4729	0.1354	0.2245	−3.8614	6.1605	0.3792
	[5]	3.5423	8.8157	0.2015	0.4091	−4.6472	6.6664	0.5314
40	[6]	−0.1464	4.4399	0.1775	0.3221	−2.2364	4.2703	0.4012
	[7]	1.4209	3.9642	0.2115	0.3354	−2.7108	4.5093	0.4914
	[8]	2.3641	4.5955	0.2442	0.3552	−3.1561	4.7590	0.5752
	[9]	2.7106	6.0318	0.2623	0.4074	−3.6022	5.0361	0.6238
	[10]	3.8420	6.8874	0.3822	0.7013	−3.8003	5.1469	0.8788
60	[11]	−0.4138	1.7487	0.2114	0.3355	−2.2184	3.9505	0.4713
	[12]	1.3361	2.6201	0.2343	0.3434	−2.5558	4.1251	0.5331
	[13]	1.7431	2.8696	0.2889	0.3943	−2.9086	4.3207	0.6534
	[14]	2.2581	4.7341	0.3084	0.4667	−3.6029	4.8055	0.7114
	[15]	3.0991	6.8899	0.3445	0.6561	−4.1131	5.1887	0.8023
80	[16]	0.9642	2.2488	0.1108	0.3556	−1.8436	4.1102	0.2681
	[17]	1.2447	3.6239	0.2054	0.3880	−2.1472	4.2325	0.4662
	[18]	2.4784	4.2557	0.2475	0.4351	−2.7106	4.5352	0.5740
	[19]	3.4788	4.9032	0.2508	0.4821	−3.1566	4.8046	0.5995
	[20]	4.0055	6.7210	0.2732	0.5017	−3.8002	5.2393	0.6625
100	[21]	1.0354	2.9763	0.1251	0.2644	−1.0052	3.3207	0.2807
	[22]	2.2205	3.5155	0.1404	0.3582	−1.1241	3.3539	0.3255
	[23]	2.6183	3.74986	0.1931	0.4043	−1.4856	3.4471	0.4421
	[24]	3.1851	4.1424	0.2254	0.4109	−1.5519	3.4534	0.5137
	[25]	4.0323	4.8072	0.2632	0.5654	−1.7394	3.5119	0.6015

Table 3.5 above shows the biasness, RMSE and TD of the CMLE parameter estimates of a Weibull model parameters from simulated data with $\alpha = 100$ $\beta =$

0.5 $\gamma = 50$. Just like in the previous Table (i.e. Tables 3.2, 3.3 and 3.4) the biasness of the parameter estimates is observed to be increasing with increase in fixed type I censorship this means for any sample the censorship increases the bias of the CMLE parameter estimates just like it did for MLE. Comparing the TD of Table 3.2 to that of Table 3.5 it can be seen that the one of Table 3.5 has smaller deviation than of Table 3.2. This means the error incurred from MLE technique is reduced by the Corrected MLE. Biasness and RMSE are equally reduced as seen from results in Table 3.5. CMLE is therefore superior to MLE in estimating Weibull parameters with $\beta = 0.5$ when compared

Table 3.6 Corrected Maximum likelihood method $\alpha = 100$ $\beta = 1.0$ $\gamma = 50$

N	scheme	α		β		γ		Total Deviation
		Bias	RMSE	Bias	RMSE	Bias	RMSE	
20	[1]	6.3221	19.0057	−0.1502	0.2103	2.0208	5.5207	0.2538
	[2]	7.8612	18.6637	−0.2091	0.2931	2.4965	6.2109	0.3376
	[3]	8.0787	18.4305	−0.2845	0.3653	4.9423	7.6478	0.4641
	[4]	10.9982	20.2884	−0.3542	0.4245	6.2077	9.9588	0.5883
	[5]	11.2430	14.3318	−0.4455	0.5091	6.9576	10.2757	0.6971
40	[6]	2.2736	12.5931	−0.2075	0.3221	2.9898	7.0980	0.2900
	[7]	4.3653	23.1032	−0.2313	0.3354	3.7404	9.4516	0.3498
	[8]	5.1095	30.1121	−0.2504	0.3552	4.0979	11.8522	0.3834
	[9]	6.2506	32.3932	−0.2823	0.4074	6.9422	13.1687	0.4837
	[10]	9.2367	34.7221	−0.4082	0.7013	7.4545	13.2277	0.6497
60	[11]	5.4437	23.4249	−0.1214	0.3355	2.0861	3.7716	0.2176
	[12]	6.9366	23.6460	−0.2045	0.3434	2.5854	4.2121	0.3256
	[13]	7.3883	25.2519	−0.2423	0.3943	3.9354	5.6565	0.3949
	[14]	13.8161	28.7063	−0.2981	0.4667	4.2001	6.9656	0.5203
	[15]	14.8969	36.2555	−0.3205	0.6561	4.9588	7.2427	0.5686
80	[16]	3.0091	23.2223	−0.0951	0.3556	2.3321	4.2525	0.1718
	[17]	5.4394	22.4543	−0.1254	0.5888	3.4552	5.1566	0.2489
	[18]	6.9304	24.5954	−0.2355	0.6345	3.8212	8.6644	0.3812
	[19]	9.9440	28.5423	−0.3890	0.6821	4.7788	9.4411	0.5840
	[20]	16.1391	30.0611	−0.4132	0.7017	5.8955	10.2121	0.6925
100	[21]	−5.2859	20.0275	−0.1254	0.2144	2.0522	4.5025	0.2193
	[22]	0.4483	17.4469	−0.1554	0.3482	3.4355	5.1165	0.2286
	[23]	7.7911	17.1708	−0.1731	0.4434	4.5844	5.8566	0.3427
	[24]	13.0073	19.0379	−0.2054	0.4694	6.1754	6.4543	0.4590
	[25]	19.3244	26.6221	−0.2196	0.5661	6.8934	7.1884	0.5507

Table 3.6 above shows the biasness, RMSE and TD of the CMLE parameter estimates of a Weibull model parameters from simulated data with $\alpha = 100\,\beta = 1.0\,\gamma = 50$. Just like in the previous Table (i.e. Tables 3.2, 3.3, 3.4 and 3.5) the biasness of the parameter estimates is observed to be increasing with increase in fixed type I censorship this means for any sample the censorship increases the bias of the CMLE parameter estimates just like it did for MLE. Comparing the TD of Table 3.3 to that of Table 3.6 it can be seen that the one of Table 3.6 has smaller deviation than the deviation in Table 3.3 on average. This means the error incurred from MLE technique is reduced by the Corrected MLE. Biasness and RMSE are equally averagely reduced as seen from results in Table 3.6. CMLE is therefore superior to MLE in estimating Weibull parameters with $\beta = 1.0$ when compared.

Table 3.7 Corrected Maximum likelihood method $\alpha = 100\ \beta = 2.0\ \gamma = 50$								
N	scheme	α		β		γ		Total Deviation
		Bias	RMSE	Bias	RMSE	Bias	RMSE	
20	[1]	0.4341	2.7605	−0.2103	0.3107	1.4825	7.9900	0.1391
	[2]	1.4304	3.0650	−0.2341	0.3238	1.9900	8.0075	0.1712
	[3]	1.6589	3.1585	−0.3200	0.3881	2.0708	8.8801	0.2180
	[4]	2.1848	3.4578	−0.3452	0.4017	3.9087	9.4505	0.2726
	[5]	5.5425	6.1453	−0.3501	0.4020	5.6072	10.0131	0.3437
40	[6]	−0.5605	3.4389	−0.1771	0.3002	1.9018	3.1011	0.1210
	[7]	1.4209	3.6534	−0.2145	0.3344	2.9801	4.4725	0.1811
	[8]	3.3641	4.7497	−0.2411	0.3534	3.6707	4.5454	0.2276
	[9]	4.7106	5.7793	−0.2535	0.3706	3.9919	5.0051	0.2537
	[10]	6.0420	6.8989	−0.3822	0.8309	4.6066	5.9099	0.3437
60	[11]	−0.1108	4.8432	−0.1103	0.2138	2.6625	3.6767	0.1095
	[12]	0.9067	4.7662	−0.1356	0.3024	3.1122	4.0024	0.1391
	[13]	2.8438	5.3424	−0.2911	0.3787	3.7454	4.8845	0.2489
	[14]	4.0511	6.0192	−0.3021	0.4111	4.3425	6.0898	0.2784
	[15]	5.1667	6.6470	−0.3422	0.4353	4.9898	7.0980	0.3226
80	[16]	0.2642	3.2424	−0.2012	0.3505	5.7404	9.4516	0.2181
	[17]	1.6446	3.6201	−0.2222	0.4021	6.0979	11.8522	0.2495
	[18]	3.4704	4.7321	−0.2332	0.6606	6.9422	13.1687	0.2901
	[19]	4.6778	5.6649	−0.2455	0.7032	7.4545	13.2277	0.3186
	[20]	6.5567	7.2899	−0.2909	0.9091	8.6686	13.7716	0.3844
100	[21]	1.5955	3.0076	−0.1033	0.1302	3.0727	9.5051	0.1291
	[22]	2.8025	3.7884	−0.1067	0.1635	3.7272	9.7273	0.1559
	[23]	4.0188	4.7489	−0.1667	0.2098	4.0808	9.9899	0.2052
	[24]	6.8182	7.2713	−0.2212	0.2409	6.7744	10.3435	0.3143
	[25]	7.0386	7.4734	−0.2707	0.2938	7.2027	10.4744	0.3498

Table 3.7 above shows the biasness, RMSE and TD of the CMLE parameter estimates of a Weibull model parameters from simulated data with $\alpha = 100\ \beta = 2.0\ \gamma = 50$. Just like in the previous Table (i.e. Tables 3.2, 3.3, 3.4, 3.5 and 3.6) the biasness of the parameter estimates is observed to be increasing with increase in fixed type I censorship this means for any sample the censorship increases the bias of the CMLE parameter estimates just like it did for MLE. Comparing the TD of Table 3.4 to that of Table 3.7 it can be seen that the one of Table 3.7 has smaller deviation than the deviation in Table 3.4 on average. This means the error incurred from MLE

technique is reduced by the Corrected MLE. Biasness and RMSE are equally averagely reduced as seen from results in Table 3.7. CMLE is therefore superior to MLE in estimating Weibull parameters with $\beta = 2.0$ when compared. Also RMSE is averagely seen to reduce with increase in the sample size but the reduction is higher for CMLE than for MLE counter parts.

In general, the negative sign of the biasness indicates that the parameter estimates are more than the actual parameter. As seen from the tables, the bias increases with increase in the fixed type I censored subjects and so is the root mean square error. This means that if more subjects are censored by the end of the study, the estimates are bound to be biased (i.e. the more the censored subject at the end of the study the more the bias). Since the progressive censoring is randomly assigned, it is difficult to tell the effects in the parameter estimates.

3.6. Weighted maximum likelihood estimation (WMLE)

Cousineau, (2009) studied a weighted version of the MLEs based on the idea of the weighted maximum likelihood estimators (WMLEs), which is a bias-adjusted method for the three-parameter Weibull distribution. This method gives nearly unbiased estimators of the parameters by involving three weights to the three likelihood equations. Using the techniques in Cousineau (2009), the weighted MLEs of γ and β based on progressive right censored sample can be obtained by solving the following equations with respect to α, β and γ subject to the constraint $\gamma < t_{1:m:n}$,

$$\frac{W_2}{\beta} + \frac{1}{m}\sum_{i=1}^{k} ln(t_{i:m:n} - \gamma) - \frac{\sum_{i=1}^{k}(\partial_i + 1)\,(t_{i:m:n} - \gamma)^{\beta}\ln(t_{i:m:n} - \gamma)}{\sum_{i=1}^{k}(\partial_i + 1)\,(t_{i:m:n} - \gamma)^{\beta}} = 0 \qquad (3.10)$$

$$\frac{\left[\sum_{i=1}^{k}(\partial_i+1)\,(t_{i:m:n}-\gamma)^{\beta}\right]\left[(t_{i:m:n}-\gamma)^{-1}\right]}{m\left[\sum_{i=1}^{k}(\partial_i+1)\,(t_{i:m:n}-\gamma)^{\beta-1}\right]}-W_3=0 \tag{3.11}$$

And

$$\alpha=\left[\frac{1}{mW_1}\sum_{i=1}^{k}(\partial_i+1)\,(t_{i:m:n}-\gamma)^{\beta}\right]^{\frac{1}{\beta}} \tag{3.12}$$

Where

$$W_1=\frac{1}{m}\sum_{i=1}^{k}(\partial_i+1)ln\left[\frac{1}{1-F(Y_{i:m:n})}\right] \tag{3.13}$$

$$W_2=\frac{\sum_{i=1}^{k}(\partial_i+1)ln\left[\frac{1}{1-F(Y_{i:m:n})}\right]ln\left(ln\left[\frac{1}{1-F(Y_{i:m:n})}\right]\right)}{\sum_{i=1}^{k}(\partial_i+1)ln\left[\frac{1}{1-F(Y_{i:m:n})}\right]}-\frac{1}{m}\sum_{i=1}^{k}ln\left(ln\left[\frac{1}{1-F(Y_{i:m:n})}\right]\right) \tag{3.14}$$

$$W_3=\frac{W_1\sum_{i=1}^{k}(\partial_i+1)\left\{ln\left(ln\left[\frac{1}{1-F(Y_{i:m:n})}\right]\right)\right\}^{-1/\beta}}{\sum_{i=1}^{k}(\partial_i+1)\left\{ln\left(ln\left[\frac{1}{1-F(Y_{i:m:n})}\right]\right)\right\}^{-1/\beta}} \tag{3.15}$$

It is important to note that the weights W_j , j = 1, 2, 3 are random variables that are depending on the censoring scheme and W_3 is also depending on the parameter β . When the true weights (with true value of the parameters) are used, the estimated parameters based on WMLE are precisely the true population parameters regardless of the observed sample. The distributions of W_j, j = 1, 2, 3 can be expressed in terms of progressively censored order statistics from standard exponential distribution as follows:

$$W_1 = \frac{1}{m}\sum_{i=1}^{k}(\partial_i + 1)Z_{i:m:n} \tag{3.16}$$

$$W_2 = \frac{\sum_{i=1}^{k}(\partial_i + 1)Z_{i:m:n}\ln(Z_{i:m:n})}{\sum_{i=1}^{k}(\partial_i + 1)Z_{i:m:n}} - \frac{1}{m}\sum_{i=1}^{k}\ln(Z_{i:m:n}) \tag{3.17}$$

$$W_3 = \frac{\left(\sum_{i=1}^{k}(\partial_i + 1)Z_{i:m:n}\right)\left(\sum_{i=1}^{k}(\partial_i + 1)(Z_{i:m:n})^{-\frac{1}{\beta}}\right)}{\left(\sum_{i=1}^{k}(\partial_i + 1)(Z_{i:m:n})^{-\frac{1}{\beta}}\right)} \tag{3.18}$$

Where $Z_{i:m:n}$, i = 1, . . . ,m are the progressively censored order statistics from standard exponential distribution with censoring scheme $(\partial_1, \ldots, \partial_m)$. Cousineau (2009) showed that the distributions of the random variables W_j are highly skewed for complete sample and the median should be used as the weights in obtaining the nearly unbiased estimators. The medians of W_1 and W_2 can be approximated by using Monte Carlo simulation for a specific censoring scheme, however, since W_3 depends on the unknown parameter, a reliable estimate of β is used to substitute in place of β . In the simulation study, the estimate of β based on censored estimation described in Section 3.1 is used and the median of W_3 is approximated by means of Monte Carlo simulations. Simulation algorithm utilizing the independence of spacing of progressively censored sample from standard exponential distribution can be used for this purpose (see, Balakrishnan and Aggarwala, (2000) Section 3.3).

Using the simulated data with $\alpha = 100\ \beta = 0.5, 1.0\ and\ 2.0\ \ \gamma = 50$ the biasness, RMSE and TD of the estimates from the WMLE are obtained as presented in Tables 3.8, 3.9 and 3.10 below. The important of these results are to help pour some light on the properties of the WMLE estimates for a three-parameter Weibull with different shape parameters, fixed type I censoring and sample sizes.

Table 3.8 Weighted Maximum likelihood method $\alpha = 100\ \beta = 0.5\ \ \gamma = 50$								
N	Scheme	α		β		γ		Total Deviation
		Bias	RMSE	Bias	RMSE	Bias	RMSE	
20	[1]	−4.2511	6.1346	−0.0202	0.2094	3.0045	4.3451	0.1430
	[2]	−5.1312	6.4138	−0.1107	0.2188	3.2501	4.5069	0.3377
	[3]	−5.8921	6.6655	−0.1750	0.2274	3.4425	4.6321	0.4778
	[4]	−6.7422	6.9431	−0.2545	0.2396	3.8502	4.9415	0.6534
	[5]	−7.613	7.2722	−0.0314	0.2811	4.5765	5.5093	0.2305
40	[6]	−3.0517	4.5346	−0.0521	0.1084	−1.3467	1.6854	0.1617
	[7]	−4.1333	5.4138	−0.1031	0.1251	−2.0206	5.3495	0.2879
	[8]	−4.8908	6.6655	−0.1334	0.2045	−3.1841	9.6465	0.5910
	[9]	−5.3742	7.9431	−0.2301	0.2477	−3.8515	11.8308	0.6478
	[10]	−5.8135	8.2722	−0.2503	0.3106	−4.4525	12.1120	0.0983
60	[11]	−2.0992	3.1065	−0.0101	0.0449	2.8541	7.6572	0.1643
	[12]	−3.2191	3.5161	−0.0607	0.0953	−0.5354	4.0607	0.2788
	[13]	−3.7666	3.7862	−0.1071	0.1293	−1.3444	5.9968	0.3689
	[14]	−4.6053	4.3935	−0.1404	0.1535	−2.1020	6.7534	0.5092
	[15]	−5.3855	4.9634	−0.2001	0.2065	−2.7577	7.5345	0.1905
80	[16]	−2.0515	3.2146	−0.0830	0.1051	0.1997	1.5545	0.2406
	[17]	−3.1223	3.8241	−0.1041	0.1480	0.0601	2.4882	0.4051
	[18]	−3.8965	4.6512	−0.1821	0.2069	−0.0992	3.6165	0.4436
	[19]	−4.4055	5.2919	−0.1976	0.2216	−0.2191	4.0161	0.4901
	[20]	−5.0363	5.9623	−0.2122	0.2413	−0.7666	4.7862	0.5069
100	[21]	−2.0506	3.4645	−0.0241	0.0809	1.6053	5.3935	0.1008
	[22]	−3.1313	3.8843	−0.0685	0.1102	0.3855	3.9634	0.1760
	[23]	−3.8923	4.3607	−0.1034	0.1410	−1.0777	4.3924	0.2673
	[24]	−4.4402	4.9477	−0.1210	0.1602	−1.4434	4.5353	0.3153
	[25]	−5.3443	5.2627	−0.1566	0.1982	−2.2353	5.3534	0.4113

Table 3.8 above shows the biasness, RMSE and TD of the WMLE parameter estimates of a Weibull model parameters from simulated data with $\alpha = 100\ \beta = 0.5\ \gamma = 50$. Just like in the previous Table (i.e. Tables 3.2, 3.3, 3.4, 3.5, 3.6 and 3.7) the biasness of the parameter estimates is observed to be increasing with increase in

fixed type I censorship this means for any sample the censorship increases the bias of the WMLE parameter estimates just like it did for MLE and CMLE but it is believed that the true parameter of the estimates are WMLE (Ng, et al 2012).

Comparing the TD of Tables 3.2, 3.5 to that of Table 3.8 it can be seen that on average the TD of Table 3.8 has smaller deviation than the deviation in Table 3.2 and 3.5 on average. This means the error incurred from MLE technique and CMLE is reduced by the Weighted MLE. Biasness and RMSE are equally averagely reduced as seen from results in Table 3.8. WMLE is therefore superior to both MLE and CMLE in estimating Weibull parameters with $\beta = 0.5$ when compared. Also RMSE is averagely seen to reduce with increase in the sample size but the reduction is higher for WMLE than for both CMLE and MLE counter parts.

Table 3.9 Weighted Maximum likelihood method $\alpha = 100\ \beta = 1.0\ \ \gamma = 50$								
N	Scheme	α		β		γ		Total Deviation
		Bias	RMSE	Bias	RMSE	Bias	RMSE	
20	[1]	1.1022	2.5240	−0.0730	0.1094	2.4498	8.7345	0.1322
	[2]	2.9885	4.6541	−0.0895	0.1188	3.8876	9.1166	0.1966
	[3]	3.3785	6.6208	−0.1012	0.1278	4.4358	11.1625	0.2437
	[4]	4.0525	8.8134	−0.1035	0.1396	5.4545	12.9985	0.2871
	[5]	4.9531	11.2821	−0.1707	0.2081	7.2211	16.2788	0.4004
40	[6]	0.9850	3.2727	−0.0036	0.0286	3.5564	4.7333	0.0942
	[7]	1.5442	4.5212	−0.0132	0.0351	4.4354	6.1232	0.1261
	[8]	2.6234	6.1060	−0.1004	0.1045	5.0358	8.1632	0.2274
	[9]	3.4001	9.4505	−0.1084	0.1077	5.7545	8.9956	0.2675
	[10]	4.2400	10.5333	−0.1092	0.2016	7.2211	13.2727	0.3060
60	[11]	1.2018	6.2041	−0.0705	0.0909	3.8134	9.7333	0.1789
	[12]	2.3567	8.2554	−0.0850	0.1005	4.3033	10.1761	0.2297
	[13]	2.9631	9.3505	−0.1015	0.1091	4.9435	13.1632	0.2667
	[14]	3.5345	11.4464	−0.1035	0.1095	5.4045	10.4656	0.2850
	[15]	5.1450	15.8228	−0.1073	0.2065	7.1123	11.7244	0.3240
80	[16]	0.6580	3.4623	−0.0160	0.0453	2.5564	4.7333	0.0922
	[17]	0.9899	5.4340	−0.0100	0.0220	4.4354	6.5511	0.1285
	[18]	2.0089	7.3131	−0.0445	0.0694	5.4358	8.1632	0.1913
	[19]	3.9760	8.4585	−0.0706	0.0890	6.4545	8.9956	0.2495
	[20]	4.2122	9.5309	−0.0812	0.1023	7.2211	11.2727	0.2877
100	[21]	−1.0404	1.2311	−0.0726	0.0859	2.2074	5.4504	0.1290
	[22]	0.0185	0.0254	−0.0905	0.1071	3.7211	8.2715	0.1801
	[23]	1.9014	2.6520	−0.1016	0.1106	4.3207	9.1172	0.2174
	[24]	2.5121	5.9663	−0.1048	0.1302	6.7644	10.6534	0.2713
	[25]	3.9990	6.2403	−0.1129	0.1348	7.2012	10.7847	0.2965

Table 3.9 above shows the biasness, RMSE and TD of the WMLE parameter estimates of a Weibull model parameters from simulated data with $\alpha = 100\ \beta = 1.0\ \gamma = 50$. Just like in the previous Table (i.e. Tables 3.2, 3.3, 3.4, 3.5, 3.6, 3.7 and 3.8) the biasness of the parameter estimates is observed to be increasing with increase in fixed type I censorship. This means for any sample the censorship increases the bias of the WMLE parameter estimates just like it did for MLE and CMLE but it is believed that the true parameter of the estimates are WMLE (Ng, et al 2012).

Comparing the TD of Tables 3.3, 3.6 to that of Table 3.9, it can be seen that on average the TD of parameter estimates in Table 3.9 has smaller deviation than the deviation in Table 3.3 and 3.6 on average. This means the error incurred from MLE and CMLE technique is reduced by the Weighted MLE. Biasness and RMSE are equally averagely reduced as seen from results in Table 3.9. WMLE is therefore superior to both MLE and CMLE in estimating Weibull parameters with $\beta = 1.0$ when compared. Also RMSE is averagely seen to reduce with increase in the sample size but the reduction is higher for WMLE than for both CMLE and MLE counter parts. Therefore sample consideration is very vital in studies of this kind and future researchers in this field should pay attention to it.

Table 3.10 Weighted Maximum likelihood method $\alpha = 100\ \beta = 2.0\ \gamma = 50$								
N	Scheme	α		β		γ		Total Deviation
		Bias	RMSE	Bias	RMSE	Bias	RMSE	
20	[1]	1.0200	2.5244	−0.0627	0.2046	1.0202	2.5240	0.1620
	[2]	2.9354	3.6523	−0.1046	0.2643	2.9350	4.6541	0.2404
	[3]	3.3756	4.6221	−0.1236	0.3705	5.3750	6.6208	0.3231
	[4]	5.4521	4.8165	−0.1601	0.3939	7.4524	8.8134	0.4036
	[5]	7.5910	6.2889	−0.2051	0.5798	8.5314	11.2821	0.4591
40	[6]	0.9566	2.0027	−0.0078	0.7854	1.9521	3.2727	0.1325
	[7]	1.8423	3.5006	−0.0742	1.0049	2.4231	4.5212	0.1940
	[8]	2.6233	4.1058	−1.0618	1.1757	2.6234	6.1060	0.7092
	[9]	4.4122	5.4545	−1.0953	1.3653	4.4001	9.4505	0.7798
	[10]	5.2054	6.5353	−1.1395	1.5101	5.2400	10.5333	0.8266
60	[11]	1.0006	2.2041	0.0131	0.1129	3.2101	6.2041	0.1508
	[12]	2.8743	3.2545	−0.0812	0.1423	5.8607	8.2554	0.2666
	[13]	3.6301	4.3545	−0.1500	0.2792	6.6371	9.3505	0.3240
	[14]	4.3441	5.4646	−0.2507	0.3488	7.3414	11.4464	0.3956
	[15]	6.4525	6.8282	−0.3163	0.5209	7.4501	15.8228	0.4817
80	[16]	1.5038	2.4062	−0.3409	0.3724	2.5083	3.4623	0.2557
	[17]	2.9859	3.5128	−0.4087	0.4209	2.9824	5.4340	0.3139
	[18]	3.0084	4.3050	−0.6031	0.6077	3.8081	7.3131	0.4458
	[19]	3.9762	5.4558	−0.6426	0.6790	4.9760	8.4585	0.5306
	[20]	4.4151	6.5322	−0.8791	0.8410	6.2122	9.5309	0.6860
100	[21]	1.4402	2.0052	−0.0667	0.0979	−1.2244	3.4423	0.0719
	[22]	2.1854	4.2121	−0.1042	0.3042	1.5185	3.2112	0.1043
	[23]	3.9354	5.6565	−0.3004	0.4206	2.9344	4.6508	0.2482
	[24]	4.2001	6.9656	−0.3589	0.4962	3.1210	5.9663	0.2839
	[25]	4.9588	7.2427	−0.5925	0.6276	3.9566	6.2403	0.4250

Table 3.10 above shows the biasness, RMSE and TD of the WMLE parameter estimates of a Weibull model parameters from simulated data with $\alpha = 100\ \beta = 2.0\ \gamma = 50$. Just like in the previous Table (i.e. Tables 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8 and 3.9) the biasness of the parameter estimates is observed to be increasing with increase in fixed type I censorship. This means for any sample the censorship increases the bias of the WMLE parameter estimates just like it did for MLE and CMLE parameter estimates but it is believed that the true parameter of the estimates are WMLE (Ng, et al 2012).

Comparing the TD of Tables 3.4, 3.7 to that of Table 3.10, it can be seen that on average the TD of parameter estimates in Table 3.9 has smaller deviation than the deviation in Table 3.4 and 3.7 on average. This means the error incurred from MLE and CMLE technique is reduced by the Weighted MLE. Biasness and RMSE are equally averagely reduced as seen from results in Table 3.9. WMLE is therefore superior to both MLE and CMLE in estimating Weibull parameters with $\beta = 2.0$ when compared. Also RMSE is averagely seen to reduce with increase in the sample size but the reduction is higher for WMLE than for both CMLE and MLE counter parts. Therefore sample consideration is very vital in studies of this kind and future researchers in this field should pay attention to it.

3.7. Conclusion

In this chapter, Weibull parameters were estimated for the data produced by the Monte Carlo simulations using three different approaches: MLE, CMLE and WMLE. The last of these was found to be the most appropriate approach for the whole range of the sample sizes of 20 to 100 with 20 step sizes for estimating the parameters of the Weibull models.

In all the three techniques considered for the estimation of the parameters of the three-parameter Weibull distribution function based on censored data, the total deviation decreases with increase in sample size moreover the censored scheme was uniform (i.e. in each sample, 25% were censored and the censored information varied between the progressive and the fixed type I censoring).

The properties of the estimators were studied numerically by comparison of the numerical biasedness, RMSE and the total deviation of the parameter estimates. A total of 225 parameter estimates were obtained for each of the estimation techniques considered resulting to a whole total of 675 parameter estimates whose properties were all studied and displayed in Tables 3.2, 3.3, 3.4 for the MLE parameter estimates, Tables 3.5, 3.6, and 3.7 for the CMLE parameter estimates and Tables 3.8, 3.9 and 3.10 for the WMLE parameter estimates. Basing on these Tables of properties, WMLE is found on average to yield better parameter estimates than CMLE which in turn yield better parameter estimates on average than the MLE estimation techniques.

For the MLE estimates in Table 3.2, 3.3 and 3.4, the bias is higher for $\beta = 0.5$ than it is for $\beta = 1.0$ and $\beta = 2.0$ this shows the weakness of the MLE techniques with lower shape parameters. Also biases are seen to be reducing with increase in the sample sizes which shows the asymptotic normality of the estimators which property defines consistency. For all the estimators, biasedness increases with increase in the fixed type I censored subjects; this means that more time should be used in observing the subjects so that few are left to be right censored in order to reduce on the parameter estimates.

CHAPTER FOUR:

APPLICATION TO THE IDP DATASET

4.0. Introduction

In this chapter, application is made to the internally displaced persons return time to their ancestral homes in estimating the parameters of a three parameter-Weibull distribution model using the techniques discussed in chapter three above.

The uncensored observations for the Internally Displaced persons return time were the subjects who have resumed their ancestral homes within the predetermined study period (seven years). The statuses of the subjects were defined as:

$$\delta_i = \begin{cases} 1 & if\ the\ i^{th} subject\ experinced\ the\ event\ of\ interest \\ 0 & if\ the\ i^{th} subject\ has\ not\ experinced\ the\ event\ of\ interest \end{cases}$$

The status contribution to the likelihood function for the subjects who have returned to their ancestral home would be $f(t_i)$ and for those who have not returned to the village would be $S(t_i)$

Lawless (2003) proposed the form of likelihood function for the survival model in the presence of censored data. The maximum likelihood method works by developing a likelihood function based on the available data and finding the estimates of parameters of a probability distribution that maximizes the likelihood function. The likelihood function for all observed and censored Subjects were defined by:

$$L(t_i, \underline{\theta}) = \prod_{i \in u} [f(t_i, \underline{\theta})] \times \prod_{i \in c} S(t_i;\ \underline{\theta}) = \prod_{i=1}^{n} [f(t_i, \underline{\theta})]^{f_{t_i}} \prod_{i=1}^{n} [S(t_i;\ \underline{\theta})]^{c_{t_i}} \qquad (4.1)$$

where f_{t_i} are the number of observed subjects until the event of interest has happened in the interval i and c_{t_i} are the number of censored individuals in the interval i each of length t, $f(t_i, \underline{\theta})$ is probability density function (pdf) in a parametric model with survivor function, $S(t_i, \underline{\theta})$ and the hazard function, $h(t_i, \underline{\theta})$ with the vector parameter $\underline{\theta} = (\alpha, \beta, \gamma)$ of the model. To obtain maximum likelihood estimates of parameters of a three-parametric Weibull model, logarithm is taken on both sides of the above equation (Likelihood function) and therefore by setting $l(t_i, \underline{\theta}) = lnL(t_i, \underline{\theta})$ (log-likelihood function) results into:

$$l(t_i, \underline{\theta}) = \sum_{i=1}^{n} f_{t_i} ln\left[f(t_i, \underline{\theta})\right] + \sum_{i=1}^{n} c_{t_i} ln\left[S(t_i, \underline{\theta})\right] \qquad (4.2)$$

It is worth noting that $S(t_i, \underline{\theta}) = 1 - F(t_i, \underline{\theta})$ and equation 4.2 is the same as the equation 3.4 but several events are considered to have happened in the interval I including censoring for equation 4.2

Also since $f(t_i, \underline{\theta}) = h(t_i, \underline{\theta}) \times S(t_i, \underline{\theta})$, then equation (4.2) becomes

$$l(t_i, \underline{\theta}) = \sum_{i=1}^{n} f_{t_i} ln\left[h(t_i, \underline{\theta})\right] + \sum_{i=1}^{n} (f_{t_i} + c_{t_i}) ln\left[S(t_i, \underline{\theta})\right] \qquad (4.3)$$

Where, the first summation is for failure and the second summation is for all censored individuals.

Letting $N_{t_i} = \left(f_{t_i} + c_{t_i}\right)$, the total number of failed and censored subjects at time t_i, of the i^{th} interval then the equation (3.3) becomes

$$l\left(t_i, \underline{\theta}\right) = \sum_{i=1}^{n} f_{t_i} ln\left[h\left(t_i, \underline{\theta}\right)\right] + \sum_{i=1}^{n} N_{t_i} ln\left[S\left(t_i, \underline{\theta}\right)\right] \tag{4.4}$$

In this study time is partitioned into intervals, which are of unit length *t* starting from zero. Moreover, failures and censoring of the subjects occur in each interval *i* of equal length of time *t*, i=1,2, …, n

For the estimation of the parameters, there is need to find out the hazard function and the survival function to be substituted in the log likelihood function and hence apply suitable iteration techniques to come out with the parameter estimates.

4.1. Survival function and Hazard function

For the parametric survival model, the survival function is defined by

$$S\left(t; \underline{\theta}\right) = \int_{t}^{\infty} f(x)dx = \int_{t}^{\infty} \left(\frac{\beta}{\alpha}\right)\left(\frac{x-\gamma}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x-\gamma}{\alpha}\right)^{\beta}} dx = e^{-\left(\frac{x-\gamma}{\alpha}\right)^{\beta}}\Bigg|_{t}^{\infty}$$

$$= e^{-\left(\frac{t-\gamma}{\alpha}\right)^{\beta}} \tag{4.5}$$

Where; $\underline{\theta} = [\alpha, \beta, \gamma]$, $f(x)$ is the probability density function of the distribution of the three-parameter Weibull distribution function for this case.

The hazard function, also called the force of mortality in Biostatistics and epidemiology especially in clinical trials is the instantaneous failure rate. Mathematically the hazard function is defined by

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{S(t)}$$

$$h(t; \underline{\theta}) = \frac{f(t; \underline{\theta})}{S(t; \underline{\theta})} = \frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{t-\gamma}{\alpha}\right)^{\beta-1} e^{-\left(\frac{t-\gamma}{\alpha}\right)^{\beta}}}{e^{-\left(\frac{t-\gamma}{\alpha}\right)^{\beta}}} = \left(\frac{\beta}{\alpha}\right)\left(\frac{t-\gamma}{\alpha}\right)^{\beta-1} \tag{4.6}$$

$= P(\text{Experiencing the event of interest in the interval } (t, t+\delta_t) | \text{survived past time, } t)$

$= P(t < T < t + \delta_t | T > t)$

4.2. MLE of the IDP return time

(MLE) is used in the estimation of the parameters of the three-parameter Weibull distribution based on internally displaced persons return time in Northern Uganda which was perceived as one of the problems for the entire continent of Africa for centuries.

The contribution of the subject status into the likelihood function is defined by

$$L(t_i, \delta_i) = \begin{cases} f(t_i; \underline{\theta}) \ if \ \delta_i = 1(uncensored) \\ S(t_i; \ \underline{\theta}) \ if \ \delta_i = 0(censored) \end{cases} \tag{4.7}$$

$$L(t_i, \delta_i) = \ [f(t_i; \ \underline{\theta})]^{\delta_i} \times [S(t_i; \ \underline{\theta})]^{1-\delta_i} \tag{4.8}$$

For the full sample in the entire period of study

$$L(t_1, t_2, \dots, t_n; \delta_1, \delta_2, \dots, \delta_n) = \prod_{i=1}^{n} L(t_i, \delta_i) \tag{4.9}$$

Substituting equation (4.8) into equation (4.9) we get equation (4.10) below

$$L(t_1, t_2, \dots, t_n; \delta_1, \delta_2, \dots, \delta_n) = \prod_{i=1}^{n} [f(t_i; \underline{\theta})]^{\delta_i} \times [S(t_i; \underline{\theta})]^{1-\delta_i} \qquad (4.10)$$

Equation (4.10) can be expressed as

$$L(t_1, t_2, \dots, t_n; \delta_1, \delta_2, \dots, \delta_n) = \prod_{i \in u} [f(t_i, \underline{\theta})] \times \prod_{i \in c} S(t_i;\ \underline{\theta}) \qquad (4.11)$$

Where $\prod_{i \in u}$,denotes the product over the uncensored observation and $\prod_{i \in c}$, the product over the censored observation.

The log-likelihood function from equation (4.11) is

$$l(t_i; \underline{\theta}) = -\ln\Big(L(t; \underline{\theta})\Big) = -\ln\left\{\prod_{i=1}^{n} [f(t_i; \underline{\theta})]^{\delta_i} \times [S(t_i; \underline{\theta})]^{1-\delta_i}\right\} \qquad (4.12)$$

But $f(t_i; \underline{\theta}) = h(t_i; \underline{\theta})S(t_i; \underline{\theta})$. Substituting for $f(t_i; \underline{\theta})$ in equation (4.12) we get

$$l(t_i; \underline{\theta}) = -\ln\left\{\prod_{i=1}^{n} [h(t_i; \underline{\theta})S(t_i; \underline{\theta})]^{\delta_i} \times [S(t_i; \underline{\theta})]^{1-\delta_i}\right\} \qquad (4.13)$$

Equation (4.13) can be expressed as

$$l(t_i; \underline{\theta}) = -\sum_{i=1}^{n} \delta_i ln\ h(t_i; \underline{\theta}) - \sum_{i=1}^{n} \delta_i ln\ S(t_i; \underline{\theta}) - \sum_{i=1}^{n} (1 - \delta_i) ln\ S(t_i; \underline{\theta}) \qquad (4.14)$$

If the total number of the internally displaced persons returning to their ancestral homes at time interval t_i is f_{t_i} and the total censored individuals at time interval t_i is c_{t_i}, then equation (4.14) becomes

$$l(t_i;\underline{\theta}) = -\sum_{i=1}^{k} f_{t_i}\, ln\ h(t_i;\underline{\theta}) - \sum_{i=1}^{k} f_{t_i}\, ln\ S(t_i;\underline{\theta}) - \sum_{i=1}^{k} c_{t_i} ln\ S(t_i;\underline{\theta}) \tag{4.15}$$

Equation (4.15) can be reduced into

$$l(t_i;\underline{\theta}) = -\sum_{i=1}^{k} f_{t_i}\, ln\ h(t_i;\underline{\theta}) - \sum_{i=1}^{k} (f_{t_i} + c_{t_i})\ ln\ S(t_i;\underline{\theta}) \tag{4.16}$$

Substituting the hazard and the survival function in equation (4.16) and simplifying result into

$$l(t;\underline{\theta}) = -(F) ln\left(\frac{\beta}{\alpha}\right) - (\beta - 1)\sum_{i=1}^{k} f_{t_i}\, ln\left(\frac{t-\gamma}{\alpha}\right) + \sum_{i=1}^{k} N_{t_i}\left(\frac{t-\gamma}{\alpha}\right)^{\beta} \tag{4.17}$$

where, $F = \sum_{i=1}^{n} f_{t_i}$ is the total number of failures in an entire study time, k is the maximum study time interval and $N_{t_i} = (f_{t_i} + c_{t_i})$ is the total number of failure and censored subjects in a given time interval.

Differentiating equation (4.17) with respect to $\alpha, \beta\ and\ \gamma$ and simplifying result into

$$\frac{\partial l(t;\underline{\theta})}{\partial \alpha} = F\left(\frac{\beta}{\alpha}\right) - \left(\frac{\beta}{\alpha}\right)\sum_{i=1}^{n} N_{t_i}\left(\frac{t-\gamma}{\alpha}\right)^{\beta} \tag{4.18}$$

$$\frac{\partial l(t;\underline{\theta})}{\partial \beta} = -\left(\frac{F}{\beta}\right) - \sum_{i=1}^{n} f_{t_i}\, ln\left(\frac{t-\gamma}{\alpha}\right) + \sum_{i=1}^{n} N_{t_i}\left(\frac{t-\gamma}{\alpha}\right)^{\beta}\ ln\left(\frac{t-\gamma}{\alpha}\right) \tag{4.19}$$

$$\frac{\partial l(t;\underline{\theta})}{\partial \gamma} = (\beta - 1)\sum_{i=1}^{n}\left(\frac{f_{t_i}}{t-\gamma}\right) + \left(\frac{\beta}{\alpha}\right)\sum_{i=1}^{n} N_{t_i}\left(\frac{t-\gamma}{\alpha}\right)^{\beta-1} \tag{4.20}$$

Further differentiating each of the equations (4.18), (4.19) and (4.20) above with respect to $\alpha, \beta\ and\ \gamma$ and simplifying, the equations below are obtained. The equations are later used to obtain the variance –covariance matrix:

Differentiating equation (4.18) with respect to $\alpha, \beta\ and\ \gamma$ and simplifying result into

$$\frac{\partial^2 l(t;\underline{\theta})}{\partial\alpha^2} = -F\left(\frac{\beta}{\alpha^2}\right) + \left(\frac{\beta+\beta^2}{\alpha^2}\right)\sum_{i=1}^{n} N_{t_i}\left(\frac{t-\gamma}{\alpha}\right)^{\beta} \tag{4.21}$$

$$\frac{\partial^2 l(t;\underline{\theta})}{\partial\alpha\partial\beta} = \left(\frac{F}{\alpha}\right) - \left(\frac{1}{\alpha}\right)\sum_{i=1}^{n} N_{t_i}\left(\frac{t-\gamma}{\alpha}\right)^{\beta} - \left(\frac{\beta}{\alpha}\right)\sum_{i=1}^{n} N_{t_i}\left(\frac{t-\gamma}{\alpha}\right)^{\beta} ln\left(\frac{t-\gamma}{\alpha}\right) \tag{4.22}$$

$$\frac{\partial^2 l(t;\underline{\theta})}{\partial\alpha\partial\gamma} = \left(\frac{\beta}{\alpha}\right)^2 \sum_{i=1}^{n} N_{t_i}\left(\frac{t-\gamma}{\alpha}\right)^{\beta-1} \tag{4.23}$$

Differentiating equation (4.19) with respect to $\alpha, \beta\ and\ \gamma$ and simplifying result into

$$\frac{\partial^2 l(t;\underline{\theta})}{\partial\beta^2} = \left(\frac{F}{\beta^2}\right) + \sum_{i=1}^{n} N_{t_i}\left(\frac{t-\gamma}{\alpha}\right)^{\beta}\left[ln\left(\frac{t-\gamma}{\alpha}\right)\right]^2 \tag{4.24}$$

$$\frac{\partial^2 l(t;\underline{\theta})}{\partial\beta\partial\alpha} = \left(\frac{F}{\alpha}\right) - \left(\frac{1}{\alpha}\right)\sum_{i=1}^{n} N_{t_i}\left(\frac{t-\gamma}{\alpha}\right)^{\beta} - \left(\frac{\beta}{\alpha}\right)\sum_{i=1}^{n} N_{t_i}\left(\frac{t-\gamma}{\alpha}\right)^{\beta} ln\left(\frac{t-\gamma}{\alpha}\right) \tag{4.25}$$

$$\frac{\partial^2 l(t;\underline{\theta})}{\partial\beta\partial\gamma} = \sum_{i=1}^{n}\left(\frac{f_{t_i}}{t-\gamma}\right) - \left(\frac{\beta}{\alpha}\right)\sum_{i=1}^{n} N_{t_i}\left(\frac{t-\gamma}{\alpha}\right)^{\beta-1} ln\left(\frac{t-\gamma}{\alpha}\right) - \left(\frac{1}{\alpha}\right)\sum_{i=1}^{n} N_{t_i}\left(\frac{t-\gamma}{\alpha}\right)^{\beta-1} \tag{4.26}$$

Differentiating equation (4.20) with respect to $\alpha, \beta\ and\ \gamma$ and simplifying result into

$$\frac{\partial^2 l(t;\underline{\theta})}{\partial\gamma^2} = (\beta-1)\sum_{i=1}^{n}\left(\frac{f_{t_i}}{(t-\gamma)^2}\right) + \left(\frac{\beta(\beta-1)}{\alpha^2}\right)\sum_{i=1}^{n} N_{t_i}\left(\frac{t-\gamma}{\alpha}\right)^{\beta-2} \tag{4.27}$$

$$\frac{\partial^2 l(t;\underline{\theta})}{\partial\gamma\partial\alpha} = \left(\frac{\beta}{\alpha}\right)^2 \sum_{i=1}^{n} N_{t_i}\left(\frac{t-\gamma}{\alpha}\right)^{\beta} \tag{4.28}$$

$$\frac{\partial^2 l(t;\underline{\theta})}{\partial\gamma\partial\beta} = \sum_{i=1}^{n}\left(\frac{f_{t_i}}{t-\gamma}\right) - \left(\frac{\beta}{\alpha}\right)\sum_{i=1}^{n} N_{t_i}\left(\frac{t-\gamma}{\alpha}\right)^{\beta-1} ln\left(\frac{t-\gamma}{\alpha}\right) - \left(\frac{1}{\alpha}\right)\sum_{i=1}^{n} N_{t_i}\left(\frac{t-\gamma}{\alpha}\right)^{\beta-1} \tag{4.29}$$

By using any one of the equations (4.17), (4.18), 4.19 and (4.20) in the DFP optimization method of the MATLAB program, the parameters estimates for which value of the likelihood function is maximum are obtained. MATLAB DFP program for the parameters estimation of the distribution model is developed. The optimal estimates of the scale, shape and location/shift parameters (α, β and γ respectively) of the three-parameter Weibull distribution are obtained by maximizing the log-likelihood function. The t-ratios of the parameters are also got. The values of parameters estimates, t-ratios, log-likelihood function and variance–covariance matrix are given below in table 4.1: -

Table 4.1. MLE Estimates of Parameters of three-parameter Weibull Distribution Using IDP

Parameters	**Estimates**	**t-ratios**
Scale, α	3.4192	50.9704
Shape, β	1.2293	86.9246
Location, γ	0.2501	3.1021
Log-likelihood	1.0724×10^3	
variance and covariance matrix	$\begin{bmatrix} 4.5\times10^{-3} & -4.1\times10^{-3} & 8.6\times10^{-3} \\ -4.1\times10^{-3} & 2.0\times10^{-4} & -3.7\times10^{-3} \\ 8.6\times10^{-3} & -3.7\times10^{-3} & 6.5\times10^{-3} \end{bmatrix}$	

From Table 4.1., the threshold parameter is 0.2501 meaning that after declaration of the safe village no IDP returned to their ancestral home until after the first three months

Making the assumption that the threshold parameter is zero, the results in the table 4.2 below are got. This corresponds to the two-parameter distribution model of the time the internally displaced persons return to their ancestral homes. It also means that as soon as the village is declared safe, the displaced persons start returning straight away. This might not sound very logical as fear was still looming among the displaced persons and it is hard to tell whether one could return straight away.

Table 4.2. MLE Estimates of Parameters of Weibull Distribution with $\gamma = 0$

Parameters	Estimates	t-ratios	Gradients
Scale, α	3.8628	29.5094	5.6843×10^{-14}
Shape, β	1.3420	27.8037	8.2414×10^{-5}
Log-Likelihood	1.1190×10^{3}		
Variance-covariance matrix	$\begin{bmatrix} 1.7135 \times 10^{-2} & 9.2087 \times 10^{-4} \\ 9.2087 \times 10^{-4} & 2.3297 \times 10^{-3} \end{bmatrix}$		

Assuming constant return rate of the internally displaced persons, the result of the parameter estimates for the IDP return time of the distribution would be as in the table 4.3 below. This is the exponential distribution model which maintained that the rate of return of the internally displaced persons is constant over time. This result is also doubtful because the factors influencing return of the displaced persons are not uniform and vary from subject to subject and might not only be the LRA cease fire.

Table 4.3. MLE Estimates of Parameters of a Weibull Models with $\gamma = 0$ and $\beta = 1$

Parameters	Estimates	t-ratios	Gradient
Scale, α	3.7591	4.4883	2.8422×10^{-14}
Log-Likelihood	1.1481×10^{3}		
variance	2.8605×10^{-2}		

4.3. CMLE of the IDP return time

From equation (3.9) the likelihood function of the corrected maximum likelihood estimation method for progressive censoring is mathematically defined by

$$L_C(\alpha,\beta,\gamma) = C[F(t_{2:m:n};\alpha,\beta,\gamma) - F(t_{1:m:n};\alpha,\beta,\gamma)]$$
$$\times \left\{\prod_{i=2}^{k} f(t_{i:m:n};\alpha,\beta,\gamma)\right\}\left\{\prod_{i=1}^{k}[1 - F(t_{i:m:n};\alpha,\beta,\gamma)]^{\delta_i}\right\}$$

Substituting the expression in equations (1.2) and (1.1) yields

$$L_C(\alpha,\beta,\gamma) = C[F(t_{2:m:n};\alpha,\beta,\gamma) - F(t_{1:m:n};\alpha,\beta,\gamma)]$$

$$\times\left\{\prod_{i=2}^{k}\left(\frac{\beta}{\alpha}\right)\left(\frac{t_i-\gamma}{\alpha}\right)^{\beta-1} e^{-\left(\frac{t_i-\gamma}{\alpha}\right)^{\beta}}\right\}\left\{\prod_{i=1}^{k}\left[e^{-\left(\frac{t_i-\gamma}{\alpha}\right)^{\beta}}\right]^{\delta_i}\right\} \tag{4.30}$$

Taking the log-likelihood of equation (4.30) yields

$$lnL_c(\alpha,\beta,\gamma) = lnC + lnA + (m-1)ln\beta - \beta(m-1)ln\alpha + (\beta-1)\sum_{i=2}^{k}(t_i-\gamma)$$

$$-\sum_{i=2}^{k}\left(\frac{t_i-\gamma}{\alpha}\right)^{\beta} - \sum_{i=2}^{k}\partial_i\left(\frac{t_i-\gamma}{\alpha}\right)^{\beta} \tag{4.31}$$

For A= $F(t_{2:m:n};\alpha,\beta,\gamma) - F(t_{1:m:n};\alpha,\beta,\gamma)$ and m subjects failed in total for the entire k study intervals.

Differentiating equation (4.31) with respect to $\alpha, \beta, and\ \gamma$ and using the results in the DFP optimization formula, the results in Table 4.4 below are obtained

Table 4.4. CMLE Estimates of Parameters of three-parameter Weibull Distribution of IDP

Parameters	**Estimates**	**t-ratios**
Scale, α	3.4560	58.3156
Shape, β	1.2895	32.8595
Location, γ	0.4051	16.5381
Log-likelihood	1.2500×10^3	
variance and covariance matrix	$\begin{bmatrix} 3.5\times10^{-3} & -5.4\times10^{-3} & 6.6\times10^{-3} \\ -5.4\times10^{-3} & 1.54\times10^{-4} & -4.5\times10^{-3} \\ 6.6\times10^{-3} & -4.5\times10^{-3} & 6.0\times10^{-4} \end{bmatrix}$	

The variance –covariance matrix of the estimates using the corrected maximum likelihood estimation is smaller than the maximum likelihood estimation counterparts this means that the CMLE is better than MLE due to the variance precision and the minimum variance obtained. The shift parameter or the threshold parameter is 0.4055 showing that for about the first four months no IDP returned to their ancestral homes

Table 4.5. CMLE Estimates of Parameters of Weibull Distribution with $\gamma = 0$			
Parameters	Estimates	t-ratios	Gradients
Scale, α	3.8700	32.3061	5.8435×10^{-12}
Shape, β	1.3500	19.9372	4.5242×10^{-7}
Log-Likelihood	1.1190×10^{3}		
Variance-covariance matrix	$\begin{bmatrix} 1.435 \times 10^{-2} & 6.249 \times 10^{-4} \\ 6.249 \times 10^{-4} & 4.585 \times 10^{-3} \end{bmatrix}$		

Suppose assumption was made that the IDP return time is constant with zero threshold parameter then the result shown in Table 4.6. below would be obtained when using the CMLE techniques. This result proves superior to that obtained by both the MLE techniques and the WMLE techniques under this assumption due to the smaller variance and the larger t-ratio. From this result, one would prefer using CMLE to both MLE and WMLE when estimating the parameter of an exponential distribution based on right censored data but more exploration of this need to be done for one to be sure of the generalization.

Table 4.6. CMLE Estimates of Parameters of a Weibull Models with $\gamma = 0$ and $\beta = 1$			
Parameters	Estimates	t-ratios	Gradient
Scale, α(mean)	3.7802	74.0646	2.5859×10^{-12}
Log-Likelihood	1.15×10^{3}		
variance	2.605×10^{-3}		

4.4. WMLE of the IDP return time

The weighted version of the MLEs is based on the idea of the weighted maximum likelihood estimators (WMLEs), which is a bias-adjusted method for the three-parameter Weibull distribution. This method gives nearly unbiased estimators of the parameters by involving three weights to the three likelihood equations of the MLEs (i.e. equations (3.5), (3.6) and (3.7)) which results in equations (4.32), (4.33) and (4.34) below that were previously introduced in chapter 3 subsection 4. Using the techniques in Cousineau (2009b), the weighted MLEs of γ and β based on progressive right censored sample can be obtained by solving the following equations with respect to α, β and γ subject to the constraint $\gamma < t_{1:m:n}$,

$$\frac{W_2}{\beta} + \frac{1}{m}\sum_{i=1}^{k} ln(t_{i:m:n} - \gamma) - \frac{\sum_{i=1}^{k}(\partial_i + 1)\,(t_{i:m:n} - \gamma)^{\beta}\ln(t_{i:m:n} - \gamma)}{\sum_{i=1}^{k}(\partial_i + 1)\,(t_{i:m:n} - \gamma)^{\beta}} = 0 \qquad (4.32)$$

$$\frac{\left[\sum_{i=1}^{k}(\partial_i + 1)\,(t_{i:m:n} - \gamma)^{\beta}\right]\left[(t_{i:m:n} - \gamma)^{-1}\right]}{m\left[\sum_{i=1}^{k}(\partial_i + 1)\,(t_{i:m:n} - \gamma)^{\beta-1}\right]} - W_3 = 0 \qquad (4.33)$$

And

$$\alpha = \left[\frac{1}{mW_1}\sum_{i=1}^{k}(\partial_i + 1)\,(t_{i:m:n} - \gamma)^{\beta}\right]^{\frac{1}{\beta}} \tag{4.34}$$

Where the weights are defined as in equations (3.13), (3.14) and (3.15) the results in Table 4.7 below are obtained

Based on these results, the WMLE is the better than both the MLE and the CMLE because of the higher t-ratios and the minimum variances. Therefore the WMLE is the best followed by the CMLE and finally the MLE. This suggests that in future estimations, WMLE should be given priority if the researcher is to obtain more accurate estimates of the three-parameter Weibull distribution parameters. Moreover, the estimates are even more precise with small sample size as seen from the simulation results.

Table 4.7. WMLE Estimates of Parameters of three-parameter Weibull Distribution of IDP

Parameters	**Estimates**	**t-ratios**
Scale, α	3.4652	53.4628
Shape, β	1.2900	82.4151
Location, γ	0.4060	17.3119
Log-likelihood	1.2510×10^3	
variance and covariance matrix	$\begin{bmatrix} 4.201\times10^{-3} & -4.55\times10^{-3} & 5.64\times10^{-3} \\ -4.55\times10^{-3} & 2.45\times10^{-4} & -4.56\times10^{-3} \\ 5.66\times10^{-3} & -4.56\times10^{-3} & 5.5\times10^{-4} \end{bmatrix}$	

If assumption was made that the IDP return time follows a Weibull distribution with the zero threshold parameter, then the resulting estimates would be as shown in Table

4.5. below of which results when compared with those obtained by MLE and CMLE prove superior due to the smaller variances estimators and the bigger t-ratios

Table 4.8. WMLE Estimates of Parameters of Weibull Distribution with $\gamma = 0$			
Parameters	Estimates	t-ratios	Gradients
Scale, α	3.8650	57.9063	5.5435×10^{-12}
Shape, β	1.3450	89.6667	5.4529×10^{-7}
Log-Likelihood	1.1190×10^{3}		
Variance-covariance matrix	$\begin{bmatrix} 4.455 \times 10^{-3} & 4.4509 \times 10^{-4} \\ 4.4509 \times 10^{-4} & 2.25 \times 10^{-4} \end{bmatrix}$		

Assuming constant return rate of the internally displaced persons, the result of the parameter estimates for the IDP return time of the distribution using the WMLE would be as shown in Table 4.9. below. This is the exponential distribution model which maintained that the rate of return of the internally displaced persons is constant over time. This result is also doubtful because the factors influencing return of the displaced persons are not uniform and vary from subject to subject and might not only be the LRA cease fire. However for the sake of estimators' comparison, it can be seen that the estimates using WMLE are better than those obtained by MLE techniques and not the result obtained by the CMLE techniques. This is because the estimated t-ratio of the CMLE estimates is larger than that of the WMLE counterpart.

Table 4.9. WMLE Estimates of Parameters of a Weibull Models with $\gamma = 0$ and $\beta = 1$			
Parameters	Estimates	t-ratios	Gradient
Scale, α (mean)	3.7708	60.6892	2.5524×10^{-12}
Log-Likelihood	1.161×10^{3}		
variance	3.8605×10^{-3}		

CHAPTER FIVE:

DISCUSSION OF RESULTS

5.1. Results

Tables 3.2, 3.3 and 3.4 show the parameter estimates of the three-parameter Weibull distribution using MLE based on Monte Carlo simulated data. From these tables, it can be seen that the total deviation increased with increase in the fixed type I censored subjects. This means that censoring increases biasness. However, the bias is higher for $\beta = 0.5$ than for $\beta = 1.0\ and\ \beta =\ 2.0$ showing that MLE is inefficient in estimating the parameters of a Weibull distribution when the shape parameter is less than two. It can also be noted that higher sample sizes produce lower bias of the parameter estimates than the lower sample sizes

For the case of the CMLE for the simulated data, the results as shown in Table 3.5, 3.6 and 3.7 maintained that censorship increases biasness but when the result is compared with that of the MLE counterparts for the same censoring scheme, CMLE provides better parameter estimates. This can be seen from their biasness and the total deviation under the same censoring scheme. It is also clear that the sample sizes influence the parameter estimates in terms of their biasness and the RMSE

From Table 3.8 to 3.10, we show the parameter estimates of the three-parameter distribution using WMLE and can be seen that the estimates are better than for both the MLE and the CMLE. It has smaller deviation than the CMLE and the MLE counterparts when a sample size of 20 is used. This means that for smaller sample size, WMLE outperform MLE and CMLE.

For the application data, based on the variance and the t-ratios of the parameter estimates, it is seen that the order of performance of the estimation techniques is WMLE followed by CMLE and then finally MLE. The estimated values of scale parameter $\alpha > 0$, shape parameter $\beta > 0$ and the location parameter $\gamma > 0$ for the internally displaced persons return time obtained by MLE, CMLE and WMLE are given in the Tables 4.1, 4.4 and 4.7 respectively along with their *t*-ratios, indicating that the estimates of scale, shape and location parameters are significant at 5% level of significance. The estimated value of β is greater than 1 for all the three estimation techniques which indicates increasing failure rate with time (the subjects are more susceptible to returning home as time passes). For the MLE, the negative values of co-variances between α and β indicate that the movements of α and β are in the opposite directions and the positive values of co-variances between α and γ indicates that the movements of α and γ are in the same directions while the negative values of co-variances between β and γ indicate that the movements of β and γ are in the opposite directions.

Table 5.1 below shows the extract of the t-ratios of the parameter estimates of the IDP return time using the three techniques together with their ranks (1,2 and 3) with the smaller rank implying the estimates is better than those whose rank are higher.

Table 5.1. Parameter comparison for the IDP data set						
parameter	MLE		CMLE		WMLE	
	t-ratio	Rank	t-ratio	Rank	t-ratio	Rank
Scale, α	50.9704	3	58.3156	1	53.4628	2
Shape, β	86.9246	1	32.8595	3	82.4151	2
Location, γ	3.1021	3	16.5381	2	17.3119	1
Total Rank		7		6		5

From Table 5.1. above, the total rank of WMLE is smaller than for the CMLE and MLE. This means the best techniques among the three estimation techniques to estimates the parameter of a three-parameter Weibull distribution model using IDP data set would be WMLE. According to the rank total, the order of performance from the best to the worst is WMLE, CMLE and MLE respectively.

Table 5.2. Parameter comparison for the IDP data set (two-parameter) $\gamma = 0$						
Parameter	MLE		CMLE		WMLE	
	t-ratio	Rank	t-ratio	Rank	t-ratio	Rank
Scale, α	29.5094	3	32.3061	2	57.9063	1
Shape, β	27.8037	2	19.9372	3	89.6667	1
Total Rank		5		5		2

When assumption is made of zero shift parameter, the t-ratios and the corresponding rank of the parameter estimates for the three estimation techniques using the IDP data set would be as in Table 5.2. above. From that table, it can be seen that the WMLE is the best in estimating both shape and scale parameter of the Weibull two parameter model. While, MLE performs better than CMLE in estimating the scale parameter and the reverse is true for the shape parameter estimation.

Table 5.3. Parameter comparison for the IDP dataset (one-parameter) $\beta = 1\ and\ \gamma = 0$						
Parameter	MLE		CMLE		WMLE	
	t-ratio	Rank	t-ratio	Rank	t-ratio	Rank
Scale, α (mean)	4.4883	3	74.0646	1	60.6892	2
Total Rank		3		1		2

For one parameter distribution model with the assumption that the shape is one and the shift is zero, then the CMLE proves to be the best in estimating the scale parameter followed by WMLE and lastly the MLE.

It can therefore be generalized that for estimating the parameter of one-parameter exponentially distributed model, CMLE is the best technique to employ. For two and three-parameter Weibull distributed model, the best technique would be the WMLE.

5.2. The Location Parameter

The resulting location parameter estimates of the three-parameter Weibull model for the IDP dataset are 0.2501, 0.4051 and 0.4060 for MLE, CMLE and WMLE respectively which are their thresholds. This means that for the MLE estimates, the IDPs start returning after 0.2501 years (3 months) and the displaced proportion (the same as the survival proportion in medical research) is one at this time. While for the CMLE estimates, the IDPs start returning after 0.4051 years (4 months and 25 days) and the displaced proportion (the same as the survival proportion in medical research) is one at 0.4051. For the WMLE estimates on the other hand, the IDPs start returning after 0.4060 years (4 months and 26 days) and the displaced proportion (the same as the survival proportion in medical research) is one at 0.4051. This threshold parameter could be due to the fear that the LRA had inflicted on the civilians which make the IDPs to wait for the confirmation of sure peace or because there was no food in the villages and returning to the villages needed planning and world food program were supporting those in camps and no access to world food program in the villages.

5.3. Displaced Proportion

The displaced function (the same as the survival function) is defined mathematically by $D\left(t;\underline{\theta}\right) = 1 - F(t;\underline{\theta})$.

Using equation (1.2) then the displaced function for a three-parameter Weibull distribution would be expressed as

$$D\left(t;\underline{\theta}\right) = exp^{\left[-\left(\frac{t-\gamma}{\alpha}\right)^{\beta}\right]}$$

For the three estimation techniques, the displaced function $D_M\left(t;\underline{\theta}\right)$, $D_C\left(t;\underline{\theta}\right)$ and $D_W\left(t;\underline{\theta}\right)$ of the MLE, CMLE and WMLE respectively are then defined by;

$$D_M(t; 3.4192, 1.2293, 0.2501) = exp^{\left[-\left(\frac{t-0.2501}{3.4192}\right)^{1.2293}\right]} for\ t \geq 0.2501$$

$$D_C(t;\ 3.4560, 1.2895, 0.4051) = exp^{\left[-\left(\frac{t-0.4051}{3.4560}\right)^{1.2895}\right]} for\ t \geq 0.4051$$

$$D_W(t; 3.4652, 1.2900, 0.4060) = exp^{\left[-\left(\frac{t-0.4060}{3.4652}\right)^{1.2900}\right]} for\ t \geq 0.4060$$

Using these three equations, the result in Table 5.4. below is obtained which is in turn used in Fig 5.1.

Table 5.4. Displaced proportions of IDP at the end of time interval t

Interval	Kaplan-Meier	WMLE	CMLE	MLE
1	0.9169	0.9023	0.9017	0.8565
2	0.6974	0.6926	0.6915	0.6447
3	0.5385	0.5024	0.5010	0.4653
4	0.3757	0.3506	0.3492	0.3262
5	0.2430	0.2372	0.2360	0.2236
6	0.1867	0.1565	0.1555	0.1504
7	0.1369	0.1009	0.1002	0.0995

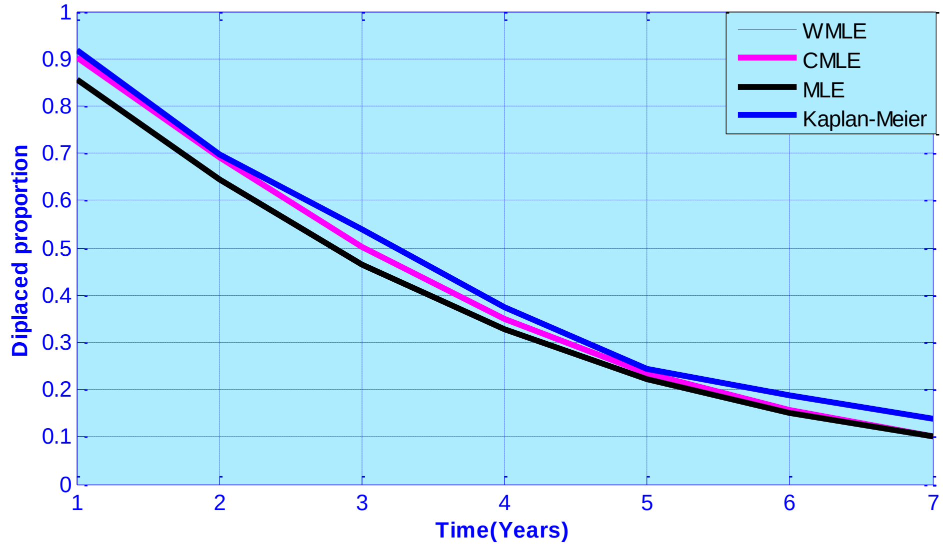

Fig. 5.1. *The graphs illustrating the IDP displaced proportion with non-zero shift*

The graphs in Fig. 5.1. show the Displaced proportions curves obtained from the MLE, CMLE and WMLE estimates of the parameters of a three-parameter Weibull distribution models and Kaplan-Meier estimates for the return time of the internally displaced persons. In the graph, CMLE and WMLE curves are meshed together and for the first time interval they are almost inseparable from the Kaplan-Meier smooth curve but thereafter a small deviation of WMLE and CMLE curves from the Kaplan-Meier curve is observed. The deviation remains until the fifth interval when it is almost negligible. For the MLE curve as in the Fig. 5.1., the deviation from the Kaplan-Meier curve is more pronounced than for both the WMLE and the CMLE. The Kaplan-Meier curve is above all the three estimates meaning MLE over underestimates the IDP return time. In future, data of this type kind had better be used with WMLE or CMLE when looking for the Weibull parametric models that depict the characteristics of the subjects under study

In the case were assumption was made that the threshold parameter is zero then the following subsequent displaced equation follows after feeding the estimates from the Tables 4.2, 4.5 and 4.8. These equations are used to obtain the results in Table 5.5. below which is in-turn used in the plotting of the graph in Fig. 5.2. below

$$D_M(t;3.8628{,}1.3420) = exp^{\left[-\left(\frac{t}{3.8628}\right)^{1.3420}\right]} \; for \; t \geq 0$$

$$D_C(t;\; 3.8700, 1.3500) = exp^{\left[-\left(\frac{t}{3.8700}\right)^{1.3500}\right]} \; for \; t \geq 0$$

$$D_W(t;3.8650{,}1.3450) = exp^{\left[-\left(\frac{t}{3.8650}\right)^{1.345}\right]} \; for \; t \geq 0$$

Table 5.5. Displaced proportions of IDP with Zero shift

Interval	Kaplan-Meier	WMLE	CMLE	MLE
0	1.0000	1.0000	1.0000	1.0000
1	0.9169	0.8495	0.8514	0.8502
2	0.6974	0.6614	0.6635	0.6622
3	0.5385	0.4905	0.4921	0.4910
4	0.3757	0.3507	0.3515	0.3509
5	0.2430	0.2432	0.2434	0.2432
6	0.1867	0.1644	0.1641	0.1642
7	0.1369	0.1085	0.1080	0.1083

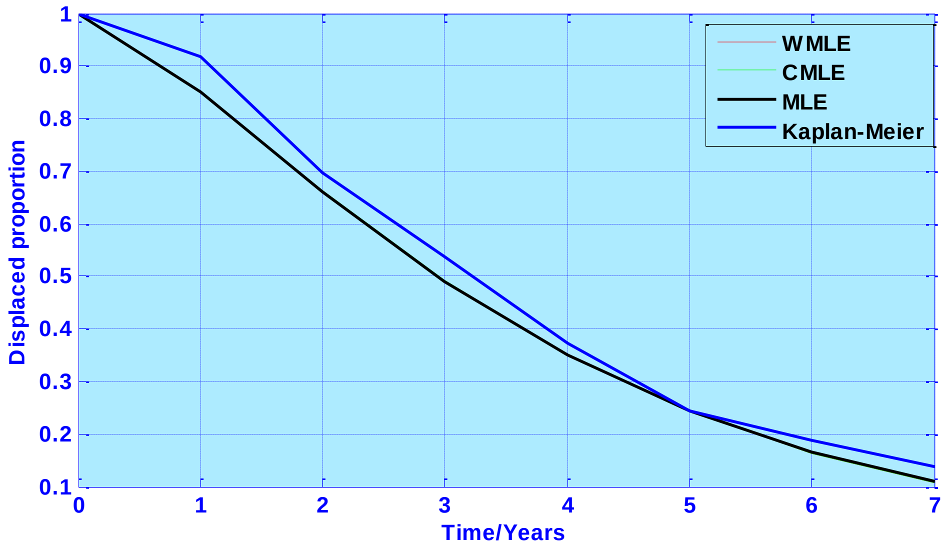

Fig. 5.2. *The graphs illustrating the curves of the IDP displaced Proportion with zero shifts*

The graphs in Fig. 5.2 show the displaced proportions curves obtained from the MLE, CMLE and WMLE estimates of the parameter of a Weibull distribution model and Kaplan-Meier estimates of IDP return time when the assumption is made that the threshold parameter is zero. All the three estimates (MLE, CMLE and WMLE) follow the same curve with MLE curve being on top of the rest. It can be seen that the parametric curve slightly underestimates the displaced proportion as compared to the Kaplan-Meier curve for the first four and half years. After the four and half years, the deviation slightly reduces and at the fifth interval, both the parametric estimates and the Kaplan-Meier smooth curves are almost inseparable.

Comparing the graphs in Fig. 5.1. and Fig. 5.2., it can be seen that the three-parameter Weibull distribution model has a smaller deviation to the Kaplan-Meier estimates than the two-parameter counterpart. This is done by comparing the deviation of the estimators curves in Fig.5.1. from the Kaplan-Meier curves and those in Fig. 5.2.

from the Kaplan-Meier curves. The deviation can also be compared by looking at the raw displaced proportions in Table 5.4. and 5.5. which tables represent the displaced proportions of the IDP of the estimators and Kaplan-Meier estimates for the three and two-parameter Weibull distribution models respectively

5.4. Conclusion

Computer modelling results show that maximal likelihood evaluates the parameters of a three-parameter Weibull distribution under appropriate choice of sample size n. for instance when the sample size was 20, the bias of the parameter estimates was too high than when the sample size was 100. For small failures m which might correspond to smaller study time, the maximum likelihood estimation is problematic and sometime might not exist and this affirmed the results obtained by lemon, (1975). The probability of non-existence of MLE decreases when the number of failures, m and the sample size, N increase. In the simulation study, it was seen that as the sample size increased from 20 to 100, the bias of the parameter estimates kept on reducing. For reliable evaluation of the parameters of a three-parameter Weibull distribution models therefore, more failures of the subjects and large samples should always be used.

When comparing the estimation techniques that were discussed in chapter 3 of this thesis, for the same censoring scheme, WMLE proves to be better than both CMLE and the MLE in-terms of their biasness and the RMSE. The biasness was seen to increase with the increase in fixed type-I censoring but with higher variances for smaller sample than it is for the larger sample sizes. In this thesis study progressive censoring was randomly assigned and its contribution to the biasness of the estimates could not been accounted for. This was because the application data was to follow a fixed type-I censoring with however some random progressively censored IDPs. The method described above could be applied for real-life problems.

For the application, the estimated values of scale, shape and location parameters of a three-parameter Weibull models displayed in Tables 4.1., 4.4. and 4.7 are positively defined and all more than zero indicating that the estimates of scale, shape and location parameters are significant at 5% level of significance. The estimated values of the shape parameters are greater than 1 for all the three estimators which indicates increasing failure rate with time (increasing return rate). This implies the validity of three-Weibull distribution in analyzing the data of this kind. The location parameters estimate for the three estimators show different threshold times of 0.2501, 0.4052 and 0.4060 years for MLE, CMLE and WMLE respectively below which nobody returned to their ancestral homes in the respective cases i.e. despite declaring the villages free of the LRA rebels and secure for human settlement, no displaced person returned before these times.

5.5. Recommendation

In this thesis, discussion is made on the estimation of parameters for a three-parameter Weibull distribution based on a mixture progressive and fixed Type-I right censored sample. Different estimation procedures are studied and compared through a Monte Carlo simulation study. Based on the simulation results, it is important to use the weighted MLEs in estimating the parameter of the Weibull distribution based on the considerations made of bias, total deviation and RMSE in general and the use of the censored estimators with a one-step bias-correction as initial estimates for the required iterative procedure. Finally longer study time should be employed in order to reduce the biasness caused as a result of the fixed type-I censoring

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APPENDICES.

APPLICATION ESTIMATION CODE

```
x=[a;b;c]; % starting value vector

nmax=50; %set maximum number of itteration

eps=[1;1;1]; % initialise error bounds eps

xvals=x; % initialise array of itteration

t=1:7; % study time for the data

f=[167,128,92,36,18,29,24]; % Returned persons to ancestral homes

N=[172,130,94,42,27,35,80]; % Total of the censored and returned
subject

A=494*(a/b)-(a/b)*sum(N.*((t-c)./a).^b); % first entry for score
function

B=-(494/b)-sum(f.*log((t-c)./a))+sum(N.*((t-c)./a).^b)*log((t-
b)./a); %second entry

C=(b-1)*sum(f./(t-c))-(b/a)*sum(((t-c)/a)^(b-1)); % third entry

g=[A;B;C]; % vector of score function

E=-494*(b/a^2)+((b+b^2)/a^2)*sum(N.*((t-c)./a).^b); % (1,1) entry of
the hessain

F=(494/a)-(1/a)*sum(N.*((t-c)./a).^b)-(b/a)*sum(N.*((t-
c)./a).^b)*log((t-b)./a); % (1,2) entry

G=((b/a)^2)*sum(N.*((t-c)./a).^b); % (1,3) hessian entry

H=(494/a)-(1/a)*sum(N.*((t-c)./a).^b)-(b/a)*sum(N.*((t-
c)./a).^b)*log((t-b)./a); % (2,1)entry

I=(494/b^2)+sum(N.*((t-c)./a).^b)*(log((t-b)./a)).^2; % (2,2)Hessian
entry

J=sum(f./(t-c))-(b/a)*sum(((t-c)/a)^(b-1))*log((t-b)./a)-
(1/a)*sum(N.*((t-c)./a).^(b-1)); %(2,3) entry
```

```
K=((b/a)^2)*sum(N.*((t-c)./a).^b); % (3,1) hessian entry
L=sum(f./(t-c))-(b/a)*sum(((t-c)/a)^(b-1))*log((t-b)./a)-
(1/a)*sum(N.*((t-c)./a).^(b-1)); %(3,2) entry
M=(b-1)*((sum(f./(t-c).^2))+(b/a^2)*sum(N.*((t-c)./a).^(b-2))); %
(3,3) Hessian entry
h=[E,F,G;H,I,J;K,L,M]; % Hessian metrix
n=0; % initialise n (number of counts)
while eps=[1e-5;1e-5;1e-5]&n<=nmax %set while condition
    y=x-(inv(h))*g; % compute next itteration
    xvals=[xvals;y]; %write next itteration arrey
    eps=abs(y-x); %compute error
    x=y; n=n+1; % update x and n
end
```

DFP OPTIMAZATION CODE

Here is the MATLAB listing:

```
function f=ch4(x,y)
t=1:7; % study time for the data
f=[167,128,92,36,18,29,24]; % Returned persons to ancestral homes
N=[172,130,94,42,27,35,80]; % Total of the censored and returned
persons
f=-(494)*(log(x1/y))-(y-1)*sum(f.*log((t-0.25)./x))+sum(N.*((t-
0.25)./x).^y); % first entry for score function
end
```

function [X,F,Iters] = dfp(N, X, gradToler, fxToler, DxToler, MaxIter, myFx)

% Function dfp performs multivariate optimization using the

% Davidon-Fletcher-Powell method.

% Input

```
% N - number of variables
% X - array of initial guesses
% gradToler - tolerance for the norm of the slopes
% fxToler - tolerance for function
% DxToler - array of delta X tolerances
% MaxIter - maximum number of iterations
% myFx - name of the optimized function
% Output
% X - array of optimized variables
% F - function value at optimum
% Iters - number of iterations
B = eye(N,N);
bGoOn = true;
Iters = 0;
% calculate initial gradient
grad1 =  FirstDerivatives(X, N, myFx);
grad1 = grad1';
while bGoOn
  Iters = Iters + 1;
  if Iters > MaxIter
    break;
  end
  S = -1 * B * grad1;
```

```
  S = S' / norm(S); % normalize vector S
  lambda = 1;
  lambda = linsearch(X, N, lambda, S, myFx);
  % calculate optimum X() with the given Lambda
  d = lambda * S;
  X = X + d;
  % get new gradient
  grad2 =  FirstDerivatives(X, N, myFx);
  grad2 = grad2';
  g = grad2 - grad1;
  grad1 = grad2;
  % test for convergence
  for i = 1:N
    if abs(d(i)) > DxToler(i)
      break
    end
  end
  if norm(grad1) < gradToler
    break
  end
%  B = B + lambda * (S * S') / (S' * g) - ...
%      (B * g) * (B * g') / (g' * B * g);
  x1 = (S * S');
```

```
  x2 = (S * g);
  B = B + lambda * x1 * 1 / x2;
  x3 = B * g;
  x4 = B' * g;
  x5 = g' * B * g;
  B = B - x3 * x4' / x5;
end
F = feval(myFx, X, N);
% end
function y = myFxEx(N, X, DeltaX, lambda, myFx)
  X = X + lambda * DeltaX;
  y = feval(myFx, X, N);
% end
function FirstDerivX = FirstDerivatives(X, N, myFx)
for iVar=1:N
  xt = X(iVar);
  h = 0.01 * (1 + abs(xt));
  X(iVar) = xt + h;
  fp = feval(myFx, X, N);
  X(iVar) = xt - h;
  fm = feval(myFx, X, N);
  X(iVar) = xt;
  FirstDerivX(iVar) = (fp - fm) / 2 / h;
```

```
end % end

function lambda = linsearch(X, N, lambda, D, myFx)

  MaxIt = 100;

  Toler = 0.000001;

  iter = 0;

  bGoOn = true;

  while bGoOn

    iter = iter + 1;

    if iter > MaxIt

      lambda = 0;

      break

    end

    h = 0.01 * (1 + abs(lambda));

    f0 = myFxEx(N, X, D, lambda, myFx);

    fp = myFxEx(N, X, D, lambda+h, myFx);

    fm = myFxEx(N, X, D, lambda-h, myFx);

    deriv1 = (fp - fm) / 2 / h;

    deriv2 = (fp - 2 * f0 + fm) / h ^ 2;

    diff = deriv1 / deriv2;

    lambda = lambda - diff;

    if abs(diff) < Toler

      bGoOn = false;

    end
```

```
  end
% end
```