ESTIMATION OF MARKET RISK VOLATILITY USING INTERQUANTILE AUTOREGRESSIVE RANGE

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Estimation of Market Risk Volatility Using Interquantile Autoregressive Range

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Declaration

I declare that this thesis is my original work and has not been presented in any other university.

Signature ………………………… Date …………………………

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Dedication

This thesis is dedicated to the my parents Julius, Jessica, brother Shadrack, sisters Eunice and Everlyne and my lovely nieces Lilian and Victoria for their prayers, support and relentless encouragements.

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Table of Contents

List of Tables

List of Figures

Abbreviations

ACF – Auto Correlation Function

ADF – Augmented Dickey Fuller

AMAPE – Average Mean Absolute Proportionate Error

AR – AutoRegression

CARR – Conditional AutoRegression Range

CAViaR – Conditional Autoregressive Value at Risk

EVT – Extreme Value Theory

EWMA – Exponentially Weighted Moving Average

GARCH – Generalized AutoRegression Conditional Heteroscedasticity

IID – Independent and Identically Distributed

IQARR – InterQuantile AutoRegression Range

NSE – Nairobi Securities Exchange

QAR – Quantile AutoRegression

QARF – Quantile AutoRegression Function

SV – Stochastic Volatility

VaR – Value at Risk

Abstract

Volatility is a market risk measure. It plays a central role in asset/derivative pricing, asset allocation/ portfolio creation, and in financial risk management. Volatility in financial applications is usually estimated using standard deviation for a given time series data. The stylized facts of financial time series data have made usage of standard deviation as an estimate for volatility, inappropriate. In order to address this problem we use the robust quantile regression procedure in estimation and also to capture the dynamic nature of volatility. In the thesis, an objective function was formulated and the function properties were investigated. The function was used to estimate quantile autoregression functions by minimization method. Volatility was then obtained by dividing the interquantile autoregression range function with a constant scale in a known distribution. The conditional quantile autoregression function is consistent and asymptotically normal. By Slutsky's Theorem, the volatility estimator was found to be consistent. A simulation study carried out also showed that the estimator was consistent. The InterQuantile Autoregressive Range method was applied to Real data to estimate the market risk volatility of Kenyan Securities Market.

CHAPTER ONE

INTRODUCTION

1.0 Introduction

This chapter discusses briefly on market risk volatility in financial markets. The importance of volatility in the performance of investments in the financial market is also discussed. In addition, statement of the problem, the study objectives, research questions, the significance and scope of the study are stated.

1.1 Background of the study

In financial terms, risk is the chance that an investment's actual return will be different from the expected. That is, it is the possibility of losing part or the entire investment. Volatility is one of the measures of market risk and it is a vital phenomenon in the financial world. It cuts across all fields of specialization in finance, including; pricing and securities valuation, financial risk management, monetary policy making and asset allocation/portfolio creation.

Various methods of estimating market risk volatility have been developed. The commonly used methods are autoregressive conditional heteroscedasticity (ARCH) which was proposed by Engle (1982); its advancement the generalized autoregressive conditional heteroscedasticity (GARCH) by Bollerslev (1986), and the stochastic volatility (SV) models by Taylor (1986). According to Brandt and Diebold (2006), range-based estimation is highly efficient as it distils volatility information from the entire intraday prices unlike GARCH and SV models which are based on the closing prices.

This indicates that range is a more exhaustive technique for estimating volatility as it uses the information contents within the prices. The price range is the difference between the highest and lowest market prices over a fixed sampling interval (Nathan et.al, 2009). It gives more information on the process followed by prices which improves on volatility estimation (Parkinson, 1980). The range-based volatility estimators are robust to microstructure noise such as bid-ask bounce (Nathan et.al, 2009).

Chou (2005) proposed the conditional autoregressive range (CARR) model. The CARR model provided more accurate volatility estimator compared to the GARCH model both in-sample and out-of-sample (Chou, 2005). The shortcoming of the model is its sensitivity to outliers. The problem is fixed by using quantile range (Nathan et.al, 2009). The concepts of volatility have been used in different spheres of finance to measure market risk.

According to modern portfolio theory, the higher the risk volatility, the higher the expected returns from an investment and vice versa (Markowitz, 1952). In selection of an investment portfolio, the investor chooses the portfolio that suits his/her risk appetite which as a result will dictates his/her utility function. The uncertainty in investment portfolios can be described by volatility.

In commodities market, Litzenberger and Rabinowitz (1995) showed that market risk volatility is positively correlated with prices. That is, when market risk volatility upsurges the prices also go up. The market volatility affects the cost of production and can lead to reduction of production (Robert, 2004). This results from decrease in supply leading to increase in demand, and hence price increase. In other markets like the shares, market risk volatility upsurge leads to decline in prices due to offloading of shares which result to excess supply and declining demand (Pandian and Jeyanth, 2009). This relation of volatility and price of the underlying asset, plays a big role in the pricing the assets. That is, volatility is a key input in pricing the asset.

In foreign exchange markets, volatility is regarded as one of the most important informational indicators for decisions on opening and closing of currency positions for importers and exporters (Maana et.al, 2010). In the forex market, volatility is measured by Relative volatility Index (www.forexrealm.com). Relative volatility Index measures the standard deviation of high and low prices over a defined range of periods. It reflects the direction in which currency price volatility changes. Since relative volatility index is determined by market dynamics and measures up the volatility instead of price, it brings important information to trading system (www.tradin-point.com/volatility). Thus the analysis of volatility in the foreign exchange market makes it possible for forex bureau and bank to trade on currencies.

In the futures and options market, volatility is critically important for call and put option assessment. Being derived from a particular asset, any volatility experienced in the underlying assets directly leads to volatility in the financial derivative (Siopis and Lyroudi, 2007). On spot market, traders are more concerned with immediate returns whereas on futures market they think further about volatility. For instance in pricing an option, we need to know the volatility of the underlying asset from the current time up to the maturity date of the option. To mitigate the negative effects of volatility in the market, risk management measures need to be employed to hedge against the risk associated to the volatility.

Financial institutions are obliged to put aside a reserve capital called Value at Risk (VaR) to curb an uncertain experience which might occur in a specific time in future since the establishment of the Basel Accord in 1996. The estimation of VaR is realized from the volatility estimates of the returns generated by the investment.

Volatility cannot be ignored in any financial undertaking since any financial decision is based on the market estimates of volatility to gauge the vulnerability of the financial markets and the economy at large.

1.2 Statement of the problem

The assumption that market risk volatility estimator follows a particular statistical distribution is a potential source of error in the modeling of financial markets (Taylor, 2005). Mostly the standard GARCH family models are widely used in market risk volatility estimation. These methods are based on the assumption that; the shape of the conditional distribution of returns is fixed over time (Taylor, 2005). This assumption is not realistic since financial returns do have stylistic characteristics which include volatility clustering, heavy tailness, low autocorrelation.

However, over time the shape of the distribution varies due to market shocks. Consequently, this implies that the previously estimated parameters before the shocks become inefficient (inappropriate) after the market shocks occur. Hence they cannot appropriately model the volatility leading to a likely source of error for the volatility estimates by these models (Taylor 2005). The conditional autoregressive value at risk (CAViaR) models of Engle and Manganelli (2004) had no distributional assumptions. The estimates obtained were consistent and asymptotically normal. Deriving market risk volatility from the same concept as above, we expect better results.

1.3 Objectives of the study

1.3.1 General objective

The general objective was to estimate volatility in financial market returns using quantile autoregression range.

1.3.2 Specific objectives

The specific objectives were:

1.3.2.1 To formulate an asymmetric objective function for volatility minimization.

1.3.2.2 To investigate the properties of the objective function.

1.3.2.3 To estimate the quantile autoregression volatility and give theoretical statistical properties of the quantile autoregression volatility estimator.

1.3.2.4 To perform a simulation study to show consistency of the volatility estimator.

1.3.2.5 To apply the interquantile autoregressive range method to estimate volatility using real life data.

1.4 Research questions

The following questions are important in the research:

1.4.1 How can volatility be minimized?

1.4.2 What are the properties of the objective function for quantile autoregression?

1.4.3 Do estimators of quantile autoregressive function have good statistical properties?

1.4.4 Is the volatility estimator obtained via interquantile range consistent?

1.4.5 Is the method applicable to real life data?

1.5 Rationale

This study is important to various users including financial institutions, investors and academia. It will add literature to the already existing knowledge in volatility estimation and open a window for further research in the area. We expect this research to be beneficial to financial institutions and individual investors (existing and potential) by providing useful insight on the financial markets on market risk volatility minimization.

1.6 Scope of the study

For empirical study, data from the Nairobi Securities Exchange 20 Share Trade Index was used.

1.7 Thesis Outline

The rest of the chapters are organized in the following ways: Chapter two is briefly on Literature Review on market risk volatility estimation, Chapter 3 is on methodology of market risk volatility estimation, Chapter 4 is on volatility estimation-where a simulation study is carried out, Chapter 5 discusses empirical study- whereby we apply the proposed method to real data and finally, Chapter 6 the Conclusion and Recommendation.

1.8 Conclusion

This chapter discussed on importance of volatility in the financial arena, background study of volatility estimation, stated the problem statement as well as formulated the objectives of the study. Research questions, scope of study and the rationale of the study were also looked into.

CHAPTER TW0

LITERATURE REVIEW

2.0 Introduction

This chapter discusses briefly on the existing literature on volatility estimation, different methods of volatility estimation and the research gap is stated.

2.1 Literature Review

The need to measure market risk began many years ago. Macaulay (1938) came up with the first mathematical method of measuring and managing financial interest rate risk called *immunization duration.* The measure considered the fluctuations of cash flows/ returns from investments to measure risk. The modern portfolio theory by Markowitz (1952) proposed standard deviation to be the risk measure of volatility from investments. The most efficient portfolio in returns rewarding is one that maximizes returns with minimized volatility (standard deviation). This is achieved by portfolio diversification and consequently, minimization of volatility became a big issue at stake so as to reap better returns from investments.

In 1953, Maurice Kendall proposed a theory that stock prices move randomly. Random walk was fit to model stock prices in his findings. This was due to the consistency of Random Walk with the efficient-market hypothesis, a principle from the modern finance theory. Engle developed a framework of volatility estimation in 1982 and named it as the Autoregressive Conditional Heteroscedastic (ARCH). The model is formulated in a way that volatility change with time and was used to model financial returns/cash inflows. He proposed that volatility is conditionally dependent on previous states of the asset. Bollerslev (1986) generalized the model after finding that the current volatility of an asset is conditionally dependent on previous states of the asset as well as the previous volatilities of the asset and Generalized Autoregressive Conditional Heteroscedastic (GARCH) was the aftermath. It gave better estimates of volatility compared to ARCH. The assumption of an underlying distribution in estimation of volatility by GARCH models leads to inefficiency (Brandt and Diebold, 2006).

Due to volatility importance in finance, researchers have embarked on finding better methods of modeling volatility. As a result, various methodologies for volatility estimation have been explored. Extensions in the volatility estimation include the Range based methods. The application of ranges in finance started with Parkinson (1980). The methods were based on the fact that the range data (a measure of dispersion) provides more information pertaining the data. The additional information is important in improving estimations in financial applications (Nathan et.al, 2009). Range experiences a drawback since it is sensitive to outliers.

More extensions to come up with a robust measure of volatility to outliers have been sought. The drive has been from the fact that financial data mostly exhibit presence of outliers hence the call for a robust volatility estimation method. Chuo (2005) developed a method that addresses the problem by using quantile range to estimate volatility. In his findings he found out that it is thus meaningful to utilize the quantile range due to its robustness rather than using standard range in volatility estimation. A boost in usage of Quantiles in estimations is that, it has the potential to capture different features of a distribution (Abdallah, 2010).

The genesis of usage of quantiles trace roots more than a century ago in 1889 when Galton used a sample of approximately a thousand subjects in calculation of conditional quartiles of height of sons given the height of their fathers according to Abdallah (2010). Little was done from then till early 1960's when the potentiality of quantiles in solving statistical problems through modeling and inference became successful via the work of Pearson and Tukey (1965). It was found out that standard deviation can be generated from a quantile framework. Further development was by Koenker and Bassett (1978) who introduced quantile regression technique, Taylor (2005) employed the quantile regression technique in volatility forecast estimation. The estimation of the model parameters was by the least squares method and it was found that parameters were consistent.

Abdalla (2010) explicitly ascertains that, a good model should explicitly approximate the exact behavior of the underlying process. Such a model is possible by adopting a measure of variance that captures the variance efficiently. The possibilities according to Abdalla, come to reality by the use of quantile based method and market risk measures. Such measures of risk include: Value at Risk, the Expected Shortfall, Interquantile range, Interquantile Autoregressive range.

2.2 Methods of Volatility Estimation

Various methods of market risk volatility estimation have been developed, as described in the following section:

2.2.1 Exponentially Weighted Moving Average

The method is a betterment of the simple moving average which assumes that all returns have equal weight in determining volatility. In reality recent returns will have more influence in estimating volatility than older returns. The basis in which exponentially weighted moving average method is based on, is by introducing a smoothing parameter $\lambda \in [0,1]$. Under this framework, each squared return is weighted by a multiplier $(1-\lambda)\lambda^{n-1}$, where $n = 1,2,3,\ldots$ is past time from current. For the yesterday's returns $n = 1$, for yesterday but one return $n = 2$ and so on, thus more weight is put on recent observations which are better predictor of the future. The volatility is algebraically expressed as:

$$\sigma_t{}^2 = \sum_{\forall\, n}(1-\lambda)\lambda^{n-1}X_{t-n}^2 \qquad (2.1)$$

$$for\ n = 1,2,3,\ldots$$

The formula degenerates to: $\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1-\lambda)X_{t-1}^2$ (2.2)

For $\lambda \in [0,1]$ where: λ is the exponential weight coefficient, σ_t is the volatility at time t and X_t is the return value at time t. A value of 0.94 for λ has been recommended by *RiskMetrics* (Morgan, 1996).

2.2.2 Autoregressive Conditional Heteroscedasticity

The Autoregressive conditional Heteroscedasticity (ARCH) model expresses the conditional variance as a linear function of lagged squared terms of the process. Let $\{X_t\}$ be a time series process, then, an ARCH model of order p, $ARCH\ (p)$, for $\{X_t\}$ is defined as:

$$X_t = \mu_t + \sigma_t e_t \qquad (2.3)$$

Where $e_t \sim iidWN(0,1)$, μ_t, σ_t are the drift and the volatility of the process at time t respectively. The drift is assumed to be approximately equal to zero ($\mu_t \cong 0$).The volatility σ_t in (2.3), is defined as;$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 + \cdots + \alpha_p X_{t-p}^2$ for $0 < \alpha_1 + \cdots + \alpha_p < 1$ and $\alpha_0 \geq 0$

2.2.3 Generalized Autoregressive Conditional Heteroscedasticity

The Generalized Autoregressive Conditional Heteroscedasticity (GARCH) model developed by Bollerslev (1986) is an advancement of the ARCH model. The algebraic expression of a process $\{X_t\}$ following a GARCH model of order (p, q), GARCH (p, q) is as in (2.3) with the volatility σ_t, defined as:

$$\sigma_t^2 = \omega + \alpha_1 X_{t-1}^2 + \cdots + \alpha_p X_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2 \tag{2.4}$$

For $0 < \alpha_1 + \alpha_2 + \cdots + \alpha_p + \beta_1 + \cdots + \beta_q < 1 \; and \; \omega \geq 0$

The shortcoming on this model is that it assumes symmetry. Whereby both negative and positive shocks are equality weighted yet a negative shock has severe effects compared to positive shocks of the same magnitude. This phenomenon is called the leverage effects.

2.3 Research Gap

Standard GARCH family models have been used widely in market risk volatility estimation. The distributional assumption of a distribution in these models is a potential source of error. Research on the no distribution assumption framework has not been adequately explored. In the thesis no distribution assumption is consider by combination with two notions in the derivation of our volatility estimate. That is, we use (i) Robust Quantile regression method, and (ii) applying the range concept.

CHAPTER THREE

ESTIMATION METHODOLOGY

3.0 Introduction

This section presents the properties of quantiles. Quantile autoregression is discussed and formulation of an objective function for minimization and investigation of the functions' mathematical properties is done here. Also the derivation of Volatility estimator from the interquantile autoregressive range function is done in this chapter.

3.1 Quantiles

A quantile of a distribution function is the inverse of that distribution at a given probability level. For independent and identically distributed random variables X with distribution F, a quantile function is mathematically defined as:

$$F^{-1}(\theta) = \inf\{\, x : F(x) \geq \theta\} = \, Q_\theta \tag{3.1}$$

Where $\theta \in (0,1)$.

Quantiles are sometimes used in estimation of mean and variance (Taylor, 2005). According to Taylor, this has been possible by the work of Pearson and Tukey (1965). According to Pearson & Tukey (1965) the ratio of the standard deviation to the interval between symmetric quantiles (quantile range), $Q_{1-\theta}$ and Q_θ in the tail of the distribution, is approximately equal to a constant for a variety of distributions. This can be expressed algebraically as;

$$\frac{\sigma}{Q_{1-\theta} - Q_\theta} = \frac{1}{K} \tag{3.2}$$

Where $K \approx \phi^{-1}(1-\theta) - \phi^{-1}(\theta) = 2\phi^{-1}(\theta)$ and σ is the volatility.

Using quantiles, other methods which can be used for scale estimation include: Interquantile range by Taylor (2005) and Interquantile Autoregression Range by Mwita (2010).

The use of quantile in estimation of volatility is called for by unique properties exhibited by quantiles.

3.1.1 Properties of Quantiles

(i) **Robustness against outliers** –This property is of significant importance in financial risk management applications since outliers in the risk management process pose the risk of distorting the final outcome of some analysis of interest.

(ii) **Equivariance to monotone transformation of quantile functions**. Let X be a random variable and consider a monotonic function h that is used to create a transformed random variable $Y = h(X)$. Then, the quantiles of the new random variable Y are obtained by transforming the quantile function of X. That is:

$$Q_\theta^Y = h\left(F_X^{-1}(\theta)\right) = h\left(Q_\theta^X\right) \tag{3.3}$$

(iii)**Easy to interpret**. The interpretation of quantiles is easy.

In this work, quantiles were used in measuring market risk volatility by using the Inter Quantile Range function (IQRF). Due to the properties exhibited by quantiles better estimates for the market risk volatility are expected compared to other methods.

3.2 Quantile Regression

Koenker and Bassett (1978) introduced *quantile regression* which is a basically an extension of the linear regression model. The regression models conditional quantiles as a function of predictor variables. Unlike the linear-regression model, Quantile Regression specifies changes in the conditional quantile, making it possible to model a phenomenon at any quantile level of interest, since it is possible to model any predetermined position of the distribution (Abdallah, 2010). As a consequence, it makes it possible to achieve a more complete understanding of how the response variable is affected by regressors, including information about shape change.

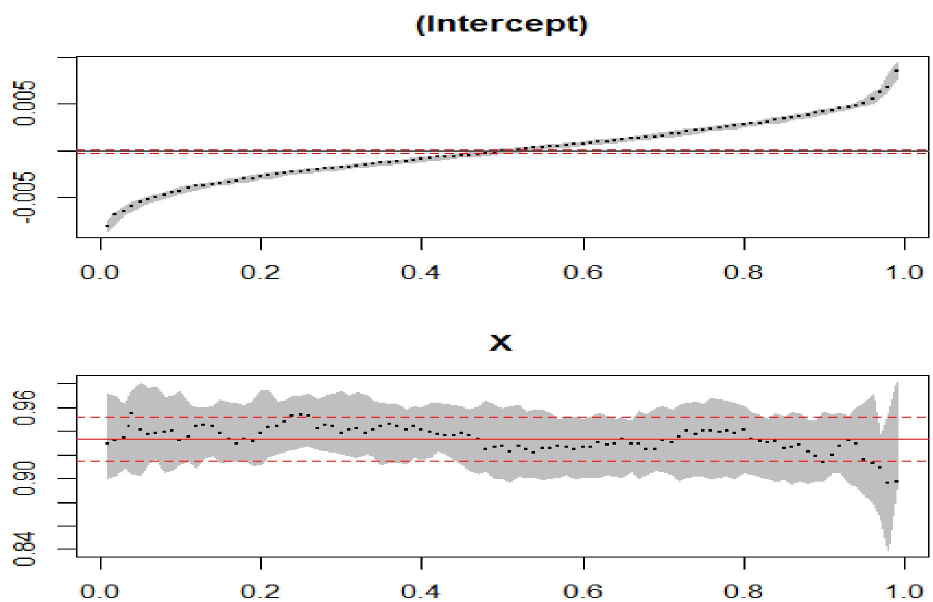

Figure 3.2.1: Quantile Regression Estimates

A summary plot (Figure 3.2.1) of quantile autoregression "grey" we see how the intercept parameter and the slope vary for different quantile levels. The parameter estimates are on the y-axis whereas; the x-axis denotes the different quantile levels. This ascertains the robustness of quantiles

3.2.1 General Quantile Regression

Consider the linear regression with d covariate $Y_t = \underset{\sim}{X}'_t\boldsymbol{\beta} + \epsilon_t$, where ϵ_t is white noise ($\epsilon_t \sim iid(0,1)\ \ \forall t,\ t = 1,2,\ldots,n$) and $\underset{\sim}{X} = (1, X_1, X_2, \ldots, X_d)'$. It follows that $E\left(Y\middle|\underset{\sim}{X}\right) = \underset{\sim}{X}'\boldsymbol{\beta}$**.** In accordance to Koenker and Bassett (1978), a quantile regression model corresponding to the linear regression above can be expressed as:

$$Y_t = \underset{\sim}{X}'_t\boldsymbol{\beta}^{(\boldsymbol{\theta})} + \epsilon_t^{(\theta)}\,, \qquad for\ 0 < \theta < 1 \tag{3.4}$$

$$t = 1,2,\ldots\ldots n$$

Where the error term $\epsilon_t^{(\theta)}$ is assumed to satisfy the axiom that; $Q_\theta\left(\epsilon_t^{(\theta)}\right) = 0$ and $\boldsymbol{\beta}^{(\boldsymbol{\theta})} = \left(\beta_0^{(\theta)}, \beta_1^{(\theta)}, \ldots, \beta_d^{(\theta)}\right)$**.** Equivalently to the linear regression, the conditional quantile is given by,

$$Q_\theta\left(Y_t\middle|\underset{\sim}{X}_t\right) = \underset{\sim}{X}'_t\boldsymbol{\beta}^{(\boldsymbol{\theta})} \tag{3.5}$$

3.2.2 Quantile Autoregression

The quantile autoregression is based on the notion that time series data exhibit some form of autocorrelation. The notion of Quantile AutoRegression (QAR) was introduced by Abberger in 1999 according to Mwita (2003). The procedure implies that, the response variable is conditioned on its past observations as the covariate(s) at specified quantile levels. The Quantile autoregression tries to model a time series process on accounts of an information dataset say,

$\Omega_t = X_t = (Y_{t-1}, Y_{t-2}, \dots, Y_1)$. Considering a Quantile Autoregression process, introduced in Mwita (2003);

$$Y_t = \pi_\theta(X_t) + \sigma(X_t)Z_t \tag{3.6}$$

Where; $t \in \mathbb{R}$, $X_t = (Y_{t-1}, Y_{t-1}, \dots, Y_{t-n})$, Z_t are assume iid with Zero θ-quantile and unit scale

$\pi_\theta(X_t)$ is the conditional θ-quantile function of Y_t given X_t

$$Q_\theta(Y_t|X_t) = \pi_\theta(X_t) \tag{3.7}$$

$\sigma(X_t)$ is the conditional scale function of Y_t given X_t.

For Heteroscedastic processes $Y_t = \sigma_t\, e_t$, and $\pi_\theta(X_t) = \sigma_t\, e_\theta$, then, taking the difference of Y_t and $\pi_\theta(X_t)$ yields equation (3.6). With $Z_t = e_t - e_\theta$ where $\{e_t\}$ is the error term series and e_θ is the θ-quantile level of $\{e_t\}$.

Equation (3.7) is implied by the assumption that $Q_\theta(Z_t) = 0$. Whereby $Q_\theta(.)$ is the conditional θ-quantile of the subject of interest.

3.3 Formulation of Objective Function

The problem of obtaining quantiles is an optimization problem where quantiles are obtained by minimizing sums of asymmetrically weighted absolute residuals (giving different weights to positive and negative residuals).

3.3.1 Quantile Regression

Considering a general quantile regression model;

$$Y_t = \underset{\sim}{X}_t' \boldsymbol{\beta}^{(\boldsymbol{\theta})} + \epsilon_t^{(\theta)}, \quad t = 1,2,\ldots \tag{3.8}$$

Where $\boldsymbol{\beta}^{(\boldsymbol{\theta})} = (\beta_{0,\theta}, \beta_{1,\theta}, \ldots, \beta_{d,\theta})'$and the regressors $\underset{\sim}{X}_t' = \left(1, X_{1,t}, X_{2,t}, \ldots, X_{d,t}\right)$.

Conditional on the information set $\Omega = \{X_t\}$, the conditional Quantile of the response variable $Y_t, \forall t$, on Ω is given by;

$$Q_\theta\left(Y_t \middle| \underset{\sim}{X}_t\right) = \underset{\sim}{X}_t' \boldsymbol{\beta}^{(\boldsymbol{\theta})}, \quad t \in \mathbb{R} \tag{3.9}$$

Since $Q_\theta\left(\epsilon_t^{(\theta)}\right) = 0$

Thus the conditional θ-quantile estimate of Y_t can be estimated as;

$$\widehat{Y}_t \equiv Q_\theta \widehat{(Y_t | X_t)} = \underset{\sim}{X}_t' \widehat{\boldsymbol{\beta}}^{(\boldsymbol{\theta})} \tag{3.10}$$

Let R_t denote the residuals obtained from our model (3.8). That is;

$$R_t = Y_t - \widehat{Y}_t \tag{3.11}$$

Define an indicator function for a random variable X as: $I(x) = \begin{cases} 1, & x \leq 0 \\ 0, & x > 0 \end{cases}$. Then, equation (3.11) can be written in indicator form as:

$$\boldsymbol{I}(R_t) = \begin{cases} 1, if\ R_t \leq 0 \\ 0\ , if\ R_t > 0 \end{cases} \tag{3.12}$$

By expanding (3.12) it can also be written as;

$$\boldsymbol{I}\left(Y_t - \underset{\sim}{X}_t'\widehat{\boldsymbol{\beta}}^{(\boldsymbol{\theta})}\right) = \begin{cases} 1, if\ Y_t - \underset{\sim}{X}_t'\widehat{\boldsymbol{\beta}}^{(\boldsymbol{\theta})} \leq 0 \\ 0\ , if\ Y_t - \underset{\sim}{X}_t'\widehat{\boldsymbol{\beta}}^{(\boldsymbol{\theta})} > 0 \end{cases} \tag{3.13}$$

Let's define the asymmetrical weights as $\theta\, and\, 1-\theta$ for positive and negative residuals respectively where $\theta \in (0,1)$.

Definition 1: Check Function

Let $\rho_\theta(x)$ denote the check function (a linear piecewise function), in general defined as;

$$\rho_\theta(x) = x\big(\theta - \boldsymbol{I}(x \leq 0)\big)\, , for\ x \in \mathbb{R} \tag{3.14}$$

$$= \begin{cases} x(\theta - 1), & if\ x \leq 0 \\ x(\theta), & if\ x > 0 \end{cases}$$

Considering our model we use equation (3.14) and replace the variable x with our model residuals R_t. This gives us the following $check\ function\ a$s;

$$\rho_\theta\left(Y_t - \underset{\sim}{X}_t'\boldsymbol{\beta}^{(\boldsymbol{\theta})}\right) = \begin{cases} (\theta - 1)\left(Y_t - \underset{\sim}{X}_t'\boldsymbol{\beta}^{(\boldsymbol{\theta})}\right), & if\ Y_t \leq \underset{\sim}{X}_t'\boldsymbol{\beta}^{(\boldsymbol{\theta})} \\ \theta\left(Y_t - \underset{\sim}{X}_t'\boldsymbol{\beta}^{(\boldsymbol{\theta})}\right), & if\ Y_t > \underset{\sim}{X}_t'\boldsymbol{\beta}^{(\boldsymbol{\theta})} \end{cases} \tag{3.15}$$

Definition 2: Objective Function

The **objective function** is a function of the loss function, obtained by taking the expectation of the loss function. Let $\boldsymbol{g}(.)$ and $\boldsymbol{E}$ denote the objective function and the expectation operator respectively, then considering the loss function in (3.15), the corresponding $objective\ function$ is defined as; $\boldsymbol{g_\theta}(\boldsymbol{x}) = \boldsymbol{E}\big(\boldsymbol{\rho_\theta}(\boldsymbol{x})\big)$**.** Replacing the variable x in the objective function with R_t, we have our objective function as;

$$\boldsymbol{g_\theta}\left(Y_t - \underset{\sim}{X}_t'\widehat{\boldsymbol{\beta}}^{(\boldsymbol{\theta})}\right) = \boldsymbol{E}\left[\rho_\theta\left(Y_t - \underset{\sim}{X}_t'\widehat{\boldsymbol{\beta}}^{(\boldsymbol{\theta})}\right)\right]$$

$$= \begin{cases} (\theta - 1)\boldsymbol{E}\left(Y_t - \underset{\sim}{X}_t'\widehat{\boldsymbol{\beta}}^{(\boldsymbol{\theta})}\right), & if\ Y_t \leq \underset{\sim}{X}_t'\widehat{\boldsymbol{\beta}}^{(\boldsymbol{\theta})} \\ \theta\boldsymbol{E}\left(Y_t - \underset{\sim}{X}_t'\widehat{\boldsymbol{\beta}}^{(\boldsymbol{\theta})}\right), & if\ Y_t > \underset{\sim}{X}_t'\widehat{\boldsymbol{\beta}}^{(\boldsymbol{\theta})} \end{cases} \tag{3.16}$$

3.3.2 Quantile AutoRegression

Considering a time series model of QAR (r)-GARCH (p, q). The GARCH part is of importance in our work as it is the one where the volatility function is derived from. The model can be expressed as; $Y_t = \pi_t + \sigma_t Z_t$, where:

$$\pi_t = \beta_{0,\theta} + \beta_{1,\theta}Y_{t-1} \ldots + \beta_{r,\theta}Y_{t-r},\ \sigma_t = \sqrt{\omega + \alpha_1 Y_{t-1}^2 + \cdots + \alpha_p Y_{t-p}^2 + \gamma_1\sigma_{t-1}^2 + \cdots + \gamma_q\sigma_{t-q}^2}$$

and Z_t are iid with and 0 θ-quantile and some scale. The process Y_t can be rewritten as;

$$Y_t = \boldsymbol{\beta}_{\boldsymbol{\theta}}'\underset{\sim}{X}_t + \sqrt{\omega + \alpha'\underset{\sim}{X}_t^2 + \gamma'\underset{\sim}{\sigma}_t^2}\, Z_t \tag{3.17}$$

Where $\boldsymbol{\beta}_\theta = (\beta_{0,\theta}, \beta_{1,\theta}, \ldots, \beta_{p,\theta})',\ \omega > 0,\ \underset{\sim}{X}_t = (1, Y_{t-1}, \ldots, Y_{t-r})', \alpha = (\alpha_1, \ldots, \alpha_q)',\ \gamma = (\gamma_1, \ldots, \gamma_q)'\ \underset{\sim}{X_t}^2 = (Y_{t-1}^2, \ldots, Y_{t-p}^2)'$and $\underset{\sim}{\sigma_t}^2 = (\sigma_{t-1}^2, \ldots, \sigma_{t-q}^2)'$

The objective function formulation follows the same procedure as on Quantile Regression. Following the same procedure we have;

(i) Theoretical Objective Function

The theoretical representation of our objective function which is the objective function for the entire population of study is defined as;

$$g_\theta(y) = E\left[\rho_\theta\left(Y_t - \pi\left(\underset{\sim}{X}_t, \beta_\theta\right)\right)\right]$$

$$= \begin{cases} \theta E\left[\left(Y_t - \pi\left(\underset{\sim}{X}_t, \beta_\theta\right)\right)\right], & if\ Y_t - \pi\left(\underset{\sim}{X}_t, \beta_\theta\right) > 0 \\ (\theta - 1)E\left[\left(Y_t - \pi\left(\underset{\sim}{X}_t, \beta_\theta\right)\right)\right], & if\ Y_t - \pi\left(\underset{\sim}{X}_t, \beta_\theta\right) \leq 0 \end{cases} \tag{3.18}$$

(ii) **Sample Version Objective Function**;

Since studying an entire population is tedious, we draw a representative sample from the population, we do our study using the sample and then generalized results to cover the entire population. The sample version of our objective function is defined as;

$$\widehat{g_\theta}(y) = \frac{1}{n}\sum_{t=1}^{n} \rho_\theta\left(Y_t - \pi\left(\underset{\sim}{X}_t, \beta_\theta\right)\right)$$

$$= \frac{\mathbf{1}}{\boldsymbol{n}}\left\{\sum_{t\in\left\{t:Y_t \leq \pi\left(\underset{\sim}{X}_t, \beta_\theta\right)\right\}} (\theta - 1)\left(Y_t - \pi\left(\underset{\sim}{X}_t, \beta_\theta\right)\right) + \sum_{t\in\left\{t:Y_t > \pi\left(\underset{\sim}{X}_t, \beta_\theta\right)\right\}} \theta\left(Y_t - \pi\left(\underset{\sim}{X}_t, \beta_\theta\right)\right)\right. \tag{3.19}$$

The objective function is our basis for model parameters estimation. This implies that the function has some desirable properties which make it useful in error minimization and maximization of the estimations. In the next section, we investigate the properties of the objective function.

3.4 Properties of Objective Function

The time series $\{Y_t\}$ is assumed to be derived from a continuous distribution function F. Then, it follows that; the loss function is also continuous and piecewise. Consequently, the objective function is a piecewise linear continuous function.

Definition 3:

A function $f : [a,b] \to \mathbb{R}$ is said to be *piecewise linear* if there exists a division $a = x_0 < \cdots < x_n = b$ such that the restriction of f to each partial interval $[x_k, x_{k+1}]$ is an affine function.

The properties of the objective function include;

(i) Lipschitz continuous

A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is lipschitz continuous if there exists a constant M such that $\forall x, y \in \mathbb{R}^n$,

$$\| f(x) - f(y) \| \leq M \| x - y \|$$

To show whether our function is Lipschitz continuous we denote the loss function $\rho_\theta \left(Y_t - \pi \left(\underset{\sim}{X_t}, \hat{\beta}_\theta\right)\right) = \rho_\theta(Y, \pi)$ (for notation convenience). The following lemma will assist us in showing that the function is Lipschitz continuous.

Lemma 1

For (Y, π) a real valued random variable, the function $\rho_\theta(Y, \pi)$ is Lipschitz continuous in π with lipschitz constant $K = 1$. i.e. $|\rho_\theta(Y, \pi_1) - \rho_\theta(Y, \pi_2)| \leq |\pi_1 - \pi_2| \; \forall Y, \pi_1, \pi_2$

Where $\pi_1, \pi_2 \in \pi$.

Proof

By definition, the check function is mathematically expressed as;

$$\rho_\theta(Y,\pi) = (Y-\pi)[\theta - I(Y-\pi \le 0)]$$

Then,

$$\rho_\theta(Y,\pi_1) - \rho_\theta(Y,\pi_2) = \left\{\left((Y-\pi_1)[\theta - I(Y-\pi_1 \le 0)]\right) - \left((Y-\pi_2)[\theta - I(Y-\pi_2 \le 0)]\right)\right\}$$

$$= \theta(\pi_2-\pi_1) - [(Y-\pi_1)I(Y-\pi_1 \le 0) - (Y-\pi_2)I(Y-\pi_2 \le 0)]$$

For $\pi_1 < Y < \pi_2$ we have;

$I(Y-\pi_1 \le 0) = 0$ and $I(Y-\pi_2 \le 0) = 1$ hence;

$$\rho_\theta(Y,\pi_1) - \rho_\theta(Y,\pi_2) = \theta(\pi_2-\pi_1) + (Y-\pi_2)$$

$$= \theta(\pi_2-\pi_1) + (Y-\pi_2) + (\pi_1-\pi_1)$$

$$= (Y-\pi_1) - (1-\theta)(\pi_2-\pi_1)$$

For $(Y-\pi_1) > 0$ and $(Y-\pi_2) > 0$, we have;

$$-(1-\theta)(\pi_2-\pi_1) \le \rho_\theta(Y,\pi_1) - \rho_\theta(Y,\pi_2) \le \theta(\pi_2-\pi_1)$$

Thus $|\rho_\theta(Y,\pi_1) - \rho_\theta(Y,\pi_2)|$ is bounded from above by at least $\theta(\pi_2-\pi_1)$ and $(1-\theta)(\pi_2-\pi_1)$.

Similarly if;

(i) $\pi_1 \le \pi_2 < Y$, then $I(Y-\pi_1 \le 0) = 0$ and $I(Y-\pi_2 \le 0) = 0$ hence;

$$\rho_\theta(Y,\pi_1) - \rho_\theta(Y,\pi_2) = \theta(\pi_2-\pi_1)$$

(ii) $Y < \pi_1 \le \pi_2$, then $I(Y-\pi_1 \le 0) = 1$ and $I(Y-\pi_2 \le 0) = 1$ hence;

$$\rho_\theta(Y,\pi_1) - \rho_\theta(Y,\pi_2) = (\theta-1)(\pi_2-\pi_1) = (1-\theta)|\pi_1-\pi_2|$$

$$|\rho_\theta(Y,\pi_1) - \rho_\theta(Y,\pi_2)| \le max(\theta, \theta-1)|\pi_2-\pi_1|$$

$$\le |\pi_2-\pi_1| = |\pi_1-\pi_2|$$

$$\leq |\pi_1 - \pi_2|$$

Hence its $lipschitz\ continuous\ with\ K = 1$

Then, for the objective function $\widehat{g_\theta}(y) = \frac{1}{n}\sum_{t=1}^{n} \rho_\theta\left(Y_t - \pi\left(\underset{\sim}{X_t}, \hat{\beta}_\theta\right)\right)$, which is the expected value of loss function and the loss function $\rho_\theta\left(Y_t - \pi\left(\underset{\sim}{X_t}, \hat{\beta}_\theta\right)\right)$ being Lipschitz continuous, it follows that; $\frac{1}{n}\sum_{t=1}^{n} \rho_\theta\left(Y_t - \pi\left(\underset{\sim}{X_t}, \hat{\beta}_\theta\right)\right)$ is also Lipschitz continuous.

(ii) Convex

Definition 4:

A function $f : Q \to \mathbb{R}$ defined on a nonempty subset Q of $\mathbb{R}^n$ and taking real values is convex, if

a) The domain Q of the function is convex

b) For every $x, y \in Q$ and every $\lambda \in [0, 1]$ we have;

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

THEOREM

Assuming that g is a smooth function (a function whose desired derivative(s) exist). Let (a, b) be an interval in the axis (we do not exclude the case of $a = -\infty$ and/or $b = +\infty$). Then;

a) A differentiable everywhere on (a, b) function g is convex on (a, b) if and only if its derivative g' is monotonically nondecreasing on (a, b);

b) A twice differentiable everywhere on *(a, b)* function g is convex on (a, b) if and only if its second derivative g'' is nonnegative everywhere on (a, b).

Proof:

Using the Theorem we show that; g is differentiable and convex on (a, b)

If g is differentiable and convex on (a, b), it implies that g' is monotonically nondecreasing. Given $x < y \in (a, b)$, and z is the convex combination of x and y in the form; $z = \frac{y-z}{y-x}x + \frac{z-x}{y-x}y$. Because of the convexity of g , we have

$$g\ (z) = g\left(\frac{y-z}{y-x}x + \frac{z-x}{y-x}y\right) \leq \frac{y\ -\ z}{y-x}g(x) + \frac{z-x}{y-x}g(y)$$

By the triangular inequality we have: $g\ (z) \leq \frac{y-z}{y-x}g(x) + \frac{z-x}{y-x}g(y)$. Due to the non-decreasing monotonicity of g', then;

$$\Rightarrow \quad \frac{g(z) - g(x)}{z - x} \leq \frac{g(y) - g(z)}{y - z}$$

Let $z = x + \varepsilon$ and enforce limit $\varepsilon \to 0$, we get $g'(x) \leq \frac{g\ (y)-g\ (x)}{y-x}$ and Let $z = y - \varepsilon$ and enforce limit $\varepsilon \to 0$, we get $g'(y) \geq \frac{g\ (y)-g\ (x)}{y-x}$. Combine the results we get $g'(y) \geq g'(x)$, which completes the proof that $g'$is monotonically non-decreasing.

Next, we prove that: g is differentiable on (a, b) and $g'$is monotonically nondecreasing on $(a, b) \Rightarrow g$ is convex on (a, b).

Let $x < y \in (a, b)$ and $z = \gamma x + (1 - \gamma) y$ with $0 < \gamma < 1$, then we have

$$g(z) \leq \gamma g(x) + (1 - \gamma) g(y) \Rightarrow \frac{g(z) - g(x)}{\gamma} \leq \frac{g(y) - g(z)}{1 - \gamma}$$

Notice that $z - x = (1 - \gamma)(y - x)$ and $y - z = \gamma(y - x)$, the inequality we want to prove is equivalent to;

$$\frac{g(z) - g(x)}{z - x} \leq \frac{g(y) - g(z)}{y - z}$$

Applying the Lagrange Mean Value Theorem, $\frac{g(z) - g(x)}{z - x} = g'(\xi)$ with $\xi \in [x, z]$ and $\frac{g(y) - g(z)}{y - z} = g'(\eta)$ with $\eta \in [z, y]$. Because g' is nondecreasing and $\xi \leq z \leq \eta$ and therefore the inequality holds. This completes the proof that g is convex.

(iii) Differentiable

We have found out that our objective function is Lipschitz continuous and convex. It follows that the function is also differentiable. The differentiability follows from **Rademacher's theorem**.

Theorem: Rademacher's theorem

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz, then f is differentiable almost everywhere.

Proof:

See the proof in (Howard, 1998)

The objective function is differentiable but non differentiable at all point where $Y_t - \pi\left(\underset{\sim}{X_t}, \hat{\beta}_\theta\right) = 0$. At the points where $Y_t - \pi\left(\underset{\sim}{X_t}, \hat{\beta}_\theta\right) = 0,$ though the objective functions is not differentiable it has directional derivatives in all directions.

Directional derivatives

A directional derivative is the change in a function for a unit step in that particular direction.

We define;

$$S(\beta) = d_\theta(Y, \pi) = \sum_{t=1}^{n} \rho_\theta\left(Y_t - \pi\left(\underset{\sim}{X_t}, \hat{\beta}_\theta\right)\right) = \sum_{t=1}^{n} \rho_\theta\left(Y_t - \beta_\theta' \underset{\sim}{X_t}\right)$$

Lemma 2

If $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j}$ is a unit vector, we define the direction derivative f_U at the point (a, b) by;

$$D_u f(x) = \lim_{h\to 0} \frac{f(x + hu) - f(x)}{h}$$

Provided that the limit exists, therefore; the directional derivative of S in the direction of u is given by:

$$D_u S(\beta) = \nabla S(\beta, u) = \lim_{h\to 0} \frac{S(\beta + hu) - S(\beta)}{h} \equiv \frac{d}{dh} S(\beta + hu)|_{h=0}$$

$$= \frac{d}{dh} \sum_{t=1}^{n} \rho_\theta\left(Y_t - (\beta_\theta' + hu)\underset{\sim}{X_t}\right)|_{h=0}$$

$$= \frac{d}{dh}\left[\sum_{t=1}^{n}\left(Y_t - \beta_\theta' \underset{\sim}{X}_t - hu\underset{\sim}{X}_t\right)\left(\theta - I\left(Y_t - \beta_\theta' \underset{\sim}{X}_t - hu\underset{\sim}{X}_t < 0\right)\right)\right]|_{h=0}$$

$$= -\sum \psi_\theta\left(Y_t - \beta_\theta' \underset{\sim}{X}_t, -u\underset{\sim}{X}_t\right) u\underset{\sim}{X}_t \,,$$

Where $\psi_\theta(u,v) = \begin{cases}\theta - I(u<0), & if\ u \neq 0\\ \theta - I(v<0), & if\ u = 0\end{cases}$

At a point $\hat{\beta}$, where the directional derivates are nonnegative $(\nabla S(\beta, u) \geq 0, \forall\, u \in \mathbb{R}^p)$ with$||u|| = 1$, then $\hat{\beta}$ minimizes $S(\beta)$

The objective function $g(.)$ has been found to be convex and continuous; then, it admits left and right derivatives. The derivatives are monotonically non-decreasing. Consequently, g is differentiable at most countably many points.

(iv) Integrable

Since, the objective function is convex, continuous and differentiable it follows that it's integrable (Botsko, 1991).

3.5 Volatility Estimation

In this section we present the procedure of volatility estimation using the interquantile autoregression range.

3.5.1 InterQuantile AutoRegression Range

The InterQuantile techniques make use of the information contained within different quantiles of a given data set. The additional information brought about by the range *(a measure of data dispersion)* helps in coming up with a better measure of volatility since Range data is more informative. The proposed model QAR-GARCH is defined as;

$$Y_t = \pi\left(\underset{\sim}{X}_t, \beta_\theta\right) + \sigma\left(\underset{\sim}{X}_t, \alpha\right) Z_t \tag{3.20}$$

Let's consider a special case of the proposed QAR-GARCH model. That is, the AR-GARCH defined as;

$$Y_t = \pi\left(\underset{\sim}{X}_t, \beta\right) + \sigma\left(\underset{\sim}{X}_t, \alpha\right) e_t \tag{3.21}$$

Where $e_t \sim iid(0,1)$ and $\{e_t\}$ is assumed to be symmetric. Let Q_θ^e be a quantile of $\{e_t\}$ and perform some manipulation to the error e_t in the AR-GARCH process. This is by redefine the e_t as;

$$e_t = Z_t + Q_\theta^e \tag{3.22}$$

Where Q_θ^e is as defined previously, and Z_t is a series of exceedances over the threshold Q_θ^e.

Suppose we have the following two models at θ-quantile and $(1-\theta)$-quantile respectively.

$$\left.\begin{array}{c} Y_t = \pi\left(\underset{\sim}{X}_t, \beta_\theta\right) + \sigma\left(\underset{\sim}{X}_t, \alpha\right) Z_t \\ Y_t = \pi\left(\underset{\sim}{X}_t, \beta_{1-\theta}\right) + \sigma\left(\underset{\sim}{X}_t, \alpha\right) Z_t \end{array}\right\} \tag{3.23}$$

From (3.23) and with the assumption that $Q_\theta(Z_t) = 0$, we have;

$Q_\theta(Y_t) = \pi\left(\underset{\sim}{X_t}, \beta_\theta\right) = \beta'_\theta \underset{\sim}{X_t}$, at θ-quantile and $Q_{1-\theta}(Y_t) = \pi\left(\underset{\sim}{X_t}, \beta_{1-\theta}\right) = \beta'_{1-\theta}\underset{\sim}{X_t}$, at $(1-\theta)$-quantile. We define the InterQuantile Autoregressive Range function as;

$$IQARR_\theta = \pi\left(\underset{\sim}{X_t}, \beta_\theta\right) - \pi\left(\underset{\sim}{X_t}, \beta_{1-\theta}\right) \tag{3.24}$$

Substituting $Z_t = e_t - Q^e_\theta \; and \; Z_t = e_t - Q^e_{1-\theta}$ in (3.23) for the respective equations will yield;

$$\left.\begin{aligned} Y_t = \pi\left(\underset{\sim}{X_t}, \beta_\theta\right) + \sigma\left(\underset{\sim}{X_t}, \alpha\right)(e_t - Q^e_\theta) \\ Y_t = \pi\left(\underset{\sim}{X_t}, \beta_{1-\theta}\right) + \sigma\left(\underset{\sim}{X_t}, \alpha\right)(e_t - Q^e_{1-\theta}) \end{aligned}\right\} \tag{3.25}$$

By subtraction we have;

$$0 = \pi\left(\underset{\sim}{X_t}, \beta_\theta\right) - \pi\left(\underset{\sim}{X_t}, \beta_{1-\theta}\right) - \sigma\left(\underset{\sim}{X_t}, \alpha\right)(Q^e_\theta - Q^e_{1-\theta}) \tag{3.26}$$

The interquantile autoregression range function at θ-quantile level $IQARR_\theta$ is defined as $\pi\left(\underset{\sim}{X_t}, \beta_\theta\right) - \pi\left(\underset{\sim}{X_t}, \beta_{1-\theta}\right)$. Making the first two terms on the right-hand side of the equation as the subject of the expression in (3.26) we have:

$$IQARR_\theta = \sigma\left(\underset{\sim}{X_t}, \alpha\right)(Q^e_\theta - Q^e_{1-\theta})$$

Our main concern is to develop interquantile autoregression framework for estimation of volatility. From above, the expression has the volatility function term. By expressing the volatility function in terms of interquantile autoregression range function at θ-quantile level completes the derivation of volatility function. The volatility function will be defined as;

$$\sigma\left(\underset{\sim}{X}_t, \alpha\right) = \frac{IQARR_\theta}{Q_\theta^e - Q_{1-\theta}^e} \tag{3.27}$$

Where Q_θ^e and $Q_{1-\theta}^e$ are the quantiles of the noise term e_t at θ and $1-\theta$ quantile levels respectively. From equation (3.27), the volatility function estimator is required. This calls for the estimation of the conditional quantile autoregression function parameters.

3.6 Parameter Estimation

Let $\mathbf{\Theta} \subset \mathbb{R}^p$ be the parameter space in which the model parameters exist and are defined in and also let $\boldsymbol{C}$ be a compact set in the parameter space $\mathbf{\Theta}$. Such that $\boldsymbol{C} \subseteq \mathbf{\Theta}$ implying that $\boldsymbol{\beta_\theta} \in \boldsymbol{C} \subseteq \mathbf{\Theta}$. Applying the minimization estimation technique to the objective function (3.18), yields the population parameter vector $\boldsymbol{\beta_\theta}$ given by;

$$\boldsymbol{\beta_\theta} = \underbrace{argmin}_{\beta \in \mathbb{R}^p} E\left[\rho_\theta\left(Y_t - \pi\left(\underset{\sim}{X}_t, \beta_\theta\right)\right)\right] \tag{3.28}$$

The check function being convex and applying Lemma 1 implies that the parameter vector $\boldsymbol{\beta_\theta}$ is a unique minimum.

3.6.1 The Existence and Uniqueness of Solution

Lemma 3

Let $\rho_\theta(Y, B) = (Y - B)(\theta - I(Y - B) \leq 0)$ where $B = \pi\left(\underset{\sim}{X}_t, \beta_\theta\right)$ and $(Y_t, X_t) \in \mathbb{R} \times \mathbb{R}^p$ with conditional density function of Y_t on X_t as $f_{X_t}(y)\colon \mathbb{R}^{p+1} \to \mathbb{R}$ and conditional quantile function of Y_t on X_t as $y_\theta = y_\theta(X_t)$. Then;

(i) $$E[\rho_\theta(Y_t,B)|X_t] - E[\rho_\theta(Y_t,y_\theta)|X_t] = \begin{cases} E\left[(Y_t - B)I_{[B,y_\theta]}\middle|X_t\right], \ \forall B \le y_\theta \\ E\left[(B - Y_t)I_{[y_\theta,B]}\middle|X_t\right], \ \forall B > y_\theta \end{cases}$$

(ii) Let $|B - y_\theta| \ge \gamma > 0$. For a suitable lower bound $L(X_t)$ of $f_{X_t}(y)$ on

$[y_\theta - \gamma, y_\theta + \gamma]$. Then;

$$E[\rho_\theta(Y_t,B)|X_t] - E[\rho_\theta(Y_t,y_\theta)|X_t] \ge \frac{L(X_t)\gamma^2}{2}$$

(iii) Assume $f(X_t,y)$ is continuous and nonnegative in the neighborhood of $[x_i, y_\theta(x_i)]$ and let $|B - y_\theta(x_i)| \ge \gamma > 0$ for some x_i, then;

$$E[\rho_\theta(Y_t,B)|X_t = x_i] - E\left[\rho_\theta\left(Y_t, y_\theta(x_i)\right)\middle|X_t = x_i\right] \ge \frac{L\gamma^2}{2}$$

For some constant $L > 0$ which is uniform lower bound of $f_{X_t}(y)$ on $[y_\theta(x_i) - \gamma^*]$

$\forall X_t$ in the neighborhood around x_i and some $\gamma^* > 0$.

Proof

For the proof is the same as the one on Mwita (2003, pg. 66)

3.6.2 Estimator of Parameters

Let $\{X_t, Y_t\}_{t=1}^{n}$be a random sample of size n drawn from the series$\left\{\underset{\sim}{X}_t, Y_t\right\}$. From the random sample $\{(X_1,Y_1),(X_2,Y_2),\dots,(X_n,Y_n)\}$ the sample version of the parameter vector estimator $\boldsymbol{\beta_\theta}$ is given by;

$$\widehat{\boldsymbol{\beta}}_{\boldsymbol{\theta}} = \underbrace{argmin}_{\beta\in\mathbb{R}^p} \ n^{-1}\sum_{t=1}^{n}\rho_\theta\left(Y_t - \pi\left(\underset{\sim}{X}_t\ ,\beta_\theta\right)\right)$$

$$= \underbrace{argmin}_{\beta} \; n^{-1} \sum_{t=1}^{n} \rho_\theta \left(Y_t - \beta'_\theta \underset{\sim}{X}_t \right) \qquad (3.29)$$

This results from the fact that $\left(\underset{\sim}{X}_t \,, \beta_\theta \right) = \beta'_\theta \underset{\sim}{X}_t$. Expanding the equation (3.29) we have;

$$\widehat{\boldsymbol{\beta}}_{\boldsymbol{\theta}} = \underbrace{argmin}_{\beta} \; n^{-1} \sum_{t=1}^{n} \left(\theta - \boldsymbol{I} \left(Y_t \le \beta'_\theta \underset{\sim}{X}_t \right) \right) \left| Y_t - \beta'_\theta \underset{\sim}{X}_t \right|$$

$$= \underbrace{argmin}_{\beta} \; n^{-1} \left[\sum_{t \in \left\{ t : Y_t \le \beta'_\theta \underset{\sim}{X}_t \right\}} (1-\theta) \left| Y_t - \beta'_\theta \underset{\sim}{X}_t \right| + \sum_{t \in \left\{ t : Y_t > \beta'_\theta \underset{\sim}{X}_t \right\}} \theta \left| Y_t - \beta'_\theta \underset{\sim}{X}_t \right| \right] \qquad (3.30)$$

3.7 Conclusion

In this chapter, we discussed on the estimation methodology using the InterQuantile Autoregressive Range (IQAR) method. The objective function has been formulated and its theoretical properties investigated. It was found that, the objective function is; Lipschitz Continuous, Convex, Differentiable and Integrable. The procedure for Volatility estimation using the IQAR method was outlined.

CHAPTER FOUR

ASYMPTOTIC PROPERTIES OF VOLATILITY ESTIMATOR

4.0 Introduction

This chapter gives the asymptotic properties of the volatility estimator, both mathematically and performing a small simulation study.

4.1 Estimators' Asymptotic Properties

4.1.1 Quantile Autoregression function estimator

We consider *consistency and asymptotic normality* of the estimator $\hat{\beta}_\theta$, the assumptions used below are as found in (Mwita, 2010).

4.1.1.1 Consistency

Considering the QAR-GARCH process, the following assumptions are useful in providing surety of consistency of $\pi\left(\underset{\sim}{X}_t, \hat{\beta}_\theta\right)$

A1. (Σ, F, P) is a complete probability space and $\{Z_t, X_t\}, t = 1,2,3, \ldots$ are random vectors on this space.

A2. The function $\pi\left(\underset{\sim}{X}_t, \beta_\theta\right): \mathbb{R}^{k_t} \times C \to \mathbb{R}$ is such that for each $\beta_\theta \in C$ a compact subset of $\mathbb{R}^p, \pi\left(\underset{\sim}{X}_t, \hat{\beta}_\theta\right)$ is measurable with respect to the Borel set C^p and $\pi\left(\underset{\sim}{X}_t, .\right)$ is continuous in C, a.s- $P, t = 1,2, \ldots$, for a given choice of explanatory variables $\{X_t\}$.

A3. i. $E\left(\left[\rho_\theta\left(Y_t - \beta_\theta' \underset{\sim}{X_t}\right)\right]\right)$exists and is finite for each β_θ in C

ii. $E\left(\left[\rho_\theta\left(Y_t - \beta_\theta' \underset{\sim}{X_t}\right)\right]\right)$ is continuous in β_θ.

iii. $\left\{\left[\rho_\theta\left(Y_t - \beta_\theta' \underset{\sim}{X_t}\right)\right]\right\}$ obeys the strong (weak) law of large numbers.

$A4.$ $\left\{n^{-1}E\left\{\left[\rho_\theta\left(Y_t - \beta_\theta' \underset{\sim}{X_t}\right)\right]\right\}\right\}$has identifiable unique maximizer.

Theorem (Consistency)

Under assumptions A1 – A4, $\hat{\beta}_\theta \to \beta_\theta$as $n \to \infty$

Proof

For the proof see White (1994, pg. 75) by using the loss function defined in equation (3.15).

4.1.1.2 Asymptotic Normality

To prove the asymptotic normality of $\hat{\beta}_\theta$, we introduce some extra notation. Let v_t be a $(r \times 1)$ vector of variables that determine the shape of the conditional distribution of $\vartheta_t = \sigma(X_t)Z_t$. Associated with v_t is a set of parameters φ. Denote the density of ϑ_t, conditional on all the past information, as $h_t(\vartheta, \varphi, v_t), \vartheta \in \mathbb{R}$. Here, v_t includes conditional variance and φ, the vector of parameters that define a volatility model. Whenever the dependence on $v_t$and φ is not relevant, we will denote the conditional density of ϑ_t simply by $h_t(\vartheta)$. Let $u_t(\varphi, \beta_\theta, s)$be an unconditional density of $s_t = (\vartheta_t, X_t, v_t)$. Finally, define the operators $\nabla \equiv \frac{\partial}{\partial \beta_\theta}$, $\nabla_i \equiv \frac{\partial}{\partial {\beta_\theta}_i}$, where ${\beta_\theta}_i$ is the i^{th} element of β_θ, and $\nabla_i \pi_t(\beta_\theta) \equiv \nabla_i \pi\left(\underset{\sim}{X_t}, \beta_\theta\right)$ and $\nabla \pi_t(\beta_\theta) \equiv \nabla \pi\left(\underset{\sim}{X_t}, \beta_\theta\right)$.

The following assumptions are important for asymptotic normality.

B1. $\nabla_i \pi_t(\beta_\theta)$is A-smooth *(a function whose derivatives for all desired orders exist and are continuous within the given domain)* with variables A_{it} and functions κ_i, $i = 1,2,\dots,p$. In addition, $max\ _i \kappa_i(d) \leq d$ for small enough.

B2. (i) $h_t(\vartheta)$ is Lipschitz continuous in ε uniformly in . That is for $\vartheta_1, \vartheta_2 \in \vartheta$ and $l \in \mathbb{R}$ we have, $|h_t(\vartheta_1) - h_t(\vartheta_2)| \leq l|\vartheta_1 - \vartheta_2|$ (implying Lipschitz continuous) and $\forall \varepsilon > 0$ there is a $\delta > 0$ s.t $|\vartheta_1 - \vartheta_2| < \delta \rightarrow |h_t(\vartheta_1) - h_t(\vartheta_2)| < \varepsilon$ (implying $h_t$is uniformly continuous in ϑ.)

(ii) For each *t* and $(\vartheta\ , v\ )$, $h_t(\vartheta, \varphi, v)$ is continuous in φ.

B3. For each $t\ \ s, u_t(\varphi, \beta_\theta, s)$ is continuous in (φ, β_θ). (following from the continuity of β_θ

B4. $\{\vartheta_t, X_t\}$ are $\alpha - mixing$ with parameter $\alpha(n)$, and there exist $\Delta < \infty$ and $r > 2$ such that $\alpha(n) \leq \Delta n^{\omega}$ for some $\omega < -\frac{2r}{r-2}$

B5. For some $r > 2$, $\nabla_i \pi_t(\beta_\theta)$ is uniformly r-dominated by functions a_{1t}.

B6. For all t and i , $E\left|sup_\beta A_{it}\right|^r \Delta_1 < \infty$. There exist a measurable functions a_{2t} such that $|u_t| \leq a_{2t}$ and for all t, $\int a_{2t} dv < \infty$ and $\int a_{1t}^3 a_{2t} dv < \infty$.

B7. there exists a matrix A such that;

$$n^{-1} \sum_{t=a+1}^{a+1} E[\nabla \pi_t(\beta_\theta) \nabla' \pi_t(\beta_\theta)] \rightarrow A$$

As $n \rightarrow \infty$ uniformly in a.

Theorem (Asymptotic Normality)

In consideration of our quantile autoregression model, if the estimator is consistent and the axioms B1, B2, B3, B4, B5, B6 and B7 hold, then we have;

$$\sqrt{n}A_n^{-\frac{1}{2}}D_n\left(\hat{\beta}_\theta - \beta_\theta\right) \xrightarrow{d} N(0,1)$$

Where;

$$A_n = \frac{\theta(1-\theta)}{n}\sum_{t=1}^{n} E[\nabla\pi_t(\beta_\theta)\nabla'\pi_t(\beta_\theta)]$$

$$D_n = \frac{1}{n}\sum_{t=1}^{n} E[h_t(0)\nabla\pi_t(\beta_\theta)\nabla'\pi_t(\beta_\theta)]$$

$$and\ \hat{\beta}_\theta = \underbrace{argmin}_{\beta}\ n^{-1}\left[\sum_{t\in\left\{t:Y_t\le \boldsymbol{\beta}'_{\boldsymbol{\theta}}\underset{\sim}{X}_t\right\}}(1-\theta)\ \left|Y_t - \boldsymbol{\beta}'_{\boldsymbol{\theta}}\underset{\sim}{X}_t\right| + \sum_{t\in\left\{t:Y_t> \boldsymbol{\beta}'_{\boldsymbol{\theta}}\underset{\sim}{X}_t\right\}}\theta\ \left|Y_t - \boldsymbol{\beta}'_{\boldsymbol{\theta}}\underset{\sim}{X}_t\right|\right]$$

Proof

To proof that the estimator is asymptotically normal we use the function $[\theta - I(x<0)]$ in place of the function $sign(x) = 2\left[\frac{1}{2} - I_{\{x\le 0\}}\right]$ in Weiss (1991) theorem 3.

4.1.2 Volatility Estimator

In this section we estimate the volatility estimator as in accordance to (3.27). The volatility estimator is given by:

$$\sigma\left(\widehat{\underset{\sim}{X_t}, \alpha}\right) = \frac{\pi\left(\underset{\sim}{X_t}, \widehat{\beta_\theta}\right) - \pi\left(\underset{\sim}{X_t}, \widehat{\beta_{1-\theta}}\right)}{Q_\theta^e - Q_{1-\theta}^e}$$

Theorem: Slutsky's Theorem

Let $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ be p-dimensional sequences with $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} c \in \mathbb{R}$, Then;

(i) $X_n + Y_n \xrightarrow{p} X + c$

(ii) $X_n Y_n \xrightarrow{p} cX$

For continuous function f in X, we have,

(iii) $f(X_n) \xrightarrow{p} f(X)$ as $n \to \infty$

The Theorem is used to show that the volatility estimator is consistent. Since $\pi\left(\underset{\sim}{X_t}, \hat{\beta}_\theta\right)$ is a consistent estimator as discussed in section 4.1.1.1, likewise $\pi\left(\underset{\sim}{X_t}, \hat{\beta}_{1-\theta}\right)$ is consistent. Suppose that $\pi\left(\underset{\sim}{X_t}, \hat{\beta}_\theta\right) \xrightarrow{p}$ a and $\pi\left(\underset{\sim}{X_t}, \hat{\beta}_{1-\theta}\right) \xrightarrow{p}$ b, then from (3.27), the volatility estimator is defined as a function of the quantile functions $\pi\left(\underset{\sim}{X_t}, \hat{\beta}_\theta\right)$ and $\pi\left(\underset{\sim}{X_t}, \hat{\beta}_{1-\theta}\right)$. Therefore, by applying the Slutsky's theorem, the estimator $I\widehat{QAR}R_\theta$ is consistent. That is;

$$I\widehat{QAR}R_\theta = \pi\left(\underset{\sim}{X_t}, \widehat{\beta_\theta}\right) - \pi\left(\underset{\sim}{X_t}, \widehat{\beta_{1-\theta}}\right) \xrightarrow{p} \mathrm{a} - \mathrm{b}$$

This implies consistency of the volatility estimator.

4.2 Simulation Study

A Monte Carlo simulation was carried out with the assumption that the error term e_t is standard normal. For simplicity and illustration purposes, an autoregressive-generalized autoregressive conditional Heteroscedastic process AR (r)-GARCH (p, q), with $r, p, q = 1$ was considered. That is, an AR (1) - GARCH (1, 1) process given by;

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \sqrt{\omega + \alpha_1 Y_{t-1}^2 + \gamma \sigma_{t-1}^2}\, e_t \tag{4.1}$$

Let $\boldsymbol{\Theta}$ be a parameter space as defined earlier, it is assumed that the parameter vectors $\boldsymbol{\beta_\theta}$ and $\boldsymbol{\beta_{1-\theta}}$ are found in the parameter space $\boldsymbol{\Theta}$ that is; $\boldsymbol{\beta_\theta}, \boldsymbol{\beta_{1-\theta}} \in \boldsymbol{\Theta}$. For computation of the Interquantile Autoregressive Range function estimator, parameter vectors, $\boldsymbol{\beta_\theta}$ and $\boldsymbol{\beta_{1-\theta}}$**,** for θ and $(1-\theta)$ conditional quantile autoregressive function estimator respectively are estimated.

For θ-quantile,

$$\widehat{\boldsymbol{\beta}}_{\boldsymbol{\theta}} = \underbrace{argmin}_{\beta \in \mathbb{R}}\, n^{-1} \sum_{t=2}^{n} \rho_\theta \left(Y_t - \left(\beta_{0,\theta} + \beta_{1,\theta} Y_{t-1} \right) \right) \tag{4.2}$$

And for $(1-\theta)$-quantile,

$$\widehat{\boldsymbol{\beta}}_{\boldsymbol{1-\theta}} = \underbrace{argmin}_{\beta \in \mathbb{R}}\, n^{-1} \sum_{t=2}^{n} \rho_\theta \left(Y_t - \left(\beta_{0,1-\theta} + \beta_{1,1-\theta} Y_{t-1} \right) \right)$$

From these parameter vector estimates $\widehat{\boldsymbol{\beta}}_{\boldsymbol{\theta}}$ and $\widehat{\boldsymbol{\beta}}_{\boldsymbol{1-\theta}}$ the conditional quantile autoregressive function estimators at θ and $(1-\theta)$ quantile levels are given by;

$$\left.\begin{aligned}\pi\left(\underset{\sim}{X}_t, \hat{\beta}_\theta\right) &= \hat{\beta}_{0,\theta} + \hat{\beta}_{1,\theta} Y_{t-1} \\ \pi\left(\underset{\sim}{X}_t, \hat{\beta}_{1-\theta}\right) &= \hat{\beta}_{0,1-\theta} + \hat{\beta}_{1,1-\theta} Y_{t-1}\end{aligned}\right\} \tag{4.3}$$

It can be noted that, even though the volatility function of the proposed QAR-GARCH model is defined with the parameters ω, α_1, γ , we shall not estimate these parameters. This is because by interquantile autoregressive range method, it suffices to estimate the parameters $\boldsymbol{\beta}$**.** This can be explained by the fact that the volatility function estimator is defined as a function of the interquantile autoregressive range function.

$$\hat{\sigma}\left(\underset{\sim}{X}_t, \alpha\right) = \frac{\pi\left(\underset{\sim}{X}_t, \hat{\beta}_\theta\right) - \pi\left(\underset{\sim}{X}_t, \hat{\beta}_{1-\theta}\right)}{q_\theta^e - q_{1-\theta}^e} \tag{4.4}$$

4.2.1 Performance Test

In this section of the thesis, a check of the asymptotic properties for the volatility function estimator was carried out. However, only consistency test of the estimator was conducted. The consistency test of the proposed volatility function estimator was performed by using the Average Mean Absolute Proportionate Error.

4.2.2 Average Mean Absolute Proportionate Error

Average Mean Absolute Proportionate Error (AMAPE) is a measure of the accuracy of a method used in estimation of some statistic. It is applicable to methods of estimation founded and formulated on the absolute value framework. For, each sample size reiteration of a hundred sub samples are generated for the consistency test purposes. Mathematically, AMAPE is defined as;

$$AMAPE\left(\hat{\sigma}(X_t,\alpha)\right) = \frac{1}{100}\sum_{i=1}^{100}\left(\frac{1}{n}\sum_{t=1}^{n}\left|\frac{\sigma^i(X_t,\alpha) - \hat{\sigma}^i(X_t,\alpha)}{\sigma^i(X_t,\alpha)}\right|\right) \tag{4.5}$$

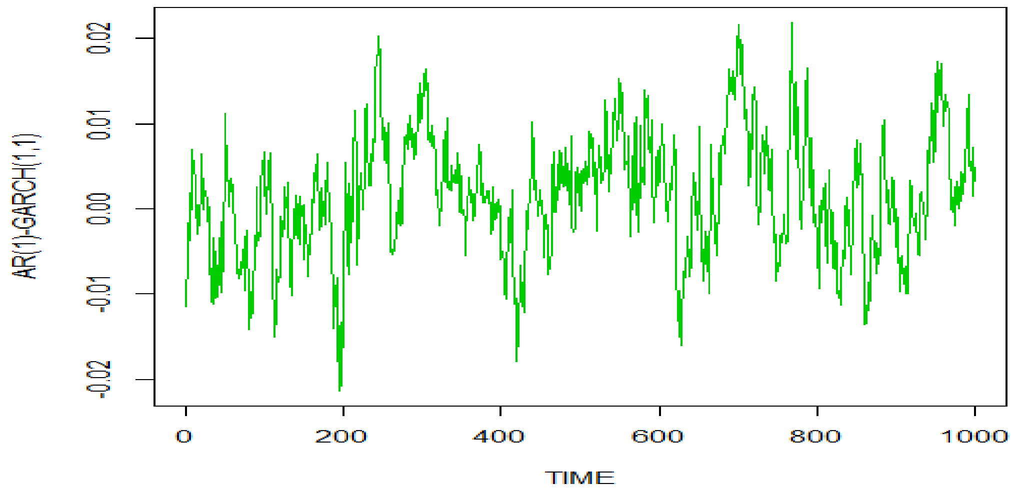

Figure 4.2.1: The simulated AR (1) – GARCH (1, 1) process of size 1000

From the plot (Figure 4.2.1), we see that there is; volatility clustering which is a stylistic features of financial data. Thus the AR (1)-GARCH (1, 1) can model financial phenomenon.

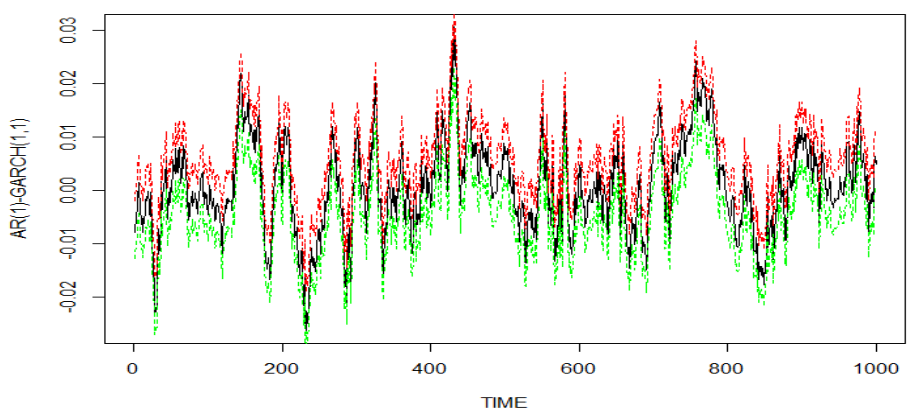

Figure 4.2.2: A simulation study with QAR functions superimposed ($\boldsymbol{\theta = 0.95}$ "red" and $\boldsymbol{\theta = 0.05}$ "green") on AR(1)-GARCH(1, 1) process

In Figure 4.2.2, a plot of simulation data for an AR (1) – GARCH (1, 1) **“black”** process with 0.05 quantile level QAR function **“green dotted line”**, and the 0.95 quantile level QAR function **“red dotted line”** superimposed. It is seen that the QAR functions at $\theta = 0.95\ and\ \theta = 0.05$ follow the same pattern as that of AR(1)-GARCH(1,1) process. This shows that quantile autoregression is a superior and a robust method of modeling time series processes. Volatility was simulated at sample sizes and at different quantiles.

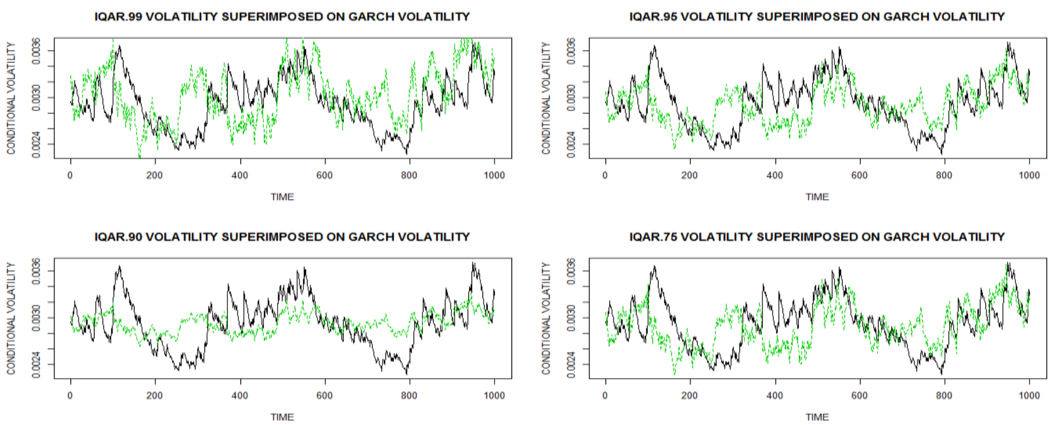

Figure 4.2.3: The Estimated Volatility “green dotted line” at different Quantiles superimposed on the True Volatility (GARCH generated) “black line” for a sample of size 1000

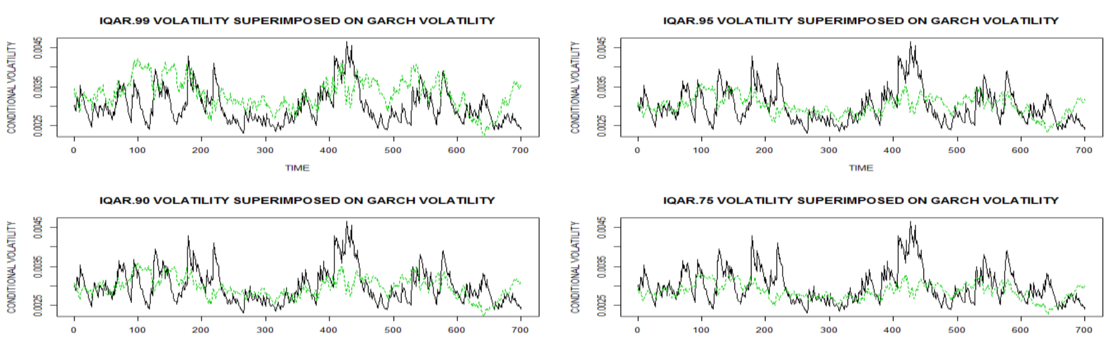

Figure 4.2.4: The Estimated Volatility “green dotted line” at different Quantiles superimposed on the True Volatility (GARCH generated) “black line” for a sample of size 700

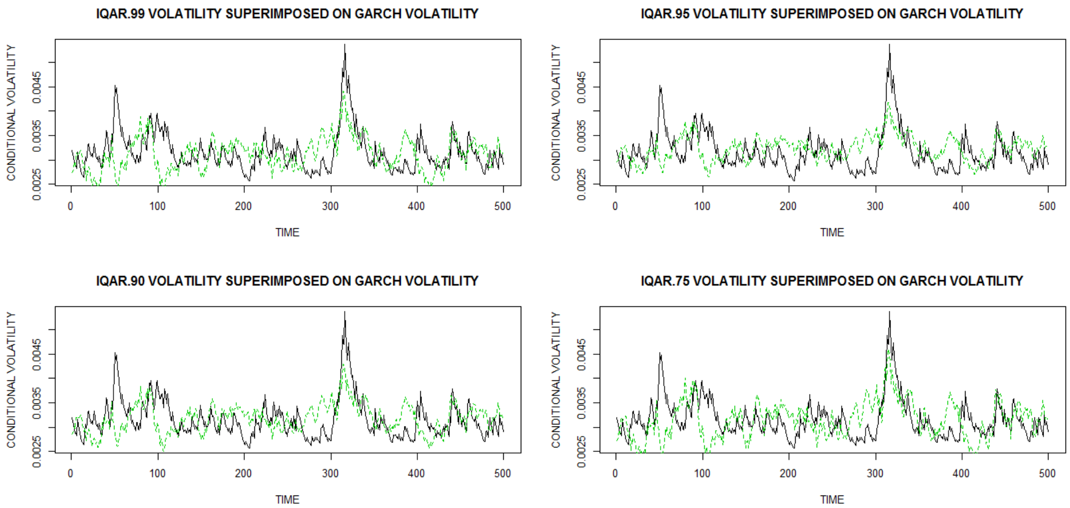

Figure 4.2.5: The Estimated Volatility "green dotted line" at different Quantiles superimposed on the True Volatility (GARCH generated) "black line" for a sample of size 500

From Figures (4.2.3, 4.2.4, 4.2.5), it is seen that the volatility by Interquantile autoregressive range follow the same pattern as the true volatility at different sample sizes and also at different quantile levels. Assuming $e_t \sim iidN(0,1)$ and taking n=500, 700 and 1000 for different quantile levels, it was found that the volatility function estimator converges to the true volatility function as the sample size is increased (refer to Equation 4.5). This is evident in the Table 4.2.1 below;

Table 4.2.1: AMAPE Table

THETA θ-Quantile Level	**SAMPLE SIZE**		
	n=500	n=700	n=1000
0.99	0.1089089	0.0978401	0.0927815
0.95	0.0972144	0.0951942	0.0910251
0.90	0.0930546	0.0902145	0.0873437
0.75	0.0881620	0.0848941	0.0798302

From the simulation results tabulated in Table 4.2.1, it is seen that the AMAPE value is decreasing down each column from (0.99 to 0.75). This can be attributed to inclusion of extreme events or extreme values in the calculation of the AMAPE at high quantile levels unlike for lower quantile levels. And also it is seen that the AMAPE value decreases across each row with increase in the sample size. This implies that, the volatility function estimator is consistent and tends to the true volatility function as the sample size increases, i.e. $\hat{\sigma}\left(\underset{\sim}{X}_t, \alpha\right) \rightarrow \sigma\left(\underset{\sim}{X}_t, \alpha\right)$ as $n \rightarrow \infty$. These results agree with the theoretical results obtained earlier in this thesis that, the proposed volatility function estimator is consistent.

4.3 Conclusion

This chapter has discussed about the estimation of volatility using the interquantile autoregressive range procedure. Theoretically, it was proved that the volatility function estimator is consistent and asymptotically normal. A Monte Carlo simulation study carried out to test for the consistency of the volatility estimator, revealed that the truly the volatility estimator is consistent.

CHAPTER FIVE

APPLICATION OF MODEL TO REAL DATA

5.0 Introduction

Data from the Nairobi Securities Exchange, NSE 20 Share Trade Index for the period 4th January 2000 to 30th December 2013 with 3519 observations were used to estimate volatility using the interquantile autoregressive range method. Also, using the same data volatility was estimated using the well-known GARCH and EWMA models.

5.1 Data Exploration

In this section, data exploration is performed to establish some properties of the data which are useful in data analysis.

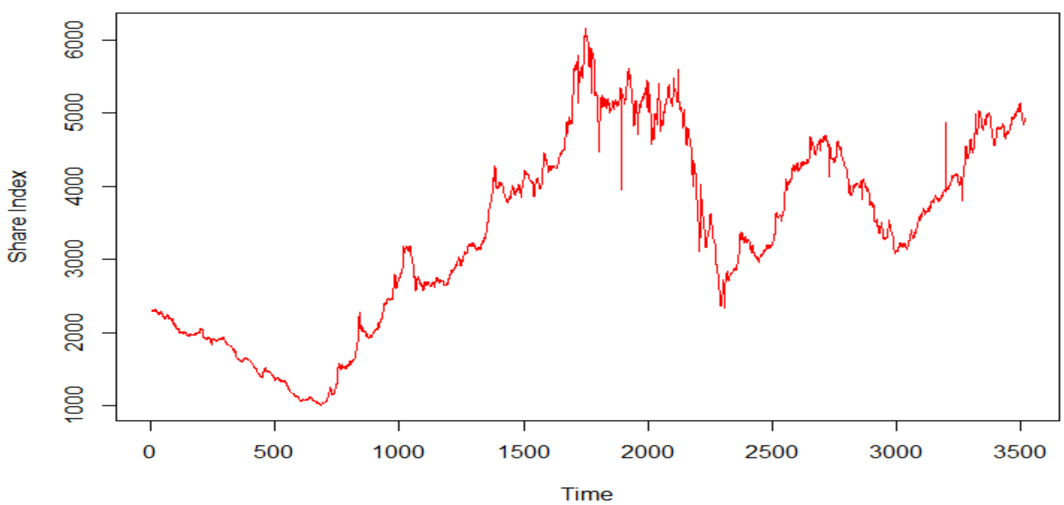

Figure 5.1.1: NSE 20 Share Index

In Figure 5.1.1, it is seen that the NSE 20 Share indices plot is random walk, that is; it exhibit undeterministic trend. Data exploration for such a non-stationary data can be very sophisticated and complex. There are existing methods of manipulating non-stationary to be stationary. In this case, we apply the data transformation technique by converting the indices data into returns using the natural logarithm. Suppose we let P_t be the NSE Trade Share Index price at time t, also let R_t denote the return at time t. Then algebraically R_t in terms of P_t is defined as:

$$R_t = \ln\left(\frac{P_t}{P_{t-1}}\right)$$

From Figure 5.1.1 and Figures (5.1.2, 5.1.3, and 5.1.4) it is seen that the NSE return data has the following stylistic properties;

(i) Returns exhibit low autocorrelation

(ii) High autocorrelation of the absolute values of the return data

(iii) Volatility clustering – volatility changes in time

(iv) Returns are heavy tailed

(v) For high thresholds, exceedances appear in clusters

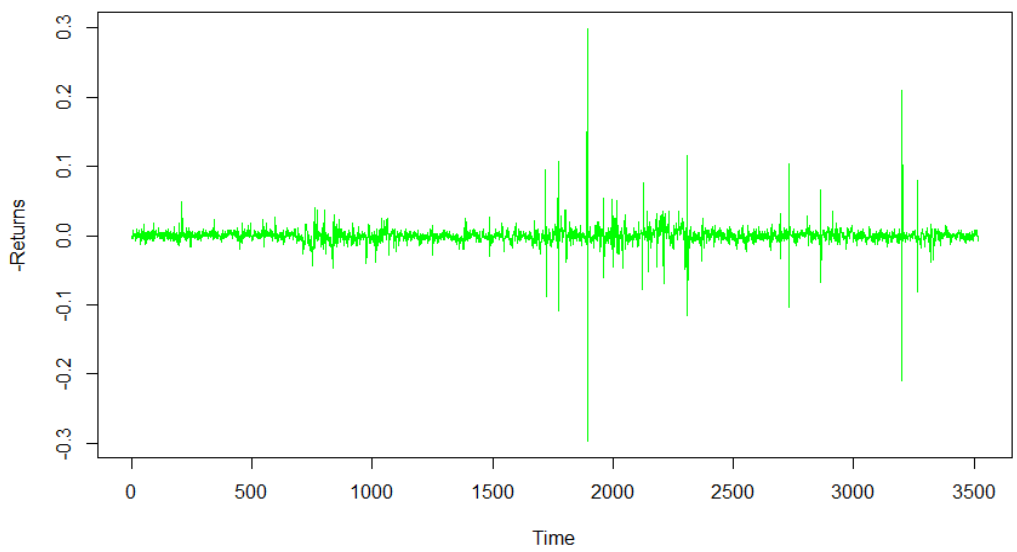

Figure 5.1.2: NSE 20 Trade Share Index Negative Returns plot

The negated version of the returns was used in the empirical study so that the losses which are of interest for financial risk management are positive. Figure 5.1.2, is a plot of the Negative Returns. In the plot (Figure 5.1.2), we see that there is volatility clustering (volatility changes with time). That is, small changes are followed by small changes and high changes are followed by high changes. And also for high thresholds, extremes appear in cluster.

From Table 5.1.1 below, we see that the kurtosis for the negative log returns is much higher than the three for normal distribution $K(x) = 169.7943 > 3$. This implies that the data is heavy tailed hence it is not normal. The Jarque-Bera statistic indicates that the distribution of the negative returns have a tail which is heaver than that of normal distribution by rejecting the null hypothesis that, the negative returns are normal distributed.

Table 5.1.1: NSE 20 trade Share Index Negative Returns Summary Statistic Table

SUMMARY STATISTICS	
Minimum	-0.296400
Mean	-0.000216
Median	-0.000007
Maximum	0.298900
Standard Deviation	0.013639
Kurtosis	169.794300
Skewness	-0.006693
Jarque-Bera Statistic	4230906
Jarque-Bera Probability	< 2.2e-16
ADF Statistic	-13.549800
ADF Probability	0.010000

The skewness $S(x) = -0.0067 < 0$ is slightly less than the zero for normal distribution, this, implies that the data is negatively skewed. Its implication is that, positive returns (profits) occur more often than negative returns (losses) in the NSE 20 share Trade Index for the period of then study. The Augmented Dickey Fuller (ADF) test for stationary reveals that the NSE 20 Share Trade Index negative returns data is stationary.

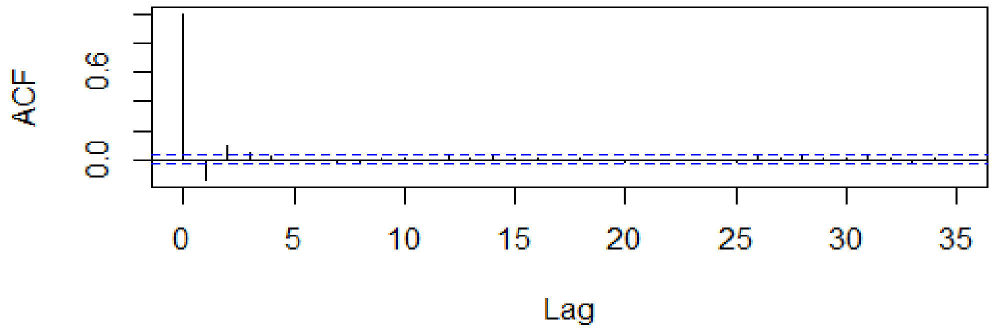

Figure 5.1.3: Autocorrelation function plot

From the plot of Auto Correlation Function (ACF), Figure 5.1.3, it is seen that there is low autocorrelation in the NSE 20 Share Trade Index negative returns. A Quantile – Quantile plot (Q-Q plot) in Figure 5.1.4 shows that the NSE 20 Share Trade Index negative returns are leptokurtic.

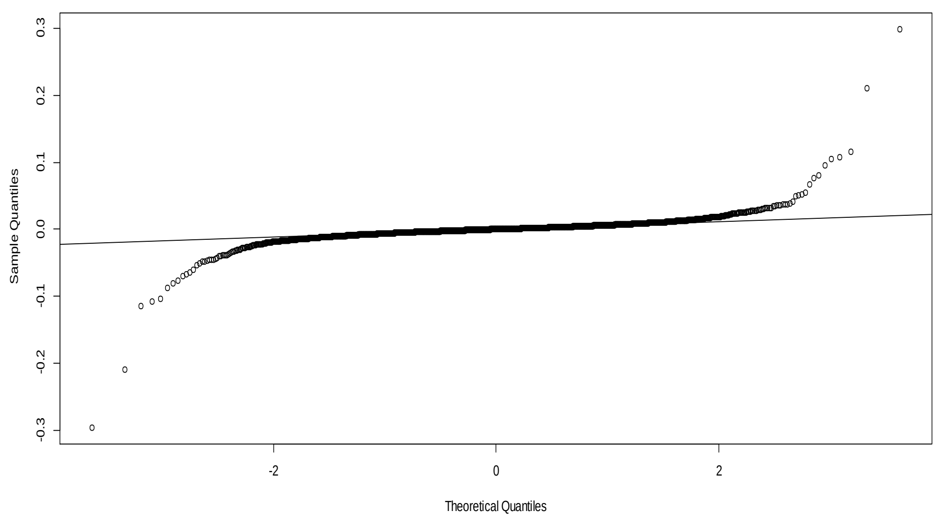

Figure 5.1.4: Quantile-Quantile Plot for the NSE.INDEX.RETURNS

From the quantile-quantile plot it is seen that the returns are heavy tailed. This is because there is concave presence (where the points lie forms a concave curve) at the right hand side of the plot.

5.2 Volatility Estimation

The Conditional Market Risk Volatility was estimated using the transformed data (NSE 20 Share Trade Index Negative Returns) assuming two cases: case one, that the error terms are independently and identically distributed (iid) with student's t-distribution and case two, that the error terms are iid with standard normal distribution. The choice of the t-distribution is that, it has mean zero and a variance of approximately one, when the sample size is sufficiently large. Since the observations of the data used are 3519, by applying the law of large numbers, this sample was categorized as a large sample size. Additionally the choice of Student's t distributed error term is that it is heavy tailed.

By the interquantile autoregression range framework in derivation of the Volatility estimator, we first estimate the symmetric Conditional Quantile Autoregressive functions at θ and $1-\theta$ levels, and then take the range. For instance, estimation at $\theta = 0.95$ we are required to also have a quantile autoregression function at $\theta = 0.05$. The plots in Figure 5.2.3, and 5.2.4, show our volatility estimates got by using the InterQuantile AutoRegression Range method (at $\theta = 0.95$ and $1-\theta = 0.05$) with student's t-distributed error term and $N(0,1)$ distributed error term respectively.

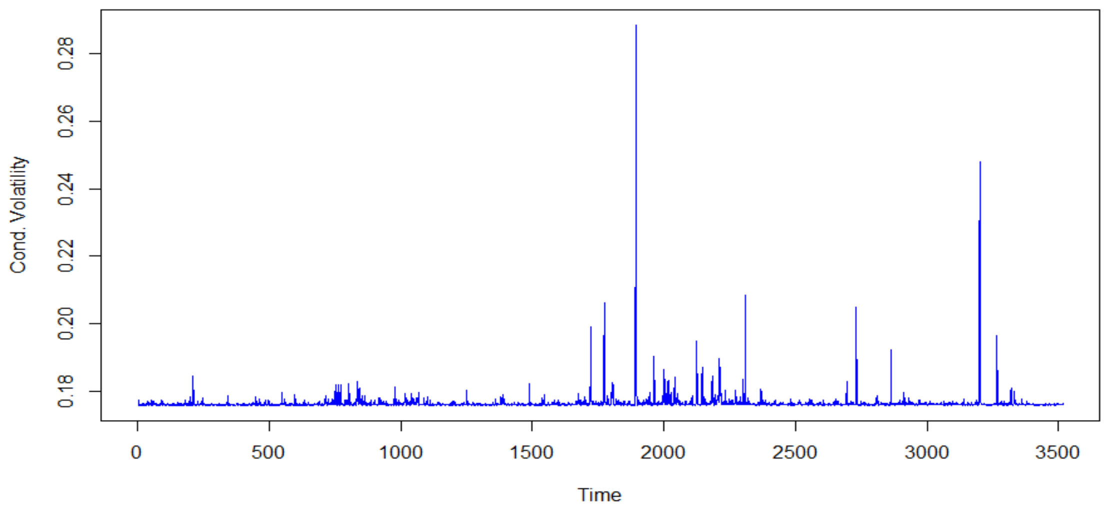

Figure 5.2.1: Conditional volatility estimates by the InterQuantile AutoRegression Range method at Theta = 0.75, with Student's t-distributed error term

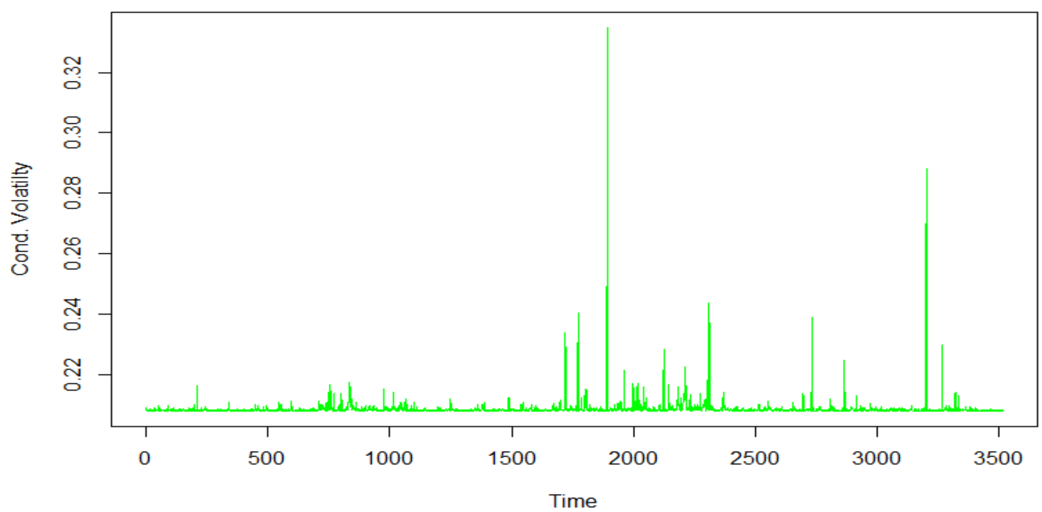

Figure 5.2.2: Conditional volatility estimates by the InterQuantile AutoRegression Range method at Theta = 0.75, with standard normal distributed error term

The choice for $\boldsymbol{\theta = 0.75}$ is intuitive, since it is known that, majority of the data points will lie around the median. Therefore, taking an Interquantile Range around to the median, one shall have more information pertaining majority of the data points. Therefore, the market risk value estimates derived from this range (around the median) yields a better estimate for compared to a range with inclusion of the extremes, for instance, a range of $\boldsymbol{\theta = 0.95}$. Nevertheless, it under estimates incases of extreme occurrences.

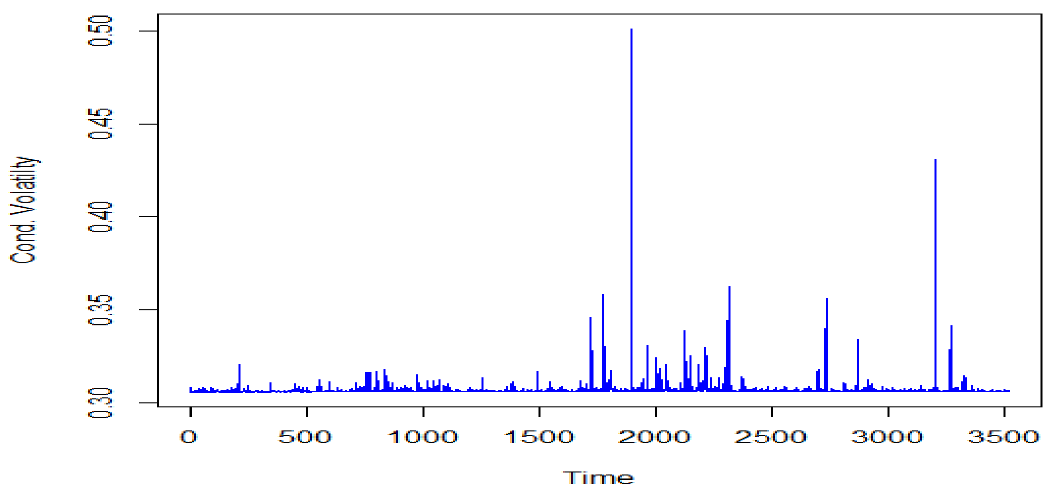

Figure 5.2.3: Conditional volatility estimates by the InterQuantile AutoRegression Range method at Theta = 0.95, with student's t-distributed error term

As seen in Figure 5.2.1 and in Figure 5.2.3 both plots are assuming t-distributed error terms, the estimated risk volatility values are higher for quantile level $\theta = 0.95$ than the estimated risk values at $\theta = 0.75$. This can be explained by the inclusion of extremes in the risk volatility estimation at quantile level $\theta = 0.95$. The same is seen for the plots in Figure 5.2.2 and Figure 5.2.4 where we assumed standard normal distributed error terms.

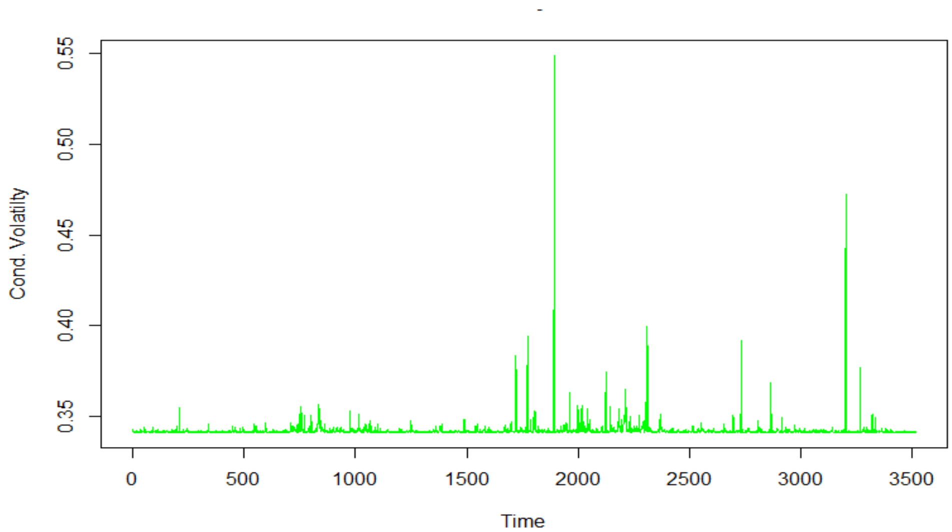

Figure 5.2.4: Conditional volatility estimates by the InterQuantile AutoRegression Range method at Theta = 0.95, with standard normal distributed error term

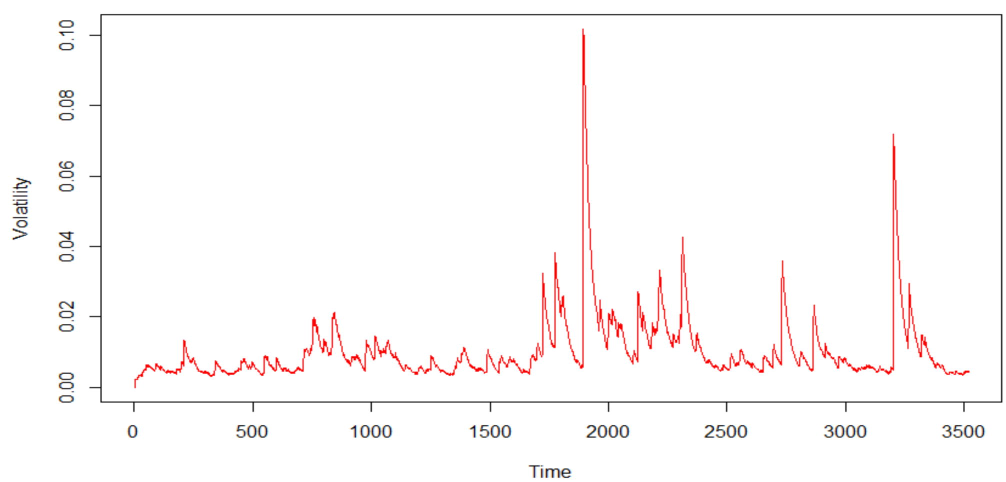

Figure 5.2.5: The NSE 20 Share Trade Index Estimated Exponentially Weighted Moving Average Volatility

To acknowledge other existing methods for market risk volatility estimation we consider the Exponentially Weighted Moving Average (EWMA) method and the Generalized Autoregressive Conditional Heteroscedastic, GARCH (1, 1) model. The respective estimated market risk volatility for the NSE 20 Share Trade Index negative returns are illustrated in Figures 5.2.5 and 5.2.6. The λ value used in the estimation of EWMA volatility is **0.94** as recommended by RiskMetrics.

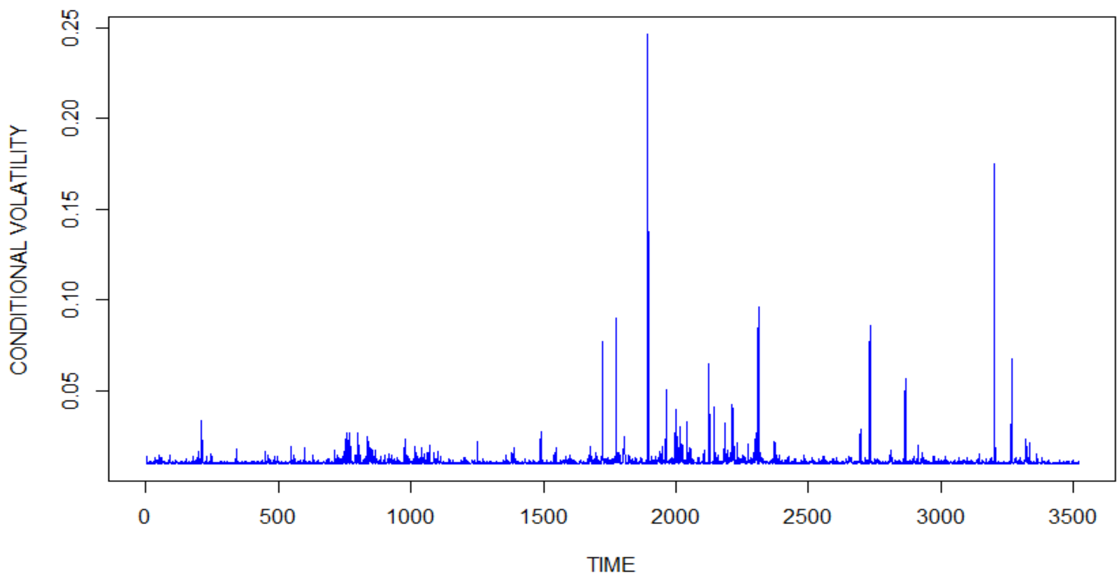

Figure 5.2.6: Estimated Volatility by GARCH (1, 1) for the NSE 20 Share Trade Index

5.3 Conclusion

In this chapter, data exploratory was performed and results discussed. The Interquantile Autoregressive range method was used in the real data in estimation of market risk volatility. Other models for volatility estimation, namely the GARCH and EWMA models, have been used to estimate market risk volatility from the NSE 20 Share Trade Index negative returns.

CHAPTER SIX

CONCLUSION AND RECOMMENDATIONS

6.0 Introduction

The discussions in this chapter are on the conclusions from the thesis and the researchers recommendations for future studies.

6.1 Conclusion

The research has come up with an estimator for estimation of market risk volatility. Since standard deviation has been used as a measure of market risk volatility in various applications in finance, the thesis provides an alternative method for market risk volatility estimation which is based on the interquantile autoregression range framework. It was found out that the quantile autoregression is dynamic hence captures the outliers (robust to outliers) in a time series data. Theoretically, consistency and asymptotic normality properties of the estimator have been derived. A simulation study carried out to check the consistency of the volatility estimator confirms that it is indeed consistent.

The method was applied to real data to estimate market risk volatility. The data which was derived from the NSE 20 Share Trade Index, and after data exploration was done, it revealed that the NSE 20 Share Trade Index negative returns were leptokurtic, negatively skewed for the period of study.

6.2 Recommendations

Financial extremes study is fundamental in managing extreme events which might occur within a financial institution. Most financial returns are asymmetric and thick-tailed in nature. The Extreme Value Theory (EVT) approach provides a framework to study the tail behavior of the fat-tailed distributions which allows for asymmetry by treating the tails of a distribution separately. For instance, in the insurance industry extreme occurrences do occur but in rare cases. The huge claim sizes are associated to extreme occurrences in the industry which may lead to ruin if the company had not prepared. To account for the unexpected extreme event, the occurrence distribution of the extremes and the associated volatility need to be estimated. These occurrences being on extreme quantiles pose a problem for future research using the quantiles approach.

Further investigations can be done on cases where the error distribution is asymmetric to see how the volatility estimator behaves. The method can also be improved by incorporating cases of data censoring, this will be important since in time series data more often than not there exist missing or censored data. Future extension of the methodology can be done on Bayesian quantile autoregression and development of a methodology for forecasting. In thesis, simulation study to check for asymptotic normality for the volatility estimator was not done; this can be done in future research.

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