Temporal and Spatial Analysis of the Black-Scholes Equation for Option Pricing

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Declaration

I, Francis Agana, declare that, to the best of my knowledge, this thesis entitled "*Temporal and Spatial Analysis of the Black-Scholes Equation for Option Pricing*", and the work presented in it are my own. I further confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this noble University.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this proposal is entirely my own work.
- I have acknowledged all the main sources of help.

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Dedication

I would like to dedicate this Research Thesis to the Almighty God for his gifts of life and health and for the abundance of His grace in my life. Secondly, I would also like to dedicate this thesis to my Parents for all their prayers, supports and words of encouragement throughout this academic journey.

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Abbreviations

ABM Arithmetic Brownian Motion

B-S Black-Scholes

BSE Black-Scholes Equation

BSM Black-Scholes Model

FDM Finite Difference Numerical Method

GBM Geometric Brownian Motion

ODE Ordinary Differential Equation

PDE Partial Differential Equation

SDE Stochastic Differential Equation

Abstract

Options are very important financial instruments and, like any other its kind, require very accurate quantitative models in order to serve its usefulness. The Black-Scholes model provides a good option pricing formula in a given market setting. In this work, we quantify the effects of the assumptions of the Black-Schole's financial market on their pricing formula. We expand their model by relaxing these assumptions and derive an option pricing formula in a more realistic financial environment. This requires the introduction of a new hedging strategy and the remodeling of stock prices and therefore the market volatility in the Black-Scholes pricing formula to reflect some activities and consequences of modern exchanges or trading markets. This work is organized such that the numerical treatment of the Black-Scholes' classical linear option pricing model is considered, followed by the derivation of an option pricing equation based on the relaxation of Black-Scholes market assumptions and then, the presentation of a numerical solution of the resulting nonlinear and generalized Black-Scholes model.

Chapter 1

Introduction

1.1 Background

In the world of finance, derivatives and most especially options have become very important and their popularity is ever increasing since their introduction. Whiles they are used by some to hedge against risks, others use options as a strategy to maximize their investments' income. A *derivative* is a financial asset or a contract whose value depends on a much more basic or underlying asset which is usually a stock on stock exchanges. Similarly, an *Option* is a financial contract that gives the holder the right to buy or sell, but not an obligation, a specified amount of asset at a fixed price(strike price) and on a specified date (maturity date). There are different types of options categorized into two main classes: a *call option* which gives the holder the right to buy and a *put option* which gives the holder the right to sell. Options traded in a typical financial market vary significantly from one type to the other and each comes with its unique features. Therefore investors choose the type of option that will favour their portfolio and help them achieve their investment objectives. For example, a *European option* is an option that can be exercised only on the maturity date whiles, on the other hand, an *American option* is an option that can be exercised on or before the maturity date. An *Asian option* is another kind of option whose payoff varies depending on the average price of the underlying asset over a given period of time.

Considering its importance in finance, it is critical and necessary that options are accurately quantified in order for them to serve their intended purposes. As noted in (Fischer Black, 1973), if options are correctly priced then the market will be fair and other important financial information such as volatility can be accurately estimated. To this effect, many option pricing models have been recorded and are still being recorded in literature with the aim of achieving this goal. Among the numerous models we have the famous Black-Scholes Model (BSM) presented by Fischer Black and Myron Scholes in the early 1970s. In their publication (Fischer Black, 1973), Black-Scholes (B-S) derived an option pricing formula based on a number of assumptions made about the financial market in which stocks and options are traded. B-S assumed an efficient and perfectly liquid market where stocks(i.e. the underlying) and European options are traded with no dividend and zero transaction costs payments. They also assumed that the risk-free interest rate and the market volatility of the underlying asset are known and constant. The assumption of an efficient market in the BSM is an important and critical one since it has direct consequences in the modeling of stock prices and therefore on options prices. An efficient market is one where all market prices fully reflect all the available and relevant information and the basis for option pricing by the no-arbitrage argument. Though a bone of contention, the *efficient market hypothesis*, developed in 1970 by Eugene Fama in (Fama, 1970), states that it is impossible to beat the market and that investors cannot predict future stock prices since the current market prices fully represent and reflect all available information making it impossible to outperform the market. Another equally important assumption of the BSM is the assumption of no transaction costs. Trading in financial markets, like any other traditional market, comes with additionally incurred costs in a form of fees, taxes, commissions, brokerage fees and labor costs and they affect the seller as much as they affect the buyer. These transaction costs also have direct effects on the market products and therefore on options prices. B-S also assumed, in the derivation of their pricing formula, that trading is done continuously in a perfectly liquid market. A liquid market is a market where one can buy or sell any quantity of product without affecting its prices. In other words, there are as many buyers as there are as many sellers and therefore the law of supply and demand have little or no price impact on the market.

The mathematical model(BSM) proved to be somewhat useful and consistent in modeling stock prices in their financial market setting. The works of B-S reiterated the importance of Partial Differential Equation (PDE) in option pricing and they showed in (Fischer Black, 1973) that option prices satisfy a given second-order differential equation which came to be known in literature as the Black-Scholes Equation (BSE). It is in fact an equation that is satisfied by options prices based on a non-dividend paying asset in a perfectly liquid and efficient market with no transaction costs. This PDE, because of its numerous assumptions and specific market setting in which it is derived, including the interest rate and the volatility being constant, can be tackled relatively well and an exact solution(option price) derived. However, as shown in (Dupire et al., 1994) and in many other papers recorded in literature, the BSM' assumptions are too strict and more often than not they do not agree with empirical stylized facts (Cont, 2001) and contemporary financial markets. This equation has served as the foundation for the development of many other financial models which incorporate and quantify different types of options under different market conditions. An Exact solution to the BSE can be found when the coefficients(the volatility and the risk free interest rate) are constants and independent of time. However this is not the case in most realistic situations. Interest rate varies with several variables in the market and the volatility is definitely not constant. The relaxation of these assumptions generate, in most cases, nonlinear PDEs which are analytically relatively difficult to tackle. Investors and researchers, therefore, allow the coefficient of the differential equation to vary, most often with time and the underlying asset, in order to derive more rigorous option pricing models and to also make use of efficient numerical algorithms in order to approximate the solutions to the resulting non-linear and generalized BSE.

An in-dept analysis of the BSE has the potential of providing very useful information to an investor who desires to make the most out of every penny invested and to a researcher who intends to make use of the BSE and modify it for one reason or the other. The BSM and thus the BSE forms the underlying model to many other financial models, its analysis will provide useful insight into the quantitative and qualitative analyses of these other models.

1.2 Problem Statement

B-S derived their option pricing equation under very strict market assumptions which do not reflect the reality in modern day exchanges across the world. The efficiency of financial markets is a major topic of discussion in literature and many market players and researchers have divergent views or opinions on it. There are as much publications for the hypothesis as there are against it and, therefore the assumption of an efficient market in the BSM can be considered acceptable or not depending on one's school of thought. What is generally not acceptable is their assumptions of a perfectly liquid market with no transaction costs. Every transaction in a typical modern financial market comes with one fee or the other and most trades are not exempted from tax payments. Brokers charge fees for brokering and/or commissions which can be very expensive depending on the quantity and the value of the transaction involved. Even the most liquid financial markets are not perfectly liquid. Market liquidity has significantly improved over the years with the use of technology but they are far from being perfectly liquid especially in the third world countries. Transaction costs and market liquidity are very important factors that affect options prices directly and indirectly and therefore any option pricing model that ignores their existence or makes strict assumptions about them is subject to very serious shortcomings in their output and in any other model in which they are used.

1.3 Justification of the Study

Option pricing theory is one of the key areas of research in mathematical finance and thus has been the focus of many researchers. This is due to its huge applications and potentials in the management of risks, in the maximization of investment returns and, to many others, due to its importance in the development of other option pricing models. The BSM and hence the BSE served as a breakthrough in the theory of option pricing and has since inspired the development of many other pricing models. The BSE equation in its original form has many deficiencies including the unrealistic market assumptions of perfect liquidity with zero transaction costs. Many attempts have been made, therefore, to extend this equation in order for it to reflect the observed market facts. And the model gets more complicated as it is readjusted to reflect reality and to optimize its output. To reflect reality is also to analyze the behaviour of the BSE subject to many physical occurrences that are unpredictably observed in the market such as the volatility of stocks, large trading and the imperfect liquidity of the market .

Therefore a spatio-temporal study of the classical BSE and the BSM in an imperfectly liquid market with non-zero transaction costs is well posed since it has the potentials to not only provide *analytic* approximations to the solution of the BSE and its extensions but also the potential to provide useful information and insight into the behaviour of the equation subject to large trading in more realistic financial markets.

1.4 Objective of the Study

1.4.1 General Objective

The main objective of the study is to do a *spatial* and a *temporal* analysis of both the classical and linear BSE and the generalized and nonlinear BSE resulting from the relaxation of B-S' assumptions of a perfectly liquid financial market with zero transaction costs.

1.4.2 Specific Objective

In particular we wish to:

i Derive and analytically solve the classical linear BSE with constant volatility and risk free interest rate.

ii Derive a generalized and nonlinear BSE with an adjusted volatility in an imperfectly liquid market with non-zero transaction costs.

iii Numerically solve the classical and linear BSE and analyze the impact of volatility and interest rate on option prices in the classical BSM.

iv Numerically solve the generalized and nonlinear BSE and analyze the effects of transaction costs, large trading and price slippage in an illiquid market.

1.5 Significance of the Study

The study is very significant considering the importance of the BSE in mathematical finance as a whole and particularly in option pricing theory. The analysis of the classical BSE and the generalized BSE under various physical market situations is very significant due to its continuous and wide applications in investment banking and in stock exchange markets. The spatial component of the equation is the underlying asset price while the temporal component is time. The BSE says that the riskless return on a delta-hedged portfolio can be expressed as the sum of the change in derivative value due to time and a term involving the second spatial derivative. For a European call, the change in value of the derivative due to time is negative and reflects the loss in the value of the derivative due to one having less time for exercising the option. Also the second spatial derivative is positive and reflects the gains in holding the call. From the viewpoint of the option issuer, e.g. an investment bank, the second spatial derivative term is the cost of hedging using a delta-hedge. Since the second spatial derivative is the greatest when the asset price of the underlying is near the strike price of the option, the sellers' hedging costs are the greatest in that circumstance. Our spatio-temporal study will therefore give some enlightenment into the changes in the BSE due to time(temporal), due to changes in the underlying asset(spatial) and due to realistic market occurrences. The study will equally bring to bear the impact of the risk-free interest rate, non-constant volatility and other occurrences in the market specifically the presence of transaction costs and large trading with imperfect market liquidity.

1.6 Scope of the study

The study will be limited only to the numerical investigation of both the classical and the generalized BSE. In particular, effects of dependency of market volatility on the option value, underlying asset price and time will be considered. We may for the purpose of clarification and illustration use practical examples and some simulations. We shall not necessarily do the analysis of models inspired by the BSM but derive an option pricing equation that holds in an imperfectly liquid market with non-zero transaction costs by relaxing some of the assumptions made by B-S in the derivation of their famous option pricing equation.

Chapter 2

Literature Review

Options have become increasingly important in finance that practically all investments and financial transactions involve one type of option or the other. Since their introduction, many attempts have been made to accurately quantify or price options and this proved to be a real challenge due to the unpredictable or stochastic nature of options prices. Arguably, the major break-through in the theory of option pricing was recorded in (Bachelier, 1900) when Louis Bachelier[1] derived an option pricing formula based on the notion of Brownian motion[2]. His works, although remained in the dark until the early 1960s, introduced and changed the course of research in the theory of option pricing and greatly influenced the works of many other researchers in the field after him.

In 1973, Fischer Black and Myron Scholes proposed in (Fischer Black, 1973) their famous option pricing model inspired by the achievements of Bachelier and others before them. In their paper they developed a model for pricing European options under very strict market assumptions and showed that option prices satisfy a given

[1]Louis Bachelier (1870-1946) seen as one of the founders of modern financial mathematics

[2]The Brownian motion is named after the Scottish botanist Robert Brown(1773-1858)

second-order PDE which came to be known in literature as the BSE. However powerful the BSM is, its assumptions are too strong and most often than not they do not agree with empirical facts: Facts such as the skewness of returns(Dupire et al., 1994), the importance of transaction costs(Leland, 1985) and dividend paying stocks(Merton et al., 1973) in option pricing, the existence of discontinuities also called *jumps* in the stock market (Cont & Tankov, 2004b) and the impact of large trading in imperfectly liquid markets(Frey, 1998). Therefore the model does not fully explain market prices and lacks very important key component to make it a complete and rigorous option pricing model. Despite these shortcomings, the BSM remains very powerful, both theoretically and practically, and still serves as the foundation and as the benchmark for the assessment of many other option pricing models.

The attempts by many researchers have been to refine the BSM in order to obtain a model that best incorporate realistic market occurrences; therefore generating more accurate option prices. Robert Merton showed in (Merton et al., 1973) that the BSM holds even under weaker assumptions than postulated in (Fischer Black, 1973). In particular, in an alternative derivation of the BSM, he showed that the model holds when the interest rate is allowed to be stochastic, when the stock pays dividend and also when the option can be exercised earlier than the maturity date(i.e. American option). On the other hand, Bruno Dupire attempted to overcome some of the shortcomings observed in the BSM by rewriting the volatility in the model as a function of both time and stock price(Dupire et al., 1994). He developed his famous local volatility model by deriving the implied volatility when option prices are observed in the market to show that the volatility is not constant as assumed in the BSM. Again, Robert Merton in (Merton, 1976) extended the BSM by incorporating *jumps* which, occasionally, are observed in the market. The *jumps* as introduced in (Merton, 1976) are due to many factors including the occurrence of an important and significant political or economic event and also the arrival of an important piece of information that has the potential of causing mild or immense market movements. This *jump* component is often characterized by a "Poisson" process as illustrated in (Cont & Tankov, 2004a). As mentioned earlier, many attempts at extending the BSM by relaxing some of its assumptions have been recorded in literature. For instance Leland observed in (Leland, 1985) that transaction costs invalidates the BSM and proposed a modified replicating strategy from that of the B-S. This new replicating strategy is based on the size of the transaction costs and on the period of revision. His strategy converges to that of the BSM as the period becomes shorter and tending to zero. In (Barles & Soner, 1998) Barles & Soner expounded on the works of Leland in a similar attempt to derive option prices with transaction costs and successfully extended the BSE into a non-linear equation that holds in a market with non-zero transaction costs. The works of (Kyle, 1985; Frey, 1998, 2000) have demonstrated the importance of the B-S' assumption of perfect market liquidity which, when does not hold, jeopardizes the BSM. In a market that is not perfectly liquid, large traders, by their trading, impact on the stock prices and therefore on the prices of options and this should be a cause for concern since most, if not all, financial markets are not perfectly liquid. To this effects, there have been several other attempts at relaxing the B-S' assumption of perfect market liquidity which invariably ends with an extension and a generalization of the BSE. In this regard we shall note key contributions of Frey in (Frey, 1998), that of Frey & Patie (Frey & Patie, 2002) and, equally as important, that of Liu & Yong (Liu & Yong, 2005). Their works focused on creating a replicating portfolio in an imperfectly liquid market and accordingly modify the constant volatility in the BSM in order for it to reflect the true nature of such a market. But little is known in literature on attempts to quantify options in imperfectly liquid markets with non-zero transaction costs. Whatever the case, the theoretical and practical importance of the BSM and therefore the BSE are incontestable.

Most market players do not, however, base their computations on the BSM but rather make use of other generalizations or extensions of the model. This is because they provide more accurate option prices than the BSM and they best incorporate empirical stylized facts some of which are illustrated in (Cont, 2001). Thus the BSM and by extension the BSE gets more and more complicated with numerous attempts at refining the model in order to get the most out of it. The BSE can be solved analytically in its original form by transforming it into the heat equation (Wilmott, Howison, & Dewynne, 1995) (Rogers & Talay, 1997). But this is not the case for some of its numerous extensions and in particular for the non-linear BSE and therefore many researchers have resolved to the use of numerical algorithms to tackle non-linear BSEs and other non-linear option pricing equations. (Company, Navarro, Pintos, & Ponsoda, 2008) & (Düring, Fournié, & Jüngel, 2004) & (Adomian, 1988) provided numerical solutions to both the linear and non-linear BSE. These are but few examples of numerical approaches to solving the BSE.

All these extensive achievements in options pricing will be taken into considerations as we attempt to not only derive an option pricing equation in an illiquid market with non-zero transaction costs but to also solve the equation, possibly, analytically or numerically.

Chapter 3

The Black-Scholes Option Pricing Equation

Derivatives and most especially options form an essential part of the financial world. An *Option* is a contract that gives the holder the right to buy some underlying assets subject to some conditions at a specified price known as the striking price. These days, it is very common to see options based on every tradable and non-tradable commodity being issued or sold. This is because of its importance in efficient risk management and in revenue maximization which forms the bottom line of every profit making organization.

The pricing of options is an essential part of its trading and many efforts have been dedicated to efficiently quantify them. One of such celebrated attempts was presented in 1900 by the French mathematician Louis Bachelier who is credited for being the founder of mathematical finance and the "father" of modern option pricing theory (Courtault et al., 2000). In his thesis (Bachelier, 1900) Louis Bachelier worked extensively on mathematical and probabilistic modeling of financial markets(i.e.

Stocks and options prices) and "initiated" or "developed" the theory of *Brownian motion* in financial modeling long before Albert Einstein's paper on Brownian motion (Einstein, 1905). He assumed that stock prices follow an Arithmetic Brownian Motion (ABM) and thus modeled stock price changes on the Paris Bourse(Paris stock exchange) as;

$$S(t) = S(0) + \sigma W(t) \tag{3.1}$$

where $W(t)$ and σ in equation(3.1) respectively denote the standard arithmetic Brownian motion and the volatility of the stock price. The payoff of a European call option is given by

$$V_c(T) = \max[S(T) - K, 0]$$

where K is the striking price. It is important to note that the options in Paris Stock exchange differed slightly from today's in the sense that the option price was fixed whiles the striking price fluctuated in accordance with the law of supply and demand. This is a complete contrast from today's option formulation but most of the underlying concepts still remain fixed. In order to price a European call option, Bachelier in his famous thesis "*Theorie de la speculation*" (Bachelier, 1900) developed and applied what he calls the "fundamental principle" which has been found to be equivalent to *martingales* as used in modern terminology in the area of mathematical finance. In his settings, the price of a simple contingent claim(i.e. European Call) is given by

$$\begin{aligned} V_c(T) &= E[\max[S(T) - K, 0]] \\ &= \int_{K-S(0)}^{\infty} (x - (K - S(0))) \frac{1}{\sigma\sqrt{2\pi T}} \exp\left(-\frac{x^2}{2\sigma^2 T}\right) dx \qquad (3.2) \\ V_c(T) &= (S(0) - K)\Phi\left(\frac{S(0)K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\phi\left(\frac{S(0) - K}{\sigma\sqrt{T}}\right) \end{aligned}$$

where T is the horizon of the option, σ is the volatility of the underlying and ϕ is the standard normal distribution(i.e. $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$). It is important to note that according to Bachelier's modelling of stock prices in equation(3.1), S is normally distributed with variance $\sigma^2 t$ and mean $S(0)$ which was a very innovative approach at the time.

Bachelier's works, though fundamental and very significant, had its' own shortcomings. On one hand the specifics of options have marginally changed since the 1900s and on the other, the usage of ABM to model stock prices implied that the prices being normally distributed could be negative. In order to remedy these flaws Paul Samuelson, upon discovering the neglected but remarkable thesis of Bachelier, proposed some modifications to the stock price model. In fact, in (P. A. Samuelson, 1965) he introduced the Geometric Brownian Motion (GBM) or the "Economic Brownian motion" as he called it and therefore modeled stock price changes as

$$S(t) = S(0) \exp\left\{ \left(\alpha - \frac{1}{2}\sigma^2 \right) t + \sigma W(t) \right\} \tag{3.3}$$

or better still

$$dS(t) = \alpha S(t)dt + \sigma S(t)W(t) \tag{3.4}$$

where $W(t)$, α and σ are respectively the standard Brownian motion, the expected growth rate(constant) and the volatility(constant) of the stock price.
The attempts at remedying the flaws in Bachelier's thesis and at presenting much better models of stocks and options prices are many (P. A. Samuelson, 1965; P. Samuelson, 1968). But by the late 1960, very significant research and important findings in Stochastic Differential Equation (SDE) had been made thus providing a very good

foundation for the groundbreaking seminal works of Fischer Black, Myron Scholes and Robert Merton (Fischer Black, 1973; Merton et al., 1973)

3.1 The Classical Black-Scholes Equation

In order to derive the BSE, Fischer Black and Myron Scholes with some assistance from Robert Merton formulated a model which was basically an improvement or a perfection of previous attempts at modeling warrants or options as enumerated in their paper (Fischer Black, 1973). This model known in literature as the BSM is based on the concept of pricing by *no-arbitrage*. An arbitrage-opportunity is said to exists in the market when one is able to make certain or sure profit by going long and short positions in assets without taking any risks. It is basically the possibility of making free money out of nothing and pricing by *no-arbitrage* principle ensures that assets are priced in such a way that an arbitrage-opportunity is not created. In any case "arbitrageurs" do exist to take advantage of such opportunities in the markets when they happen which in the long run ensures arbitrage-opportunities are eliminated.

So many pricing models are based on the no-arbitrage principle and the BSM is one of such models. Their pricing model and thus their pricing formula is consistent with absence of arbitrage and does not create such opportunities in the market. The following lemma is one of the fundamental results in stochastic calculus and is very important in handling SDE.

Lemma 3.1.1 ***(Itô's formula)***

Assume that a process X has a stochastic differential given by

$$dX_t = \mu_t dt + \sigma_t dW(t)$$

where μ and σ are adapted processes(i.e. observable) at time t. Let also $f(t, X_t)$ be a function that is twice differentiable. Then f has a stochastic differential given by

$$df(t, X_t) = \frac{\partial f(t, X_t)}{\partial t} dt + \frac{\partial f(t, X_t)}{\partial x} dX + \frac{1}{2} \frac{\partial^2 f(t, X_t)}{\partial x^2} (dX)^2$$

subject to the following rule of multiplication

$$\left\{ \ (dt)^2 = 0 \quad , dt \cdot dW(t) = 0 \quad , (dW(t))^2 = dt \ \right\}$$

3.2 Black-Scholes Model

In order to compute a theoretical fair price of a simple European Call(i.e. an option that gives the holder the right to buy a single share of a common stock) Black and Scholes formulated a model which is consistent with the no-arbitrage principle under the following assumptions:

Assumption 1 ***(Efficient market)***
The prices of products fully reflect all available and relevant information and it is impossible to beat or to outsmart the market and predict future prices of products.

Assumption 2 ***(Non-dividend paying stock)***
Assume that the stock pays no dividend over the life of the option and that trading takes place continuously.

Assumption 3 ***(Frictionless Market)***
Trading takes place continuously in time with no transaction costs while short-selling is also allowed. Thus the dynamics of the stock price is as illustrated in equation(3.4)

Assumption 4 *We can borrow or buy any fraction of the price of a security at the short term interest rate.*

From the assumptions illustrated above, the call option V is therefore dependent on the underlying stock $S(t)$ and on time t and from Itô's lemma, it implies that

$$\begin{aligned} dV(S(t),t) = \left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S} + \frac{\partial V(S(t),t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S(t),t)}{\partial S^2}\right) dt \\ + \sigma S(t)\frac{\partial V(S(t),t)}{\partial S} dW(t) \end{aligned} \tag{3.5}$$

These assumptions make it possible and easy to create a *delta hedged* portfolio(i.e. a portfolio strategy with the aim of canceling the risks associated with price movements in the underlying) by mathematically eliminating the stochastic part of equation(3.5). Thus Black and Scholes formulated a portfolio as follows; Suppose that $S(t) = S$ is the current price of the stock and form a portfolio consisting of a unit of the underlying stock and $\left(-\frac{1}{\frac{\partial V(S(t),t)}{\partial S}}\right)$ units of options. Therefore the value of the portfolio strategy(i.e. hedging portfolio) Π is given by

$$\Pi = S(t) - \frac{V}{\frac{\partial V}{\partial S}}$$

where V is the shorthand notation for $V(S(t),t)$. The total change in the value of the hedging portfolio is thus given by

$$d\Pi = d\left(S(t) - \frac{V(S(t),t)}{\frac{\partial V(S(t),t)}{\partial S}}\right) \implies d\Pi = dS(t) - \frac{dV(S(t),t)}{\frac{\partial V(S(t),t)}{\partial S}}$$

In our subsequent derivations, we shall suppress the parameters (t) of $S(t)$ for clarity of expression. From Itô's formula, the process V has the stochastic differential given by

$$
\begin{aligned}
d\Pi &= dS - \frac{dV}{\frac{\partial V}{\partial S}} \\
&= dS - \frac{\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2}{\frac{\partial V}{\partial S}} \\
&= dS - \frac{\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}(\mu S dt + \sigma S dW(t)) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\mu S dt + \sigma S dW(t))^2}{\frac{\partial V}{\partial S}} \\
&= dS - \frac{\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\left(\mu S dt + \sigma S\left(\frac{dS}{S} - \mu dt\right)\frac{1}{\sigma}\right) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt}{\frac{\partial V}{\partial S}} \\
&= dS - \frac{\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\left(\mu S dt + \sigma S\left(\frac{dS}{S} - \mu dt\right)\frac{1}{\sigma}\right) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt}{\frac{\partial V}{\partial S}} \\
d\Pi &= dS - \frac{\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt}{\frac{\partial V}{\partial S}}
\end{aligned}
$$

$$
d\Pi = -\left(\frac{\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2}{\frac{\partial V}{\partial S}}\right)dt \tag{3.6}
$$

Equation(3.6) does not contain a stochastic component(i.e. the Weinner part has been eliminated). Therefore the portfolio is risk-free and its return must be equal to the return on the risk-free asset else an "arbitrage opportunity" would be created. If we let r be the interest rate on the risk-free asset, then

$$
d\Pi = \left(S - \frac{V}{\frac{\partial V}{\partial S}}\right) r dt \implies -\left(\frac{\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2}{\frac{\partial V}{\partial S}}\right)dt = \left(S(t) - \frac{V}{\frac{\partial V}{\partial S}}\right) r dt
$$

And hence

$$
-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rS\frac{\partial V}{\partial S} - rV
$$

$$\frac{\partial V(S(t),t)}{\partial t} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V(S(t),t)}{\partial S^2} + rS(t)\frac{\partial V(S(t),t)}{\partial S} - rV(S(t),t) = 0 \quad (3.7)$$

Equation(3.7) is known in literature as the BSE which option prices must satisfy. Option prices are solutions to the BSE and in order to solve the above equation, B-S introduced a change of variable in order to transform it into the famous *heat equation* which can be easily solved subject to the boundary conditions (Coppex, 2009).

3.3 Generalized Black-Scholes Equation

The classical BSE(equation 3.7) was derived under very strong market assumptions. As noted by Dupire (Dupire et al., 1994) if the volatility was constant as assumed in the classical BSE, then upon the observation of option prices in the market, the *implied volatility* will be the same for all the observations. This is, however, not the case since the volatility depends on time and on the underlying. Also Leland (Leland, 1985) noted the importance of transaction costs in deriving options prices. In order to remedy this flaw in the BSM the assumption of constant volatility is relaxed and, in order to also maintain the completeness(i.e. the ability to hedge) of the model, the volatility is taken to depend on time t and on the underlying $S(t)$ (Dupire et al., 1994)(Björk, 2004)(Hull, 2012). We therefore assume, as in the BSM, that the underlying stock price follows the general form of a SDE which is given by

$$dS(t) = \mu(S(t),t)dt + \sigma(S(t),t)dW(t) \quad (3.8)$$

where μ, σ are respectively the drift and the diffusion coefficients whiles $W(t)$ is the standard Brownian motion. Then from Itô's formula(lemma(3.1.1)) the process $V(S(t), t)$ has the stochastic differential given as:

$$
\begin{aligned}
dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS(t) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS(t))^2 \\
&= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\left(\mu(S(t), t)dt + \sigma(S(t), t)dW(t)\right) + \frac{1}{2}\sigma(S(t), t)^2\frac{\partial^2 V}{\partial S^2}dt \\
dV &= \left(\frac{\partial V}{\partial t} + \mu(S(t), t)\frac{\partial V}{\partial S} + \frac{1}{2}\sigma(S(t), t)^2\frac{\partial^2 V}{\partial S^2}\right)dt + \sigma(S(t), t)\frac{\partial V}{\partial S}dW(t)
\end{aligned}
$$

where V is a shorthand notation of the $V(S(t), t)$. Let V be an option and lets create a portfolio Π consisting of a single derivative, e.g. a simple European call, and $-\theta$ number of the underlying stock S thus the value at time t of the portfolio is given by:

$$\Pi = V - \theta S(t) \tag{3.9}$$

And hence

$$
\begin{aligned}
d\Pi &= dV - \theta dS(t) \\
&= \left(\frac{\partial V}{\partial t} + \mu(S(t), t)\frac{\partial V}{\partial S} + \frac{1}{2}\sigma(S(t), t)^2\frac{\partial^2 V}{\partial S^2} - \mu(S(t), t)\theta\right)dt \\
&+ \sigma(S(t), t)\left(\frac{\partial V}{\partial S} - \theta\right)dW(t)
\end{aligned} \tag{3.10}
$$

A "*delta hedging*" strategy is considered in order to offset the risks associated with price movements in the underlying asset by eliminating the stochastic part of the process. We therefore let $\theta = \frac{\partial V}{\partial S}$ and hence equation(3.10) generates:

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma(S(t), t)^2\frac{\partial^2 V}{\partial S^2}dt \tag{3.11}$$

The portfolio is therefore risk-free and hence must earn the same rate of return r as other riskless securities such as a government bond hence

$$
\begin{aligned}
d\Pi = r\Pi dt \implies d\Pi &= \frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma(S(t),t)^2\frac{\partial^2 V}{\partial S^2}dt \\
r\left(V - S(t)\frac{\partial V}{\partial S}\right)dt &= \frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma(S(t),t)^2\frac{\partial^2 V}{\partial S^2}dt
\end{aligned}
$$

Whence

$$
\frac{\partial V(S(t),t)}{\partial t} + rS(t)\frac{\partial V(S(t),t)}{\partial S} + \frac{1}{2}\sigma(S(t),t)^2\frac{\partial^2 V(S(t),t)}{\partial S^2} - rV(S(t),t) = 0 \quad (3.12)
$$

Equation(3.12) is referred to as the generalized BSE and it can be observed that it does not feature the drift coefficient $\mu(S_t, t)$ and hence the only variable that impacts on the equation is the generalized volatility which is a function of time t and the underlying stock S. A Special case of equation(3.12) is when $\sigma(S_t, t) = \sigma S$ in which case equation(3.12) becomes the classical BSE(i.e. equation(3.7)).

3.4 Solving The Black-Scholes Equation

Option prices are solutions to the BSE. The price of a simple European call or put option can be derived analytically by solving the classical BSE subject to terminal and boundary conditions. For instance, for a simple European call option, the BSE is given by

$$
\frac{\partial V_c(S(t),t)}{\partial t} + \frac{1}{2}\sigma^2 S(t)^2\frac{\partial V_c^2(S(t),t)}{\partial S^2} + rS_t\frac{\partial V_c(S(t),t)}{\partial S} - rV_c(S(t),t) = 0 \quad (3.13)
$$

subject to the terminal and boundary condition

$$\left.\begin{aligned} V_c(0,t) = 0 \\ V_c(S(T),T) = \max(S(T)-K,0) \\ V_c(S(t),t) \to S \text{ as } S \to \infty \\ V_c(L,t) = L - Ke^{-r(T-t)}, (\forall L \geq S(T-t)) \end{aligned}\right\} \tag{3.14}$$

where K is the strike price, t and T are respectively the current and the expiry time, σ is the volatility of the underlying asset and r the risk-free interest rate. There are various ways of solving the BSE and deriving option prices (Wilmott, Dewynne, & Howison, 1993)(Yue-Kuen, 1998). One approach which we shall illustrate is to reduce the equation, by change of variable, to a general parabolic equation and therefore into a diffusion equation as illustrated in (Coppex, 2009) and (Wilmott et al., 1993). By Fourier transformation method the heat equation is solved completely.

Definition 3.4.1 ***Parabolic PDE***

A second order PDE of the form

$$A\frac{\partial^2 f}{\partial x^2} + 2B\frac{\partial^2 f}{\partial x \partial t} + C\frac{\partial^2 f}{\partial t^2} + D\frac{\partial f}{\partial x} + E\frac{\partial f}{\partial t} + Ff + G = 0 \tag{3.15}$$

is called a parabolic PDE if it satisfies the condition that

$$B^2 - AC = 0$$

Definition 3.4.2 ***Fourier transform***

Let $f(x)$ be a squared integrable function(i.e. $\int_{-\infty}^{\infty} f(x)dx < \infty$). Then the Fourier

Transform $\mathcal{F}(f(x))(\xi)$ of $f(x)$ is defined by

$$\mathcal{F}(f(x))(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx \tag{3.16}$$

where $i^2 = -1$

Given the Fourier Transform of a function $f(x)$, one can derive the function $f(x)$ by applying the inverse Fourier Transform.

Definition 3.4.3 ***Inverse Fourier transform***

Let $\mathcal{F}(f(x))$ be the Fourier transform of the function $f(x)$ as defined above. Then, the inverse Fourier Transform is defined as follows;

$$\mathcal{F}^{-1}(\mathcal{F}(f(x))(\xi))(x) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(f(x))(\xi)e^{i\xi x}d\xi \tag{3.17}$$

where $i^2 = -1$

The following theorem is an important result of Fourier transform.

Theorem 3.4.4 ***(Katznelson, 2004)***

Let $f_1(x)$ and $f_2(x)$ be two functions of a variable x. Then, the Fourier transform of the convolution product of f_1 and f_2 is equal to the product of the fourier transforms of f_1 and f_2. This means that

$$\mathcal{F}[(f_1 * f_2)] = \mathcal{F}[(f_1)]\mathcal{F}[(f_2)] \tag{3.18}$$

where

$$(f_1 * f_2)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x-y)f_2(y)dy \tag{3.19}$$

In a similar derivation of the exact solution of the classical BSE in (Coppex, 2009), let V be the shorthand notation for $V(S(t), t)$ the value of a simple European option. We do the following change of variables:

$$\tau = (T - t)\frac{\sigma^2}{2} \tag{3.20a}$$

$$V(S(t), t) = Kv(x, \tau) \implies \frac{\partial V}{\partial t} = -K\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} \tag{3.20b}$$

$$x = \ln\left(\frac{S}{K}\right) \implies \frac{\partial V}{\partial S} = \frac{K}{S}\frac{\partial v}{\partial x} \tag{3.20c}$$

$$\frac{\partial^2 V}{\partial S^2} = -\frac{K}{S^2}\frac{\partial v}{\partial x} + \frac{K}{S^2}\frac{\partial^2 v}{\partial x^2} \tag{3.20d}$$

By inserting the change of variable of equations(3.20) into equation(3.13), the following result is obtained:

$$\begin{aligned} -K\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} + \frac{1}{2}\sigma^2 S(t)^2\left(-\frac{K}{S^2}\frac{\partial v}{\partial x} + \frac{K}{S^2}\frac{\partial^2 v}{\partial x^2}\right) + rS(t)\frac{K}{S}\frac{\partial v}{\partial x} - rKv(x, \tau) = 0 \\ \frac{\partial v}{\partial \tau} - \frac{\partial^2 v}{\partial x^2} - \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} + \frac{2r}{\sigma^2}v(x, \tau) = 0 \end{aligned} \tag{3.21}$$

Equation(3.21) is of the form of a parabolic equation(3.21) which can be rewritten as:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + a\frac{\partial v}{\partial x} + bv(x, \tau) \tag{3.22}$$

where $a = \left(\frac{2r}{\sigma^2} - 1\right)$ and $b = -\frac{2r}{\sigma^2} = -(a + 1)$.

Let us assume that the solution to equation(3.22) takes the following form:

$$v(x, \tau) = f(\tau)g(x)h(x, \tau) \tag{3.23}$$

Then

$$\frac{\partial v}{\partial \tau} = \frac{\partial}{\partial \tau}(fgh) = \left(\frac{\partial f}{\partial \tau}\right) gh + fg\left(\frac{\partial h}{\partial \tau}\right) \tag{3.24a}$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(fgh) = f\left(\frac{\partial g}{\partial x}\right) h + fg\left(\frac{\partial h}{\partial x}\right) \tag{3.24b}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2}{\partial x^2}(fgh) = f\left(\frac{\partial^2 g}{\partial x^2}\right) h + fg\left(\frac{\partial^2 h}{\partial x^2}\right) + 2f\left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial h}{\partial x}\right) \tag{3.24c}$$

Thus replacing the partial derivatives above into equation(3.22), we derive the following result

$$\begin{aligned}\left(\frac{\partial f}{\partial \tau}\right) gh + fg\left(\frac{\partial h}{\partial \tau}\right) = f\left(\frac{\partial^2 g}{\partial x^2}\right) h + fg\left(\frac{\partial^2 h}{\partial x^2}\right) + 2f\left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial h}{\partial x}\right) \\ +af\left(\frac{\partial g}{\partial x}\right) h + afg\left(\frac{\partial h}{\partial x}\right) + bfgh\end{aligned} \tag{3.25}$$

Let us assume that the functions $f(\tau) = c_1 \exp[\hat{f}(\tau)]$ and $g(x) = c_2 \exp[\hat{g}(\tau)]$, where c_1 and c_2 are constants, satisfy equation(3.25) and hence

$$\begin{aligned}f\left(\frac{\partial \hat{f}}{\partial \tau}\right) gh + fg\left(\frac{\partial h}{\partial \tau}\right) = fg\left(\frac{\partial^2 \hat{g}}{\partial x^2} + \left(\frac{\partial \hat{g}}{\partial x}\right)^2\right) h + fg\left(\frac{\partial^2 h}{\partial x^2}\right) + 2fg\left(\frac{\partial \hat{g}}{\partial x}\right)\left(\frac{\partial h}{\partial x}\right) \\ +afg\left(\frac{\partial \hat{g}}{\partial x}\right) h + afg\left(\frac{\partial h}{\partial x}\right) + bfgh \\ \left(\frac{\partial h}{\partial \tau}\right) = h\left[-\left(\frac{\partial \hat{f}}{\partial \tau}\right) + \frac{\partial^2 \hat{g}}{\partial x^2} + \left(\frac{\partial \hat{g}}{\partial x}\right)^2 + a\left(\frac{\partial \hat{g}}{\partial x}\right) + b\right] + \left(\frac{\partial^2 h}{\partial x^2}\right) + \left(\frac{\partial h}{\partial x}\right)\left[2\left(\frac{\partial \hat{g}}{\partial x}\right) + a\right]\end{aligned} \tag{3.26}$$

Our aim is to obtain an equation similar to the heat equation and in order to achieve this, it is require that the following equalities hold:

$$2\left(\frac{\partial \hat{g}}{\partial x}\right) + a = 0 \implies \hat{g}(x) = -a\frac{x}{2} + k_1 \tag{3.27a}$$

$$-\left(\frac{\partial \hat{f}}{\partial \tau}\right) + \frac{\partial^2 \hat{g}}{\partial x^2} + \left(\frac{\partial \hat{g}}{\partial x}\right)^2 + a\left(\frac{\partial \hat{g}}{\partial x}\right) + b = 0 \implies \hat{f}(x) = \left(b - \frac{a^2}{4}\tau\right) + k_2 \tag{3.27b}$$

Where k_1 and k_2 are constants. Inserting the derived values of f and g into the solution in equation(3.23), we deduce that

$$v(x,\tau) = k_3 e^{-(a^2/4+a+1)\tau} e^{-(a/2)x} h(x,\tau) \tag{3.28}$$

Hence the BSE can be transformed, by the change of variable, into a diffusion equation given as follows

$$\frac{\partial h(x,\tau)}{\partial \tau} = \frac{\partial^2 h(x,\tau)}{\partial x^2}, \text{ where } x \in \mathbb{R},\ \tau \in \left[0, \frac{\sigma^2}{2}T\right] \tag{3.29a}$$

$$S = Ke^x \tag{3.29b}$$

$$\tau = (T-t)\frac{\sigma^2}{2} \tag{3.29c}$$

$$V(S,T) = Ke^{-\left(\frac{a^2}{4}+a+1\right)\tau} e^{-\left(\frac{a}{2}\right)x} h(x,\tau) \tag{3.29d}$$

$$a = \frac{2r}{\sigma^2} - 1. \tag{3.29e}$$

The heat diffusion equation in equation(3.29a) can be solved using the Fourier transforms as defined in definition(3.16) by applying the theorem on the Fourier transform of a convolution product as illustrated in theorem(3.4.4). To do this, let $\mathcal{F}(h(x,\tau)) = \tilde{h}(\xi,\tau)$ be the Fourier Transform of $h(x,t)$ with respect to x. Hence the Fourier transform of equation(3.29a), we have that

$$
\begin{aligned}
\mathcal{F}\left(\frac{\partial h(x,\tau)}{\partial \tau}\right) &= \mathcal{F}\left(\frac{\partial^2 h(x,\tau)}{\partial x^2}\right) \\
\frac{\partial \tilde{h}(\xi,\tau)}{\partial \tau} &= -\xi^2 \tilde{h}(\xi,\tau) \\
\tilde{h}(\xi,\tau) &= \tilde{h}(\xi,0)e^{-\xi^2\tau} \\
\tilde{h}(\xi,\tau) &= \tilde{h}(\xi,0)\tilde{g}(\xi,\tau)
\end{aligned} \tag{3.30}
$$

where $\tilde{g}(\xi,\tau) = e^{-\xi^2\tau}$. It thus follows from the inverse Fourier transform (i.e. definition(3.17)) that

$$
\begin{aligned}
g(x,\tau) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\xi^2\tau}e^{i\xi x}d\xi \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\left[-\tau\left(\xi-\frac{ix}{2\tau}\right)^2-\frac{x^2}{4\tau}\right]}d\xi \\
&= \frac{1}{\sqrt{2\pi\tau}}e^{-\frac{x^2}{4\tau}}\int_{-\infty}^{+\infty} e^{-y^2}dy \\
g(x,\tau) &= \frac{1}{\sqrt{2\tau}}e^{-\frac{x^2}{4\tau}}
\end{aligned} \tag{3.31}
$$

and also that

$$
\mathcal{F}^{-1}(\tilde{h}(\xi,0)) = h(x,0) \tag{3.32}
$$

Hence by the convolution theorem in theorem(3.4.4) we have that

$$
\mathcal{F}(h)\mathcal{F}(g) = \mathcal{F}(f * g) \tag{3.33}
$$

and by the definition of convolution product, we have that

$$
\begin{aligned}
h(x,\tau) &= (h * g)(x,\tau) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(x-\xi,\tau) g(\xi,\tau) d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\tau}} e^{-\frac{(x-\xi)^2}{4\tau}} h(\xi,0) d\xi \\
h(x,\tau) &= \frac{1}{\sqrt{4\tau\pi}} \int_{\mathbb{R}} \exp\left[-\frac{(x-\xi)^2}{4\tau}\right] h(\xi,0) d\xi
\end{aligned}
\tag{3.34}
$$

We can, therefore, apply our solutions above and the boundary and terminal condition to equations(3.29). By our change of variable in equations(3.29), we have that $\tau = 0 \implies t = T$ which corresponds to the expiration date. Thus the payoff of a simple European option is given by

$$
V(S,T) = max[\epsilon(S-K), 0] \tag{3.35}
$$

where ϵ takes the value 1 or -1 and indicates respectively whether it is a call option or a put option. From equation(3.29d) we have that

$$
V(S,T) = Ke^{-\left(\frac{a^2}{4}+a+1\right)\tau} e^{-\left(\frac{a}{2}\right)x} h(x,\tau) \implies h(x,\tau) = \frac{V(S,T)}{K} e^{\left(\frac{a^2}{4}+a+1\right)\tau} e^{\left(\frac{a}{2}\right)x}
$$

And thus

$$
\begin{aligned}
h(x,0) &= \frac{V(K,T)}{K} e^{\left(\frac{a}{2}\right)x} \\
h(x,0) &= \frac{max[\epsilon(S-K),0]}{K} e^{\left(\frac{a}{2}\right)x} \\
h(x,0) &= max\left[\epsilon\left(e^{\left(\frac{a}{2}+1\right)x} - e^{\left(\frac{a}{2}\right)x}\right), 0\right]
\end{aligned}
\tag{3.36}
$$

Replacing $h(x,0)$ in equation(3.36) into equation(3.34), we have that

$$\begin{aligned}
h(x,\tau) &= \frac{1}{\sqrt{4\tau\pi}}\int_{\mathbb{R}} \exp\left[-\frac{(x-\xi)^2}{4\tau}\right] h(\xi,0)d\xi \\
&= \frac{1}{\sqrt{4\tau\pi}}\int_{\mathbb{R}} \exp\left[-\frac{(x-\xi)^2}{4\tau}\right] max\left[\epsilon\left(e^{\left(\frac{a}{2}+1\right)\xi}-e^{\left(\frac{a}{2}\right)\xi}\right),0\right] d\xi \\
h(x,\tau) &= \frac{1}{\sqrt{4\tau\pi}}\int_{\mathbb{R}} \exp\left[-\frac{(x-\xi)^2}{4\tau}\right] \epsilon\left(e^{\left(\frac{a}{2}+1\right)\xi}-e^{\left(\frac{a}{2}\right)\xi}\right) d\xi \\
&= \frac{1}{\sqrt{4\tau\pi}}\epsilon\int_{\mathbb{R}} \exp\left[-\frac{(x-\xi)^2}{4\tau}\right] e^{\left(\frac{a}{2}+1\right)\xi}d\xi - \frac{1}{\sqrt{4\tau\pi}}\epsilon\int_{\mathbb{R}} \exp\left[-\frac{(x-\xi)^2}{4\tau}\right] e^{\left(\frac{a}{2}\right)\xi}d\xi \\
h(x,\tau) &= \epsilon e^{\left(\frac{a}{2}+1\right)\left(x+\frac{a\tau}{2}+\tau\right)}\Phi\left(\epsilon\frac{x+\tau a+2\tau}{\sqrt{2\tau}}\right) - \epsilon e^{\left(\frac{a}{2}\right)\left(x+\frac{a\tau}{2}\right)}\Phi\left(\epsilon\frac{x+\tau a}{\sqrt{2\tau}}\right)
\end{aligned} \tag{3.37}$$

and hence from equation(3.29d), we conclude that

$$\begin{aligned}
V(S,T) &= Ke^{-\left(\frac{a^2}{4}+a+1\right)\tau}e^{-\left(\frac{a}{2}\right)x}h(x,\tau) \\
&= Ke^{-\left(\frac{a^2}{4}+a+1\right)\tau}e^{-\left(\frac{a}{2}\right)x}\left[\epsilon e^{\left(\frac{a}{2}+1\right)\left(x+\frac{a\tau}{2}+\tau\right)}\Phi\left(\epsilon\frac{x+\tau a+2\tau}{\sqrt{2\tau}}\right) - \epsilon e^{\left(\frac{a}{2}\right)\left(x+\frac{a\tau}{2}\right)}\Phi\left(\epsilon\frac{x+\tau a}{\sqrt{2\tau}}\right)\right. \\
&= \epsilon Ke^{x}\Phi\left(\epsilon\frac{x+\tau a+2\tau}{\sqrt{2\tau}}\right) - \epsilon Ke^{-(a+1)\tau}\Phi\left(\epsilon\frac{x+\tau a}{\sqrt{2\tau}}\right) \\
&= \epsilon K\frac{S}{K}\Phi\left(\epsilon\frac{\ln(S/K)+(T-t)(r-\sigma^2/2)}{\sigma\sqrt{(T-t)}}\right) \\
&\quad - \epsilon Ke^{-r(T-t)}\Phi\left(\epsilon\frac{\ln(S/K)+(T-t)(r+\sigma^2/2)}{\sigma\sqrt{(T-t)}}\right) \\
V(S,T) &= \epsilon S\Phi(\epsilon d_1) - \epsilon Ke^{-r(T-t)}\Phi(\epsilon d_2)
\end{aligned} \tag{3.38}$$

We conclude that the fair price of a simple European Option is given by

$$\begin{aligned}
V(S,T) &= \epsilon S\Phi(\epsilon d_1) - \epsilon K e^{-r(T-t)}\Phi(\epsilon d_2) \\
d_1 &= \frac{\ln(S/K) + (T-t)(r-\sigma^2/2)}{\sigma\sqrt{(T-t)}} \\
d_2 &= \frac{\ln(S/K) + (T-t)(r+\sigma^2/2)}{\sigma\sqrt{(T-t)}} \\
\epsilon &= \begin{cases} 1 \text{ For Call option} \\ -1 \text{ For Put option} \end{cases}
\end{aligned} \tag{3.39}$$

Chapter 4

Numerical Treatment of Option Pricing In Financial Markets: A Case Study of Classical Black-Scholes Model

In this chapter, the B-S option pricing equation is subjected to numerical analysis. The spatial and temporal behaviour of the classical BSE are investigated numerically and the results compared with its analytic solutions for both European call and put options. The numerical method employed in the study is based on a semi-discretization finite difference technique known as the method of lines coupled with a fourth order Runge-Kutta-Fehlberg integration scheme.

A derivative is a financial asset whose value depends on the value of a much basic underlying asset and it is usually traded on derivative exchanges around the world or in over-the-counter markets (Hull, 2012). An Option is a derivative asset because it derives its value from the value of another basic underlying asset. In financial markets, an option is a security or a contract that gives the holder the right to trade in a fixed number of assets at a price and on a date that are pre-specified in the contract (Björk, 2004). The pre-specified price is called the strike price or the exercise price and the date, the expiration date or the maturity date. The holder is said to exercise the option if he chooses to transact the trade. A call option earns the owner the right to buy the assets whiles a put option earns the owner the right to sell the assets (Yue-Kuen, 1998). European-style options can only be exercised on the expiration date in contrast to American-style options where the holder can exercise his right anytime before/on the maturity date (Angermann & Wang, 2007). The value of an option is directly dependent on the strike price and on the value of the underlying asset. For instance if the strike price is less than the price of the underlying asset, a holder of a call option will normally choose to exercise the option and then gain by selling the underlying asset at the market price. Otherwise the investor will choose not to exercise his right. On the contrary, a put option will normally be exercised if the strike price is greater than the value of the underlying asset and vice versa. The relationship between the value of the underlying asset, the strike price and the value of an option is summarily captured in the table(4) as shown below:

Options are very important financial instruments and trading in it provides a number of benefits. They can provide some leverage to your investments and they can, as well, protect and enhance an investor's portfolio in rising and falling markets. Regardless of the reasons for trading in options and the strategies employed, it is important to understand that different factors or variables determine the value of an option. The expiration date impacts on the value of an option in that the closer

Moneyness	Call Option	Put Option
In-the-Money-Option	The price of the underlying asset is greater than the strike price of the option	The price of the underlying asset is lesser than the strike price of the option
At-the-Money-Option	price of the underlying asset is equal to the strike price of the option	price of the underlying asset is equal to the strike price of the option
Out-of-he-money-the-Money-Option	Price of the underlying asset is lesser to the strike price of the option	price of the underlying asset is greater than the strike price of the option

Table 4.1: Relationship of the underlying asset to the Strike Price

the latter approaches its' expiration date the less valuable it gets. Also the more volatile an asset is the riskier it is to trade in it or in derivatives based on it but the more volatile an asset is, the more profitable it can is.

In 1973, as mentioned earlier, famous economists Fisher Black and Myron Scholes (Fischer Black, 1973) derived the famous option pricing model and equation for valuing European Plain Vanilla Options with the help of the economists Robert Merton. The main idea behind the formulation of the model was based on the fact that, under certain market assumptions, investors can create a hedging portfolio to eliminate all the risks associated with price movements in the underlying asset (Cox, Ross, & Rubinstein, 1979). In an efficient market with no arbitrage opportunity, a portfolio with zero market risk must have an expected rate of return equal to the risk-free interest rate of a bond. Else it may be possible to make arbitrage profits. The famous BSM formulation establishes an equilibrium condition between the expected return on the option, the expected return on the stock and the risk-free interest rate. The final valuation formula of option prices from the BSM depends on a few observable variables except the volatility parameter(i.e. refer to section(3.4)). Therefore the accuracy of the prices can be ascertained by direct empirical tests with the market (Hull, 2012)(Cont, 2001). It is therefore reasonable to adopt the existing theory and methods of handling PDEs as a fundamental approach to the study of the classical BSE. Several authors (Raul Kangro, 2001)(Courtadon, 1982)(Zvan, Forsyth, & Vetzal, 1998) have employed different numerical methods to tackle the classical BSE and other option pricing equations with varying degrees of accuracy. Bohner et al (Bohner & Sánchez, 2014), for instance, utilized the non-perturbative analytical approach based on Adomain decomposition method (Adomian, 1988) to obtain the valuation of a European-call option. Meanwhile the numerical approximations to option pricing equations of contingent claims are quickly becoming one of the most accepted techniques in derivative security valuation (Rogers & Talay, 1997) since an exact solution may be impossible to obtain in certain instances. One of the most common methodology is the Finite Difference Numerical Method (FDM) for well posed PDEs . We describe our technique to obtaining the numerical solution to the BSE in the following sections.

4.1 Model Problem

Let $V(S(t), t)$ denote the price of an option at time t, where $S(t)$ is the price of the underlying asset at the same time t. The PDE for the classical BSM for option pricing is given by:

$$\frac{\partial V(S(t), t)}{\partial t} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V(S(t), t)}{\partial S^2} + rS(t)\frac{\partial V(S(t), t)}{\partial S} - rV(S(t), t) = 0 \quad (4.1)$$

subject to the following terminal boundary conditions

$$\left.\begin{aligned} V(S,T) &= \max(S(T)-K,0) \\ V(L,t) &= L - Ke^{-r(T-t)}, (\forall L > S) \\ V(0,t) &= 0 \end{aligned}\right\} \text{ (For Call Options)} \tag{4.2}$$

and

$$\left.\begin{aligned} V(S,T) &= \max(K-S(T),0) \\ V(L,S) &= 0, (\forall L > S) \\ V(0,t) &= Ke^{-r(T-t)} \end{aligned}\right\} \text{ (For Put Options)} \tag{4.3}$$

where K is the strike price, L the maximum price attained by the underlying asset, T is the expiry time of the European option, σ the volatility of the underlying asset and r is the risk-free interest rate. It is noteworthy that even though the volatility and the risk-free interest rate are assumed to be constant in the classical framework, empirical evidence shows that these parameters are non-constants. A holder of a European call option will only exercise the option when the asset price is higher than the strike price(i.e. $S(t) > K$) on the expiry date. Otherwise the call option is worthless. The value of a European call option is thus known at maturity, namely it is either 0(i.e. worthless) or $S(T)-K$ which is the gross profit the holder makes from the transaction. The boundary conditions in equation(4.2) follow from economical arguments. If the underlying asset price S is zero then the value of the call option is equally zero. And as the asset price $S(t)$ tends to infinity, there is a high chance that the holder will eventually exercise the option. Therefore the value of the option at any point in time will simply be the current asset price minus the present value of the strike price (Hull, 2012)(Björk, 2004). On the other hand, a European put option is valuable if the strike price is more than the price of the underlying asset otherwise it is worthless, so $V(L, S(t)) = 0 \; \forall S(t) > K$. Hence if the asset price is less than the strike price then the gross profit made by the Put option's holder is $K - S(T)$.

4.2 Analytic Solution

In the previous chapter (i.e. section(3.39)), we derived an analytic solution to the BSE. In this section, we provide an expression for the exact solution to equation(4.1) subject to conditions(4.2) for a European call option and conditions(4.3) for a European put option as illustrated earlier.

Theorem 4.2.1 ***(For A European Call Option)***
The solution $V(S(t), t) = V_c(S(t), t)$ to the classical BSE (i.e. equation(4.1)) subject to the terminal boundary conditions(4.2) is given by

$$V_c(S(t), t) = S(t)\Phi(b_1) - Ke^{-r(T-t)}\Phi(b_2) \tag{4.4}$$

where

$$\Phi(S) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{S} e^{\frac{y^2}{2}} \, dy \tag{4.5}$$

is the cumulative distribution function for the standard normal distribution and

$$b_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad ; \quad b_2 = \frac{\ln(\frac{S}{K}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{4.6}$$

Theorem 4.2.2 ***(For A European Put Option)***
The solution $V(S(t), t) = V_p(S(t), t)$ to the classical BSE (i.e. equation(4.1)) subject

to the terminal boundary conditions(4.3) is given by

$$V_p(S,t) = Ke^{-r(T-t)}\Phi(-b_2) - S(t)\Phi(-b_1) \tag{4.7}$$

where Φ, b_1 *and* b_2 *are respectively given in equation(4.5) and equation(4.6).*

Remark : The results in theorem(4.2.1) and in theorem(4.2.2) can be obtained by first transforming the BSE into the heat equation and solve completely using the Fourier transformation method (Coppex, 2009)(Yue-Kuen, 1998).

4.3 Numerical Procedure

In this section we consider the numerical treatment of the classical BSE. Equation(4.1) subject to conditions(4.2 & 4.3) that may be considered as a terminal boundary problem and then solved numerically using the semi-discretization finite difference method known as the method of lines (Rogers & Talay, 1997)(Burden & Faires, 2010). Let $S(t)$ be the value of the underlying asset at time t. A partition of the spatial interval $0 \leq S(t) \leq L$ into N equal parts is introduced such that the grid size $\Delta S(t) = \frac{L}{N}$ and grid points $S_i(t) = (i-1)\Delta S(t)$, $1 \leq i \leq N+i$. Here L is the maximum price attained by the underlying asset(i.e. the spatial variable). Let S be the shorthand notation of $S(t)$. The discretization is based on a linear Cartesian mesh and uniform grid on which finite differences are taken. The first and second spatial derivatives in equation(4.1) are approximated by the second order central finite differences. Consider the Taylor series expansion below:

$$V(S_{i+1},t) = V(S_i,t) + \frac{\partial V(S_i,t)}{\partial S}(\Delta S) + \frac{1}{2}\frac{\partial V(S_i,t)}{\partial S^2}(\Delta S)^2 + 0(\Delta S)^3$$

$$V(S_{i-1},t) = V(S_i,t) - \frac{\partial V(S_i,t)}{\partial S}(\Delta S) + \frac{1}{2}\frac{\partial V(S_i,t)}{\partial S^2}(\Delta S)^2 - 0(\Delta S(t))^3$$

hence

$$\begin{aligned} V(S_{i+1},t) - V(S_{i-1},t) &= 2\frac{\partial V(S_i,t)}{\partial S}(\Delta S) \\ \frac{\partial V(S_i,t)}{\partial S} &= \frac{V(S_{i+1},t) - V(S_{i-1},t)}{2(\Delta S)} + 0(\Delta S)^3 \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} V(S_{i+1},t) + V(S_{i-1},t) &= 2V(S_i,t) + \frac{\partial V(S_i,t)}{\partial S^2}(\Delta S)^2 \\ \frac{\partial V(S_i,t)}{\partial S^2} &= \frac{V(S_{i+1},t) - 2V(S_i,t) + V(S_{i-1},t)}{(\Delta S)^2} + 0(\Delta S)^3 \end{aligned} \tag{4.9}$$

Let us assume that $V_i(t)$ is the approximation of $V(S_i,t)$, then the semi-discrete system for our problem in question becomes

$$\frac{dV_i(t)}{dt} + \frac{1}{2}\sigma^2 S_i^2 \frac{V_{i+1}(t) - 2V_i(t) + V_{i-1}(t)}{(\Delta S)^2} + rS_i \frac{V_{i+1}(t) + V_{i-1}(t)}{\Delta S} - rV_i(t) = 0 \tag{4.10}$$

with the following terminal boundary conditions;

$$\left.\begin{aligned} V_i(T) &= \max(S_i - K, 0), \text{ For Call Option} \\ V_i(T) &= \max(K - S_i, 0), \text{ For Put Option} \end{aligned}\right\}, \; 1 \le i \le N+1 \tag{4.11}$$

We now modify the equations corresponding to the first and the last grid points to incorporate the boundary conditions as follows:

$$\left.\begin{aligned} V_1 &= 0 \\ V_{N+1} &= L - Ke^{-r(T-t)} \end{aligned}\right\} \text{(For Call Options)} \tag{4.12}$$

and

$$\left.\begin{aligned} V_1 &= Ke^{-r(T-t)} \\ V_{N+1} &= 0 \end{aligned}\right\} \text{(For Put Options)} \tag{4.13}$$

There is, therefore, only one independent variable in equation(4.10) so the equation is a first Ordinary Differential Equation (ODE) with known terminal conditions. The resulting problem can be easily solved iteratively using the fourth order Runger-Kutta-Fehlberg integration technique (Rogers & Talay, 1997)(Cartwright & Piro, 1992) programmed using Maple. From the process of numerical computation, the effect of volatility and risk-free interest rate on the spatio-temporal structure of option price value are obtained and presented graphically as follows:

Table 4.2: Comparison of the Results Obtained for the Valuation of a European Call Option for $K = 40, L = 65, r = 0.05, \sigma = 0.324366, T = 3/4, 1/2, 1/4, 1/6, 1/12$

Expiry Time (T)	**Bhoner et al. (Bohner & Sánchez, 2014)**	**Numerical Solution**	**Exact Solution**	**Relative Error (%)**
9 months	—	26.4643314	26.6424145	0.66842
6 months	—	25.9815452	26.0397637	0.22358
3 months	25.4965	25.4934348	25.4993226	0.02309
2 months	25.3319	25.3295481	25.3321022	0.01008
1 month	25.1663	25.1650698	25.1663199	0.00496

Table 4.3: Comparison of the Results Obtained for the Valuation of a European Put Option for $K = 40, x = 25, r = 0.05, \sigma = 0.324366, T = 3/4, 1/2, 1/4, 1/6, 1/12$

Expiry Time (T)	**Numerical Solution**	**Exact Solution**	**Relative Error (%)**
9 months	13.5997714	13.7593793	1.16000
6 months	14.0471639	14.0833176	0.25671
3 months	14.5139067	14.5067172	0.04956
2 months	14.6737878	14.6683086	0.03735
1 month	14.8358189	14.8336802	0.01442

Table 4.4: Effects of Increasing Volatility and Interest Rate on the Valuation of European Call Option where $K = 40, L = 65, T = 1/12$

Volatility(σ)	**Interest Rate (r)**	**Numerical Solution**	**Exact Solution**	**Relative Error(%)**
0.1	0.05	25.1650698	25.1663199	0.00497
0.3	0.05	25.1650698	25.1663199	0.00497
0.5	0.05	25.1650698	25.1669735	0.00756
0.1	0.07	25.2293829	25.2326541	0.01296
0.1	0.09	25.2922439	25.2988771	0.02622
0.1	0.1	25.3230575	25.3319483	0.03510

Table 4.5: Effects of Increasing Volatility and Interest Rate on the Valuation of European Put Option where $K = 40, x = 25, T = 1/12$

Volatility(σ)	**Interest Rate (r)**	**Numerical Solution**	**Exact Solution**	**Relative Error(%)**
0.1	0.05	14.8319779	14.8336800	0.01147
0.3	0.05	14.8343137	14.8336800	0.00427
0.5	0.05	14.8490424	14.8344402	0.09844
0.1	0.07	14.7639201	14.7673459	0.02320
0.1	0.09	14.6953982	14.7011221	0.03894
0.1	0.1	14.6610523	14.6680517	0.04772

4.4 Results and Discussion

In order to gain insight into the valuation of options using our numerical experiment on the classical Black-Scholes PDE, some realistic parameter values for volatility (σ), strike price (K), risk-free interest rate (r) and the expiry time T are used. Similarly, the same parameter values are utilized to compute the exact solution of the equation. Tables (4.2) and (4.3) show a comparison between the numerical solution and the exact solution for the European call and put options price at different expiry dates.

Generally, the relative errors are small. In addition, a comparison of our numerical results for a European call option in table (4.2) with the one obtained in (Bohner & Sánchez, 2014) which uses the Adomian decomposition method (Adomian, 1988) shows a good output of the numerical procedure especially when the expiry time becomes smaller. It is noteworthy that the option valuation results in tables (4.2) and (4.3) confirm the accuracy of our numerical procedure. Figures(4.1) and (4.2) illustrate the spatio-temporal behaviour of European call and put options. It is observed that an increase in the underlying asset price causes the price of a call option to increase and that of a put option to decrease as expected in accordance with option pricing theory. Meanwhile, the time value depends on where the underlying asset price is in relation to the option's strike price at the expiry date and the option can be in, out or at the money. The option is at the money, when the strike price of the option is equal to the current price of the underlying asset and it has zero intrinsic value as shown in the figures (4.1) and (4.2). Figures (4.3) and table (4.4) illustrate the effect of volatility on the call option. The value of the call option is increased by high volatility. Volatility is the property of a stock that describes its tendency to undergo price changes. More volatile stocks undergo larger or more frequent price changes. Volatility is of interest to options traders because it is a vital factor in determining the market price of stock and thus options. When volatility is higher, the call option is more likely to end up in-the-money or out-of-the-money. Moreover, when it ends up in-the-money, it is likely to be over the strike price by a greater amount. With high volatility, movements in the stock price are big; both upward movements and downward movements. If the stock moves up by a huge margin, the call option holder will benefit greatly. Option buyers make money when stocks undergo significant price changes. Because volatile stocks are much more likely to undergo large price changes, option buyers pay a much higher premium for options of volatile stocks. In a similar trend, it is important to note that the value of a put option also increases with high volatility as clearly demonstrated in figure (4.4) and table (4.5). Moreover, call options value rises when interest rate rises as shown in figure (4.5) and table (4.4). This may be attributed to the increase in the interest rate component of the BSM. Bear in mind that the risk free interest rate is the opportunity cost of investing in other financial instruments such as stocks or options. The higher the interest rates, the higher the interest income would be. This makes the call option more attractive and more expensive. For put options, the opposite holds true as shown in figure (4.6) and table (4.5), that is, the higher the interest rates the lower the put option price. This is because if interest rates are high the holder will have to hold the asset for a longer time to deliver it under the put option. Simply selling the asset and using the proceeds to invest at a higher rate would be a better option. This makes the put option less attractive and hence less costly when interest rates are high. Figures (4.7) and (4.8) depict the effects of increasing strike price of the underlying asset on the option values. A decrease in the value of European call option is observed as the strike price increases. This is expected, since the strike price may become higher than the price of underlying asset, hence, making call option unattractive. The revise is the case of a put option value, an increase in the strike price increases the value of a European put option since this will guarantee a gross profit, hence, making the call option more attractive and more expensive.

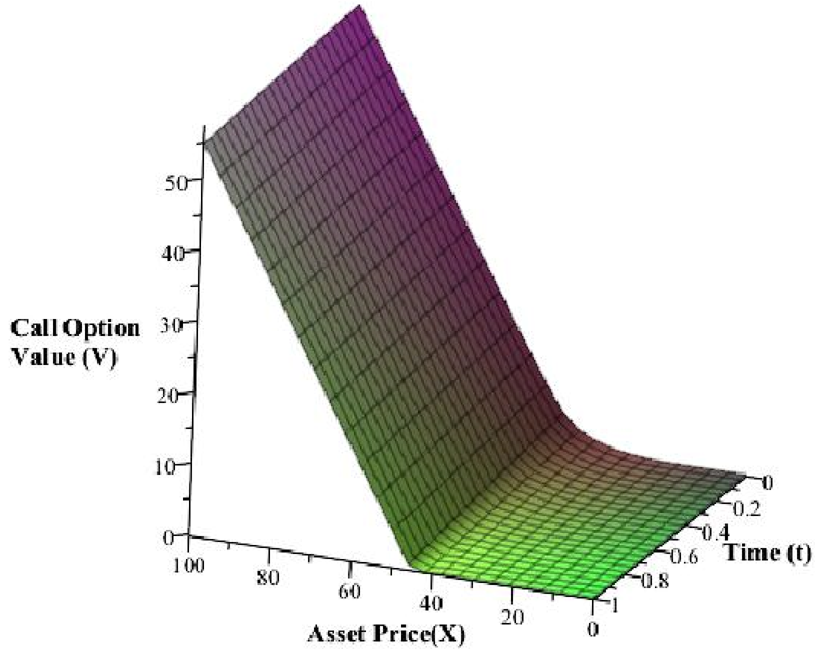

Figure 4.1: Call Option Value for $\sigma = 0.1$, $r = 0.05$, $T = 1$ year, $K = 40$, $L = 100$

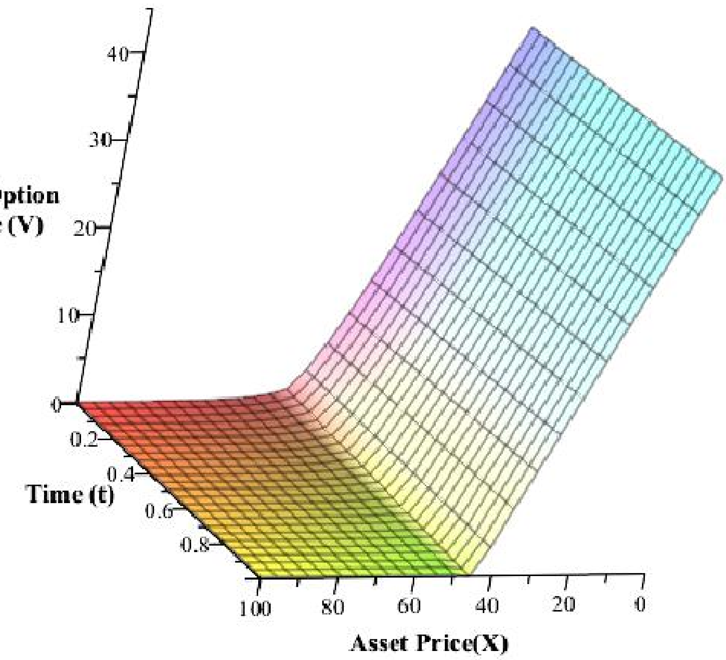

Figure 4.2: Put Option Value for $\sigma = 0.1$, $r = 0.05$, $T = 1$ year $K = 40$, $L = 100$

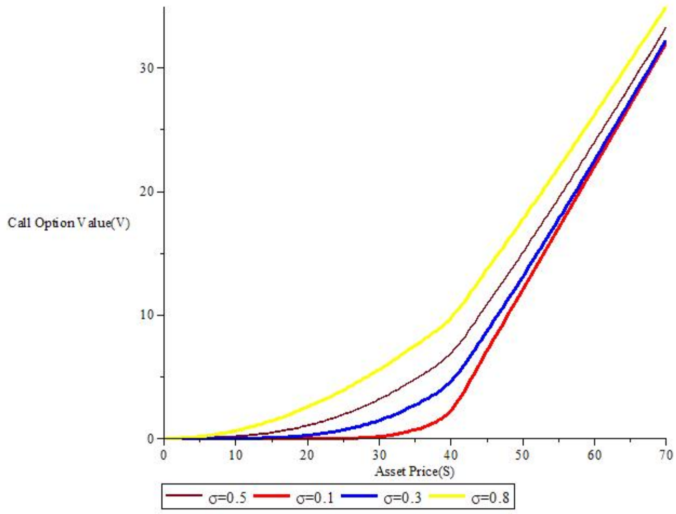

Figure 4.3: Effects of Increasing Volatility(σ) on Call Option: $r = 0.05$, $T = 1$ year $K = 45$, $L = 100$

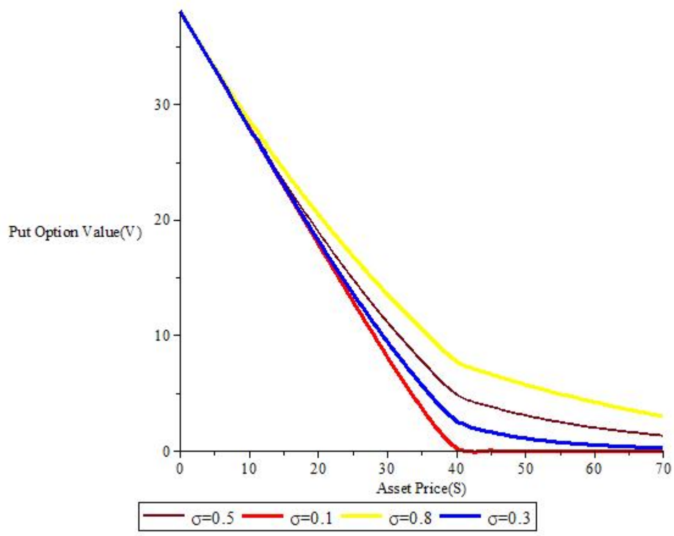

Figure 4.4: Effects of Increasing Volatility(σ) on Put Option: $r = 0.05$, $T = 1$ year $K = 40$, $L = 100$

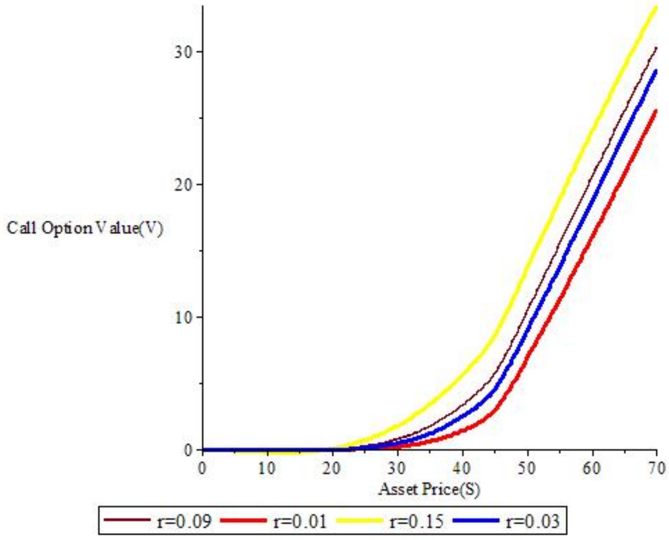

Figure 4.5: Effects of Increasing Interest Rate(r) on Call Option: $\sigma = 0.2$, $T = 1$ year $K = 40$, $L = 100$

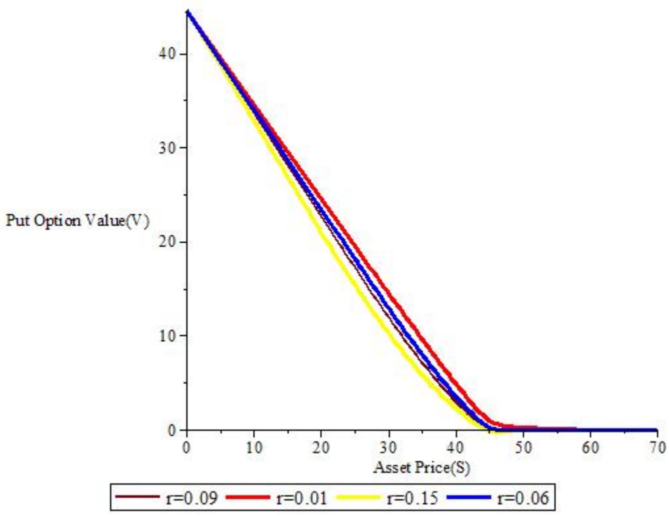

Figure 4.6: Effects of Increasing Interest Rate(r) on Put Option: $\sigma = 0.2$, $T = 1$ year $K = 45$, $L = 100$

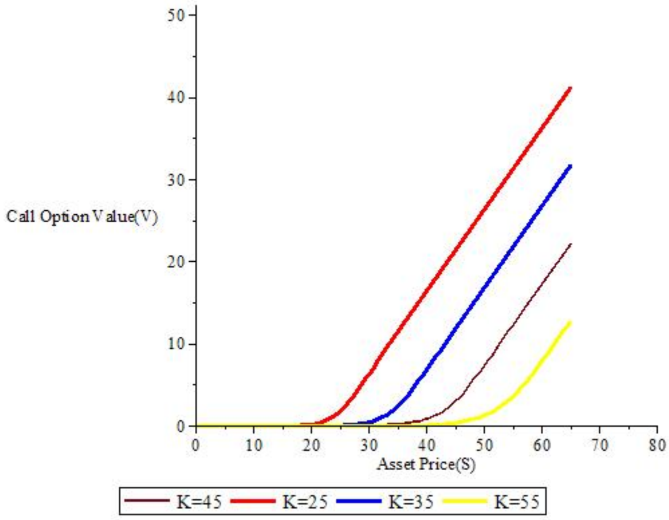

Figure 4.7: Effects of Increasing Strike Price(K) on Call Option: $\sigma = 0.02$, $r = 0.05$, $T = 1$ year, $L = 100$

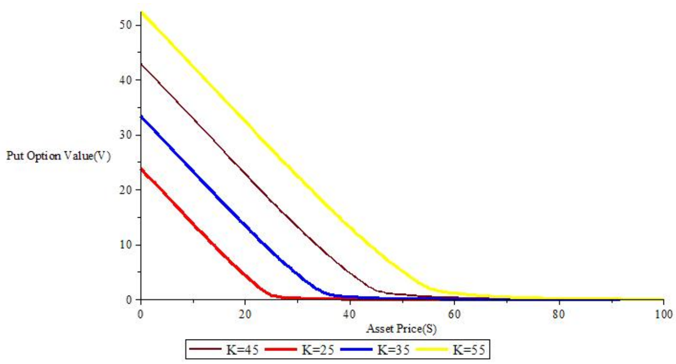

Figure 4.8: Effects of Increasing Strike Price(K) on Put Option: $\sigma = 0.1$, $r = 0.05$, $T = 1$ year, $L = 100$

Chapter 5

Numerical Treatment of A Generalized Black-Scholes Model For Option Pricing in Financial Markets

The classical BSE was derived under strict and unrealistic market assumptions. The existence of transaction costs, large traders and price slippage in illiquid financial markets impact heavily on option prices and therefore cannot be ignored. Particularly, the existence of non-zero transaction costs and large traders in imperfectly liquid financial markets require the implementation of very powerful and dynamic hedging strategies with some feedback effects on the market volatility and drift and therefore on the prices of option . In such scenarios, the underlying asset needs to be remodeled to incorporate these feedback effects and an adjusted volatility and drift introduced to accurately quantify options prices. This, invariably, leads to a nonlinear Black-Scholes PDE with an adjusted volatility which is a function of a second derivative of option prices. In this chapter, our aim is to derive an option pricing equation inspired by the BSM which holds in an imperfectly liquid market with large traders and non-zero transaction costs and to numerically tackle the resulting non-linear PDE using a semi-discretization finite difference method known as the method of lines.

The BSE (Fischer Black, 1973) for option pricing was derived under the assumption of a frictionless and perfectly liquid market and therefore only suitable in a perfectly liquid financial market where there are no transaction costs such as fees or taxes. The model was equally derived under the assumptions that trading takes place continuously, interest rate for borrowing and lending are equal, all parties have immediate access to any information, ask price is equal to bid price and credits of any size are available at anytime to an investor. As Barles & Soner (Barles & Soner, 1998) observed, the assumption of continuous trading in a market with non-zero transaction costs would be ruinously very expensive therefore invalidating the B-S arbitrage argument. Empirical evidence shows that large trading on stocks for various reasons have non negligible impact on stock prices and therefore on options prices and on the costs of replication (Frey, 1998).
The BSM provides not only a rational option pricing formula but also a hedging portfolio that replicates the contingent claim under strict market assumptions. However, in real life situations, these assumptions are never fulfilled. In arbitrage pricing, the risk tolerance of an investor is irrelevant because the latter is not exposed to any risk. If an opportunity exists, investors with different levels of risk aversion would follow the same strategies to profit from the arbitrage (Yue-Kuen, 1998). On the

other hand, in the preference based approach, an investor follows a trading strategy that optimizes his or her returns. So the value of an option is the expected cost of creating and hedging a portfolio that replicates the payout of an option. The BSM (Fischer Black, 1973) creates a hedging portfolio which replicates the options payoff on the maturity date but the creation of such a portfolio to exactly replicate the options payoff in an illiquid market with non-zero transaction costs is a difficult task. Non-zero transaction costs have great influence on the trading strategy of an investor and they change the expected cost of replicating an option but its effect is bounded (Leland, 1985) (Barles & Soner, 1998). In such a situation, finding an exact replicating portfolio becomes very complicated and therefore many researchers tend to find a surreplication portfolio which tend to be cheaper than an exact replication (Barles & Soner, 1998). Similarly, because markets are not perfectly liquid large trading in the underlying asset for hedging purposes has some feedback effects on stock prices and therefore options (Bank & Baum, 2004)(Frey, 1998). In general, transaction costs, price slippage and large trading in illiquid markets are extremely important factors that influence the prices of options (Ankudinova & Ehrhardt, 2008)(Company, Jódar, & Pintos, 2010)(Esekon, 2013)(Kyle, 1985).

Moreover, there are have been several successful attempts at remodeling the BSE by separately taking into account the effects of market illiquidity, transaction costs and price slippage on option prices. This, invariably, results in the extension of the classical and linear BSE into a strongly nonlinear model where both the volatility and the drift depend on time and/or the underlying asset price of the derivatives of the option itself (Barles & Soner, 1998)(Kyle, 1985)(Liu & Yong, 2005). Leland (Leland, 1985) proposed a modified option replication strategy that features transaction costs. In doing this, he used a modified BSE with an adjusted volatility and, inspired by the Black-Scholes replication strategy, derived a modified version whose error is not correlated with the market. Soner and Barles (Barles & Soner, 1998) showed that the value of the portfolio replicating the long position in continuous time tends to the stock price for any finite transaction cost. Fray and Patie (Frey & Patie, 2002), on the other hand, studied the feedback effect of large trading in illiquid markets. Particularly, they considered the case where hedging strategies affect the underlying asset's price process and numerically analyzed a generalized and nonlinear BSM in order to investigate the risk management for derivatives in illiquid market. Liu & Yong (Liu & Yong, 2005) also examined the effects of price impact in replicating a European contingent claim. Several other authors (Çetin, Soner, & Touzi, 2010)(Frey, 2000)(Liu & Yong, 2005) have also studied the impact of the market on options prices and derived various modified nonlinear BSM under various prevailing economic rationale with respect to imperfect market liquidity, transaction costs and price slippage. In this work, the combined effects of non-zero transaction costs and the effects of large traders in imperfectly liquid markets on the prices of options is investigated and an option replicating strategy to a European call option is derived. A numerical solution to the nonlinear BSE arising from the impact of the combination of non-zero transaction costs and price slippage is also studied.

5.1 Model Set-Up

Consider a financial market with mainly two traded assets: a risky asset, for instance a stock, and a risk-free asset which in this model shall be a standard Bond. The market is formulated such that the Bond market can be traded in any large quantity without affecting its price(i.e. perfectly liquid market) whiles the stock is a non dividend paying asset and is traded in an illiquid market with non-zero transaction costs for any traded share. In this work, a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ on which a one-dimensional Brownian motion is defined. The price process of the bond is given by;

$$dB(t) = B(t)r(t, S(t))dt \tag{5.1}$$

where $r(t, S(t))$ is the interest rate which is dependent on the time t and on the underlying stock price S. The price process of the underlying stock, on the other hand is given by;

$$\frac{dS(t)}{S(t)} = \mu(t, S(t))dt + \sigma(t, S(t))dW(t) + \lambda(t, S(t))dN(t) + \kappa(t, S(t))dN(t) \tag{5.2}$$

where $\mu(t, S(t))$ and $\sigma(t, S(t))$ are respectively the drift and the volatility of the stock in the absence of transaction costs and the market's impact on the stock. We assume the existence of a large trader in the stocks' market whose trading strategy attracts fees and taxes and impacts on the underlying asset price. In accordance with economic theory, our model is such that the price of the stock falls(rises) when the large trader sells(buys) shares of the underlying asset. Thus $\lambda(t, S(t)) \geq 0$ and $\kappa(t, S(t)) \geq 0$ are positive real functions that describe respectively the impact on the market with respect to imperfect liquidity and the impact of transaction costs on the drift and the volatility of the underlying stock price. Note also that $N(t)$ represents the hedgers trading strategy for replication purposes. In a similar argument as in (Liu & Yong, 2005), the hedgers trading strategy is a Itô process given by;

$$\begin{aligned} dN(t) &= \eta(t)dt + \zeta(t)dW(t) \quad t \geq 0 \\ N(0) &= N_0 \end{aligned} \tag{5.3}$$

where $\eta(t)$ and $\zeta(t)$ are adapted processes to be determined. We shall suppress the expressions (t) and $(t, S(t))$ in the notations of process for simplicity in our subsequent equations. Thus equation(5.2) becomes;

$$\frac{dS}{S} = [\mu + \lambda\eta + \kappa\eta]dt + [\sigma + \lambda\zeta + \kappa\zeta]dW(t) \tag{5.4}$$

From equations(5.1),(5.4) & (5.3), by suppressing the parameters of the functions for simplicity sake, the traders budget equation $\Pi(t)$ follows the following SDE;

$$\begin{aligned} d\Pi &= r\Pi dt + NS(\mu - r)dt + NS\sigma dW + NS\lambda dN(t) + NS\kappa dN(t) \\ d\Pi &= [r\Pi + NS(\mu - r + \lambda\eta + \kappa\eta)]dt + NS(\sigma + \lambda\zeta + \kappa\zeta)dW(t) \end{aligned} \tag{5.5}$$

Hedging a contingent claim entails finding a self financing portfolio Π at time t such that at the maturity date $T \geq t$, the portfolio equals the payoff($h(S(T))$) of the contingent claim(i.e.$\Pi(T) = h(S(T))$). Therefore, in order to perfectly hedge the corresponding European option, we have to solve equations (5.3),(5.4) & (5.5) subject to the terminal boundary condition $\Pi(T) = h(S(T))$;

$$\begin{aligned} dN(t) &= \eta dt + \zeta dW(t) \\ N(0) &= N_0 \\ \frac{dS}{S} &= [\mu + \lambda\eta + \kappa\eta]dt + [\sigma + \lambda\zeta + \kappa\zeta]dW(t) \\ d\Pi &= [r\Pi + NS(\mu - r + \lambda\eta + \kappa\eta)]dt + NS(\sigma + \lambda\zeta + \kappa\zeta)dW \\ \Pi(T) &= h(S(T)) \end{aligned} \tag{5.6}$$

Let us suppose that $V(t, S(t))$ is a continuously differentiable function and a solution to the system of equation in equation(5.6), $\Pi(t) = V(t, S(t))$. Itô's lemma leads to;

$$\begin{aligned}
d\Pi &= [r\Pi + NS(\mu - r + \lambda\eta + \kappa\eta)]dt + NS(\sigma + \lambda\zeta + \kappa\zeta)dW = dV \\
dV &= \left(\frac{\partial V}{\partial t} + S\frac{\partial V}{\partial S}[\mu + (\lambda + \kappa)\eta] + \frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2}(\sigma + (\lambda + \kappa)\zeta)^2\right)dt \\
&+ S\frac{\partial V}{\partial S}[\sigma + (\lambda + \kappa)\zeta]dW(t)
\end{aligned} \tag{5.7}$$

By identification of similar components, it is deduced that

$$N(t) = \frac{\partial V(t, S(t))}{\partial S} \tag{5.8}$$

whence

$$\begin{aligned}
rV + S\frac{\partial V}{\partial S}(\mu - r + \lambda\eta + \kappa\eta) &= \frac{\partial V}{\partial t} + S\frac{\partial V}{\partial S}[\mu + (\lambda + \kappa)\eta] + \frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2}(\sigma + (\lambda + \kappa)\zeta)^2 \\
rV - rS\frac{\partial V}{\partial S} &= \frac{\partial V}{\partial t} + \frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2}(\sigma + (\lambda + \kappa)\zeta)^2
\end{aligned} \tag{5.9}$$

We apply Itô's lemma to equation(5.8)

$$\begin{aligned}
dN(t) &= d\left(\frac{\partial V(t, S(t))}{\partial S}\right) \\
\eta dt + \zeta dW(t) &= \left(\frac{\partial^2 V}{\partial t \partial S} + S\frac{\partial^2 V}{\partial S^2}[\mu + (\lambda + \kappa)\eta] + \frac{1}{2}S^2\frac{\partial^3 V}{\partial S^3}(\sigma + (\lambda + \kappa)\zeta)^2\right)dt \\
&+ S\frac{\partial^2 V}{\partial S^2}[\sigma + (\lambda + \kappa)\zeta]dW(t)
\end{aligned} \tag{5.10}$$

Hence by identification we deduce that

$$
\begin{aligned}
\zeta(t) &= S\frac{\partial^2 V}{\partial S^2}[\sigma + (\lambda+\kappa)\zeta] \implies \zeta(t) = \frac{\sigma S \frac{\partial^2 V}{\partial S^2}}{1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}} \\
\eta(t) &= \frac{\partial^2 V}{\partial t \partial S} + S\frac{\partial^2 V}{\partial S^2}[\mu + (\lambda+\kappa)\eta] + \frac{1}{2}S^2\frac{\partial^3 V}{\partial S^3}(\sigma + (\lambda+\kappa)\zeta)^2 \implies \\
\eta(t) &= \frac{\frac{\partial^2 V}{\partial t\partial S} + \mu S\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}S^2\frac{\partial^3 V}{\partial S^3}\left[\sigma + (\lambda+\kappa)\left(\frac{\sigma S\frac{\partial^2 V}{\partial S^2}}{1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}}\right)\right]^2}{1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}} \\
\eta(t) &= \frac{1}{1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}}\left(\frac{\partial^2 V}{\partial t\partial S} + \mu S\frac{\partial^2 V}{\partial S^2} + \frac{\sigma S^2\frac{\partial^3 V}{\partial S^3}}{2\left(1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}\right)^2}\right)
\end{aligned} \tag{5.11}
$$

Thus the underlying stock in the presence of transaction costs and price impact in an illiquid market is modeled as;

$$
\begin{aligned}
\frac{dS(t)}{S(t)} &= \mu(t,S(t))dt + \sigma(t,S(t))dW(t) + \lambda(t,S(t))dN(t) + \kappa(t,S(t))dN(t) \\
&= (\mu + \lambda\eta + \kappa\eta)dt + (\sigma + \lambda\zeta + \kappa\zeta)dW(t) \\
&= \left[\mu + \frac{\lambda+\kappa}{1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}}\left(\frac{\partial^2 V}{\partial t\partial S} + \mu S\frac{\partial^2 V}{\partial S^2} + \frac{\sigma S^2\frac{\partial^3 V}{\partial S^3}}{2\left(1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}\right)^2}\right)\right]dt \\
&\qquad + \left[\sigma + (\lambda+\kappa)\left(\frac{\sigma S\frac{\partial^2 V}{\partial S^2}}{1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}}\right)\right]dW(t) \\
&= \left[\frac{\mu + (\lambda+\kappa)\frac{\partial^2 V}{\partial t\partial S}}{1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}} + \frac{\sigma(\lambda+\kappa)S^2\frac{\partial^3 V}{\partial S^3}}{2\left(1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}\right)^2}\right]dt + \left[\frac{\sigma}{1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}}\right]dW(t) \\
\frac{dS(t)}{S(t)} &= \hat{\mu}(t,S(t))dt + \hat{\sigma}(t,S(t))dW(t)
\end{aligned} \tag{5.12}
$$

Where

$$
\hat{\sigma}(t,S(t)) = \frac{\sigma(t,S(t))}{1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}} \tag{5.13}
$$

and

$$\hat{\mu}(t,S(t)) = \left[\frac{\mu(t,S(t)) + (\lambda+\kappa)\frac{\partial^2 V}{\partial t \partial S}}{1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}} + \frac{\sigma(t,S(t))(\lambda+\kappa)S^2\frac{\partial^3 V}{\partial S^3}}{2\left(1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}\right)^2}\right] \tag{5.14}$$

are respectively the modified volatility and the modified drift in order to incorporate transaction costs and market price impact and thus extending the BSE to a generalized model. The parameters σ and μ are respectively the constant volatility and drift in the classical BSM. We present formally a generalization of the BSM in a form of generalized and nonlinear BSE which arises from the relaxation of the unrealistic assumptions of a frictionless and perfectly liquid financial market in the BSM. Our model or derivation extends the analysis of (Barles & Soner, 1998),(Frey & Patie, 2002) and (Liu & Yong, 2005) and we obtain their analysis as a special case of our model. Again for clarity reasons, we shall suppress the parameters of the functions.

Theorem 5.1.1 *Suppose that $\mu, r, \sigma, \kappa, \lambda : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions such that $\kappa \geq 0$, $\lambda \geq 0$ and $\sigma \geq \epsilon > 0$ $\forall$ $(t, S(t)) \in [0,T] \times \mathbb{R}$. Then there exist a unique solution $V(t, S(t))$ to the generalized and nonlinear Black-Scholes PDE for option pricing*

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2\left[1-[\lambda(t,S(t))+\kappa(t,S(t))]S\frac{\partial^2 V}{\partial S^2}\right]^2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{5.15}$$

subject to the corresponding terminal and boundary condition, the positive functions $\lambda(t, S(t))$ and $\kappa(t, S(t))$ are respectively the price impact function in an illiquid market and transaction costs determinant. The stock $S(t)$ price, in such market, follows *the SDE below;*

$$\frac{dS(t)}{S(t)} = \hat{\mu}(t, S(t))dt + \hat{\sigma}(t, S(t))dW(t) \tag{5.16}$$

where

$$\begin{aligned} \hat{\sigma}(t, S(t)) &= \frac{\sigma}{1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}} \\ \hat{\mu}(t, S(t)) &= \left[\frac{\mu(t, S(t)) + (\lambda+\kappa)\frac{\partial^2 V}{\partial t \partial S}}{1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}} + \frac{\sigma(\lambda+\kappa)S^2\frac{\partial^3 V}{\partial S^3}}{2\left(1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}\right)^2}\right] \end{aligned} \tag{5.17}$$

Here σ is the constant volatility assumed in the BSM.

Theorem(5.1.1) is an extension and generalization of the BSM by relaxing the unrealistic assumption of a frictionless and perfectly liquid financial market. Most importantly it is the generalization of the results of in (Barles & Soner, 1998), (Frey & Patie, 2002) and (Liu & Yong, 2005) which can be obtain as special cases of theorem(5.1.1). For instance, if we assume the existence of price impact of large trading in an illiquid financial market with no transaction costs(i.e. $\kappa(t, (S(t)) = 0)$, equation(5.15) reduces to that obtained by Liu & Yong in (Liu & Yong, 2005) and by Frey & Patie in (Frey & Patie, 2002) with $\lambda(t, S(t)) = \rho$ the liquidity parameter. Also if we assume that the financial market is perfectly liquid(i.e. $\lambda(t, S(t)) = 0$) but trading in the underlying asset attracts non-zero transaction costs, our model then reduces to that of Barles & Soner in (Barles & Soner, 1998) where $\kappa(t, S(t)) = \frac{Sk^2 e^{r(T-t)}}{2}$ with k being the proportional transaction costs. The classical BSE is equally obtained when we assume a frictionless(i.e. $\kappa(t, S(t)) = 0$) and perfectly liquid market(i.e. $\lambda(t, S(t)) = 0$).

Our hedging strategy is very similar to that of Liu & Young (Liu & Yong, 2005) as the impact of transaction costs is absorbed in the market impact parameter $\lambda(t, S(t))$ and therefore does not change the replication strategy employed in an imperfectly liquid market. Thus from equation(5.8), we deduce that the large trader at time $t = 0$ trades $N(0) = \frac{\partial V(0,S(0))}{\partial S}$ and subsequently follows the trading strategy $N(t)$.

We shall now present a numerical study of the generalized and nonlinear Black-Scholes PDE in equation(5.15) subject to the terminal boundary conditions below;

$$\left.\begin{aligned} V(S(T),T) &= \max(S(T)-K,0) \\ V(t,L) &= L - Ke^{-r(T-t)}, (\forall L > S(T)) \\ V(t,0) &= 0 \end{aligned}\right\} \text{(For Call Options)} \tag{5.18}$$

and

$$\left.\begin{aligned} V(S(T).T) &= \max(K-S(T),0) \\ V(t,L) &= 0, (\forall L > S) \\ V(t,0) &= Ke^{-r(T-t)} \end{aligned}\right\} \text{(For Put Options)} \tag{5.19}$$

where K is the strike price, T the expiry time of the European option and L the maximum price attained by the stock.

5.2 Numerical Approximation

Equations(5.15) subject to conditions(5.18) & (5.19) may be considered as a terminal boundary value problem and solved numerically using the semi-discretization finite difference method known as the method of lines (Rogers & Talay, 1997)(Burden & Faires, 2010). A partition of the spatial interval $0 \leq S \leq L$ into N equal parts is introduced such that the grid size $\Delta S = L/N$ and grid points $S_i = (1-i)\Delta S$, $1 \leq i \leq N+1$. Here, $L > K$ is the maximum price attained by the underlying asset. A finite difference discretization on a uniform grid of Cartesian mesh is employed. Both first and second spatial derivatives in equation(5.15) are approximated with second-order central finite differences. Let $V_i(t)$ be the approximation of $V(t, S_i)$ and consider $\lambda_i(t, S(t)) = \rho$ and $\kappa_i(t, S(t)) = \frac{S_i k^2 e^{r(T-t)}}{2}$. Assuming constant interest rate, then the semi-discrete system for the problem becomes

$$\begin{aligned} \frac{\partial V_i}{\partial t} + \frac{\sigma^2 S_i^2}{2\left[1 - [\lambda + \kappa] S_i \left(\frac{V_{i+1} - 2V_i + V_{i-1}}{(\Delta S)^2}\right)\right]^2} \left(\frac{V_{i+1} - 2V_i + V_{i-1}}{(\Delta S)^2}\right) \\ + r S_i \left(\frac{V_{i+1} - V_{i-1}}{2\Delta S}\right) - r V_i = 0 \end{aligned} \tag{5.20}$$

with the terminal conditions for the European option given as:

$$\left.\begin{aligned} V_i(T) &= \max(S_i - K, 0), \text{ For Call Option} \\ V_i(T) &= \max(K - S_i, 0), \text{ For Put Option} \end{aligned}\right\}, \; 1 \leq i \leq N+1 \tag{5.21}$$

The equations corresponding to the first and the last grid points are modified to incorporate the boundary conditions as follows:

$$\left.\begin{aligned} V_1 &= 0 \\ V_{N+1} &= L - K e^{-r(T-t)} \end{aligned}\right\} \text{(For Call Options)} \tag{5.22}$$

and

$$\left.\begin{aligned} V_1 &= K e^{-r(T-t)} \\ V_{N+1} &= 0 \end{aligned}\right\} \text{(For Put Options)} \tag{5.23}$$

Equation(5.20) only has one independent variable so it is a first order nonlinear ODE with known terminal conditions. Therefore the resulting problem can be easily solved iteratively using the forth order Runge-Kutta-Fehlberg integration technique (Cartwright & Piro, 1992). From the process of numerical computation, the effects of volatility and risk-free interest rate on the spatial and temporal structure of option

Table 5.1: Effects of Increasing transaction costs and increasing liquidity parameter on the Valuation of a European Call Option where $x = 40, K = 40, L = 70, T = 1, \sigma = 0.3, r = 0.05$

Call Option Prices

κ \ ρ	0.00	0.01	0.02	0.03
0.00	5.2610002	5.3099366	5.3585060	5.40671429
0.01	5.2715813	5.3204343	5.3689216	5.41704892
0.02	5.3032086	5.3518128	5.4000552	5.44794122
0.03	5.3555365	5.4037317	5.4515709	5.49905913

Table 5.2: Effects of Increasing transaction costs and increasing liquidity parameter on the Valuation of a European Put Option where $x = 40, K = 40, L = 70, T = 1, \sigma = 0.3, r = 0.05$

Put Option Prices

κ \ ρ	0.00	0.01	0.02	0.03
0.00	3.3164669	3.3654220	3.4140102	3.4622374
0.01	3.3270523	3.3759239	3.4244300	3.4725763
0.02	3.3586922	3.4073153	3.4555766	3.5034817
0.03	3.4110415	3.4592557	3.5071140	3.5546214

price value are obtained and presented graphically and in tabular form below:

5.3 Results and Discussion

Our result extends the BSM in a financial market that is not perfectly liquid with transaction costs. Our results extends that of Frey & Patie (Frey & Patie, 2002), Liu & Young (Liu & Yong, 2005) and that of Barles & Soner (Barles & Soner, 1998) which in turn are obtained as special cases of our results. From tables(5.1) & (5.2), it is obvious that option valuation in an imperfectly liquid financial market and in the presence of transaction costs is totally different from the one predicted by the classical BSM. Interestingly, as the liquidity and/or transaction cost parameter increases, both call and put options increase in value. This is because an illiquid market in the presence of transaction costs has direct impact on the volatility function of the underlying stock. A similar trend is observed with the adjusted volatility and transaction costs(see tables(5.1) and (5.2)). Hence a trader, in an illiquid market, generally needs to buy(short) more asset and borrow(lend) more to replicate a call(put) option. The excess cost a trader incurs is found to be significant even with small price impact. It is also found that the incurred transaction costs for a put option is higher than that of an identical call option with the same adjusted volatility. Both call and put prices increase with an increase in the strike price with or without transaction costs. The spatio-temporal behaviour of European Call and Put options in the presence of transaction costs are illustrated in figures(5.1) and (5.2). It is observed that an increase in the price of the underlying asset increases the call option value and decreases the put option value as expected. Figures(5.3) and (5.4) illustrate the effect of transaction costs and liquidity parameter on the call and put options. The value of both the call and put options increase with a rise in transaction costs; hence the option holder will benefit greatly since the call and put options are more likely to end up in-the-money. Meanwhile, as the call and put get more and more in-the-money, the excess cost increases and therefore the adjusted costs.

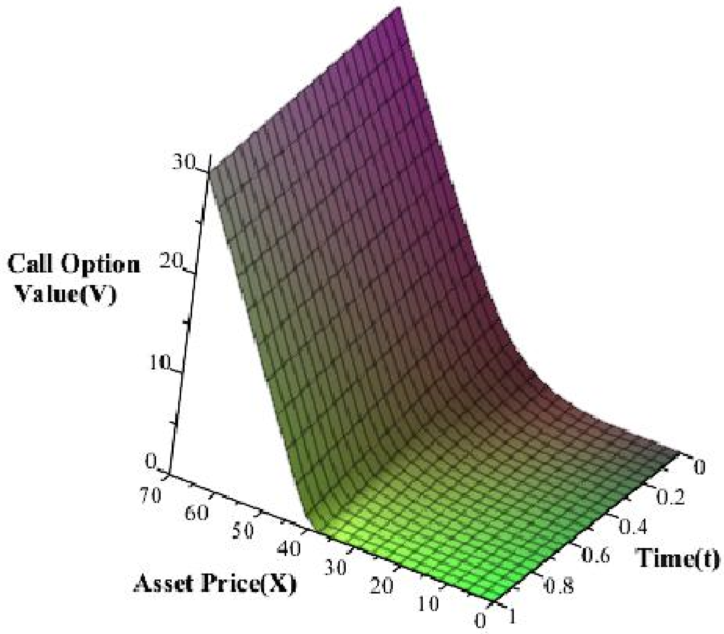

Figure 5.1: Call Option Value for $\sigma = 0.3$, $r = 0.05$, $T = 1$ year, $K = 40$, $L = 65$

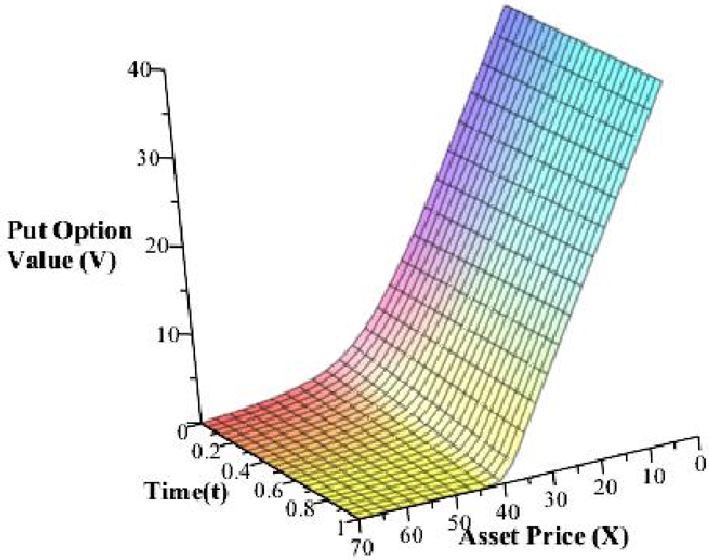

Figure 5.2: Put Option Value for $\sigma = 0.3$, $\rho = 0.05$, $r = 0.05$, $T = 1$ year $K = 40$, $L = 100$

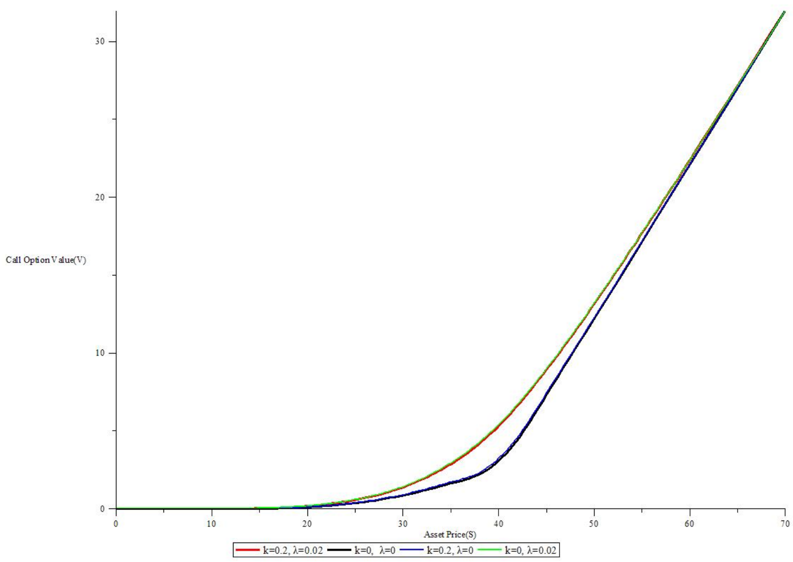

Figure 5.3: Call Option Value for $\sigma = 0.3, r = 0.05, T = 1\text{year}, K = 40, L = 70, S = 40$

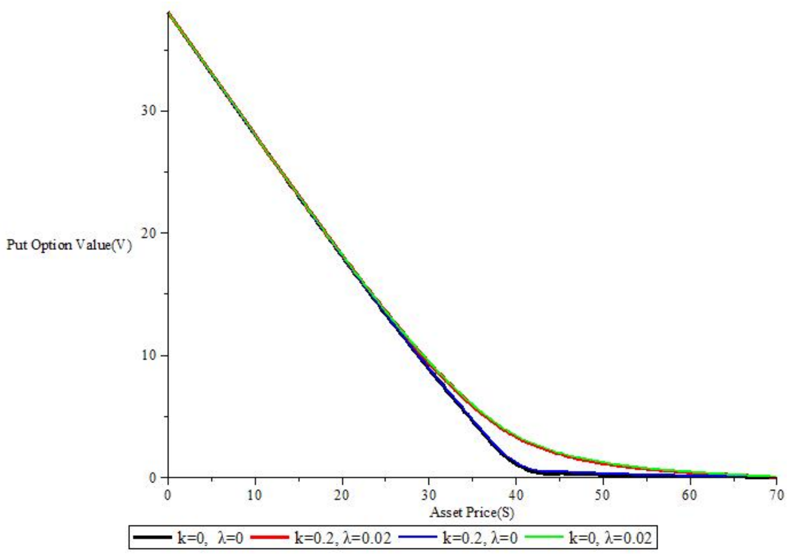

Figure 5.4: Put Option Value for $\sigma = 0.3, r = 0.05, T = 1\text{year}, K = 40, L = 70, S = 40$

Chapter 6

Conclusion

The BSM is undoubtedly, one of the most important results in mathematical finance. This model has been and still is the focus point of many research in option pricing theory ever since its introduction in 1973. This is because the BSM provides the fair value for the price of a vanilla option subject to a number of market assumptions. B-S assumed an efficient market where the volatility is constant. They also assumed that trading is done in a perfectly liquid financial market where no dividends, no transaction costs nor commissions payments are made. In this study, the focus was on the effects of the BSM' assumptions of zero transaction costs and large trading in a perfectly liquid market on options prices. Today's financial markets, especially in the third world countries, are certainly not perfectly liquid despite the numerous technological advancements and applications in the sector. The use of technology in trading, on one hand, has significantly improved the number of transactions made but, on the other hand, it has significantly increased the costs of transactions due the immense number of shares traded daily. Therefore these assumptions are too strict for a good option pricing model and thus any model with such strict assumptions will not give a true reflection of the market prices. Many researchers and option traders rely heavily on the BSE either as the underlying pricing formula for trading or as the starting point for developing new pricing formulas for much more complicated options. In this regard, the attempts have been to extend the BSM by relaxing the strict and unrealistic market assumptions made in the model in order to derive more efficient and realistic pricing formulas. Black and Scholes proposed not just a pricing model but also an option replication portfolio which is a strategy that enables the trader to replicate the payoff of an option.

In this study, we derived an option pricing equation, inspired by the BSE, in a market with non-zero transaction costs where the existence of large traders and therefore large trading has some feedback effects in a form of price impact on stocks and thus on options prices. The resulting equation is a non-linear and a generalized BSE which extends the BSM and the latter is obtained when the parameters indicating the existence of non-zero transactions costs and a perfect market liquidity are null. The derivation of the generalized model and equation are also dependent on the assumption of an efficient market which was a key assumption in formulation of the classical BSM. The application of the no-arbitrage principle is, in part, due of the assumption of an efficient market which in itself is a subject of many divergent views in literature. Therefore the creation of a replication strategy for the trade of stocks and options should form an integral part of every option pricing model and the tracking error observed closely in order to ascertain the performance of such portfolio and hence performance of option pricing models.

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