PAN AFRICAN UNIVERSITY
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On the Location of a Free Boundary for American Options

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B. Sc (Maths, Stat) - Hons [Makerere Univ]

A research thesis submitted to Pan African University, Institute of Basic Science, Technology and Innovation in Partial Fulfillment of the Requirement for the award of the degree of

Master of Science in Mathematics

(Financial Option)

2017

Declaration

I declare that this work is my original work and that it has never been submitted to any institution of higher learning for any award or assessment.Where information from others sources have been used, citation has been made.

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Acknowledgements

Counsel in the heart of man is like deep water; but a man of understanding will draw it out: - Proverbs 20:5

In acknowledgement of the mystery of God, and of the Father, and of Christ, in whom are hid all the treasures of wisdom and knowledge, I bless Jehovah God for his abundant love for me. I bless God for everything during the pursuance of this degree and all he has taken me through.

Secondly, I want to thank the African Union and specifically the Pan African University, Institute of Basic Sciences, Technology and Innovation for the opportunity of study for this masters and research project.

I also bless God for my supervisors **Prof. Diaraf SECK** and **Dr. Philip Ngare** for the immense invaluable efforts they have invested in directing me to the success of this research. I honestly can not thank YOU enough but to say my best and leave the rest to God for all the ways you have patiently taught and directed me even when I was *seemingly* un willing to learn. Thank you very much.

I also want to bless God for my colleagues Ronald, Mariam, Aloysius and the whole PAUSTI 2015 cohort, especially PAUSTI Christian fellowship for the conducive environment and love during this period of study.

Lastly, but not least, I bless God for my family who always encouraged me to be focused and go further for all so as to live out what God had already purposed for me to live; according to his eternal purpose which he laid in in Christ Jesus even before the foundations of the world were set in place; the very redemptive work of humanity that he accomplished at the cross of the Christ.

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Abstract

We study the free boundary problem of the American type of options. We consider a continuous dividend paying put option and provide much simpler way of approximating the option payoff and value. The essence of this study is to apply geometric techniques to approximate option values in the exercise boundary. This, being done with the nature of the exercise boundary in mind, more accurate results are guaranteed. We define a transformation (map) from a unit square to the free boundary. We then examine the transformation and its properties. We take a linear case for a transformation as well as a nonlinear case which would be more fitting for option values. We consider stochasticity (an Ito process) as we define this transformation and this yields better approximations for option values and payoffs. We also numerically compute optimal option prices using the same transformation. We finally demonstrate that our transformation performs better than most semi-analytic results.

Chapter 1

Introduction

1 Background

Option pricing is one of the major areas of financial engineering (also commonly referred to as Financial Mathematics or Mathematical Finance). There are various types of options spanning from European, Asian, American, Bermudan among others. However, the two major traded types are the American above all followed by the European kind of options. The latter have been studied and pricing formulas established. For the former, work has been done but more is needed to obtain appropriate pricing measures for this kind of options. The American type of options differs from the European in only the exercise time. Much as the European options can only be exercised at their expiration time, American options can be exercised at any time even before their expiration time. This relatively makes their pricing a bit complicated compared to the European type. Despite the fact that Black-Scholes equation for pricing options has attracted a lot of attention from both a theoretical as well as a practical point of view, there is still a significant challenge to meet as regards the American type of options. One of the interesting problems in the study of American option pricing is the analysis of the early exercise and the optimal stopping times especially for options on assets that are paying continuous dividends. This kind of problem can be studied as a problem of solving a certain free boundary problem for the Black?Scholes equation ((Morebeke, 1976)). However, the exact analytical expression for the free boundary problem is not known yet ((Kim, 1990)). Many authors have developed various approximate models leading to approximate expressions for valuing American call and put options.

2 Statement of the problem

Definition 2.1. *(σ-algebra)*

A σ-algebra on a set X is a collection Σ of subsets of X that includes the empty subset, is closed under complementation, closed under countable unions and closed under countable intersections.

The American option value is obtained as the expected discounted payoff at exercise. However as said earlier on, unlike for European options, the exercise time for American Options is just any random stopping time between the current time, say t and the expiration time, say T . Suppose we denote by V (S, t) the option value at time t and by f (S, t) the pay-off at the same t and S is the price of the underlying stock. Then the stopping times, τ would be defined by;

$$T_{[t,T]} = \{\tau; t \leq \tau \leq T; \tau \in \mathcal{F}_t\}$$

where $\mathcal{F}_t$ is the σ-algebra generated by W_t; a wiener process. This set of points can indeed be a solution set to an optimal stopping problem given by;

$$V(S,t) = \sup E^Q(e^{-r(t-T)}f(S_\tau)|S_t = S)$$

Now T$\tau = t$ being a stopping time implies that the pay-off will always be greater or equal to the option value. Now the free boundary that we seek for is the set of points at which

$$V(S,t) = f(S,t)$$

This set of points could be obtained from solving the so-called complete free boundary problem for American Options.

3 Justification

Locating the free boundary of the Black-Scholes PDE for American options is highly significant in option valuation. The location of a set of points where the option value V is equivalent to the pay off f will induce better optimal points for options and related derivatives. Normally in practice, most traded options follow the American option trend but their prices are usually just approximated using chosen numerical techniques. The location of the BS-PDE is thus a very vital area of research in option pricing, specifically American option pricing.

4 Objectives

4.1 Main Objective

To locate a free boundary for the Black-Scholes PDE for American Options

4.1.1 Specific Objectives

(i) To obtain a (an asymptotic) solution to the American option analytic valuation

(ii) To solve the American put and call option value variational inequalities (optimal stopping problem)

(iii) To assess the viability of our results in mathematical finance

5 Scope of the study

This study will entirely be devoted to establishing a free boundary (or an asymptotic one). This study will not involve the abstract concepts that may come along with this solution set (free boundary). The purpose of this study therefore will include devising solutions (asymptotic) to the corresponding variational inequalities, having relaxed some assumptions and constraints on these equations. We shall also study the existence, uniqueness as well as economic significance of the suggested solutions.

Chapter 2

Literature Review

The problem of discovering free boundaries dates back to the start of the 19th century. The location of a free boundary for American options is undoubtedly one of the most outstanding problems in mathematical economics ((McKean, 1965); (Byun, 2005)). Rarely, even in mathematical physics where PDEs play a very vital role, do we obtain explicit solutions to PDEs, specifically those to do with solving for a free boundary. However, the vitality of the Black-Scholes equation in mathematical finance and in particular option pricing has attracted attention from researchers in different areas who have tried to study the free boundary for the Black-Scholes PDE for over four centuries now since Black & Scholes breakthrough paper on Option pricing see (Black & Scholes, 1973). McKean (1965) is believed to be the first (Barone-Adesi & Whaley, 1987); (Brennan & Schwartz, 1978) to have studied a free-boundary problem of a parabolic type (which is our interest) arising in optimal stopping of the American option pricing. Moerbeke (1976) furthered the work of McKean having studied the properties of this free boundary. A summary of the essential results on the American option pricing problem is given by (Myneni, 1992). (Jacka, 1991) discussed the pricing of an American put option as an optimal stopping problem. They also verify the essential uniqueness of the solution to the free boundary problem and also identify an integral equation satisfied by the stopping boundary. Kim, 1990 presents an analytic valuation formula for American options written on assets that pay continuous dividends. Black & Scholes demonstrated that European options could be valued with the help of a closed form equation. However, this is hardly the case for American options since their exercise time can never be known. A great remark here is that as long as the underlying asset pays no dividends, American call options can never be exercised early and hence will always have the same value as their equivalent European call options. Moreover, even with some dividends provided they are of a discrete nature, early exercise is still dubious (Merton, 1973); (Brenner, Courtadon, & Subrahmanyam, 1985). This eases the work of premature exercise analysis since it would only be optimal if it exercised just before the dividend payment. Nevertheless, this is never the case for American puts, calls on futures, contracts and foreign currencies. This is also inapplicable in cases when the dividends paid by the underlying asset are of a continuous type (Merton, 1965). This is basically due to the fact that there then will always be only a single possibility of early exercise ((Carr, Jarrow, & Myneni, 1992)). This explains the major setback for the impossibility of using European option formulas on American type options written on futures, contracts and foreign currencies. More so, up until now, there are no generally accepted (or even proposed) analytic solutions that exist for those option types. The whole problem is rooted in the fact that the optimal exercise boundary must be determined as a part of the solution (Karoui & Karatzas, 2008); (Geske, 1979). On these very grounds, researchers in this area specifically the valuation of American puts have resorted to numerical methods or some kind of approximation methods. An example is Brenner, Courtadon, and Subrahmanyam (1985) have applied the implicit finite-difference method on the work of Brennan and Schwartz (1977, 1978) as a way of accounting for the early exercise possibility of American options on futures contracts. Geske (1984) devised a valuation formula for American puts which is in terms of a series of functions of compound-options. A series of numerical techniques have been developed in the past all for the purpose of approximating the American option value. Most of these techniques have instances in which they are best suited for use as well as times when they fail to bring out the best results. Kim (1990) attempted to work on the analytic valuation of the American type of options and they defined integral equations that define the option value. These integral equations are of Volterra type II kind. Little, Pant, and Hou (2000) have also derived another alternative integral equation for the American put option optimal exercise boundary. Byun (2005) studied the properties of the integral equations derived in Kim (1990). Many numerical and approximation methods (integral representation) have been invented basing on these very integral equations but the journey to obtaining analytic solutions still seems to be a very distant one (Little, Pant, & Hou, 2000). Faye et al. (2015) have approximated the optimal exercise boundary using the mathematical concepts of topological optimization. They developed a numerical algorithm that starts off with a choice of time and outputs the set of values in the free boundary and as well as the approximate optimal value for the American option in question. This is clearly a significant advancement towards obtaining analytic descriptions of optimal times and values for American options.

This work is organized as follows. Section 3 reviews the various valuation techniques that have been employed in literature. The next chapter reviews the American put option which is our major focus in this work including sub-sections devoted to the analysis of the problem as an optimal stopping problem with our own results included, in which we utilize series expansions to determine an analytic expression for the optimal exercise boundary. This sub-section is followed by another of like nature devoted to the analysis of the same as a variational inequality in which we discuss the regularity results of this pricing problem. Section 5 carries on with the development of the pricing transformation from a unit square to the optimal exercise boundary before section 6 details the numerical work done here in. Section 7 is the last and sums it up with suggestions for different approaches to handling this same problem.

Chapter 3

Methodology

1 Pricing Techniques

1.0.2 Finite Differences Method (FDM)

With FDM, three major methods are considered i.e. the explicit, implicit and Crank-Nicolson Methods. With discretization of the pricing equations in regard to the selected scheme, functions are re-written as per the convenience of the user and the options are valued according to the output of the defined functions.

1.0.3 Binomial Option Pricing

Here in, we consider the American put option being priced using the binomial tree (often referred to as the Cox-Ross and Rubinstein (CRR) model). At each node of the lattice-like binomial tree, we compute the value of the option. The valuation of American options proceeds as follows:

- At each node, we check for early exercise

- If the value of the option is greater when exercised, we assign that value to the node. Otherwise, we assign the value of the option un exercised
- We work backward through the three as usual

With the Binomial pricing model, the option value can either go up by a value , say u or down by a value, say d. Notice that the American put option value must be greater than or equal to the payoff function i.e. we assume there exists no opportunity for arbitrage. Mathematically,

$$P \geq \max(0, K - S)$$

otherwise there is arbitrage.

Example 1.1. *Suppose one buys stock for S and an option for, say P and immediately exercise it by selling the stock at a price, say E. Then*

$$E - (P + S) > 0$$

General Binomial Pricing Method for American Put Option

So in general, denote by P_n^m the n-th possible value of the put option as time step $m\Delta t$, the American put option value is given by

$$P_n^m = \max\left\{\max(K - S_n^m, 0), e^{-r\Delta t}(pP_{n+1}^{m+1} + (1-p)P_n^{m+1})\right\} \qquad (3.1)$$

where S_n^m is the n-th possible value of the stock price at time step $m\Delta t$. Hence

$$P_n^N = \max(K - S_n^N, 0); n = 0, 1, 2, \cdots, N; K = \text{the strike prices some times denoted as K} \qquad (3.2)$$

1.0.4 Monte Carlo Techniques

This method is very popular among scholars of the American option as several paths have been taken to value an American option by the help of Monte Carlo simulation techniques.

Major Pros and Cons of Monte Carlo Simulation

Monte Carlo simulation methods have proved to beat most methods when it comes to valuation of path dependent options. Their supremacy proceeds from the following facts;

- It can simulate the underlying asset price path by path
- It can calculate the payoff associated with the information for each simulated path
- It can also utilize the average discounted payoff to approximate the expected discounted payoff, which is the value of path-dependent options

However, Monte Carlo methods become cumbersome to work with when it comes to American options. This is due to the difficulty in derivation of the holding value (or the continuation value) as at each time point, it has to be based on one single subsequent path. Scholars have suggested to resort to multiple-tier Monte Carlo simulation as a means of estimating the holding value for American options, which rellay seems to make the work easier but still the method is infeasible for a large number of early exercise time points.

2 American put option

Modern finance theory states that the fair price of an American put option with expiration at infinite time is give by

$$V(S) = \sup_{\tau} E_S(e^{-r\tau}(K - S_\tau)^+) \tag{3.3}$$

the supremum being taken over all stopping times τ of the geometric Wiener process $(S_t)_{t\geq 0}$. This process satisfies the SDE given by;

$$dS_t = rS_t dt + \sigma S_t dW_t; S_0 = s$$

under P_s. The corresponding unique solution to the SDE defining S is given by;

$$S_t = S_0 \exp\{\sigma W_t + (r - (\sigma^2/2))t\}; t \geq 0, x > 0$$

Now noticeably as $S \to 0$, there is little or even no possibility of the payoff increasing thus continuation is u called for in that case. However, this also points to the existence of a set of stopping times τ_B (that can be deemed optimal to the equation (3.3)) such that basing on the strong Markov property, we come up with the following set of expressions for the unknown set B, which in this case is referred to as the free boundary.

$$\mathcal{L}V = rV \quad ; S > B \tag{3.4}$$

$$V(S) = (K - S)^+ \quad ; S = B \tag{3.5}$$

$$V'(S) = -1 \quad ; S = B \tag{3.6}$$

$$V(S) > (K - S)^+ \quad ; S > B \tag{3.7}$$

$$V(S) = (K - S)^+ \quad ; 0 < S < B \tag{3.8}$$

where $\mathcal{L}$ is the infinitesimal generator defined by;

$$\mathcal{L} = rS\frac{\partial}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2}{\partial S^2}$$

Now the price of the option would then comprehensively be given by;

$$P(T, S_t) = \sup_{\tau \in [0,T]} \mathbb{E}e^{-r\tau}V\left[S_t e^{\left(r-\delta-\frac{\sigma^2}{2}\right)\tau+\sigma W_\tau}\right] \tag{3.9}$$

where

$$V(S_t) = (K - S_t)^+$$

Some of the desirable properties about this price are such that;

- For $S_t \in [0, +\infty)$, $t \to P(t, S_t)$ is increasing.
- Also for $t \in [0, T]$, $S_t \to P(t, S_t)$ is non-increasing and convex, of course this is as a result of the monotone and convex nature of Phi
- $\forall\ (t, S_t) \in [0, T] \times [0, +\infty)$, $P(t, S_t) \geq V(S_t) = P(T, S_T)$

Theorem 2.1. *For any market, if the risk-free rate r is positive, then for every $t < T$,*

$$V_E(t, S_t) \leq V_A(t, S_t)$$

where $V_E(t, S_t)$ *and* $V_A(t, S_t)$ *are the values of the European and American put options respectively.*

This theorem is clearly reflected by the decomposition of the American option value into the European option value as well as the early exercise premium. Since the premium can at worst be zero but not negative, this establishes the theorem 2.1. as can be seen from the figure 3.1.

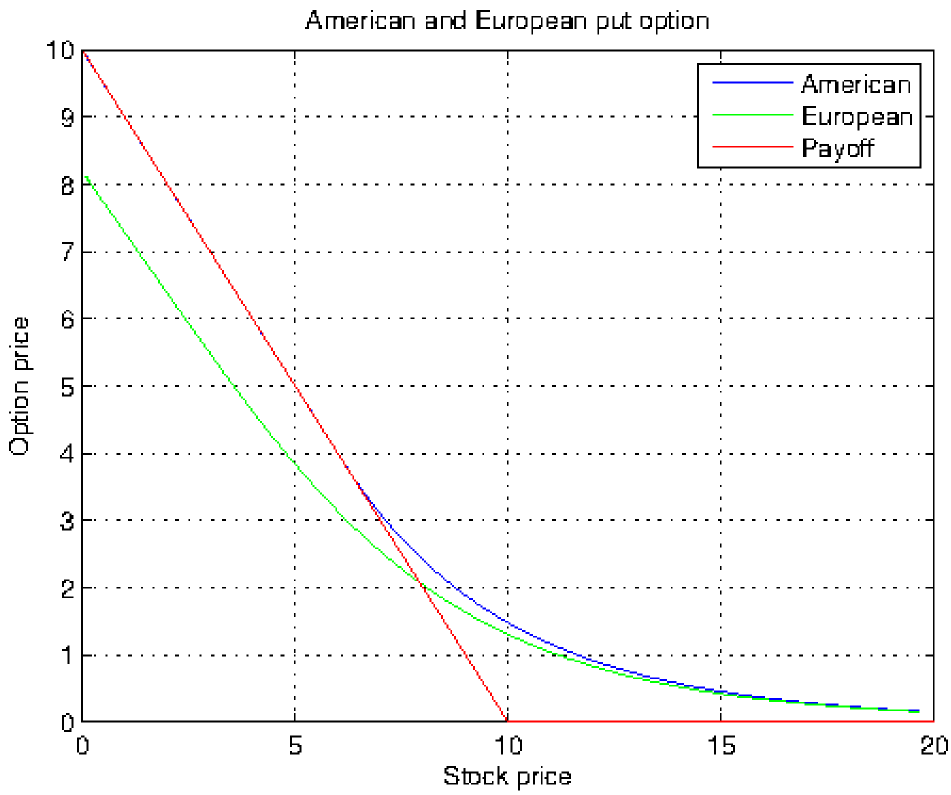

Figure 3.1: American and European option payoffs

First of all, in developing this figure, we assumed a strike price of 10, expiration time $T = 1$, interest rate $r = 0.2$, and volatility $\sigma = 0.5$ and plotted the variation of the option price against the stock price for both an American as well as European option prices. Now first we recognize that the pay off for the American option is always greater than that of its European counterpart. Also the plot of both American and European option prices produced by our transformation is similar to those from other techniques of approximating the option prices. Figure 3.1 emphatically depicts the difference between the European and American option prices as well as payoffs. This is a way explains the popularity of the American option as it tends to pay more to its holder than the European type. The relationship between the stock price and the option price is also seen to be an inverse one as expected. Another great point to note is that the payoff of a European option can hardly be equal to the price of the underlying asset thus exercise only at expiration time.

However for the American option, exercise can clearly be done at any time prior to the expiration time as the pay off can at any time be equal or even greater than the underlying asset price. Lastly, note that when the payoff reaches 0, the stock price is the same as the strike price $K = 10$ (still as is expected from theory). So then lets get into the standard pricing results about this option, having looked at its behavior graphically.

2.0.5 Optimal stopping problem

Suppose that $X = \{X_t\}_{t\geq 0}$ is a time-homogeneous strong Markov process defined on the measure space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Define also the measures $\{P_x, x \in \mathbb{R}\}$, P_x being the law of distribution for the process X form $P(X_0 = x) = 1$. The transition probability would then be;

$$P(X_t \in B | X_0 = x) = P_x(X \in B)$$

and hence the family $(\Omega, \mathcal{F}, \mathcal{F}_{tt\geq 0}.\{P_x, x \in \mathbb{R}\})$, together with the process X form a Markov family. Now with out loss of generality, define $\omega : [0, \infty) \rightarrow \mathbb{R}$ on which the σ-algebra $\mathcal{F}$ is defined, $\mathcal{F}$ being spanned by $\{\omega \in \mathbb{R}[0, \infty) : \omega(t_1)\mathbb{F}_1, \omega(t_2) \in F_2, \cdots \omega(t_n) \in F_n; n \in \mathbb{N} \, \& F_i \subset \mathbb{R}\}$. Then we can defined the Markov process X as the projection process

$$X_t(\omega) = \omega(t); t \geq 0, \omega \in \Omega$$

Now in this case we assume that the shift operator θ_t is well defined. Now our intention is to solve (or attempt to) the American (put) option optimal stopping problem. However, we start off with a general easier case before we embark on to the very problem addressed here in.

2.0.6 The general optimal stopping problem

In this sub section we address the problem of the optimal stopping problem (OSP). We discuss regularity results and dwell on that in the next section to attempt to solve the American put option stopping problem. Through out this section, we presume the following assumptions hold.

(a) $\mathcal{F}_{\prime}$ contains all P-null sets from $\mathcal{F}$ and that also the filtration $\mathcal{F}_{tt\geq 0}$ is right continuous i.e. $\mathcal{F}_{+} := \bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t$

(b) Right continuous sample paths of X i.e. $X_t \rightarrow X_s$ whenever $X_0 = x$ and $t \downarrow s; P - a.s$ (t decreases towards s)

(c) Over stopping times, the sample paths of X are left continuous i.e. $For X_0 = x, P - a.s., X_{\tau_n} \rightarrow X_\tau$ whenever $\tau_n \uparrow \tau$ (τ_n increases towards τ)

The general optimal stopping problem can be defined as;

$$V(x) = \sup_{0 \leq \tau \leq T} E_x G(X_\tau) \tag{3.10}$$

$G : \mathbb{R} \to \mathbb{R}$ being a measurable function, and supremum is taken over all stopping times satisfying $P(\tau < \infty) = 1$. G and V are usually referred to as Gain and Value functions respectively. Notice also that

$$E_x(\sup_{0 \leq \tau \leq T} |G(X_t)|) < \infty; x \in \mathbb{R}, G(X_T) := 0 \text{ provided } T = \infty$$

otherwise equation (3.10) would be void.

Definition 2.1. Super harmonic function

Let $F : \mathbb{R} \to \mathbb{R}$ be a measurable function and $a > 0$. Then F is said to be a-super harmonic if

$$E_x e^{-a\sigma} F(X_\sigma) \leq F(x); x \in \mathbb{R}$$

with $P(\sigma < \infty) = 1$. F is said to be just super harmonic if $a = 0$.By the super harmonicity of F, it implies that $\{F(X_t)_{t\geq 0}\}$ is a super martingale just as $\{e^{-a\sigma} F(X_t)_{t\geq 0}\}$ is a super martingale when F is a-super harmonic.

Now in solving the equation (3.10), we need to basically accomplish two main tasks namely; finding an optimal time τ^* and finding the value of V. Here, we accomplish this by showing that V is the smallest super harmonic function that dominates G and by proving that the stopping time $\tau_D := \inf\{t \geq 0; X_t \in D\}$ is optimal in equation (3.10) whenever $P(\tau_D < \infty) = 1; x \in \mathbb{R}$.

Theorem 2.2. *Let V be a lower semi continuous function and G be a bounded* *upper semi continuous functions, Consider the optimal stopping problem*

$$V(x) = \sup_{\tau \geq 0} E_x G(X_\tau)$$

Then V is the smallest super harmonic function that dominates G and τ_D is optimal for this OSP provided that $P_x(\tau_D < \infty) = 1$.

Proof: Now $V(X_\sigma)$ is measurable since V is measurable (as it can written as a sequence of continuous functions basing on its lower semi continuity). Assume that $X_0 = x$ and fix $x \in \mathbb{R}$, then for each stopping time σ, $V(X_\sigma) = \sup_{\tau \geq 0} E_{X_\sigma} G(X_\tau)$ and by the string Markov property of X, $V(X_\sigma) = \operatorname{ess\,sup}_{\tau \geq 0} E_x(G(X_\tau \circ \theta_\sigma)|\mathcal{F}_\sigma)$, the essential supremum of the expectation E_x. Since the family $\{E_x(G(X_\tau \circ \theta_\sigma)|\mathcal{F}_\sigma); \tau \text{ is a stopping time}\}$ is upwards directed, then there exists a sequence of stopping times $\{\tau_n; n \geq 1\}$ such that $V(X_\sigma) = \lim_{n\to\infty} E_x(G(X_{\tau_n} \circ \theta_\sigma)|\mathcal{F}_\sigma)$ with $\{E_x(G(X_\tau \circ \theta_\sigma)|\mathcal{F}_\sigma); n \geq 1\}$ increasing P_x-a.s. and hence by the monotone convergence theorem, $E_x V(X_\sigma) = \lim_{n\to\infty} E_x(G(X_{\tau_n} \circ \theta_\sigma + \sigma) \leq V(x)$ which completes the first part of the proof. Now we also prove the optimality of τ_D to the equation (3). Fix $\varepsilon > 0$ and define the sets $C_\varepsilon := \{x \in \mathbb{R}; V(x) > G(x) + \varepsilon\}$ and $D_\varepsilon := \{x \in \mathbb{R}; V(x) \leq G(x) + \varepsilon\}$. Notice that $C_\varepsilon \uparrow C$ and $D_\varepsilon \downarrow D$ as $\varepsilon \downarrow 0$. Recall that the stopping times are defined as

$$\tau_{D_\varepsilon}; = \inf\{t \geq 0; X_t \in D_\varepsilon\} \tag{3.11}$$

We have to prove that

(i) $V(x) = E_x V(X_{\tau_{D_\varepsilon}}); \forall x \in \mathbb{R}$

(ii) $\tau_{D_\varepsilon} \uparrow \tau_D$ as $\varepsilon \downarrow 0$

Now to prove part (i) above, suppose we define $c; = \sup_{x\in\mathbb{R}}\{G(x) - E_x V(X_{\tau_{D_\varepsilon}})\} < \infty$, then c is finite since $V(X_{\tau_{D_\varepsilon}}) \leq G(X_{\tau_{D_\varepsilon}}) + \varepsilon$ and G is bounded. Then $G(x) \leq E_x V(X_{\tau_{D_\varepsilon}}) + c; \forall x \in \mathbb{R}$ Now given $\delta > 0$ with $\delta \leq \varepsilon$, choose $y \in \mathbb{R}$ such that $G(y) - E_y V(X_{\tau_{D_\varepsilon}}) \geq c - \delta$ by the supremum definition. Thus $V(y) \geq G(y) + \delta \geq G(y) + \varepsilon \Rightarrow y \in D_\varepsilon \Rightarrow \tau_{D_\varepsilon} = 0 \Rightarrow E_y V(X_{\tau_{D_\varepsilon}}) = E_y V(X_0) = V(y) \Rightarrow 0 \geq G(y) - V(y) \geq c - \delta$. Now given that δ is arbitrary and small, then $c \leq 0$, thus $G(x) \leq E_x V(X_{\tau_{D_\varepsilon}})$ and since $E_x V(X_{\tau_{D_\varepsilon}})$ is super harmonic and yet V is the smallest one that dominates G, then

$$V(x) \leq E_x V(X_{\tau_{D_\varepsilon}}) \tag{3.12}$$

Now because V is super harmonic, we also have the inverse inequality

$$V(x) \geq E_x V(X_{\tau_{D_\varepsilon}}) \tag{3.13}$$

Hence from inequalities (3.12) and (3.13), we have our first result in part (i) above. Now for the second part (ii), note that $V(X_{\tau_{D_\varepsilon}}) \leq G(X_{\tau_{D_\varepsilon}}) + \varepsilon$ over the set D_ε and since $D_\varepsilon \downarrow D$ then there exists a stopping time $\tau_0 \leq \tau_D$ such that $\tau_D \uparrow \tau_0$, hence $V(X_{\tau_{D_\varepsilon}}) \to V(\tau_0), G(X_{\tau_{D_\varepsilon}}) \to G(\tau_0); \forall x \in \mathbb{R} P_x a.s.$ as $\varepsilon \downarrow 0$ by the left continuity of X. Recall that V is lower semi continuous and thus $V(X_{\tau_0} \leq \lim_{\varepsilon\downarrow 0} \inf V(X_{\tau_{D_\varepsilon}}) \leq \lim_{\varepsilon\downarrow 0} \sup G(X_{\tau_{D_\varepsilon}}) \leq G(X_{\tau_0}) \Rightarrow V(X_{\tau_0}) = G(X_{\tau_0})$ and hence $X \in D$ at time τ_0 by the definition of D. This proves that $\tau_D \leq \tau_0 \Rightarrow \tau_D = \tau_0$ ∎

2.0.7 American put option: optimal stopping problem

Now we stretch our strides to now handle the major aim of (attempting to solve) solving the OSP for the American put option. Here we apply the Markovian approach of solving optimal stopping problems and develop an asymptotic solution for the same.

The American put option OSP is one of the group of discounted OSPs and it is defined by;

$$Q(t) = \sup_{t \leq \tau \leq T} E[e^{-r(\tau - t)}(K - S_\tau)_+ \mathcal{F}_t] \tag{3.14}$$

with S the stock price following the stochastic differential equation (SDE)

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{3.15}$$

$r > 0$ being the market interest rate and $\sigma > 0$ the stock volatility, W_t the Wiener process. Solving equation (3.15) for a good solution yields

$$S_t = S_0 exp\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}; \forall t \geq 0 \tag{3.16}$$

Clearly, S_t is follows an Ito process, hence a strong Markov process. Now the infinitesimal generator is

$$L; = rx\frac{\partial}{\partial x} + \frac{\sigma^2}{2}x^2\frac{\partial^2}{\partial x^2} \tag{3.17}$$

since S is time homogeneous. Consider now the OSP

$$V(t, x) = \sup_{0 \leq \tau \leq T - t} E_{t,x} e^{-r\tau}(K - S_{t+\tau})_+ \tag{3.18}$$

Also let $S_t = x$. Then equations (3.18) and (3.14) can be related by the equality

$$Q(t) = V(t, S_t); \forall t \in [0, T]$$

Now the set of stopping times for the equation (3.18) would be defined as;

$$\tau_D = \inf\{0 \leq u \leq T - t; (t + u, S_{t+u}) \in D\} \tag{3.19}$$

since $V(t, x)$ is continuous on $[0, T] \times (0, \infty)$ and $G(x) = (K - x)_+$ is such that $|G(x)| \leq Me^{M|x|}; M > 0$. The set D is then called the stopping set and defined as $D = \{(t, x) \in [0, t] \times (0, \infty); V(t, x) = G(x)\}$ and its complement usually referred to as the continuation region is then defined as $C := \{(t, x) \in [0, t] \times (0, \infty); V(t, x) > G(x)\}$. Now the boundary between C and D is what we refer to as **the optimal exercise boundary**. Obtaining the boundary set between C and D will surely accomplish the American put option pricing problem and this is what we attempt to do in the next sections.

2.0.8 Analytic expression for exercise boundary

Kuske & Keller (1998) have obtained an asymptotic analytic expression for the exercise boundary in their celebrated paper *Optimal exercise boundary for an American put option.* The expression is obtained from solving the integral representation of the exercise boundary (eq 3.4, pp 4) through applying Greens theorem to it. The exercise boundary according to that paper is given by

$$e^{-\rho t} - 1 + I(t) = \frac{1}{\sqrt{2\pi}} \left[e^{b(t)+(1-(\rho-1))t} \int_{-\frac{b(t)+(3-\rho)t}{\sqrt{2t}}}^{\infty} e^{-\frac{z^2}{2}} dz - \int_{-\frac{b(t)-(\rho-1)t}{\sqrt{2t}}}^{\infty} e^{-\frac{z^2}{2}} dz \right] \tag{3.20}$$

where

$$I(t) = \int_0^t \left[-\frac{b(t) - b(s)}{2(t - s)} + \frac{1}{2}(\rho - 1) + b'(s) \right] (e^{\rho s} - 1) G[b(t), t; b(s), s] ds$$

and $G(x,t;\xi,s)$ being Greens function given by;

$$G(x,t;\xi,s) = \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{[z-\xi+(\rho-1)(-t-s)]^2}{4(t-s)}}; s < t$$

and also $\rho = \frac{2r}{\sigma^2}$, z is a normal random variable and $b(t) = \log(\beta(T)/K), \beta(.)$ being the exercise boundary, K is the strike price. Now from equation (3.20), we evaluate the integrals on the right hand side.

2.0.9 Integral evaluation

Now we wish to compute the integral

$$J_a = \int_a^{+\infty} e^{-x^2} dx$$

But let us first compute

$$J_0 := J = \int_0^{+\infty} e^{-x^2} dx$$

Applying double integration techniques along with the change of variables, we have that;

$$J^2 = \left(\int_0^{+\infty} e^{-x^2} dx\right)^2 = \left(\int_0^{+\infty} e^{-x^2} dx\right)\left(\int_0^{+\infty} e^{-z^2} dz\right) = \int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+z^2)} dxdz$$

Now define the transformation $(x,z) = T(r,\theta)$ with $x = r\cos\theta; z = r\sin\theta; r > 0$ and

$$\theta \in \left[0, \frac{\pi}{2}\right]$$

(*notice that the choice of* θ *being in the first quadrant is supported by the fact that the second integral takes on the whole circle and thus yielding the whole area. Using full circles on both would yield ambiguous results as laws of multivariate* *calculus suggest.*)

and then it yields the Jacobian matrix given by;

$$DT = \begin{pmatrix} \cos\theta & \sin\theta \\ \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

and the absolute value of the determinant is given by $|DT| = r$, then we have;

$$J^2 = \left(\int_0^{+\infty} \int_0^{\frac{\pi}{2}} re^{-r^2} drd\theta \right) = \frac{\pi}{4} \Rightarrow J = \frac{\sqrt{\pi}}{2}$$

Hence for J_a, it suffices that

$$J_a = J - \int_0^a e^{-x^2} dx = \frac{\sqrt{\pi}}{2} - \int_0^a e^{-x^2} dx$$

The latter integral can be shown to be approximately equal to;

$$\int_0^a e^{-x^2} dx = ae^{-\frac{a^2}{8}}$$

and hence

$$J_a = \frac{\sqrt{\pi}}{2} - ae^{-\frac{a^2}{8}}$$

Now from the above integral in equation (3.20), we have that

$$J_{a_i} = \frac{\sqrt{\pi}}{2} - a_i e^{-\frac{a_i^2}{8}}; i = 1, 2;\ a_1 = -\frac{b(t) + (3-\rho)t}{\sqrt{2t}} \quad \text{and} \quad a_2 = -\frac{b(t) - (\rho - 1)t}{\sqrt{2t}}$$

which yields

$$\begin{aligned} J_{a_1} &= \frac{\sqrt{\pi}}{2} - a_1 e^{-\frac{a_1^2}{8}} \\ &= \frac{\sqrt{\pi}}{2} - \left(-\frac{b(t) + (3-\rho)t}{\sqrt{2t}} \right) e^{-\frac{\left(-\frac{b(t)+(3-\rho)t}{\sqrt{2t}}\right)^2}{8}} \end{aligned}$$

and

$$J_{a_2} = \frac{\sqrt{\pi}}{2} - a_2 e^{-\frac{a_2^2}{8}}$$
$$= \frac{\sqrt{\pi}}{2} - \left(-\frac{b(t)-(\rho-1)t}{\sqrt{2t}}\right) e^{-\frac{\left(-\frac{b(t)-(\rho-1)t}{\sqrt{2t}}\right)^2}{8}}$$

and J_{a_1} and J_{a_2} substituted in to equation (3.20), we obtain

$$e^{\rho t} - 1 + I(t) = \frac{1}{\sqrt{2\pi}}\left[\frac{\sqrt{\pi}}{2} e^{[b(t)+(1-(\rho-1))t]} + \frac{b(t)+(3-\rho)t}{\sqrt{2\pi}} e^{\frac{16tb(t)+16t^2(1-(\rho-1))-(b(t)+(3-\rho)t)^2}{16t}} - \frac{\sqrt{\pi}}{2} + \left(\frac{(\rho-1)t-b(t)}{\sqrt{2t}} e^{-\frac{1}{16t}[b(t)+(\rho-1)]^2}\right)\right]$$

where

$$I(t) = \int_0^t \left[\frac{1}{4\pi(\sqrt{t-s})} e^{-\frac{1}{16(t-s)^2}[b(t)-b(s)+(\rho-1)(t+s)]}\right] ds; s < t$$

which can be approximated with appropriate integral schemes thus equation (3.20) can be evaluated to obtain an analytic expression for the exercise boundary $b(t)$. Now we move on to study the regularity results, having re-written this problem as a variational inequality in the forth coming section.

2.0.10 The Variational Inequality

Here, we review the existence and uniqueness results for the American option variational inequality. We also derive the transformation for approximating optimal prices (values) basing on a unit square.

2.0.11 Regularity results on the variational inequality

Definition 2.2. *The finite dimensional variational inequality problem $VI(F,K)$ seeks to determine a vector $x^* \in K \subset \mathbb{R}^n$ such that*

$$F(x^*)^T.(x - x^*) \geq 0; \forall x \in K$$

where F is a given continuous function from K to $\mathbb{R}^n$ and K is a closed convex set.

Definition 2.3. Compactness

A space (set) X is said to be compact if for each open cover of X there exists a finite sub collection that also covers X. In other words, X is compact if each open cover of X contains a finite sub cover.

Definition 2.4. Monotonicity

Monotonicity can be defined as either the entire increasing or decreasing nature of a function or sequence. Mathematically, $F(x)$ is monotone on K if

$$[F(x) - F(y)]^T.(x - y) \geq 0; \forall x, y \in K$$

Strong monotonicity of $F(x)$ on K then is such that given $\mu > 0$,

$$[F(x) - F(y)]^T.(x - y) \geq \mu ||x - y||^2; \forall x, y \in K$$

and strict monotonicity of $F(x)$ on K is such that;

$$[F(x) - F(y)]^T(x - y) > 0; \forall x, y \in K, x \neq y$$

Definition 2.5. Lipschitz continuous

$F(x)$ is Lipschitz continuous on K is there exists a $C > 0$ such that

$$||F(x) - F(y)|| \leq C||x - y||; \forall x, y \in K$$

Definition 2.6. Coercivity

$F(x)$ is said to be coercive if it satisfies the coercivity condition

$$\frac{(F(x) - F(y))^T(x - y)}{||x - y||} \rightarrow \infty$$

as $||x|| \rightarrow \infty$ for $x \in K$ and for some $y \in K$.

2.0.12 General variational inequality

Definition 2.7. *Let $X : [0, +\infty) \times \Omega \to \mathbb{R}^n$ defined on a probability space (Ω, Σ, P) be an Ito diffusion satisfying the stochastic differential equation of the form*

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

where W is an m-dimensional Wiener process and $\mu : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}n \times m$ are the drift and diffusion fields respectively. For a point $x \in \mathbb{R}^n$, let P^x denote the law of X with $X_0 = x$ and let E^x be the expectation with respect to P^x. The infinitesimal generator of X is the operator A, which is defined to act suitable functions $f : \mathbb{R}^n \to R$ by

$$Af(x) = \lim +t \to 0 \frac{E^x[f(X_t)] - f(x)}{t}$$

Definition 2.8. *Dynkin's formula is a theorem giving the expected value of any suitably smooth statistic of an Ito diffusion at a stopping time. At times, it is understood as the stochastic generalization of the (second) fundamental theorem of calculus.*

Statement

Let X be the $\mathbb{R}^n$ valued Ito diffusion solving the stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

For a point $x \in \mathbb{R}^n$, let P^x denote the law of X with $X_0 = x$ and let E^x be the expectation with respect to P^x. Let A be the infinitesimal generator of X, defined *by its action on compactly supported C^2, then*

$$Af(x) = \sum_i b_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

Now if τ is a stopping time with $E^x[\tau] < +\infty$ and f is C^2 with compact support, then Dynkins formula holds such that

$$E^x[f(X_\tau)] = f(x) + E^x\left[\int_0^\tau Af(X_s)ds\right]$$

Consider the *Dynkin*-operator $\mathcal{D}$ associated with the stochastic differential equation given by;

$$\mathcal{D} = \frac{\partial}{\partial t} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial}{\partial x_i} \tag{3.21}$$

where

$$a(t,x) = [a_{ij}(t,x)]_{1\le i,j\le d} = \sigma(t,x) \times \sigma^t(t,x)$$

Now suppose X is a solution to the SDE 3.21 on $[0,T]$ and let $\beta_t = e^{\left\{-\int_0^t r(r,X_s)ds\right\}}$, then for $F \in C^{1,2}$ on $[0,T] \times \mathbb{R}^d$,

$$\beta_t F(t,X_t) = F(0,X_0) + \int_0^t \beta_s \nabla F(s,X_s)\sigma(s,X_s)dW_s + \int_0^t \beta_s(\mathcal{D}F - rF)(s,X_s)ds$$

with

$$\nabla F(s,X_s)\sigma(s,X_s)dW_s = \sum_{i=1}^{d}\frac{\partial F}{\partial x_i}(s,X_s)\sum_{j=1}^{d}\sigma_{ij}(s,X_s)dW_s^j$$

Now for the process $(\beta_t F(t,X_t))_{0\le t\le T}$ to be the Snell envelope of the discounted payoff process $(\beta_t f(t, X_t))_{0\leq t\leq T}$ we need

$$\mathcal{D}F - rF \leq 0$$

$$F \geq f$$

$$F(T,.) = f(T,.)$$

$$\mathcal{D}F - rF = 0 \text{ on the set } \{F > f\}$$

which can be summarized as;

$$\begin{cases} \max(\mathcal{D}F - rF; f - F) = 0 \\ F(T,.) = f(T,.) \end{cases}$$

2.0.13 American options variational inequality

Now for the American option (put specifically), the *Dynkin*-operator is given by;

$$\frac{\partial}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \mu\frac{\partial}{\partial x}$$

and

$$dX_t = \mu dt + \sigma dW_t \tag{3.22}$$

where $X_t = \log(S_t)$ with $\mu = r - \delta - \frac{\sigma^2}{2}$. If X^* is the solution to equation 3.22, then $X_t^* = x + \mu t + \sigma W_t; X_0^* = x$. The American put price denoted by $P(t,x) = F(t, \log x)$ where

$$F(t,x) = \sup_{\tau\in\mathcal{T}_0,T-t} \mathbb{E}e^{-r\tau} f(X_\tau^*) \tag{3.23}$$

with $f(x) = (K - e^x)^+$.

Now notice that all the partial derivatives above i.e. $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial t}, \frac{\partial^2 F}{\partial x^2}$ are locally bounded. Also F, as is defined in equation 3.23 satisfies the variational inequality

$$\begin{cases} \max(\mathcal{D}F - rF; f - F) = 0 \\ F(T,.) = f(T,.) \end{cases}$$

$dtdx$ a.e. in $(0,T) \times \mathbb{R}$.

Theorem 2.3. Uniqueness (Narguney; (2002))

Suppose that $F(x)$ is strictly monotone on K. Then the solution to the variational inequality $VI(F,K)$ is unique, if one exists.

Proof:

Suppose $x \neq y$ are both solutions to the variational inequality $VI(F,K)$. Then

$$F(x)^T.(z - x) \geq 0; \forall z \in K \tag{3.24}$$

$$F(y)^T.(z - y) \geq 0; \forall z \in K \tag{3.25}$$

After substituting y for z in 3.24 and x for z in 3.25 and adding the two resulting inequalities, we obtain

$$(F(x) - F(y))^T.(y - x) \geq 0 \tag{3.26}$$

but inequality 3.26 contradicts the definition of strong monotonicity. Hence

$$x = y$$

■

Theorem 2.4. Existence and Uniqueness of solutions to Variational Inequalities (Narguney; (2002))

Assume that $F(x)$ is strongly monotone, then there exists precisely one solution x^* to the variational inequality $VI(F,K)$.

Proof: (Heuristic[1])

Strong monotonicity implies coercivity and also strong monotonicity implies strict monotonicity. Thus for an unbounded feasible set K, existence and uniqueness are established by strong monotonicity. Also, if K is compact, the continuity of F assures us of existence of a solution and strict monotonicity suffices for uniqueness.

Hence since for the American options variational inequality $VI(F, K)$, $K = [0, T]$ is compact and $F = F(t, x) = \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E} e^{-r\tau} f(X^*_\tau); f(x) = (K - e^x)^+$ is continuous, $VI(F, K)$ admits a unique solution. ∎

3 The Option Price Transformation

We now embark on establishing a transformation for the optimal boundary. We define this transformation from a unit square a set R_ε whose area we approximate using quadrature numerical integration techniques. The area of R_ε is, here, analogous to the measure of the disjoint sets A_i and thus we can easily evaluate the American option using the value equation supplied in literature. Here, we evaluate for the put option as the call option can also be equally evaluated or rather employ the put-call parity relationship. We also consider the value equation supplied in Kim, 1990 (pp. 560). Notice that

$$\lim_{\tau \to 0} V(S, \tau, B(\tau)) = f \tag{3.27}$$

[1]For a more concrete proof of theorem 2.4, see Nagurney; 2002 or any other good text on Variational Inequalities

$V()$ being the option value and the pay off f is given by;

$$f = max[0, (K - S)^{+}] \tag{3.28}$$

$$\forall \varepsilon > 0, \text{there} \quad \text{exists} \eta_{\varepsilon} > 0 \text{ s.t } \forall \tau > 0, |\tau| \leq \eta_{\varepsilon}$$

$$\Rightarrow |V(S, \tau, B(\tau)) - f| \leq \varepsilon$$

$$\Rightarrow V(S, \tau, B(\tau)) \in [-\varepsilon + f, \varepsilon + f]$$

whenever $\tau \in [-\eta_{\varepsilon}, \eta_{\varepsilon}]$. This basically explains that the option value is always going to lie with in some value from the payoff. In brief, we can say it as the option value can not exceed the payoff of the option otherwise the option is not worth any thing then. Having noticed that the option value is at most the option payoff, we now analytically define some parts of the free boundary. This is intended in mathematically explaining why (and how) the option value is at most the option payoff. So set

$$S_1 = \{(S, \tau); B(\tau) < S \leq \infty, \tau \in [-\eta_{\varepsilon}, \eta_{\varepsilon}] \cap (0, T] = [0, \eta_{\varepsilon}] \cap (0, T)\} \tag{3.29}$$

and also that

$$V(S, \tau, B(\tau)) \in [-\varepsilon + f, \varepsilon + f]$$

Now the standard American put option valuation equation is given by;

$$K - B(\tau) = P(B(\tau), \tau) + \int_0^{\tau} \left[rKe^{-r(\tau-\xi)} N(-d_2(B(\tau), \tau - \xi, B(\xi))) \right\}$$

$$-\delta B(\tau) e^{-\delta(\tau-\xi)} N(-d_1(B(\tau), \tau - \xi, B(\xi))) \big] \, d\xi \tag{3.30}$$

And

$$\lim_{S \to B(\tau)} V(S, \tau, B(\tau)) = K - B(\tau); \forall \tau \in (0, T]$$

Set $g = K - B(\tau)$ then;

$$\forall \varepsilon > 0, \text{there} \quad \text{exists}\, \delta_\varepsilon > 0, \forall S \,\text{such that}\, |S - B(\tau)| \leq \delta_\varepsilon$$

$$\Rightarrow |V(S, \tau, B(\tau)) - g| \leq \varepsilon$$

Again, set

$$S_2 = \left\{ (S, \tau); \begin{array}{c} B(\tau) < S \leq \infty; \\ \\ \tau \in (0, T); \\ \\ S \in [B(\tau), B(\tau) + \delta_\varepsilon] \\ \\ V(S, \tau, B(\tau)) \in [-\varepsilon + g, \varepsilon + g] \end{array} \right\} \quad (3.31)$$

$$\lim_{S \to 0} V(S, \tau, B(\tau)) = 0; \forall \tau \in (0, T]$$

$$\forall \varepsilon > 0 \,\text{there} \quad \text{exists}\, \gamma_\varepsilon; \forall S \; (S > B(\tau))$$

$$\text{such that } |S| \leq \gamma_\varepsilon \Rightarrow |V(S, \tau, B(\tau))| \leq \varepsilon$$

Lets again set S_3 to be defined by;

$$S_3 = \left\{ (S, \tau); \begin{array}{c} \tau \in (0, T] \\ \\ B(\tau) < S \leq +\infty \\ \\ S \in [0, \gamma_\varepsilon]; \\ \\ V(S, \tau, B(\tau)) \in [0, \varepsilon] \end{array} \right\} \quad (3.32)$$

$\lim_{S \to B(\tau)} V_S(S, \tau, B(\tau)) = -1$ and thus

$$\forall \varepsilon > 0; \text{there} \quad \text{exists}\lambda_\varepsilon > 0$$

$$\text{such that}\, \forall S > B(\tau); |S - B(\tau)| \leq \lambda_\varepsilon$$

$$\Rightarrow |V_s + 1| \leq \varepsilon$$

where $V_s = \frac{\partial V}{\partial S}(S, \tau, B(\tau))$. Set also that

$$S_4 = \left\{ (S, \tau); \begin{array}{c} \tau \in (0, T] \\ \\ B(\tau) < S \leq +\infty \\ \\ S \in [-\lambda_\varepsilon + B(\tau), \lambda_\varepsilon + B(\tau)]; \\ \\ V_S \in [-1 - \varepsilon, -1 + \varepsilon] \end{array} \right\} \tag{3.33}$$

The optimal exercise boundary is $B(\tau)$ and the free boundary is the set given by;

$$\mathcal{B} = \left\{ (\tau, B(\tau)); \tau \in (0, \tau], B(0) = \lim_{\tau \to 0} B(\tau) = \left\{ \begin{array}{c} K; \delta \leq r \\ \\ \frac{r}{\delta}; \delta > r \end{array} \right\} \right\} \tag{3.34}$$

Recall that $A_i = (\tau_i, \tau_{i+1}]; A_i \cap A_j = \phi$ for $i \neq j; \bigcup_{i=0}^{n} A_i = (0, T]$. Now

$$\mathcal{B} = \bigcup_{i=1}^{n} \mathcal{B}_i$$

where $\mathcal{B}_i = \{(\tau, B(\tau)); \tau \in A_i\}$ Now consider the coordinates (t, y) and the curve $f := l$ with a trajectory given by the definition $y = l(\tau)$, the curve of f, can be somewhat like as in figure (3.2) below.

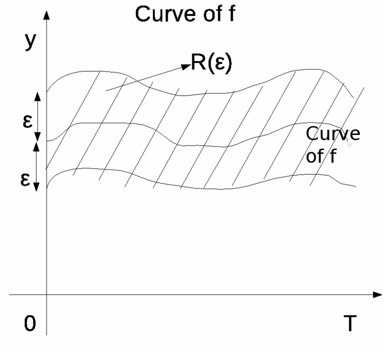

Figure 3.2: Plot of pay off function, f

$$\text{Now } y = l(\tau);$$

$$\tau \in [0, T]$$

$$R_\varepsilon = \{\tau, l(\tau), \tau \in [0, T] \text{ and } -\varepsilon \leq l(\tau) \leq \varepsilon\}$$

The aim is to locate $\mathcal{B}$ and the area of R_ε is given by;

$$Area(R_\varepsilon) = \int_{\tau_i}^{\tau_{i+i}} \int_{-\varepsilon}^{\varepsilon} l(\tau) d\tau dy = 2\epsilon \int_{\tau_i}^{\tau_{i+1}} l(\tau) d\tau \tag{3.35}$$

The next question now is how to compute or express R_ε^i. We approximate $Area(R_\varepsilon)$ through employing the Gaussian quadrature method of integration on a unit square.

3.1 Quadrature approximation of Area (R_ε)

In this sub section, we apply quadrature techniques to obtain an approximate for the area of R_ε. Quadrature approximation has merits such as the ease with which it can be applied as well as its accuracy. Still the fact that all nodes lie with in the interior of the main interval guarantees that even integrals with functions that tend to infinite value at one end of the interval can be handled (of course given that the integral is defined there). Recall that

$$Area(R_\varepsilon) = \int_{\tau_i}^{\tau_{i+i}} \int_{-\varepsilon}^{\varepsilon} l(\tau) d\tau dy = 2\epsilon \int_{\tau_i}^{\tau_{i+1}} l(\tau) d\tau \tag{3.36}$$

and computing the value of the integral, we employ the Gaussian quadrature approximation technique.

$$\int_{\tau_i}^{\tau_{i+1}} l(\tau) d\tau \cong \sum_{i=0}^{n} A_i l(\tau_i) \tag{3.37}$$

and $\tau_0 = -\varepsilon, \tau_n = \varepsilon$ where

$$A_i = \int_{\tau_i}^{\tau_{i+1}} \left[\prod_{j=0, j\neq i}^{n-1} \left(\frac{\tau - \tau_j}{\tau_i - \tau_j} \right) \right] d\tau \tag{3.38}$$

and now we notice that for $n = 2$, we have that

$$\int_{\tau_i}^{\tau_{i+1}} l(\tau) d\tau \cong \sum_{i=0}^{n} A_i l(\tau_i) = A_0 l(x_0) + A_1 l(x_1)$$

and

$$A_0 = \int_{\tau_i}^{\tau_{i+1}} \left[\left(\frac{\tau - \tau_1}{\tau_0 - \tau_1} \right) \left(\frac{\tau - \tau_2}{\tau_0 - \tau_2} \right) \right] d\tau; A_1 = \int_{\tau_i}^{\tau_{i+1}} \left[\left(\frac{\tau - \tau_0}{\tau_1 - \tau_0} \right) \left(\frac{\tau - \tau_2}{\tau_1 - \tau_2} \right) \right] d\tau$$

and hence

$$\begin{aligned} \int_{\tau_i}^{\tau_{i+1}} l(\tau) d\tau \cong \sum_{i=0}^{n} A_i l(\tau_i) = l(\tau_0) \int_{\tau_i}^{\tau_{i+1}} \left[\left(\frac{\tau - \tau_1}{\tau_0 - \tau_1} \right) \left(\frac{\tau - \tau_2}{\tau_0 - \tau_2} \right) \right] d\tau \\ + l(\tau_1) \int_{\tau_u}^{\tau_{i+1}} \left[\left(\frac{\tau - \tau_0}{\tau_1 - \tau_0} \right) \left(\frac{\tau - \tau_2}{\tau_1 - \tau_2} \right) \right] d\tau \end{aligned}$$

Consequently with increase in the n value, there will be an increase in the terms on the expansion thus increasing accuracy. So then recall the equation (3.35) which is;

$$Area(R_\varepsilon) = 2\varepsilon \int_{\tau_i}^{\tau_{i+1}} l(\tau) d\tau$$

which is the required exercise boundary i.e.

$$Area(R_\varepsilon) = 2\varepsilon \int_{\tau_i}^{\tau_{i+1}} l(\tau) d\tau = 2\varepsilon \left\{ \sum_{i=0}^{n} l(\tau_i) \int_{\tau_i}^{\tau_{i+1}} \left[\prod_{j=0, j\neq i}^{n-1} \left(\frac{\tau - \tau_j}{\tau_i - \tau_j} \right) \right] d\tau \right\} \tag{3.39}$$

Now having obtained the area of the co-domain of our desired transformation, we now move on to establish this transformation from a unit square to the set R_ε whose area has been computed in the preceding sub section. This transformation is designed in way that it ought to satisfy some conditions especially at the grid bounds of the unit square. Other values within the square can be consequently computed with reference to the bound values using any mathematically efficient numerical scheme as shall be seen in section 4. We define the transformation to be

$$T : [0, 1]^2 \to R_\varepsilon$$

such that

$$T : (x, z) \to T(x, z) = \left\{ \begin{matrix} T_1(x, z) \\ T_2(x, z) = (\tau, z);\ such\ that\ |z - l(\tau)| \leq \varepsilon \end{matrix} \right\} \tag{3.40}$$

and on the boundaries of the unit square, it is clear that

$$\left\{ \begin{matrix} T(0,0) = (\tau_i, l(\tau_i) - \varepsilon) \quad T(0,1) = (\tau_i, l(\tau_i) + \varepsilon) \\ T(0,1) = (\tau_{i+1}, l(\tau_{i+1} - \varepsilon); \quad T(1,1) = \tau_{i+1}, l(\tau_{i+1} + \varepsilon) \\ \text{i.e.} \\ T_1(0,0) = \tau_i \qquad\qquad T_2(0,0) = l(\tau_i - \varepsilon) \\ T_1(0,1) = \tau_i \qquad\qquad T_2(0,1) = l(\tau_i + \varepsilon) \\ T_1(1,0) = \tau_{i+1} \qquad\qquad T_2(1,0) = l(\tau_{i+1} - \varepsilon) \\ T_1(1,1) = \tau_{i+1} \qquad\qquad T_2(1,1) = l(\tau_{i+1} + \varepsilon) \end{matrix} \right\} \tag{3.41}$$

So a transformation that satisfies both of these equations (3.40) and (3.41) would be our appropriate result to use in the analytic approximation of the option optimal prices alongside their corresponding optimal times.

3.2 Linear transformation

Now note that;

$$T_1(0,0) = \tau_i, \;\; T_1(1,0) = \tau_{i+1}$$

and also that T_1 is the part of the transformation that evaluates values on the x-axis of the unit square. Hence we have the values as depicted in table (3.1) and interpolating these results linearly in x yields;

Table 3.1: Interpolation Table for transformation $T : T_1$

(x,z)	$(0,0)$	$(x,0)$	$(1,0)$
$T_1(x,z)$	$T_1(0,0) = \tau_i$	$T_1(x,0)$	$T_1(1,0) = \tau_{i+1}$

$$\frac{1-0}{\tau_{i+1}-\tau_i} = \frac{x-0}{T_1(x,0)-\tau_i}$$

$$\Rightarrow T_1(x,0) - \tau_i = x(\tau_{i+1} - \tau_i)$$

$$\Rightarrow T_1(x,0) = \tau_i + x(\tau_{i+1} - \tau_i)$$

Consequently, a similar expression can be obtained for various other values of y provided they are assumed constant and only x varying. In general, the transformation for values along the x-axis of the unit square is given by;

$$T_1(x,z) = \tau_i + x(\tau_{i+1} - \tau_i) \tag{3.42}$$

And also for the second piece of the transformation i.e. in the y-direction we apply interpolation. However here we note that the $\mathcal{T}_{i_2}$ varies on two indices i and the x concurrently since the optimal value would have to depend on the optimal times that is in an economically meaningful sense. Having taken that into consideration, we use the following table 3.2 and and on linearly interpolating;

Table 3.2: Interpolation Table for transformation $T : T_2$

(x,z)	$(1,0)$	$(1,z)$	$(1,1)$
$T_2(x,z)$	$T_1(1,0) = l(\tau_{i+1}) - \varepsilon$	$T_2(0,z)$	$T_1(1,0) = l(\tau_{i+1}) + \varepsilon$

$$\frac{1-z}{l(\tau_{i+1}) + \varepsilon - T_2(1,z)} = \frac{1-0}{l(\tau_{i+1}) + \varepsilon - l(\tau_{i+1}) + \varepsilon}$$

$$\Rightarrow l(\tau_{i+1}) + \varepsilon - T_2(1,z) = 2\varepsilon(1-z)$$

$$\Rightarrow T_2(1,z) = l(\tau_{i+1}) + \varepsilon - 2\varepsilon(1-z)$$

Also, a similar expression can be obtained for various other values of x provided they are assumed constant and only y varying. hence in general, the transformation for values along the y-axis of the unit square is given by;

$$T_2(x,z) = l(\tau_{i+1}) + \varepsilon - 2\varepsilon(x-z)] \tag{3.43}$$

In a summary, the transformation would then be defined as in the next proposition which is one of the major results of this work.

Proposition 3.1. Linear transformation

Define $\mathcal{T}_i : [0,1]^2 \to R^i_\varepsilon$ *by*

$$\mathcal{T}_i(x,z) = \left\{ \begin{array}{c} \mathcal{T}_{i_1}(x,z) = \tau_i + x(\tau_{i+1} - \tau_i) \\ \\ \mathcal{T}_{i_2}(x,z) = l(\tau_{i+1}) + \varepsilon - 2\varepsilon(x-z) \end{array} \right\}$$

where $[0,1]^2$ *is a unit square, then* $\mathcal{T}_i$ *is a bijection.*

$\mathcal{T}_{i_1}$ *in this case represents the optimal times as* $\mathcal{T}_{i_2}$ *represents the optimal value of the option at node* i *whose location is geometrically* (x,z)*.*

Proof: The first part of the proof is to prove that $\mathcal{T}_i$ is one-to-one and the second is to prove that it is onto. Consider the associated vector transformation of $\mathcal{T}_i$, say $\mathcal{H}_i$ i.e.

$$\mathcal{H}_i = \begin{Bmatrix} \mathcal{H}_i^1(x,z) = x(\tau_{i+1} - \tau_i) \\ \mathcal{H}_i^2(x,z) = -2\varepsilon(x-z) \end{Bmatrix}$$

Notice that

$$Ker(\mathcal{H}_i) = \{0\} \Rightarrow Ker(\mathcal{T}_i) = 0$$

hence $\mathcal{T}_i$ is one-to-one. So we now prove that $\mathcal{T}_i$ is onto. Now, for every $(\tau, l(\tau)) \in R_\varepsilon^i$, there exists a point $(x,z) \in [0,T]^2$ for which we have that

$$\mathcal{T}_i(x,z) = \begin{Bmatrix} \mathcal{T}_{i_1}(x,z) = \tau_i + x(\tau_{i+1} - \tau_i) \\ \mathcal{T}_{i_2}(x,z) = l(\tau_{i+1}) + \varepsilon - 2\varepsilon(x-z) \end{Bmatrix}$$

Hence $\mathcal{T}_i$ is onto. Therefore $\mathcal{T}_i$ being onto as well as one-to-one implies that $\mathcal{T}_i$ is a bijection. ■

3.3 Non-linear transformation

Now, we have a linear transformation that could be used to approximate the payoff values of the option over time. Nevertheless, we remark that the approximations from it would be too inaccurate as option payoffs are known not to be linear over time otherwise. It is rather evident that option prices and their corresponding pay-offs follow Ito processes and not log normal processes (even though the two are somewhat related). So we need to consider this in the approximation of payoff values from the bound values of the same. We thus employ techniques borrowed from (Kim, 1990) the area of stochastic interpolation[2]; *the type of interpolation in which we approximate functional values for random (stochastic) data*; with some modifications so as to suit our problem here. Consider an Ito process

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{3.44}$$

where μ is the drift term (also called the mean of the process) and σ is the volatility, W_t is the standard Wiener process and S_t is the price of the underlying asset on which the option is written. Since we have to consider the variation of the point (optimal times) whose payoff value we seek from both ends, we propose a method that takes this into consideration and there after demonstrate its accuracy as it proves to be better than most known. Consider $(\tau_0, l(\tau_0) = l_0)$ and $(\tau_n, l(\tau_n) = l_n)$ and that we wish to know the optimal payoff corresponding to time τ_k, the technique below can help swipe away the high variations and approximate an appropriate value. This technique is basically given by;

$$l_k = l(\tau_k) = \frac{1}{2}\left(\frac{l_0 P_k}{1 - P_k} + \frac{l_n(1 - P_k)}{P_k}\right) \tag{3.45}$$

where

$$P(\tau \leq k) = P_k = \frac{1}{\tau_k \sigma \sqrt{2\pi}} exp\left\{\frac{(\ln \tau_k - \mu)^2}{2\sigma^2}\right\} \tag{3.46}$$

the log-normal probability density function. Using the technique given in equation (3.45), we can reliably approximate the payoff values and thus define a trans-

[2]first introduced by (Gandin, 1970)

formation from the unit square using this approximation. So then we now organize the preceding results into a proposition which follows right away.

Proposition 3.2. *Suppose the price S_t of an underlying asset for a given American put option follows an Ito process, then the option pay off can be approximated by the transformation*

$$\mathcal{T}_i(x,z) = \left\{ \begin{array}{c} \mathcal{T}_{i_1}(x,z) = \tau_i + x(\tau_{i+1} - \tau_i) \\ \\ \mathcal{T}_{i_2}(x,z) = l(\tau_{i+1}) + \varepsilon - 2\varepsilon(x-z) \end{array} \right\}$$

where

$$l(\tau_{i+1}) = l_{i+1} = \frac{1}{2}\left(\frac{l_0 P_{i+1}}{1-P_{i+1}} + \frac{l_n(1-P_{i+1})}{P_{i+1}}\right) \tag{3.47}$$

and $[0,1]^2$ is a unit square. $\mathcal{T}_{i_1}$ in this case are the optimal times as $\mathcal{T}_{i_2}$ are the optimal value of the option at node i whose location is geometrically (x_k, y_k) and R^2_ε the optimal exercise boundary.

Proof: Notice that the proof can be done in the very exact way as in proposition (3.1) with a change of $l(.)$ into a stochastic representation now. Never the less, we provide a heuristic one here. Suppose the hypothesis in proposition 3.2 holds, then we prove that this transformation is indeed a better approximation compared to the previous one. Consider the extreme points (τ_0, l_0) and (τ_n, l_n), the the probability that we approximate and obtain the functional value for $\tau_0 \leq \tau_k \leq \tau_n$ is P_k and the functional value l_k is such that

$$l_k = \frac{l_0 P_k}{1-P_k}$$

from the start of the interval. Also from the interval end it would then be given as;

$$l_k = \frac{l_n(1 - P_k)}{P_k}$$

Hence from either end; we have that

$$l_k = \frac{1}{2}\left(\frac{l_0 P_k}{1 - P_k} + \frac{l_n(1 - P_k)}{P_k}\right)$$

$\blacksquare$

So the task remains to demonstrate that our results concur with this in all ways. But before embarking on that we desire to note some properties of a good approximation for the put price that we can perhaps look out for from our results.

Chapter 4

Numerical Results

In this section we demonstrate the numerical results of this project. We develop algorithms for simulation of optimal option prices from the transformation. Now we proceed to derive and demonstrate the numerical approximations of the method in sub section 3. So the task remains to demonstrate that our results concur with this in all ways. But before embarking on that we desire to note some properties of a good approximation for the put price that we can perhaps look out for from our results.The algorithm used to simulate the forthcoming option price plot is given as;

- Select T, the expiration time of the option, r the interest rate and δ, the volatility.
- $\tau \in [0, T], \varepsilon = \frac{1}{10000}$, Choose $\eta_\varepsilon = 1$
- S_t, K, the price of the underlying asset and the strike price respectively are both given and known.

- $B(0) = \lim_{\tau \to 0} B(\tau) = \begin{cases} K; \delta \leq r \\ \\ \frac{r}{\delta}; \delta > r \end{cases}$

- $f = \lim_{\tau \to 0} V(S_t, \tau, B(\tau)) = max[0, (K - S_t)\phi(S_t - B(0))]$

- For $\tau \in (0, \eta_\varepsilon] \cap (0, T)$ and $V(S_t, \tau, B(\tau)) \in [-\varepsilon + f, \varepsilon + f], B(\tau) < S_t \leq \infty$,

$$S_1 = (S_t, \tau)$$

- Plot S_1

and the exercise boundary is obtained graphically as depicted in figure (4.1).

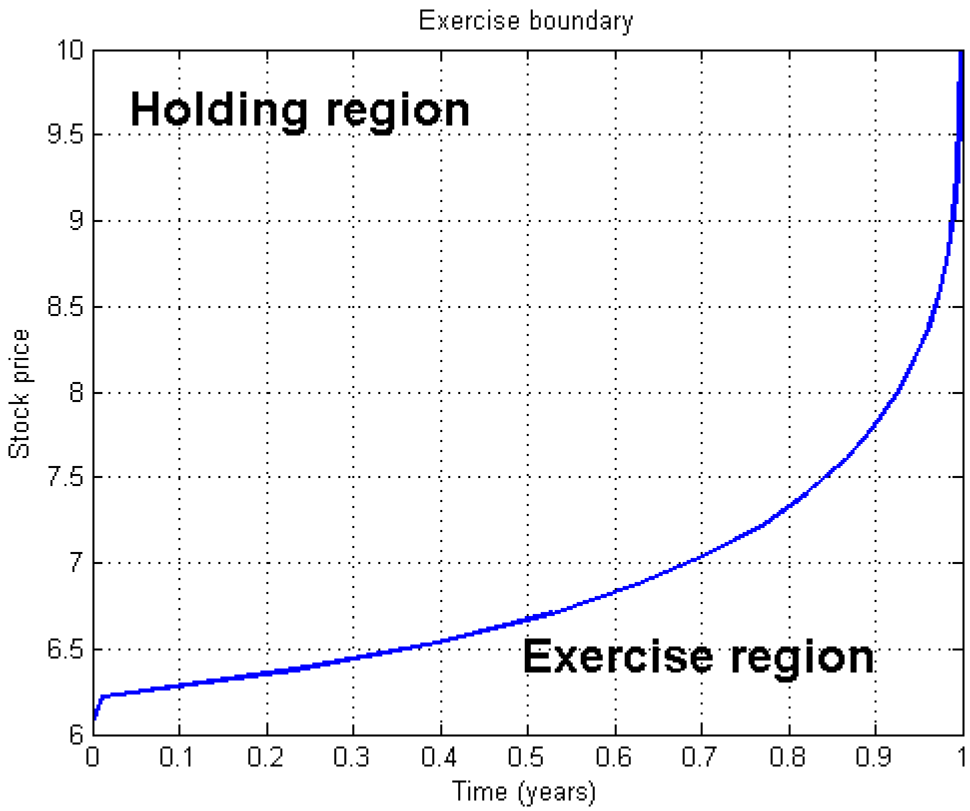

Figure 4.1: American option exercise boundary

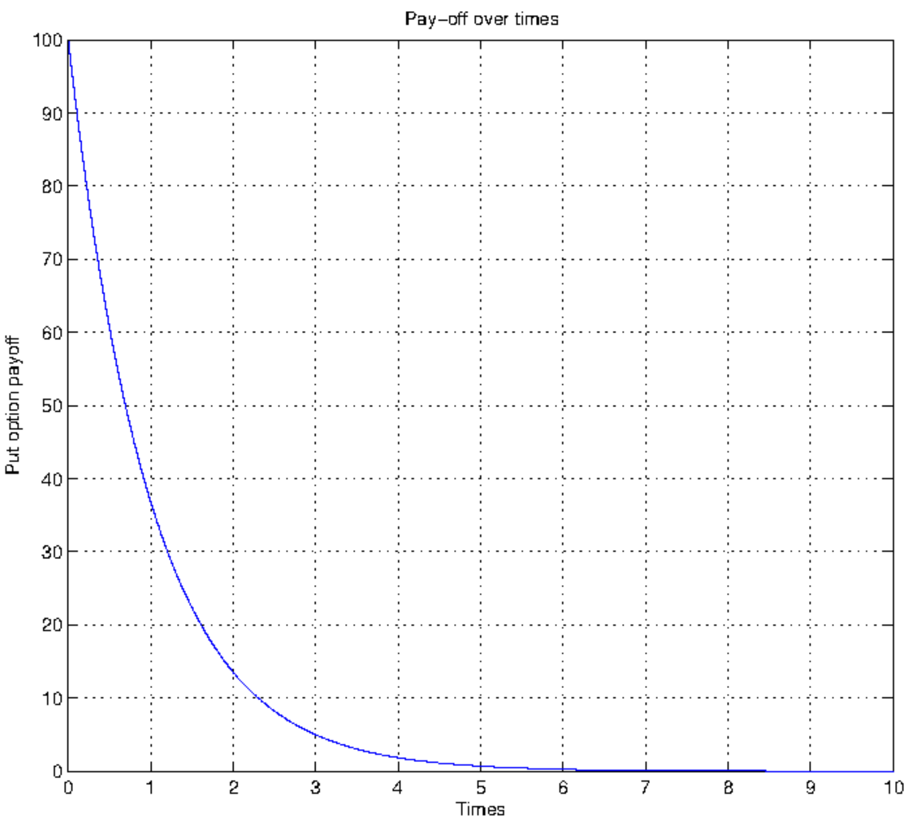

Figure 4.2: American put payoff

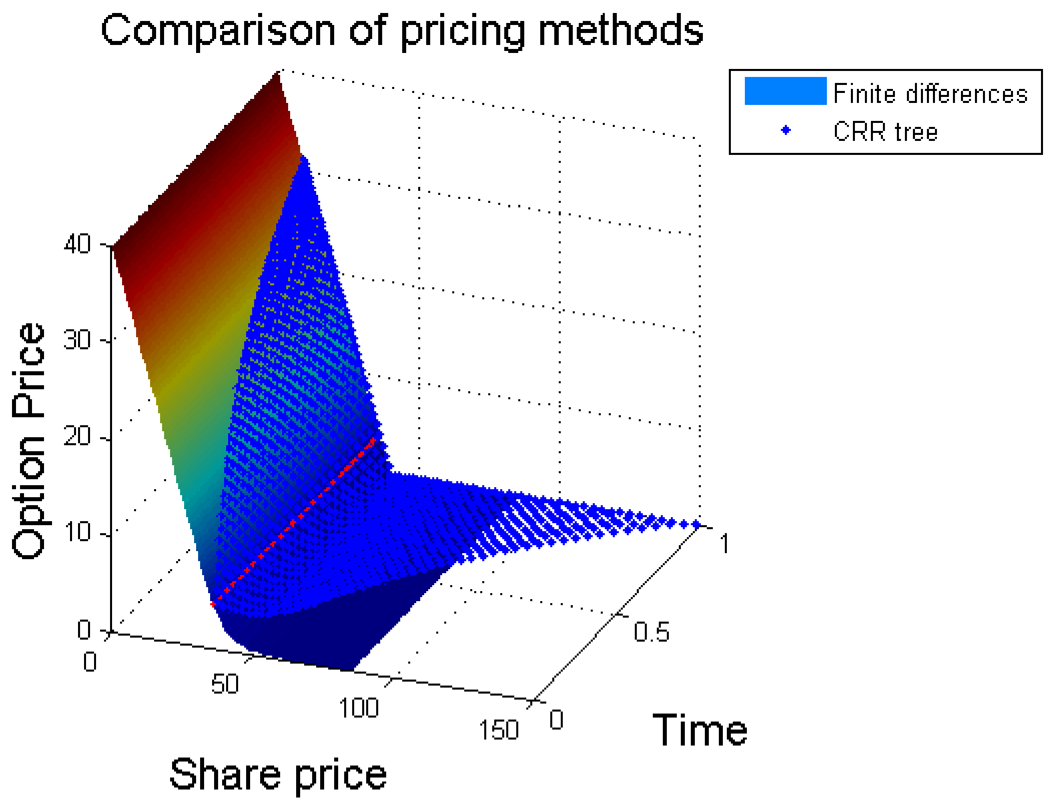

Figure 4.3: Comparison: FDM Vs CRR

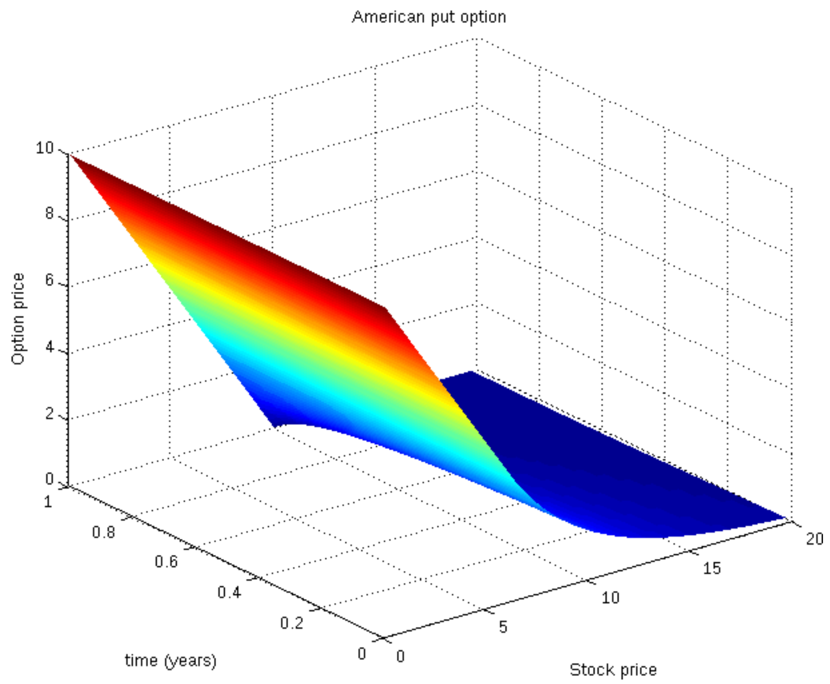

Figure 4.4: 3D American put option

Now for the exercise boundary plot, as the stock price grows over time there is at first a gradual expansion in the size of the exercise region with a concurrent reduction in the holding region. Also at about $t = \frac{T}{2} = 0.5$, the expansion of the exercise region starts to grow at an exponential rate. Now we next study a 3D plot of option prices against stock prices over time. This is depicted in figure 4.4. This plot clearly informs us of the various relationships among these three. Notice that the structure of the exercise boundary (region) is depicted along the x-axis (time) of this plot which is analogous to the structure obtained for the exercise boundary plot which is graphed as stock prices over time.

Notice also that the shape of the variation of the payoff for an American option is also reflected here in (Option prices axis) over time. This, in a nut shell is a plot that summarizes all the plots into one. So all conclusions made regarding the other previous two plots still hold under figure 4.4.

Chapter 5

Results & Discussion

Here, we have provided a far much simpler way of approximating option values as well as payoffs basing on a unit square. Most approximation techniques provided in literature tend to be sophisticated and some what cumbersome at specific times of the option. However, our method stays put in regard to application through out the entire life of the option. We have demonstrated that an option value can be approximated through basing on the unit square to acquire far better accurate results. This beats most approximation techniques already in existence. This method also exceeds others in terms of simplicity of application coupled with accuracy of results. The major objective of this work has been achieved as it was majorly providing an easier way of approximating the payoff using a transformation from a unit square to the exercise boundary. This has been achieved. The transcendence of our method is evidenced by the fact that when approximating pay offs, one works with in a known set, the unit square. More over, our method can easily be run on a computer and the average running time is so minimal as compared to the Binomial Scheme and the Finite Difference Scheme.

Chapter 6

Recommendations & Conclusion

Further work may be needed to be done in this area to improve the results such as considering better and more efficient non linear approximation (interpolation) schemes such as ordinary Kriging, universal Kriging. Notice that considering these approximation schemes would better the results (in terms of accuracy) as variance is minimized. However, such methods were beyond the scope of this work.

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