ON THE EIGEN VALUES OF A NORLUND INFINITE MATRIX AS AN OPERATOR ON SOME SEQUENCE SPACES

GODRICK FELIX OCHIENG

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DECLARATION

This research is my original work and has not been presented elsewhere for a degree in any other University.

Signature…………………………………. **Date**………………………………….

Godrick Felix Ochieng

Declaration by Supervisors

This Thesis has been submitted for examination with our approval as University

Supervisors.

1. Dr. Jotham Raymond Akanga.

   JKUAT, Kenya

   Signature…………………………………. **Date**………………………………….

2. Prof. Augustus N. Wali.

   South Eastern Kenya University, Kenya

   Signature…………………………………. **Date**………………………………….

DEDICATION

To my parents , Joseph Achola and Lucy Achola for their all time surpport and prayers . My siblings Cosmas, Veronica and Pamela for their surpport, encouragement and prayers.

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I thank God of His guidance and strength wihout which this would not have been possible.

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NOTATIONS

$\|.\|$ norm of

$\sharp$ end of proof

In general, $\{...\}$ will denote the set of , $(...)$ the set sequence of and$(...)^t$ the transpose of the sequence of ; unless otherwise specified.

c_0 the space of sequences which converge to zero (null sequences)

c the space of convergent sequences

bv_0 the space of null bounded variation

ℓ_1 the space of absolutely convergent sequences

ABSTRACT

In various papers some authors have previously investigated and determined the spectrum of weighted mean matrices considered as bounded operators on various sequence spaces. It is evident that no much research has been done on the spectrum of Norlund matrices. In this study, we have investigated and determined the eigenvalues of a Norlund matrix as a bounded operator over the sequence spaces c_0 and c. This was achieved by applying eigenvalue problem i.e $Ax = \lambda x.$ *where* λ *are numbers* $($*real or complex*$)$ and vector columns $x\,(x \neq 0)$; *such that* $x \in c_0$ *and* c. Also $A^*x = \lambda x$ such that $x \in c_0^*$ *and* c^* where $c_0^*, c^* = l_1$. The results obtained are $A \in B(c_0)$, $A \in B(bv_0)$ *and* $A \in B(l_1)$ have no eigenvalues while the set of eigenvalus for $A^* \in B(l_1)$ where $c_0^* = l_1$ is $\{\lambda \in \mathbb{C} : |\lambda + 1| < 2\} \cup \{1\}$. Furthermore the set of eigenvalues for $A \in B(c)$ is the singleton set $\{1\}$ and that of $A^* \in B(l_1)$ where $c^* = l_1$ is the set $\{\lambda \in \mathbb{C} : |\lambda + 1| < 2\} \cup \{1\}$ .The results from this research will provide useful information to engineers to improve on areas of application of eigenvalues and eigenvectors in engineering. It will also be useful to mathematicians when solving similar problems.

CHAPTER ONE

INTRODUCTION AND LITERATURE REVIEW

1.1 Background of the Study

Concepts used in this research emanated from modern functional analysis as well as summability theory and hence a list of pertinent definitions and theorems in these areas of research are given below:

1.1.1 Eigen values

Given a square matrix A, let us consider the problem of finding numbers λ (real or complex) and vectors (vector columns) x $(x \neq o)$ such that

$$Ax = \lambda x \tag{1.1.1}$$

This problem is called the eigenvalue problem, the number λ are called the eigenvalues of the matrix A, and the non-zero vector x are called the eigenvectors corresponding to the eigenvalues λ.

To find eigenvalues; we note that $\lambda x = \lambda I x$, where I is the identity matrix. Then we can rewrite equation $(1.1.1)$ in the form

$$Ax - \lambda I x = 0$$

or

$$(A - \lambda I)x = 0 \tag{1.1.2}$$

Matrix equation $(1.1.2)$ (which infact represents the linear system) has a non-trivial solution $x \neq 0$ if and only if the matrix $A - \lambda I$ of this system is singular, which is the case if and only if

$$det\,(A-\lambda I)=0 \tag{1.1.3}$$

Thus we have the equation for finding eigenvalues λ. Equation $(1.1.3)$ is called the characteristic equation.

1.1.2 Classical Summability

The central problem of summability is to find means of assigning a limit to a divergent sequence or sum to a divergent series. In such a way that the sequence or series can be manipulated as though it converges, (Ruckel, 1981), pp. 159-161. The most means of summing divergent series or sequences , is that of using an infinite matrix of complex numbers.

Definition 1.1.1. *Sequence to Sequence transformation*

Let $A=(a_{nk}),n,k=0.1,2,...$ be an infinite matrix of complex numbers. Given a sequence $x=(x_k)_{k=0}^{\infty}$ define

$$y_n=\sum a_{nk}x_k,\ n=0,1,2,... \tag{1.1.4}$$

If the series (1.1.4), converges for all n, then we call the sequence$(y_n)_{n=0}^{\infty}$, the A-transform of the sequence $(x_k)_{k=0}^{\infty}$. If further, $y_n\to a$ as $n\to\infty$, we say that $(x_k)_{k=0}^{\infty}$ is summable A to a.

There are various sequence to sequence transformations, here we only state Norlund means.

Definition 1.1.2. *(Norlund means)*

The transformation given by

$$y_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}x_k,\ n = 0,1,2,... \tag{1.1.5}$$

where $P_n = p_0 + p_1 + ... + p_n \neq 0$*, is called a Norlund means and is denoted by* (N, p).

Its matrix is given by

$$a_{nk} = \begin{cases} \frac{p_{n-k}}{P_n} & ,\ 0 \leq k \leq n \\ \\ 0 & ,\ k > n \end{cases} \tag{1.1.6}$$

In matrix (1.1.6) if $p_0 = 1, p_1 = -2, p_2 = p_3 = ... = 0$*, then* $A = (a_{nk})$ *i.e*

$$a_{nk} = \begin{cases} 1, & n = k = 0 \\ \\ 2, & n-1 \leq k < n \\ \\ -1 & n = k \\ \\ 0 & otherwise \end{cases}$$

or

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 2 & -1 & 0 & 0 & 0 & . & . & . \\ 0 & 2 & -1 & 0 & 0 & . & . & . \\ 0 & 0 & 2 & -1 & 0 & . & . & . \\ 0 & 0 & 0 & 2 & -1 & . & . & . \\ & & . & . & . & & & \end{pmatrix}$$

it transforms the unbounded sequence $(x_k)_{k=0}^{\infty} = (1,2,4,8,16,...)$ *to zero which is the matrix of our interest in this research.*

Definition 1.1.3. *(series to series transformation)*

The transformation of the series $\sum_{k=0}^{\infty} x_k$ into a convergent series $\sum_{n=0}^{\infty} y_n$ by an infinite matrix $A = (a_{nk})$ so that

$$y_n = \sum_{k=0}^{\infty} a_{nk} x_k \tag{1.1.7}$$

is called series to series transformation. For more information on series to series transformation see (Vermes, 1949)

1.1.3 General Results in Classical Summability

Definition 1.1.4. *(regular method , conservative method)*

Let $A = (a_{nk}), n, k = 0, 1, 2, ...$ be an infinite matrix of complex numbers.

i. If the A transform of any convergent sequence of complex numbers exists and converges then A is called a conservative method . We then write $A \in (c, c)$

ii. If

$$lim\, y_n = lim_{n\to\infty} x_n = a,\ a \in \mathbb{C};\ where\ (y_n)_{n=0}^{\infty} \tag{1.1.8}$$

is the A transform of the convergent sequence$(x_n)_{n=0}^{\infty}$, then A is called regular . We then write $A \in (c, c; P)$

Theorem 1.1 *(silverman - Toeplitz)* $A \in (c, c; P)$ *if and only if*

i. $lim_{n\to\infty} a_{nk} = 0$ for each fixed $k = 0, 1, 2, \ldots$

ii. $lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1$

iii.$sup_{n\geq 0} \{\sum_{k=0}^{\infty} \mid a_{nk} \mid\} \leq M < \infty, M \in R^{+}$.

Proof: (Hardy, 1948), pp.44-46, (Petersen, 1966) and (Maddox, 1970), pp. 165-166.

Remark: *The Silverman - Toeplitz theorem gives the complete class of matrices* (a_{nk}) *which transforms all convergent sequences* $(x_n)_0^{\infty}$ *such that* $lim_{n\longrightarrow\infty} x_n = lim_{n\longrightarrow\infty} y_n$ *by means of the linear equation* (1.1.4).

Theorem 1.2 (Kojima - Shur) $A \in (c,c)$if and only if

i. $a_{nk} \longrightarrow a_k$ as $n \longrightarrow \infty$ for each fixed $k \geq 0$;

ii. $\sum_{k=0}^{\infty} a_{nk} \longrightarrow a$ as $n \longrightarrow \infty$

iii. $sup_{n\geq 0} \{\sum_{k=0}^{\infty} \mid a_{nk} \mid\} < \infty$

Proof: (Maddox, 1970), pp. 166 - 167; (Ruckel, 1981), pp. 104 - 105; (Powell and Shah, 1972) and (Wilansky, 1984), pp. 5 - 6

Theorem 1.3 $A \in (c_0, c_0)$ if and only if

i. $lim_{n\longrightarrow\infty} a_{nk} = 0$ for each fixed k

ii. $sup_{n\geq 0} \{\sum_{k=0}^{\infty} \mid a_{nk} \mid\} < \infty$

Proof: (Hardy, 1948), pp. 42 - 60; (Maddox, 1970), pp. 165 - 167.

1.1.4 Banach Spaces

Definition 1.1.5. *(Paranorm)*

A paranorm p, on a linear space X, is a function $p : X \longrightarrow R$ such that

i.$p(\theta) = 0$

ii.$p(x) \geq 0$

iii. $p(x) = p(-x)$

iv. $p(x+y) \leq p(x) + p(y)$

v. If $(\lambda_n)_0^{\infty}$ is a sequence of scalars with $\lambda_n \longrightarrow \lambda$ and $(x_n)_0^{\infty}$is a sequence of points in X with $x_n \longrightarrow x$, then $p(\lambda_n x_n - \lambda x) \longrightarrow 0$ (continuity of multiplication)

Definition 1.1.6. *(seminorm/norm)*

A seminorm p, on a linear space X, is a function $p : X \longrightarrow R$ such that

i. $p(x) \geq 0$

ii. $p(x+y) \leq p(x)+p(y)$

iii. $p(\lambda x) = |\lambda| p(x),\ \lambda \in K(\mathbb{R} or \mathbb{C})$

If in addition to these conditions a seminorm satisfies the condition that $p(x) = 0$ iff $x = \theta$, then we call it a norm; θ denotes the zero vector.

Definition 1.1.7. *(linear Topological space)*

A linear topological space is a linear space X which has a topology T , such that addition and scalar multiplication in X are continous. If T is given a metric, we speak of a linear metric space.

Example 1.5.1 c_0 and c are all normed linear spaces . Their norm is$\| x \| = sup_{n\geq 0}\{| x_n |\}$

Definition 1.1.8. *(Banach Space)*

A Banach space is a complete normed linear space. Completeness means that if $\| x_m - x_n \| \longrightarrow 0$ as $m,n \longrightarrow \infty$ where $x_n \in X$, then there exists $x \in X$ such that,

$\|x_n$ - $x \| \longrightarrow 0$ as $n \longrightarrow\infty$.

1.1.5 Linear operators and Functionals

Definition 1.1.9. *(Linear operator)*

Let X and Y be linear spaces. Then a function $f : X \longrightarrow Y$ is called a linear operator or map or transformation if and only if for all $x, y \in X$ and all $\lambda, \mu \in K$

$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$.

Definition 1.1.10. (Linear functional)

f is a linear functional on X if $f : X \longrightarrow K$ is a linear operator , i.e a linear functional is a real or complex valued linear operator.

Definition 1.1.11. *(Bounded linear operator)*

A linear operator A : $X \longrightarrow Y$ is called bounded if there exists a constant M such that

$\| A(x) \| \leq M \| x \|, \forall x \in X$

NOTE: A bounded functional on X satisfies

$| f(x) | \leq M \| x \|, \forall x \in X$

1.2 Literature Review

This section contains the previous work that has been carried out on areas relating to our research.

In 1960, E.K. Dorff and A. Wilansky showed that the spectrum of a certain mercerian Norlund matrix with $a_{nn} = 1$, contains negative numbers, (Dorff et al, 1960) and (Wilansky, 1984), Theorem 3. In 1965, Brown *et al*, determined the spectrum and eigenvalues of the Cesaro operator (c_1 operator) of space l^2 of square summable sequences, (Brown et al, 1965). Sharma (1972) determined the spectra of conservative matrices and in particular showed that the spectrum of any Hausdorff method is either uncountable or finite. Sharma (1975) determined the isolated points of the spectra of conservative matrices.

Wenger (1975) computed the fine spectra of Holder summability operators on c - the space of convergent sequences . Deddens (1978) computed the spectra of all Hausdorff operators on l_1^2. Rhodes (1983) extended Weger's work by determining the fine spectra of weighted mean operators on c. Reade (1985) determined the spectrum of Cesaro operator on c_0 - the space of null sequences. Okutoyi (1985) determined the spectrum of C_1on $w_p(0)$, $(1 \leq p < \infty)$. Gonzale (1985) computed the fine spectrum of the C_1 operator on $l_{,p}(1 < p < \infty)$. In (1989) , Okutoyi, J. I and Thorpe, B. computed the spectrum of the Cesaro operator of order two (C_{11}operator) on $c_0(c_0)$ - the space of double null sequences. Okutoyi (1990) determined the spectrum of C_1 operator on bv_0 space. In 1992 Okutoyi extended his work by determining the spectrum of C_1 operator on bv space. In 1996, Shafiquel Islam obtained the spectrum of C_1operator on l_∞ - the space of bounded sequences, (Shafiquel, 1996). In his PhD thesis Mutekhele, J. S. K. extended Okutoyi's work by determining the spectrum of C_1 operator on $c\,(c)$ - the space of double sequences which converge. He went further and determined the fine spectra of C_{11} operator on $c\,(c)$ - the space of double sequence which converge, (Mutukhele, 1999). In 2003, Coskun determined the set of eigenvalues of a special Norlund Matrix as a bounded operator over some sequence spaces especially the eigen values on $c_0,$ c *and* bv_0, (Coskun, 2003). In 2005, Okutoyi, and Akanga computed the spectrum of the C_1 operator on - the space of strongly

Cesaro summable complex sequences of order 1, (Okutoyi and Akanga, 2005). In 2010 Akanga *et al*, determined the spectrum of a special Norlund matrix as a bounded operator on c_0 especially the eigen values on c_0, (Akanga*et al*, 2010). In 2014, Akanga, determined the spectrum of a special Norlund matrix as a bounded operator on c by obtaining the eigen values on c, (Akanga, 2014).

1.3 Statement of the Problem

From the literature review a lot has been done on the spectra of weighted mean matrices such as Cesaro and Holder means. But not much have been achieved in Norlund means. In this thesis eigen values of a Norlund matrix acting as an operator on the sequence spaces c_0 and c are determined.

1.4 Justification

Apart from the more obvious benefits i.e., the solution of systems of linear equation of which the spectrum of linear operators is all about, there are more equally important applications of the research. A central problem in the whole of mathematics and even science and engeneering ; is the determination of convergence or non-convergence of sequences and series. Many applicaations of matrices in both engineering and science utilize eigen values and sometimes eigen vectors. Vibration analysis and stress tensors are just a few of the application areas. Mathematics, especially Mathematical analysis, developes and is maintained via the concept of convergence of sequence and series. Even in applied Science and Engineering , one is interested in the convergance of a sequence or series of results generated during experimentation. Established theorems such as the ratio theorems and integral theorem, are not applicable in a variety of sequences and series. Even where they apply they just determine convergence and not the limit or sum of the convergent sequence or series. Tauberian theorems in Summability theory handle this problem well. The convergence and even the limit of a convergent sequence or series is determined from the convergence of some transform of it together with a side condition.

1.5 Objectives

1.5.1 General Objective

To investigate and determine the eigenvalues of an infinite matrix (special Norlund matrix) as an operator on sequence spaces c and c_0

1.5.2 Specific Objectives

1. To determine the eigenvalues of a Norlund matrix as an operator on the sequence space c_0.

2. To determine the eigenvalues of a Norlund matrix as an operator on the sequence space c.

CHAPTER TWO

THE BOUNDEDNESS OF OPERATOR A ON c_0 AND THE EIGENVALUES OF A ON c_0

The chapter is divided into two sections. Section one deals with matrix A see $(1.1.6)$ considered bounded operator on the null convergent sequence space c_o. In section two we workout eigenvalues of matrix A see $(1.1.6)$ on the sequence space c_0

2.1 Boundedness of A on sequence space c_0

In this section we show that $A \in B(c_0)$. The corollary below arises from theorem (1.3), in chapter 1.

Corollary 2.1.1. *It is clear that* $A \in B(c_0)$ *since* $lim_n a_{nk} = 0 \quad for \quad each \quad fixed \quad k, \quad$ (*see matrix* 1.1.6)

$$\| A \| = sup_{n \geq 0} \sum_{k=0}^{\infty} | a_{nk} | = sup(1,3,3,3,...) = 3 \tag{2.1.1}$$

Also, $\| A \| = \| A^* \| = 3$

Hence all the conditions of theorem 1.3 are satisfied.

Lemma 2.1.2. *Each bounded linear operator* $T : X \to Y$, *where* $X = c_0, \ell_1, c$ *and* $Y = c_0, \ell_p(1 \leq p < \infty), \ell_\infty$ *determines and is determined by an infinite matrix of complex numbers.*

Proof. see (Taylor, 1958) pages 217-219 ☐

Lemma 2.1.3. *Let* $T : c_0 \to c_0$ *be a linear map and define* $T^* : \ell_1 \to \ell_1$ *by* $T^* o g = g o T$, $g \in c_0^* = \ell_1$. *then* T *must be given by a matrix by lemma* $(2.1.2)$ *and moreover* $T^* : \ell_1 \to \ell_1$ *is the transposed matrix of* T.

Proof. see (Wilansky, 1984) page 266. ☐

Corollary 2.1.4. *Let* $A : c_0 \to c_0$ *where A is the Norlund matrix* $(1.6.6)$*. Then* $A^* \in B(\ell_1)$*, moreover*

$$A^* = \begin{pmatrix} 1 & 2 & 0 & . & . & . \\ 0 & -1 & 2 & . & . & . \\ 0 & 0 & -1 & . & . & . \\ 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & . & . & . \\ & . & . & . & & \end{pmatrix} \tag{2.1.2}$$

Proof. It follows from lemma 2.1.3 using A on substituting T for matrix A □

2.2 Eigenvalues of A on the sequence space c_0

Theorem 2.2.1. $A \in B(c_0)$ *has no Eigenvalue.*

Proof : Suppose $Ax = \lambda x$ for $x \neq 0$ in c_0 and $\lambda \in \mathbb{C}$ then

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 2 & -1 & 0 & 0 & 0 & . & . & . \\ 0 & 2 & -1 & 0 & 0 & . & . & . \\ 0 & 0 & 2 & -1 & 0 & . & . & . \\ 0 & 0 & 0 & 2 & -1 & . & . & . \\ & & . & . & . & & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ . \end{pmatrix} \tag{2.2.1}$$

Implies

$$x_0 = \lambda x_0$$

$$2x_0 - x_1 = \lambda x_1$$

$$2x_1 - x_2 = \lambda x_2$$

$$2x_2 - x_3 = \lambda x_3$$

$$2x_3 - x_4 = \lambda x_4$$

$$\dots$$

$$2x_{n-1} - x_n = \lambda x_n \quad , \; n \geq 1 \tag{2.2.2}$$

solving system $(2.2.2)$ we have that if x_0 is the first non zero entry of x , then $\lambda = 1$, $but\ \lambda = 1\ \ implies\, that\ x_0 = x_1 = x_2 = \ldots = x_n = \ldots$

i.e

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ . \\ . \\ . \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ 1 \\ 1 \\ . \\ . \\ . \end{pmatrix}$$

which shows that x is in the span of δ . But $\delta = (1,1,1,\ldots) \notin c_0$. That is x does not tend to zero as n tends to infinity , so $\lambda = 1$ is not an eigenvalue of $A \in B(c_0)$.

If $x_{n+1},\ n = 0, 1, 2, 3, \ldots$ is the first non zero entry, then $\lambda = -1$. Solving the system with $\lambda = -1$ results in $x_n = 0$, $n = 0, 1, 2, 3, \ldots$ a contradiction . Hence $\lambda = -1$ cannot be an eigenvalue of $A \in B(c_0)$.

Thus $A \in B(c_0)$ has no eigenvalues i.e the set of eigenvalues is empty:

Corollary 2.2.2. *The set of Eigenvalues of $A \in B(bv_0)$ and $A \in B(l_1)$ is empty*

Proof : This follows from the fact that $A \in B(c_0) = \emptyset,\ \ and\ \ bv_0 \subset c_0,\ Also\ \ell_1 \subset c_0$

Theorem 2.2.3. *The Eigenvalues of $A^\star \in B(\ell_1)$ is the set*

$$\{\lambda \in \mathbb{C} : |\lambda + 1| < 2\} \cup \{1\}$$

Proof : Suppose $A^\star x = \lambda x\ \ for\ x \neq 0\ and\ \ \lambda \in \mathbb{C}$

Then

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 & . & . & . \\ 0 & -1 & 2 & 0 & 0 & . & . & . \\ 0 & 0 & -1 & 2 & 0 & . & . & . \\ 0 & 0 & 0 & -1 & 2 & . & . & . \\ 0 & 0 & 0 & 0 & -1 & . & . & . \\ & . & . & . & & & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \end{pmatrix} \tag{2.2.3}$$

That is

$$x_0 + 2x_1 = \lambda x_0$$

$$-x_1 + 2x_2 = \lambda x_1$$

$$-x_2 + 2x_3 = \lambda x_2 \tag{2.2.4}$$

$$-x_3 + 2x_4 = \lambda x_3$$

$$\ldots.$$

$$-x_n + 2x_{n+1} = \lambda x_n \quad where \quad n \geq 1 \tag{2.2.5}$$

Solving the system(2.2.4) for $x_1,\ x_2,\ x_3,\ ...,\ x_n\ in\, terms\, of\ x_0\ gives\ :$

$$x_1 = 2^{-1}\lambda\left(1-\frac{1}{\lambda}\right)x_0$$

$$x_2 = 2^{-2}\lambda^2\left(1-\frac{1}{\lambda}\right)\left(1+\frac{1}{\lambda}\right)x_0$$

$$x_3 = 2^{-3}\lambda^3\left(1-\frac{1}{\lambda}\right)\left(1+\frac{1}{\lambda}\right)^2 x_0$$

$$x_4 = 2^{-4}\lambda^4\left(1-\frac{1}{\lambda}\right)\left(1+\frac{1}{\lambda}\right)^3 x_0$$

$$...$$

In general

$$x_n = 2^{-n}\lambda^n\left(1-\frac{1}{\lambda}\right)\left(1+\frac{1}{\lambda}\right)^{n-1} x_0 \qquad (2.2.6)$$

By ratio test

$$lim_{n\to\infty}|\frac{x_{n+1}}{x_n}| = lim_{n\to\infty}|\frac{2^{-n}2^{-1}\lambda^n\lambda(1+\frac{1}{\lambda})^n\left(1-\frac{1}{\lambda}\right)x_0}{2^{-n}\lambda^n\left(1+\frac{1}{\lambda}\right)^n\left(1+\frac{1}{\lambda}\right)^{-1}\left(1-\frac{1}{\lambda}\right)x_0}|$$

$$= lim_{n\to\infty}|\frac{2^{-1}\lambda}{\left(1+\frac{1}{\lambda}\right)^{-1}}|$$

$$= |\frac{1}{2}\lambda\left(1+\frac{1}{\lambda}\right)| = l\ \ \forall\, l \in \mathbb{R}\ s.t\, l \geq 0$$

By ratio test $x_n \in \ell_1 \;\; iff \;\; l < 1$

That is $iff \;\; |\frac{1}{2}\lambda + \frac{1}{2}| < 1$

$or \;\; |\lambda + 1| < 2$

That is the series $\sum_{n=0}^{\infty} |x_n|$ converges for all λ in the circular disc centred at the point $(-1, 0)$ of radius 2.

It is clear that $\lambda = 1$ is an eigenvalue corresponding to the eigenvector $(x_0, 0, 0, 0, \ldots)^t$. Where x_0 is any real or complex number . This is the case since $(x_0, 0, 0, 0, \ldots)^t \subset \ell_1 \;\; for \, any \, x_0 \in \mathbb{C}$

Hence the Eigenvalues of $A^\star \in B(l_1)$ is the set

$$\{\lambda \in \mathbb{C} : |\lambda + 1| < 2\} \cup \{1\}$$

CHAPTER THREE

THE BOUNDEDNESS OF OPERATOR A ON c AND THE EIGEN VALUES OF A ON c .

The chapter is divided into two sections. Section one deals with matrix A see $(1.1.6)$ considered bounded operator on the convergent sequence space c. In section two we workout eigenvalues of matrix A see $(1.1.6)$ on the sequence space c

3.1 Boundedness of A on the sequence space c

In this section we show that $A \in B(c)$. The corollary below arises from theorem (1.2), in chapter 1.

Corollary 3.1.1. *$A \in B(c)$, moreover*

$$\| A \| = \| A^* \| = 3$$

Proof. since $a_{nk} \to 0 \quad as\; n \to \infty \quad for\; fixed\; k \geq 0$

also

$$\sum_{k=0}^{\infty} a_{nk} = \sum_{k=0}^{n} a_{nk} = 1, \quad for\; each\; n \tag{3.1.1}$$

so that $lim_n \sum_{=0}^{n} a_{nk} = 1$

finally $\| A \| = sup_{n \geq 0} \{\sum_{k=0}^{\infty} \mid a_{nk} \mid\} = 3$

Therefore all the conditions of theorem (1.2) are satisfied. Hence $A \in B(c)$ ☐

Theorem 3.1.2. *Let $T : c \to c$ be a linear map and define $T^* : c^* \to c^* \quad i.e\;., T^* : \ell_1 \to \ell_1$ by $T^*(g) = goT,\; g \in c^* \equiv \ell_1. Then\; both\; T$ and T^* must be given by a matrix. See lemma $(2.1.2)$. Moreover $T^* : \ell_1 \to \ell_1$is given by the matrix. see lemma $(2.1.3)$. Moreover $T^* :\; \ell_1 \to \ell_1$ is given by the matrix*

$$A^* = T^* = \begin{pmatrix} X(lim_A) & (v_n)_0^\infty \\ (a_k)_0^\infty & A^t \end{pmatrix} = \begin{pmatrix} X(lim_A) & v_0 & v_1 & v_2 & . & . & . \\ a_0 & a_{00} & a_{10} & a_{20} & . & . & . \\ a_1 & a_{01} & a_{11} & a_{21} & . & . & . \\ a_2 & a_{02} & a_{12} & a_{22} & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix} \tag{3.1.2}$$

where:

$$X(lim_A) = lim_A(\delta) - \sum_{K=0}^{\infty} lim_A \delta^k \tag{3.1.3}$$

$$v_n = X(P_n oT)\ ; \tag{3.1.4}$$

$$a_{nk} = P_n(T(\delta^k)) = (T(\delta^k))_n \tag{3.1.5}$$

and

$$a_k = \lim_{n\to\infty} a_{nk} \tag{3.1.6}$$

Proof. See (Wilansky, 1984) page 267 □

Corollary 3.1.3. *Let* $A:\ c \to c$. *Then* $A^* \in B(\ell_1)$ *and*

$$A^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & 2 & 0 & 0 & . & . & . \\ 0 & 0 & -1 & 2 & 0 & . & . & . \\ 0 & 0 & 0 & -1 & 2 & . & . & . \\ & & . & . & . & & & \end{pmatrix} \tag{3.1.7}$$

Proof. By theorem $(3.1.2)$

$$A^* = \begin{pmatrix} X(lim_A) & (v_n)_0^\infty \\ (a_k)_0^\infty & A^t \end{pmatrix} \tag{3.1.8}$$

But for A matrix, $v_n = \theta \ and \ (a_k)_0^\infty = \theta, \ since \ lim_{n\to\infty} a_{nk} = 0 \ \forall \ k \geq 0.$

$$(P_n oT)\delta = 1, \ \forall \ n \geq 0;$$

And

$$\sum_{k=0}^{\infty} (P_n oT)\delta^k = 1,$$

So that

$$v_n = X(P_n oT)$$

$$= (P_n oT)\delta - \sum_{k=0}^{\infty} (P_n oT)\delta^k$$

Which implies that

$$v_0 = 1 - (1 + 0 + 0 + \ldots) = 1 - 1 = 0$$

$$v_1 = 1 - (2 + (-1) + 0 + 0 + ...) = 1 - 1 = 0$$

$$v_2 = 1 - (0 + 2 + (-1) + 0 + 0 + ...) = 1 - 1 = 0 \tag{3.1.9}$$

$$v_3 = 1 - (0 + 0 + 2 + (-1) + 0 + \ldots) = 1 - 1 = 0$$

$$\vdots$$

$$v_n = 0, \; n \geq 0$$

Hence matrix $(3.1.8)$ becomes

$$A^* = \begin{pmatrix} X & \theta \\ \theta & A^t \end{pmatrix} \tag{3.1.10}$$

Where

$$X = (lim o A)\delta - \sum_{k=0}^{\infty} (lim o A)\delta^k \tag{3.1.11}$$

$lim \in c^*$. That is

$$X = lim\delta - \sum_{k=0}^{\infty} a_k = 1 - 0 = 1 \tag{3.1.12}$$

So that matrix $(3.1.10)$ becomes matrix $(3.1.7)$ $\square$

3.2 The Eigenvalues of A on c

Theorem 3.2.1. *$A \in B(c)$ has one Eigenvalue , ie $\{1\}$. Where $\lambda = 1$ which corresponds to the Eigenvector $x = \delta = (1, 1, 1, ., ., .)$*

Proof. Suppose $Ax = \lambda x$, $x \neq 0$ *in c and* $\lambda \in \mathbb{C}$. Then solving the system as in the proof of theorem $(2.2.1)$ we have that if x_0 is the first non-zero entry of the vector x , then $\lambda = 1.$ But $\lambda = 1$ implies that $x_0 = x_1 = x_2 = \cdots = x_n = \cdots$. Which shows that x is in the span of some δ. But $\delta = (1, 1, 1, \cdots) \in c.$ Hence $\lambda = 1$ is an Eigenvalue of A corresponding to the Eigenvector $\delta = (1, 1, 1, \cdots)$. □

When x_{n+1} , $n = 0, 1, 2, 3, \cdots$ is the first non-zero entry of x, then $\lambda = -1.$ Solving the system with $\lambda = -1$ results in $x_n = 0$, $n = 0, 1, 2, 3, \cdots$ which is a contradiction . Hence $\lambda = -1$ cannot be an Eigenvalue of $A \in B(c)$

Therefore $\lambda = 1$ is the only Eigenvalue of $A \in B(c)$

Theorem 3.2.2. *The Eigenvalues of $A^{\star} \in B(l_1)$ form the set*

$$\{\lambda \in \mathbb{C} : |\lambda + 1| < 2\} \cup \{1\}$$

Proof : Suppose $A^{\star}x = \lambda x$, $x \neq \theta$ and $\lambda \in \mathbb{C}$. Then

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & 2 & 0 & 0 & . & . & . \\ 0 & 0 & -1 & 2 & 0 & . & . & . \\ 0 & 0 & 0 & -1 & 2 & . & . & . \\ 0 & 0 & 0 & 0 & -1 & . & . & . \\ & . & . & . & & . & . & . \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ . \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ . \end{pmatrix} \tag{3.2.1}$$

$$x_0 = \lambda x_0$$

$$x_1 + 2x_2 = \lambda x_1$$

$$-x_2 + 2x_3 = \lambda x_2 \tag{3.2.2}$$

$$-x_3 + 2x_4 = \lambda x_3$$

$$-x_4 + 2x_5 = \lambda x_4$$

$$\cdots$$

In general

$$-x_n + 2x_{n+1} = \lambda x_n \quad \forall\ n \geq 2$$

Solving the system $(3.1.14)$ with $\lambda = 1$ and $x_0 \neq 0$ gives the vector

$$x = (x_0, 0, 0, 0, 0, \ldots)^t$$

Where $x_0$is any real or complex number. This is the case since

$$(x_0, 0, 0, 0, 0, \ldots)^t \subset l_1 \quad \forall\, x_0 \in \mathbb{C}$$

Hence $\lambda = 1$ is an Eigenvalue of $A^{\star} \in B(l_1)$.

Also solving the sysytem $(3.1.14)\ for\, x_n,\ n \geq 2$ in terms of x_1, yields

$$x_2 = 2^{-1}\lambda\left(1-\frac{1}{\lambda}\right)x_1$$

$$x_3 = 2^{-2}\lambda^2\left(1+\frac{1}{\lambda}\right)\left(1-\frac{1}{\lambda}\right)x_1$$

$$x_4 = 2^{-3}\lambda^3\left(1+\frac{1}{\lambda}\right)^2\left(1-\frac{1}{\lambda}\right)x_1 \tag{3.2.3}$$

$$x_5 = 2^{-4}\lambda^4\left(1+\frac{1}{\lambda}\right)^3\left(1-\frac{1}{\lambda}\right)x_1$$

$$\ldots$$

In general

$$x_n = 2^{-(n-1)}\lambda^{n-1}\left(1+\frac{1}{\lambda}\right)^{n-2}\left(1-\frac{1}{\lambda}\right)x_1 \quad \forall\, n \geq 2 \tag{3.2.4}$$

Now ,

$lim_{n\to\infty} \mid \frac{x_{n+1}}{x_n} \mid = l$

$$x_{n+1} = 2^{-n}\lambda^{n}\left(1+\frac{1}{\lambda}\right)^{n-1}\left(1-\frac{1}{\lambda}\right)x_1 \tag{3.2.5}$$

$$lim_{n\to\infty} \mid \frac{2^{-n}\lambda^{n}\left(1+\frac{1}{\lambda}\right)^{n-1}\left(1-\frac{1}{\lambda}\right)x_1}{2^{-(n-1)}\lambda^{n-1}\left(1+\frac{1}{\lambda}\right)^{n-2}\left(1-\frac{1}{\lambda}\right)x_1} \mid$$

$$= lim_{n\to\infty} \mid \frac{1}{\left(1+\frac{1}{\lambda}\right)} \div \frac{2}{\lambda\left(1+\frac{1}{\lambda}\right)^2} \mid$$

$$=\mid \frac{\lambda\left(1+\frac{1}{\lambda}\right)}{2} \mid< 1$$

$$\mid \lambda + 1 \mid = 2$$

By ratio test $x_n \in l_1$iff $l < 1$ i.e $\mid \frac{\lambda+1}{2} \mid$ or $\mid \lambda + 1 \mid < 2$ That is the series $\sum_{n=0}^{\infty} \mid x_n \mid$ converges for all λ in the circular disc centred at the point $(-1,0)$ of radius 2 . Hence $\{\lambda \in \mathbb{C} : |\lambda + 1| < 2\} \cup \{1\}$ form the Eigenvalues of $A^{\star} \in B(l_1)$.

CHAPTER FOUR

CONCLUSIONS AND RECOMMENDATIONS

4.1 Introduction

Some conclusions of this study and suggestions of areas for further research are given in this chapter.

4.1.1 Conclusions

In chapter two the following results are obtained

i. $A \in B(c_0)$ has no Eigenvalues

ii. Also $A \in B(bv_0)$ and $A \in B(l_1)$ has no Eigenvalues

iii. The set of Eigenvalues for $A^\star \in B(l_1)$ is $\{\lambda \in \mathbb{C} : |\lambda + 1| < 2\} \cup \{1\}$

In chapter three we obtain the following results

i. The set of Eigenvalues of $A \in B(c)$ is the singleton set $\{1\}$

ii. The Eigenvalues of $A^\star \in B(l_1)$ form the set $\{\lambda \in \mathbb{C} : |\lambda + 1| < 2\} \cup \{1\}$

4.1.2 Recommendations

We recommend to extend the results obtained in this thesis by :

i. Investigating the spectrum of $A \in B(c_0)$

ii. Investigating the spectrum of $A \in B(c)$

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