PRACTICAL APPLICATION OF THE GEOMETRIC GEOID FOR HEIGHTING OVER NAIROBI COUNTY AND ITS ENVIRONS

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Abstract
Geoid determination is one of the main current geodetic problems in Kenya. This is because a geoid model is required to convert ellipsoidal heights to orthometric heights that are used in practice. A local geometric geoid covering Nairobi County and its environs has been determined by a geometric approach. Nineteen points levelled by both Global Positioning System (GPS) and precise levelling techniques in the area of study have been used. Seven triangulation points have been used for the determination of transformation parameters between World Geodetic System 1984 (WGS84) and Arc-Datum 1960 coordinates in order to express the local geoid height as a function of position. The geoid height is expressed as a function of the local plane coordinates through a biquadratic surface polynomial, using 14 GPS/levelling points. Five points have been used for testing the results. The experience with Nairobi County and its environs geometric geoid indicates that interpolation of geoid heights in a small area by a biquadratic polynomial is simple and it works well. The geoid heights obtained by biquadratic polynomial (interpolation) compare favourably on the test points with root mean square and standard deviation of $\pm$1cm in the area of study. This accuracy is sufficient for most engineering projects.

Key words: Geoid, GPS, coordinate transformation, height determination, Nairobi County

1.0 Introduction

The geoid may be defined as the equipotential surface of the Earth's gravity field that coincides, on average, with mean sea level in the open ocean, taking into account the Sea Surface Topography (SST). The geoid is represented by its departure (undulation) usually denoted by N, with respect to a reference ellipsoid, which essentially, is a mathematically defined reference figure of the Earth. The global reference surface currently in use is the Geodetic Reference System 1980 (GRS80, Moritz, 1980a).

The defining parameters of GRS80 are practically identical to World Geodetic System 1984 (WGS84). WGS84 is the Geodetic Reference System used for GPS positioning. A GRS80 referenced gravimetric geoid is thus compatible with GPS derived elevations. This allows recovery of orthometric heights or their differences from GPS using the gravimetric geoid model. A gravimetric geoid is the geoid computed using observed gravity.

The geoid represents the most obvious mathematical formulation of a horizontal surface at sea level. This is why the use of the geoid simplifies geodetic problems and makes them accessible to geometrical intuition. The disadvantage is that the potential W inside the earth and hence the geoid (W=constant), depends on the density (ρ) because of Poisson's equation, $\Delta W=-4\pi k\rho+2\omega^2$ (Moritz, 1980b; Heiskanen and Moritz, 1967), therefore, in order to determine or to use the geoid, the density of the masses at every point between the geoid and the ground must be known, at least theoretically. This is clearly impossible, and therefore some assumptions concerning the density must be made, which is unsatisfactory theoretically, even though the practical influence of these assumptions is usually very small (Heiskanen and Moritz, 1967).

In geodetic positioning the geoid is approximated with a reference ellipsoid. This reference ellipsoid through its deviations from the geoid, further serves as a means of describing the geoid. The linear difference between the reference ellipsoid and the geoid at a point is referred to as geoidal undulation or geoid height. Determination of the geoid therefore refers to the determination of geoidal undulation (N) from a reference ellipsoid (WGS84 in this case). The knowledge of geoid is normally required principally for practical purposes e.g. surveying, cartography, navigation, determination of the orbit of the artificial satellites, study of the earth's crust and other geophysical studies. Although a local geoid model is important for the development of a country, Kenya is still in the process of developing a national geoid model. However, Gachari and Olliver (1998) developed a gravimetric geoid model covering the Eastern Africa (Kenya, Uganda and part of Tanzania) using Ohio State University model 1991A (Rapp et al., 1991), terrestrial gravity and satellite altimetry data.

Gravity observations in Kenya began around 1955 and observations have been carried out by various organizations, notably petroleum companies (Lwangasi, 1991). These data are unfortunately scanty in format and distribution in addition to being not readily available for research work, especially in Nairobi County and its environs where less or no exploration activities have been done. Hence the geometric geoid determination approach is applied in this study. The study is limited to Nairobi County and its environs (1° 8′ 00″ S ~ 1° 25′ 18″ S and 36° 38′ 55″ E ~ 36° 58′ 24″ E) due to lack of GPS/levelling data sets in other parts of Kenya. However, we note the ongoing GPS observations covering the entire country by Survey of Kenya.

GPS is a new technology in surveying. It is fast and efficient in determination of positional data (coordinates) based on the WGS84. It measures heights above WGS84 reference ellipsoid. These heights are called ellipsoidal heights (h). However, orthometric heights (H) are the functional heights for mapping, engineering works, navigation, and other geophysical applications (Ayhan, 1993). The geoid is the actual reference surface for orthometric heights. These heights are normally obtained by spirit levelling which is a very tedious and expensive process. To exploit the capabilities of GPS for heighting purposes, the geoidal undulations must be determined in an area. A biquadratic polynomial has been used to convert ellipsoidal heights into orthometric heights in the area of study. For the interpolated geoid heights to be used effectively for local work, their positions should be known in the local coordinate system. This therefore calls for the determination of transformation parameters between WGS84 and the local (Arc-Datum 1960) coordinates.

2.0 Datum Transformation

A Datum may be defined as any conventional framework into which observations are referred in a given locality or globally. A geodetic datum therefore refers to a specifically oriented reference ellipsoid upon which geodetic observations are based. Eight parameters are required to define a geodetic datum: two to specify the dimensions of the ellipsoid, three to specify the location of its centre with respect to the centre of the earth and three to specify the orientation of the ellipsoid.

Datum transformation then refers to the determination of parameters that relate two reference systems. In the specific case of GPS positioning, the interest is in the transformation of coordinates of points between WGS84 system and other systems. Usually in such case (s), the WGS84 system is considered as the reference with respect to which the transformation of the other system(s) would be taken.

The existing geodetic control networks in various parts of the world are usually rotated to regionally defined datums. The reference ellipsoids of some of those datums happen to be directly compatible with the geocentric WGS84 ellipsoid whereas sizeable differences occur for other datums having non-geocentric reference ellipsoids e.g. the Arc-Datum 1960 used in Kenya. This therefore necessitates the determination of transformation parameters between WGS84 and other local systems to enable the use of GPS for local surveys.

The effect of a transformation varies from simple changes of location and direction, to a uniform change in scale and finally to change in shape and size of degree of linearity. There is therefore a whole family of transformations, some of which are applicable to two-dimensional space and others for use in three-dimensional space (system). The former refers to transformation between plane coordinates e.g. (X, Y) and (x, y), while the latter refers to the transformation between triplet sets of coordinates e.g. (X, Y, Z) and (x, y, z).

A similarity three dimensional coordinate transformation between X, Y, Z and x, y, z coordinate frames is the seven parameter transformation which allows for three rotations, three translations and one uniform scale change. The seven parameters are required to rigorously transform coordinates from one three dimensional Cartesian system with a different origin, scale and spatial orientation. The seven parameter model (Bursa-Wolf) works well for a large area. However, for a small area the solution of the rotation parameters is not stable. Hence the use of three-parameter Molodensky model in this study. In this approach the rotation parameters are assumed to be zero or of such small magnitudes that they are insignificant. The scale factor is assumed to be unity. Hence only the translation parameters are considered. Molodensky's model can be expressed as shown in equation 1.

$$(\rho + h)d\phi = -X_O \sin\phi\cos\phi - Y_O \sin\lambda\cos\phi + Z_O \cos\phi + \frac{e^2 \sin\phi\cos\phi}{\left(1-e^2\sin^2\phi\right)^{\frac{1}{2}}}da$$
$$+\sin\phi\cos\phi\left(2\upsilon + e'^2\rho\sin^2\phi\right)(1-f)df$$

$$(\upsilon + h)\cos\phi d\lambda = -X_O\sin\lambda + Y_O\cos\lambda$$
$$dh = X_O\cos\phi\cos\lambda + Y_O\cos\phi\sin\lambda + Z_O\sin\phi - \left(1-e^2\sin^2\phi\right)^{\frac{1}{2}}da + \frac{a(1-f)\sin^2\phi}{\left(1-e^2\sin^2\phi\right)^{\frac{1}{2}}}df$$

(1)

where, $d\phi$, $d\lambda$ and dh are differences in ellipsoidal coordinates between two datums, da and df are the differences between the parameters of the reference ellipsoids used in the two datums (Arc-Datum1960 and WGS84), X_O, Y_O and Z_O are the translation parameters.

A network of seven triangulation points in Nairobi County and its environs (Figure 1) was used to determine transformation parameters between WGS84 and Arc Datum 1960 coordinates. The WGS84 coordinate were provided by Kenya Institute of Surveying and Mapping (KISM) while the local coordinates on Arc-datum 1960 were provided by Survey of Kenya. The local plane (UTM) coordinates were first transformed to geodetic coordinates. The geodetic coordinates were then used to determine three-dimensional Cartesian coordinates based on Arc-Datum 1960 together with the dimensions of modified Clarke 1880 reference ellipsoid.

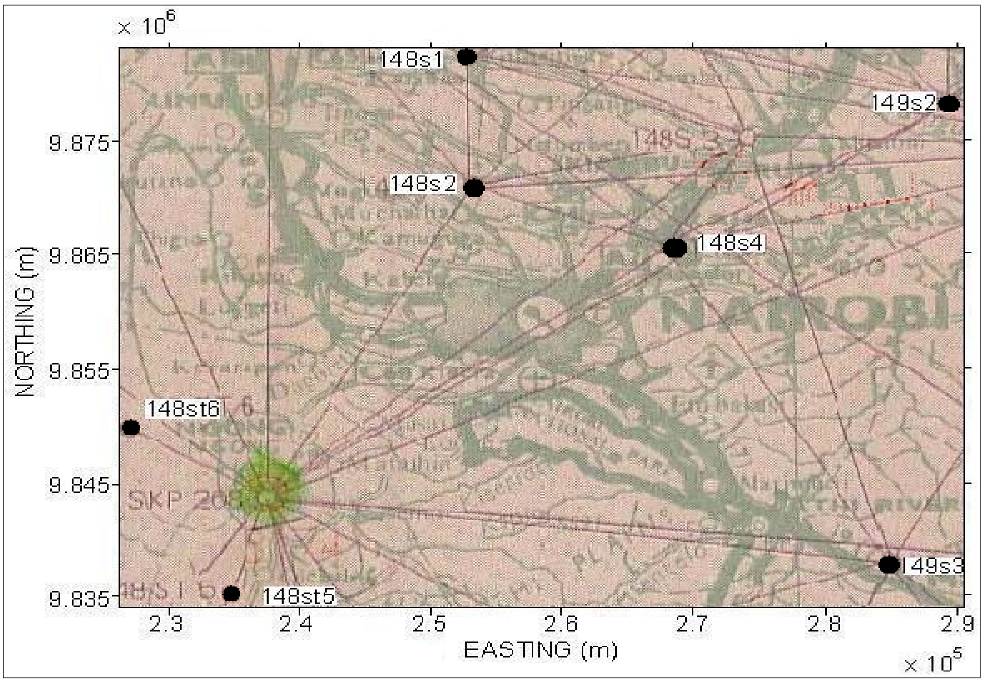

Figure 1: Spatial distribution of the triangulation points

3.0 Geometric Geoid Determination

The position of a point in space may be uniquely and best defined using some form of a coordinate system. Two types of coordinate systems are commonly used. These are Cartesian and ellipsoidal coordinate systems. The relationship between the two coordinate systems is shown in Figure 2. In the recent past, GPS positioning and its application has experienced rapid growth as well as remarkable improvement in accuracy attainable. This has in turn resulted in a growing need for high-resolution geoid models.

Geoid heights may be combined with GPS-derived ellipsoidal heights to provide the physically meaningful Orthometric heights for various applications in engineering, mapping, and scientific research.

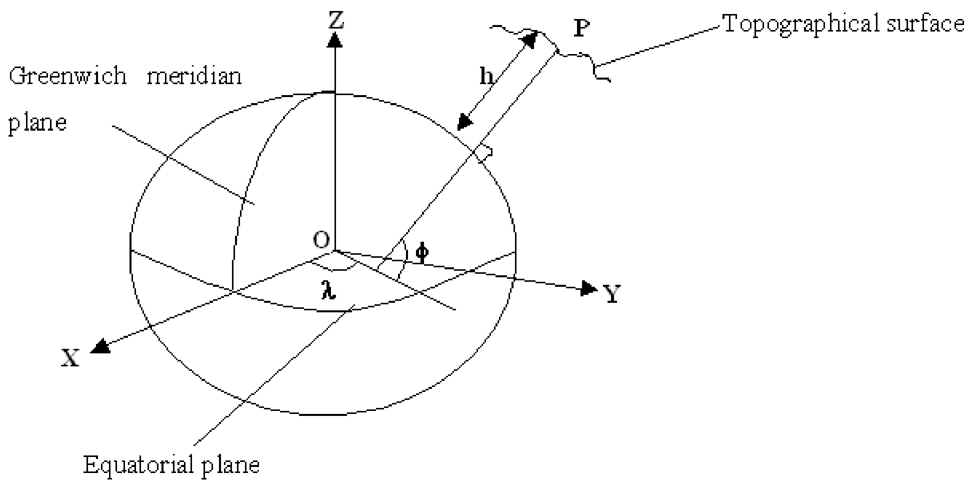

Figure 2: Relationship between Cartesian and ellipsoidal coordinates

where, X, Y and Z are the Cartesian coordinates; ϕ, λ and h are the ellipsoidal coordinates (latitude, longitude and ellipsoidal height respectively); P is a point on the topographical surface and O is the centre of mass of the earth.

The Separation between the geoid and the ellipsoid is known as geoid height (or geoid undulation). Knowledge of this parameter is necessary to enable the GPS derived ellipsoidal height h to be converted accurately, to the physically meaningful orthometric height H commonly used in many practical applications. Figure 3 gives a general relationship between the geoid height, ellipsoidal height and orthometric height (H). Global geoid models are available but in many cases they are too generalized and not accurate enough for this conversion. However, they are a useful input in the determination of local or regional gravimetric geoids.

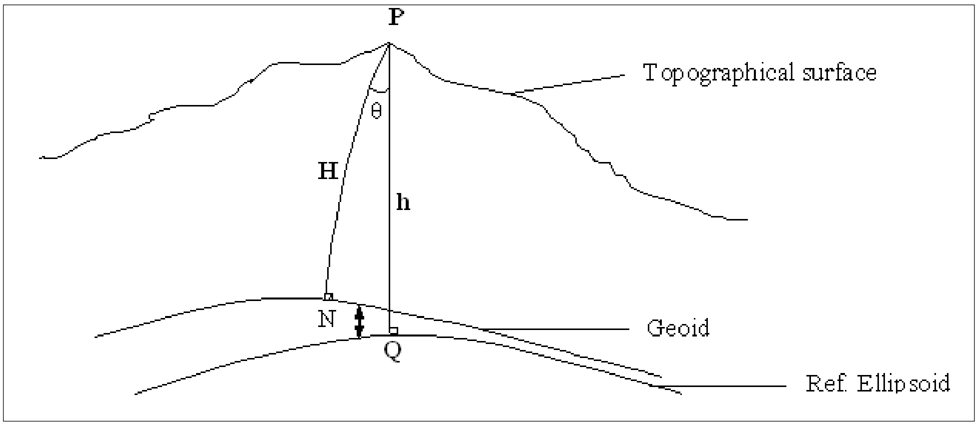

Figure 3: Relationship between geoid height (N), ellipsoidal height (h) and orthometric height (H)

In this approach, if the deflection of the vertical (θ) is ignored for practical applications, then the orthometric height H , may be obtained as,

$$H = h - N \, , \qquad (2)$$

From which the geoid undulation can be determined as,

$$N = h - H \, . \qquad (3)$$

In a relatively small and flat area the local geoid can be determined by a combination of GPS derived heights and levelled heights, called the geometric approach (e.g. Chen and Luo, 2004). A plane or low order polynomial is usually used to model the geoid (Featherstone et al., 1998). A geometric approach has been applied for the determination of geoid height in the study area. The spatial distribution of the GPS/levelling points within the triangulation network is given in Figure 4.

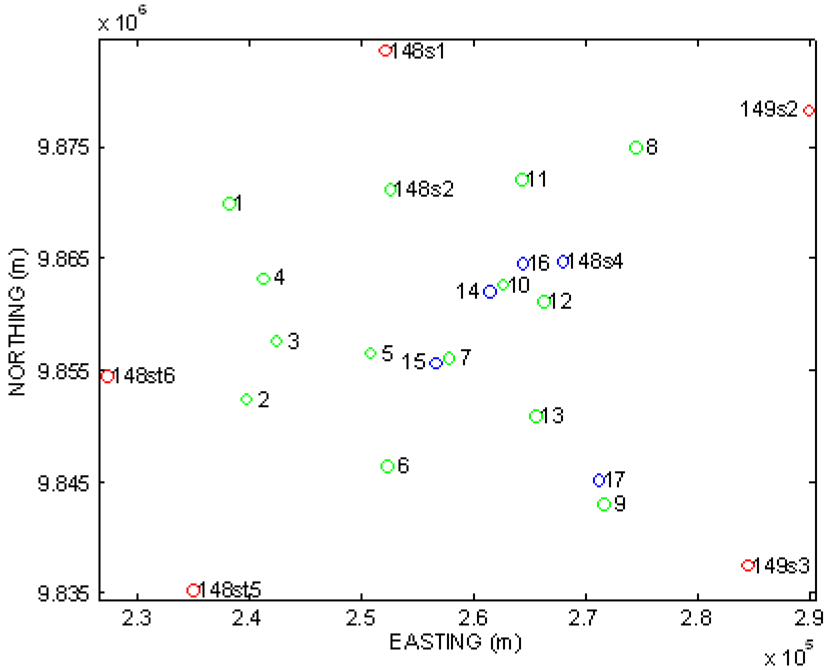

Figure 4: Spatial distribution of the GPS/levelling points within the triangulation network

It should be noted that five triangulation points (represented by small red circles in Figure 4 have not been included in geoid determination due to lack of precise levelling data. The heights of these points were determined using trigonometric heighting method, which is not accurate enough for this study. The geoid height is expressed as a function of the local plane coordinates through a biquadratic surface polynomial using 14 GPS/levelling points (represented by small green circles in Figure 4). Five points (represented by small blue circles in Figure 4) have been used for testing the results.

4.0 Results and Discussion

The geometric geoid of Nairobi County and its environs is shown in Figure 5 while the computed transformation parameters and their accuracies are given in Table 1. A comparison between the geometric geoid heights and interpolated geometric geoid heights (using biquadratic polynomial) at the five test points is given in Table 2.

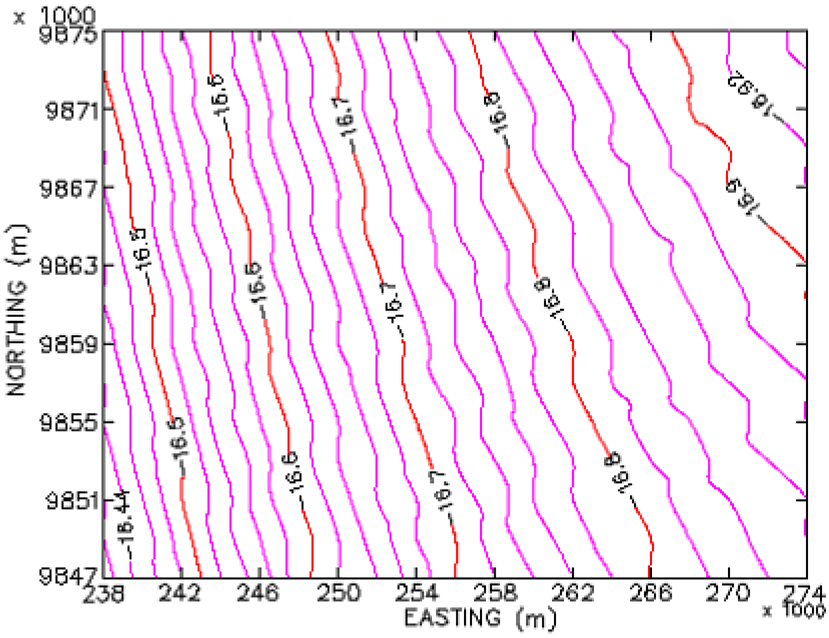

Figure 5: Geometric geoid of Nairobi County and its environs with respect to WGS84 (units are in m)

Table 1: Transformation parameters between Arc-Datum 1960 and WGS84 in Nairobi area

Parameter	Value (m)	Accuracy (m)
X_O	-150.104	$\pm$ 0.001
Y_O	2.635	$\pm$ 0.001
Z_O	-298.512	$\pm$ 0.000

Table 2: Comparison of the geometric geoid heights and interpolated geometric geoid heights (using biquadratic polynomial) at the five test points

Test Point	Geometric geoid height	Interpolated geoid height (m)	Geoid height difference (m)
148s4	-16.860	-16.872	+0.012
14	-16.821	-16.807	-0.014
15	-16.742	-16.732	-0.010
16	-16.836	-16.844	+0.008
17	-16.816	-16.828	+0.012
RMS			± 0.011
SD			± 0.013

The application of the transformation parameters determined by Molodensky model (Table 1) give local coordinates to sub-meter level compared to the local values determined by conventional methods. To analyse geoid heights obtained, orthometric and geodetic heights of 5 test points have been used to obtain Geometric geoid heights upon which the interpolated geoid heights by biquadratic polynomial are assessed. From Table 2 it is observed that the geoid heights obtained by polynomial compare favourably with the geometric geoid heights at the test points with root mean square (RMS) of ± 1cm and standard deviation (SD) of ± 1cm. This accuracy is good enough for most levelling applications (e.g. engineering surveying, topographic surveying and other related fields). It is mentioned in Uren and Price (1994) that for longitudinal sections, it is sufficiently accurate to record levelling readings to the nearest 1cm.

The biquadratic polynomial can therefore be used for geoid interpolation in a small area. However, for a large area (country or continent) a bicubic polynomial should be considered. This is because the change in slope of the geoid to the reference ellipsoid in a small area is generally uniform and gentle as opposed to a larger area. The biquadratic polynomial was also found to perform better than bilinear polynomial in the preliminary investigations.

5.0 Conclusions

The Nairobi County and its environs geoid has been determined by geometric method. The geoid heights increase eastwards in the area of study. For the interpolation of geoid heights in the area of study, a biquadratic polynomial has been used. RMS and standard deviation of the differences between observed and predicted values at the test points are equal, i.e. ± 1cm. Therefore it is possible to obtain fairly accurate orthometric heights from GPS heights. This accuracy is sufficient for most engineering projects that require levelling data. It should be noted that the accuracy referred to here is for the interpolation not the geoid.

It was also in the interest of this research to make the use of the determined geoid possible for local topographic and engineering surveying. It is appreciated that most if not all of the survey work in Nairobi County and its environs have been done in the local coordinates based on Arc–Datum 1960. There is therefore a need for the determination of transformation parameters between WGS84 and Arc– Datum 1960. The transformation parameters between WGS84 and Arc– Datum 1960, determined by Molodensky's model are adequate for application in the area of study. The transformation parameters give local coordinates to sub-meter level in Nairobi County and its environs.

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