THE SPECTRUM OF THE NORLUND Q OPERATOR ON c_0 SPACE

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ABSTRACT

In this paper, we determine the spectrum of the Norlund Q operator on . In which case we show that the spectrum comprises of all complex number λ, such that $\left|\lambda-\frac{1}{2}\right| \leq \frac{1}{2}$. We achieve this by solving the system $(Q-\lambda I)x=y$ for x in terms of y to obtain the matrix of $(Q-\lambda I)^{-1}$. We then subject the matrix to analysis using summability methods to determine the conditions for $(Q-\lambda I)^{-1} \in B(c_0)$.

Key words: Boundedness, operator, spectrum, norm, convergence, sequences, matrix

Notations: $\mathbb{C}$, $\sigma(T), \mathbf{R}, c_0, \|T\|, C_1,$ will denote respectively, the set of complex numbers, the spectrum of T, the set of real numbers, the set of sequences converging to zero, the norm of an operator T, the Cesaro matrix of order 1.

1.0 INTRODUCTION

Given a regular matrix A, Mercerian theorems are concerned with determining the real or complex values α for which $\alpha I + (I - \alpha) A$ That is $\lim\left(\alpha Ix + (I - \alpha) Ax\right) = t$, implies that lim x =t. For $\alpha \neq 1$ (Wenger,1975). The problem is equivalent to determining the resolvent set for A, or, determining the spectrum $\sigma(A)$ of A. Therefore a study of spectra of operators may lead to the development of their mercerian theorems.

A study of the structure of Norlund means shows that they are a generalisation of weighted means and Holder means which comprises of Cesaro means. Therefore a study of the spectra of Norlund operators may be regarded as a generalization of similar results obtained for weighted means as well as Holder means.

Quite a lot has been done on the analysis of Norlund means. In 1956, Alexander Peyerimhoff wrote on the convergence fields of Norlund means,(Peyerimhoff,1956). T. Patt(1959) wrote on the absolute summability of Norlund means of a Fourier series. David Borwein and F.P.Cass(1968) wrote on strong Norlund summability. B.Kuttner and B.Thorpe(1969) obtained some results on strong Norlund summability of a Cauchy product series. In 1984, David Borwein and F.Peter Cass obtained some results on Norlund matrices as bounded operators on ℓ_p ,(Borwein and Cass,1984). D.Borwein and B.Thorpe(1985),determined conditions for inclusion between Norlund summability methods.

We now state some of the results achieved concerning the spectra of certain matrix operators on some sequence spaces. From this we see that very little has been done concerning the spectra of Norlund means or matrices. Hence there is a need to shift our interest to this particular class of infinite matrices. We also summarise some key results from functional analysis and summability theory, especially those that are crucial to our study.

In 1960, E. K.Dorff and A.Wilansky showed that the spectrum of a certain mercerian Norlund matrix with $a_{nn} = 1$, contains negative numbers(Dorff and Wilansky,1960). C. Coskun(2003), determined the set of eigenvalues of a certain Norlund matrix as a bounded operator over some sequence spaces. In 1965, A. Brown, P. R. Halmos, A. L. Shields, determined the spectrum of the Cesaro operator on ℓ_2 of square summable sequences(Brown et al,1965). They showed that the spectrum is all $\lambda \not\in$, such that, $|\lambda - 1| \leq 1$. They used the fact that the key to determining the spectrum $\sigma(C_1)$ of C_1 is the identity $(I - C_1)(I - C_1^*) = I - D$, where D is diagonal. In 1968, D. W. Boyd extended the work by determining the spectrum of the same operator on $L^p(R^+)$, for $p \neq 2$ - the space of p- Lebesgue integrable functions on $\Re^+$ (Boyd,1968). He in particular showed that the spectrum is the set

$\left\{\lambda \in \mathbb{C} : \mathrm{Re}\left(\frac{1}{\lambda}\right) = \frac{p-1}{p}\right\}$,which for $p > 1$, is a circle with center $\frac{p}{2(p-1)}$ and the same radius. And for p=1, is the imaginary axis. The method of proof involved exhibiting integral operators which are proved to be the resolvents of Cesaro operator for $\mathrm{Re}\left(\frac{1}{\lambda}\right) < \frac{p-1}{p}$, and $\mathrm{Re}\left(\frac{1}{\lambda}\right) > \frac{p-1}{p}$ respectively. In 1972, N. K. Sharma determined the spectra of conservative matrices and in particular showed that the spectrum of any Hausdorff method is either uncountable or finite. In the latter case it is shown that the spectrum consists of either one point or two points. They also obtain the sharpest possible Mercerian theorem for Euler methods. In their proof the used the properties of an analytic function in D = $\left\{z \in \mathbb{C} : \left|z - \frac{1}{2}\right| < \frac{1}{2}\right\}$ and which commutes with the Cesaro operator of order 1(Sharma,1972). In 1975, Robert B. Wenger computed the fine spectra of Holder summability operators on c-the space of convergence sequences (Wenger, 1975). He use the state diagram in his proof. In 1978, James A. Deddens computed the spectra of all Hausdorff operators on ℓ_+^2 - the space of square summable sequences (Deddens, 1978). They were able to show that $\sigma(H_\alpha) = \{z^\alpha : |z-1| \le 1\}$ and $\|H_\alpha\| = 2^\alpha$. The technique of his proof involved standard operator theory and the use of analytic functions. In 1983, B. E. Rhoades extended Wenger's results by determining the fine spectra of weighted mean operators on c (Rhoades, 1983). In 1985, J. B. Reade determined the spectrum of the Cesaro operator on c_0 - the space of null sequences (Reade, 1985). He showed that the spectrum consists of all complex numbers λ, such that $\left|\lambda - \frac{1}{2}\right| \le \frac{1}{2}$. The proof used standard operator theory. In 1985, Manuel Gonzalez computed the fine spectrum of the C_1 operator on ℓ_p $(1 < p < \infty)$(Gonzalez, 1985). In 1989, J. Okutoyi and B. Thorpe computed the spectrum of the operator on () – the space of double null sequences (Okutoyi and Thorpe, 1989). They identified the spectrum as the set, In 1990, J. Okutoyi determined the spectrum of the operator on (Okutoyi, 1990). He showed that the spectrum consists of all complex numbers such that He obtained the results by finding the eigenvalues of the adjoint operator on (the dual space of). And then showing that the operator lies in Bfor all outside the closure of this set of eigenvalues. In 2005, Okutoyi and Akanga computed the spectrum of the operator on (Okutoyi and Akanga, 2005). They used methods similar to those of the preceding case and were able to show that the spectrum is the same.

1.1 Definition: Norm of an Operator

Let X be a Banach space. B(X,X)=B(X)-the linear space of all bounded linear operators T on X into itself, is a Banach space with the norm

$$\|T\| = \sup_{\|x\|} \|Tx\|,$$

[see Dunford and Schwartz, 1967, page 475].

1.2 Definition: (Adjoint Operator T^*)

The adjoint T^* of a linear operator T $\in$ B(X,Y) is the mapping from Y^* the dual space of Y to X^* the dual space of X, defined by T^* o f = f o T, fY^*.

1.3 Definition: (Resolvent operator, $R_\lambda = R_\lambda(T)$)

Let X be a non empty Banach space and suppose that $T: X \to X$. With T we associate the operator $T_\lambda = T - \lambda I, \lambda \in \square$, I the identity operator on X. If T_λ has an inverse, we denote it by and call it the resolvent operator of T.

1.4 Definition: (Resolvent set $\rho(T)$, the spectrum , $\sigma(T)$ of T)

Let X be a non empty Banach space and suppose that $T: X \to X$. The resolvent set $\rho(T)$ of T is the set of all complex numbers λ for which exists as a bounded operator with domain X. The spectrum of T is the complement of $\rho(T)$ in $\mathbb{C}$.

$$C_1 = (a_{nk}) \begin{cases} \frac{1}{n+1}, 0 \le k \le n \\ 0, k > n \end{cases}$$

1.5 Definition: (C_1 matrix)

The C_1 matrix of order 1 is defined as

1.6 Definition: **(ℓ_p Space)**

This is the space of sequences, (x_k), such that, $\sum_{k=0}^{\infty} |x_k|^p < \infty$.

1.7 Definition: **(Space bv_0)**

This is the space of sequences of bounded variation, i.e., sequences x, such that,

$$\sum_{k=0}^{\infty} |x_{k+1} - x_k| < \infty, \text{ with } x_k \to 0 \text{ as } k \to \infty.$$

1.8 Definition: **(Space $w_p\ (1 \le p < \infty)$)**

This is the space of strongly Cesaro summable complex sequences of order 1 index p, i.e, the set of all sequences $x = (x_k)_{k=1}^{\infty}$, such that there exists a number ℓ depending on x for which $\sum_{k=0}^{\infty} |x_k - \ell|^p < \infty.$

Theorem 1.1 (Wilansky, 1984): Let T $\in$ B(X), where X is any Banach space. Then the spectrum of T^* is identical with the spectrum of T. Furthermore $R_\lambda\left(T^*\right) = \left(R_\lambda\ (T)\right)^*$ for $\lambda \in \rho\left(T\right) = \rho\left(T^*\right)$.

Theorem 1.2 (Stieglitz and Tietz, 1977): A matrix $A = (a_{nk}) \in (c_0, c_0)$ iff

(i) $\lim_n a_{nk} = 0$ for each $k \ge 0$

(ii) $\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$, moreover $\|A\| = \sup_n \sum_{k=0}^{\infty} |a_{nk}|$.

Lemma 1.1 (Taylor and Lay, 1980): Each bounded linear operator T : X $\to$ Y, where X = c_0, ℓ_1, c and Y = $c_0, \ell_p\ (1 \le p < \infty), c, \ell_\infty$ determines and is determined by an infinite matrix of complex numbers.

Lemma 1.2 (Wilansky, 1984): Let $T : c_0 \to c_0$ be a linear map and define $T^* : \ell_1 \to \ell_1$ by

$T^* og = goT, g \in c_0^* = \ell_1$. Then T must be given by a matrix by lemma (1.1) and moreover $T^* : \ell_1 \to \ell_1$ is the transposed matrix of T.

$T^{*}og = goT, g \in c_0^{*} = \ell_1$. Then T must be given by a matrix by lemma (1.1) and moreover $T^{*} : \ell_1 \to \ell_1$ is the transposed matrix of T.

2.0 THE SPECTRUM OF Q MATRIX AS AN OPERATOR ON c_0

2.1 Definition: (Norlund means (N,p))

The transformation given by

$$y_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} x_k , n = 0,1,2,...$$

(2.1)

where $P_n = p_0 + p_1 + p_2 + ... + p_n \neq 0$, is a complex series ; is called a Norlund means and is denoted by (N,p). The corresponding matrix is given by

$$A = (a_{nk}) = \begin{cases} \frac{p_{n-k}}{P_n}, k \le n \\ 0, k > n \end{cases}$$

See Wilansky,1984, pages 24-33.

If $p_n = 1, n = 0,1$ and $p_n = 0, n \geq 2$, then the matrix $A = (a_{nk})$ in (2.2) becomes matrix Q given by the formula,

$$Q = (q_{nk}) = \begin{cases} 1, n = k = 0 \\ \frac{1}{2}, n-1 \le k \le n \\ 0, otherwise \end{cases} \tag{2.3}$$

So that

$$Q = (q_{nk}) = \begin{pmatrix} 1 & 0 & 0 & 0 & . & . & . \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & . & . & . \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & . & . & . \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & . & . & . \\ 0 & 0 & 0 & \frac{1}{2} & . & . & . \\ & . & . & . & & & \end{pmatrix}$$

This is the matrix of our interest in this paper.
2.1 **Remark:** The Q matrix is not a Hausdorff matrix. This is so since Q does not commute with the Cesaro matrix of order 1, C_1 matrix (Wilansky, 1984) page 24. Corollary 2.1: $Q \in B(c_0)$, moreover $\|Q\|_{c_0} = 1$.

Proof: It is clear from theorem (1.2) and matrix (2.3) since $\lim_n q_{nk} = 0$ for each $k \geq 0$. Moreover

$$\|Q\|_{c_0} = \sup_n \sum_{k=0}^{\infty} |q_{nk}| = \sup(1,1,1,...) = 1.$$

Corollary 2.2: Let $Q : c_0 \to c_0$, then $Q^* \in B(\ell_1)$ and in addition

$$Q^* = \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & . & . & . \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & . & . & . \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & . & . & . \\ & . & . & . & & & \end{pmatrix} \qquad (2.4)$$

Proof: It is clear from lemma (1.2).

Theorem 2.1: $Q \in B(c_0)$ has no eigenvalues.

Proof: Suppose $Qx = \text{l}\ x, x \neq \text{q}$ such that $x \in c_0$ and $\text{l} \notin$. Then solving the system of equations involved, we have that if x_0 is the first non-zero entry of vector x, then $\text{l} = 1$. But implies that So that does not tend to zero as . Hence is not an eigenvalue of . If is the first non-zero entry of x, then . Solving the system with results in which is a contradiction. Hence cannot be an eigen-value of . Hence, the result.

2.1 Main Results

Theorem 2.2: The eigenvalues of $Q^* \in B(\ell_1)$ forms the set

$$\left\{ \lambda \in \square : \left| \lambda - \frac{1}{2} \right| < \frac{1}{2} \right\} \cup \left\{ 1 \right\}$$

Proof: Let $Q^{*}x = \mathrm{l}\ x, x \neq \mathrm{q}$ and $\mathrm{l} \notin$ .Then

$$\begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & . & . & . \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & . & . & . \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & . & . & . \\ & & . & . & . & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \end{pmatrix}$$

Solving system (2.5) for x in terms of x ^{0}, gives

$$x_n = (2\mathrm{l}\ )^{n} \left(1 - \frac{1}{2\mathrm{l}}\right)^{n-1} \left(1 - \frac{1}{\mathrm{l}}\right) x_0, n \geq 1$$

Now,

$$\lim_{x \to \infty} \left| \frac{x_{n+1}}{x_n} \right| = \lim_{x \to \infty} \left| \frac{(2\mathrm{l}\ )^{n+1} \left(1 - \frac{1}{2\mathrm{l}}\right)^{n} \left(1 - \frac{1}{\mathrm{l}}\right)}{(2\mathrm{l}\ )^{n} \left(1 - \frac{1}{2\mathrm{l}}\right)^{n-1} \left(1 - \frac{1}{\mathrm{l}}\right)} \right| = \left| 2\mathrm{l}\ \left(1 - \frac{1}{2\mathrm{l}}\right) \right| = m$$

,

say for some $m \in \mathrm{R}$, such that $m \geq 0$.

By the ratio test $(x_n) \in \ell_1$ if and only if m<1. That is, if and only if

$$\left| 2\mathrm{l}\ \left(1 - \frac{1}{2\mathrm{l}}\right) \right| < 1$$

or

$$\left|\lambda - \frac{1}{2}\right| < \frac{1}{2}.$$

It is clear that $\lambda = 1$ is an eigenvalue corresponding to the eigenvector $(x_0, 0, 0, \ldots)^t$, where x_0 is a non-zero real or complex number. Hence the result.

Theorem 2.3: The spectrum $\sigma(Q)$ of $Q \in B(c_0)$ is the set

$$\left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \le \frac{1}{2}\right\}.$$

Proof: By virtue of theorem (2.2) and theorem (1.1), it is enough to show that $(Q-\lambda I)^{-1} \in B(c_0)$ for all $\lambda \notin$, such that $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}.$.

To this end we solve the system $(Q-\lambda I)x = y$ for x in terms of y to obtain:

$$x_n = -\frac{1}{2^n \lambda^{n+1}\left(1-\frac{1}{2\lambda}\right)^n \left(1-\frac{1}{\lambda}\right)} y_0 - \frac{1}{2^{n-1}\lambda^n\left(1-\frac{1}{2\lambda}\right)^n} y_1 - \frac{1}{2^{n-k}\lambda^{n-k+1}\left(1-\frac{1}{2\lambda}\right)^{n-k+1}} y_k, 0 \le k \le n \tag{2.6}$$

System (2.6) yields the matrix of $(Q-\lambda I)^{-1}$ which we denote by M. That is

$$M = \begin{pmatrix} \frac{1}{\lambda\left(1-\frac{1}{\lambda}\right)} & 0 & 0 & 0 & \cdots \\ -\frac{1}{2\lambda^2\left(1-\frac{1}{2\lambda}\right)\left(1-\frac{1}{\lambda}\right)} & -\frac{1}{\lambda\left(1-\frac{1}{2\lambda}\right)} & 0 & 0 & \cdots \\ -\frac{1}{2^2\lambda^3\left(1-\frac{1}{2\lambda}\right)^2\left(1-\frac{1}{\lambda}\right)} & -\frac{1}{2\lambda^2\left(1-\frac{1}{2\lambda}\right)^2} & -\frac{1}{\lambda\left(1-\frac{1}{2\lambda}\right)} & 0 & \cdots \\ -\frac{1}{2^3\lambda^4\left(1-\frac{1}{2\lambda}\right)^3\left(1-\frac{1}{\lambda}\right)} & -\frac{1}{2^2\lambda^3\left(1-\frac{1}{2\lambda}\right)^3} & -\frac{1}{2\lambda^2\left(1-\frac{1}{2\lambda}\right)^2} & -\frac{1}{\lambda\left(1-\frac{1}{2\lambda}\right)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{2.7}$$

So that

$$M=(m_{nk})=\begin{cases} -\dfrac{1}{2^n\lambda^{n+1}\left(1-\dfrac{1}{2\lambda}\right)^n\left(1-\dfrac{1}{\lambda}\right)}, k=0 \\ -\dfrac{1}{2^{n-k}\lambda^{n-k+1}\left(1-\dfrac{1}{2\lambda}\right)^{n-k+1}}, 1\le k\le n \\ 0, k>n \end{cases} \quad (2.8).$$

Note that

$$M^{-1}=\begin{pmatrix} (1-\lambda) & 0 & 0 & 0 & . & . & . \\ \frac{1}{2} & \left(\frac{1}{2}-\lambda\right) & 0 & 0 & . & . & . \\ 0 & \frac{1}{2} & \left(\frac{1}{2}-\lambda\right) & 0 & . & . & . \\ 0 & 0 & \frac{1}{2} & \left(\frac{1}{2}-\lambda\right) & . & . & . \\ & . & . & . & & & \end{pmatrix}$$

A simple calculation shows that

$$MM^{-1}=M^{-1}M=I$$

Which confirms that M is the inverse of matrix $(Q-\lambda I)$.

Columns of matrix M converge to zero if $\left|\dfrac{(n+1)^{th}\ term}{n^{th} term}\right|<1, k\ge 0.$

For $k\ge 0$, we have that $\lim_{n\to\infty} m_{nk}=0$, when $\left|\lambda-\frac{1}{2}\right|>\frac{1}{2}$. This deals with condition (i) of theorem (1.2).

Using theorem (1.2) on matrix M it is easy to see that,

$$\sup_n \sum_{n=0}^{\infty}|m_{nk}|<\infty$$

, provided ~~$\lambda \notin$~~ , is such that $\left|\lambda-\frac{1}{2}\right|>\frac{1}{2}$. So that $M=(Q-\lambda I)^{-1}\in B(c_0)$ if

$\lambda \in \mathbb{C}$ is such that $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$. All this implies that $M = (Q - \lambda I)^{-1} \notin B(c_0)$ for all $\lambda \in \mathbb{C}$, such that $\left|\lambda - \frac{1}{2}\right| \le \frac{1}{2}$ $\sqcup$.

Application

The results found in this paper may be used in solving systems of linear equations which arise during experiments in science and engineering.

3.0 CONCLUSION

In this paper we have determined the following results:

(i) The set of eigenvalues for $Q \in B(c^0)$ is null.

(ii) The spectrum of $Q \in B(c^0)$ is the set $\lambda \in \mathbb{C}: \left|\lambda - \frac{1}{2}\right| \le \frac{1}{2}$.

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