Suborbital Graphs Corresponding to the Action of Dihedral Group and Cyclic Group on the Vertices of a Regular Polygon

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DECLARATION

This research is my original work and has not been presented for a degree in any other University.

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DEDICATION

To my beloved mother Mrs R. N. Mwai, my brother Mr J. K. Mwai and my late father Mr. J. K. Mwai for their encouragement through my period of study.

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SYMBOLS AND ABBREVIATIONS

Γ	The suborbital graph corresponding to the suborbit $\triangle$
$\begin{pmatrix} n \\ r \end{pmatrix}$	n combination r
$\mid fix(g) \mid$	Number of elements fixed by g $\in$ G
$\triangle$	Suborbit of G on X
$\triangle^*$	The G - suborbit paired with $\triangle$
C_n	Cyclic group of order n
D_n	Dihedral group of degree n and order 2n
e	The indentity in a group
O	The suborbital of G on X $\times$ X
p	A prime number
$PGL(n,q)$	The projective general linear group
$PSL(n,q)$	The projective special linear group
S_n	Symmetric group of degree n and order n factorial
$Stab_G(x)$	Stabilizer of a point x in X
$X^{(r)}$	Set of all unordered r - element subsets from X = $\{1,2,\ldots, n\}$

LIST OF FIGURES

ABSTRACT

The main aim of this research was to determine transitivity, primitivity, ranks, subdegrees, and suborbital graphs of cyclic group C_n and dihedral group D_n acting on vertices of a regular $n-$ gon. These areas have not received much attention, in fact most of the researchers have been focused on testing whether the action of specific degrees of the dihedral group are primitive or transitive on the vertices of a regular $n-$ gon. This research extends the work of Hamma to the general degree n for both C_n and D_n. With regard to the suborbital graphs of these two groups, nothing appears in literature and so to some extent the results obtained in this research can be regarded as new. In this research it has been shown that C_n and D_n act transitively on the vertices of a regular $n-$ gon . Also C_n and D_n act imprimitivily on the vertices of a regular $n-$ gon if n is not prime. The rank of C_n is shown to be n and the rank of D_n is shown to be $\frac{n}{2}+1$ when n is even and $\frac{n+1}{2}$ when n is odd. It is also shown that the suborbits of C_n are not all selfpaired; only 2 are selfpaired when n is even and 1 when n is odd, the rest are paired with each other such that $\triangle_i$ of C_n is paired with $\triangle_{n-i}$, but all the suborbits of D_n are selfpaired. The subdegrees of C_n are shown to be all singletons, and the subdegrees of D_n are shown to be 1,1,2,2, $\cdots$,($\frac{n}{2}-1$) twos when n is even and 1,2,2,2, $\cdots$,($\frac{n-1}{2}$) twos when n is odd. Further it is shown that for a suborbital O_{i-1} in C_n, $(a,b) \in O_{i-1}$ if and only if $|b-a| = \begin{cases} i-1 & if\, b > a \\ n-(i-1) & if\, a > b \end{cases}$, and that all suborbital graphs of C_n are connected if and only if n is prime. The suborbitals of D_n are shown to be union of the paired suborbitals of C_n, and the corresponding suborbital graphs are connected if and only if n is prime. Finally it is shown that the number of components of the suborbital graph Γ_{i-1} for both groups is $d = gcd(n, i-1)$ and its girth is $r = \frac{n}{d}$, when $d \neq \frac{n}{2}$ and zero if $d = \frac{n}{2}$.

CHAPTER ONE
INTRODUCTION AND LITERATURE REVIEW

1.1 Background of the Study

Some basic concepts in group theory, graph theory, suborbital graphs and a list of theorems which will be used in the actual research are discussed in the preceding subsections.

1.1.1 Group Theory

The groups which will be considered in this work are **permutation groups.** Let $X = \{1, 2 \cdots, n\}$,a permutation of X is a one to one mapping of X onto itself. The **symmetric group** of degree n is the group of all permutations of X under the binary operation of composition of mappings. It is denoted by S_n and is of order $n!$.
Let G be a group, then G is **cyclic** if there exist $a \in G$ such that $G =< a >= \{a^n | n \in \mathbb{Z}\}$.
A **dihedral group** is the group of symmetries of a regular polygon and it is denoted by D_n where $n \geq 3$, and has order $2n$.The vertices of the regular polygon will be denoted as the set $X = \{1, 2, \cdots, n\}$.
The conventional way of writing $D_n = \left\langle x, y \,|\, x^n = y^2 = e, yx = x^{n-1}y = x^{-1}y \right\rangle$, thus D_n is the group generated by the elements x, y subject to the conditions $x^n = y^2 = 1$; $yx = x^{n-1}y = x^{-1}$y, and the $2n$ distinct elements of D_n are 1, x, x^2, $\ldots$, x^{n-1}, y, xy, x^2y, $\ldots$, $x^{n-1}y$. Here x is a rotation about the centre of the polygon through angle $2\Pi^c/n$; it generates a cyclic subgroup C_n of order n. The element y is a reflectional symmetry along the line joining a vertex to the centre of opposite edge if n is odd; or a reflectional symmetry along the line from a vertex to an opposite vertex or from the centre of an edge to the centre of the opposite edge if n is even.

1.1.2 Group Actions

Let X be a set and G a group. Then G acts on a set X on the left if $\forall g \in G$ and $x \in X$ there exists a unique $gx \in X$ such that if $g_1, g_2 \in G$, $(g_1 g_2)(x) = g_1(g_2(x))$, and $1x = x$, where 1 denotes the identity in G. The action of G on X from the right can be defined in a similar way.
If G acts on a set , then X is partitioned into disjoint equivalence classes called orbits or transitivity classes of the action. For each $x \in X$, the orbit containing x is denoted by $Orb_G(x)$, therefore $Orb_G(x) = \{gx \,|\, g \in G\}$. If the action of a group G on a set X has only one orbit, then G is said to act transitively on X. Hence G acts transitively on X if for every pair of points x, $y \in X$, there exists $g \in G$ such that $gx = y$.

The **stabilizer** of $x \in X$, denoted by $Stab_G(x)$, is the set of all elements in G that fix x i.e. $Stab_G(x) = \{g \in G \mid gx = x\}$. This set is also denoted by G_x and it can be shown that $Stab_G(x) \leq G$.
Let G act on a set X. The set of elements of X fixed by $g \in G$ is called the fixed point set of g, denoted by $Fix(g)$. Thus $Fix(g) = \{x \in X \mid gx = x\}$.

Theorem 1.1.1. *(Orbit - Stabilizer Theorem - Rose 1978, p.72)*
Let G be a group acting on finite set X and $x \in X$. Then
$|Orb_G(x)| = |G : Stab_G(x)|$.

Let G be a transitive group acting on a set X. A subset Y of X is said to be a **block** for the action if, for each $g \in G$, either $gY = Y$ or $gY \cap Y = \emptyset$. All $1-$ element subsets of X, $\emptyset$, and X are obvious blocks and they are called the trivial blocks. If they are the only blocks then G acts **primitively** on X , otherwise G acts **imprimitively**.

Example 1.1.2. Le t $G = D_4 = \{e, (1234), (13)(24), (1432), (12)(34), (14)(23), (24), (13)\}$ acting on the set $X = \{1,2,3,4\}$. Then G acts imprimitively on X since $Y = \{1,3\}$ is a non - trivial block.

Theorem 1.1.3. *(Scott, 1964;Passman, 1968, p.15)*
If G acts on a set X, where G is a transitive group of prime degree, then G is primitive.

A **Maximal Subgroup** of a group G is a subgroup M not equal to G such that there is no proper subgroup N of G properly containing M. The following theorem can be used to test the primitivity of an action

Theorem 1.1.4. *Let G be a transitive permutation group acting on a set X and let $x \in X$. Then G is primitive if and only if G_x is a maximal subgroup or equivalently G is imprimitive if and only if G_x is not a maximal subgroup of G.*

If a finite group G acts on a set X with n elements, each $g \in G$ corresponds to a permutation σ of X, which is uniquely as a product of **disjoint cycles**. If σ has α_1 cycles of length 1, α_2 cycles of length 2,... , α_n cycles of length n; we say that σ and hence g has cycle type $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Theorem 1.1.5. *(Harary, 1969, p.98)*
Let G be a finite group acting on a set X. The number of orbits of G is
$\frac{1}{|G|}\sum_{g \in G} |fix(g)|$.

(This theorem is referred to as Cauchy - Frobenius Lemma).

1.1.3 Graph theory

A simple graph is an ordered pair $H = (V,E)$, where V is a finite non - empty set of objects called vertices and E is a (possibly empty) set of $2-$ element subsets of V called edges. The set V is called the vertex set of H and E is called the edge set of H. If $e = \{u,v\} \in E(H)$, vertices u and v are adjacent in H and that e joins or connects u and v. The edge e is said to be incident with u (and v), and vice versa.
A directed graph (or digraph) (V,E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u,v) is said to start at u and end at v.
Degree or valency $d_H(v)$ of a vertex v of a graph H is the number of vertices of H adjacent to v. A vertex of degree 0 is an isolated vertex. If H is a graph with vertices $v_1, v_2, \ldots, v_n$, then the degree sequence of H is the sequence $d_H(v_1), d_H(v_2), \ldots, d_H(v_n)$, it is usually ordered in such a way that $d_H(v_1) \leq d_H(v_2) \leq \ldots \leq d_H(v_n)$. A graph in which every vertex has the same degree is called regular.
A **walk** of length k joining u and v in H is a sequence of vertices and edges of H of the form $v_0, e_1, v_1, e_2, v_2, \ldots v_{k-1}, e_k, v_k$, where $v_0 = u, v_k = v$ and $e_i = \{v_{i-1}, v_i\}$ for $i = 1,2,\ldots,k$. A walk joining u and v is closed if $u = v$, and is a path if no two vertices of the walk (except possibly u and v) are equal; a closed path is called a circuit or cycle. The length of the shortest cycle (if any) in H is called the **girth** of H.
A graph H is **connected** if every pair of vertices of H is joined by some path; otherwise, H is disconnected. A connected component of H is a maximal connected subgraph of H. Each vertex and edge of H belongs to precisely one component of H.

1.1.4 Suborbital Graphs

Let G be transitive on X and let G_x be the stabilizer of a point $x \in X$. The orbits $\triangle_0 = \{x\}, \triangle_1, \triangle_2, \ldots, \triangle_{r-1}$ of G_x on X are called the **suborbits** of G. The rank of G is r and the sizes $n_i = \mid \triangle_i \mid$ $(i = 0,1,2,\ldots,r-1)$, often called the "lengths " of the suborbits, are known as subdegrees of G. Note that both r and the cardinalities of the suborbits $\triangle_i$ $(i = 0,1,2,\ldots,r-1)$ are independent of $x \in X$.
Let $\triangle$ be an orbit of G_x. Define $\triangle^* = \{gx | g \in G, x \in g \triangle\}$, then $\triangle^*$ is also an orbit of G_x and is called the G_x-orbit (or the G - suborbit) paired with $\triangle$. Clearly $\mid \triangle \mid = \mid \triangle^* \mid$. If $\triangle^* = \triangle$, then $\triangle$ is called a **selfpaired** orbit of G_x.

Theorem 1.1.6. *(Cameron, 1975 p.422)*

If G is primitive, with subdegrees $1 = n_0, n_1, \ldots, n_{r-1}$(in increasing order of magnitude), then $n_1 n_{i-1} \geq n_i$ for $i = 1,\ldots,r-1$. Now if there exist an index $i > 0$ such that $n_i > n_1 n_{i-1}$, then G is imprimitive.

Theorem 1.1.7. *(Wielandt, 1964, section 16.5)*
G_x has an orbit different from $\{x\}$ and paired with itself if and only if G has even order.

Observe that G acts on $X \times X$ by $g(x,y) = (gx, gy), g \in G\, x,y \in X$. If $O \subseteq X \times X$ is a G - orbit on $X \times X$, then for a fixed $x \in X$, $\triangle = \{y \in X \,|\, (x,y) \in O\}$ is a $G_x - Orbit$ on X. Conversely, if $\triangle \subseteq X$ is a G_x-orbit, then $O = \{(gx, gy) \,|\, g \in G, y \in \triangle\}$ is a $G - Orbit$ on $X \times X$. We say that $\triangle$ corresponds to O. The G - orbits on $X \times X$ are called suborbitals. Let $O_i \subseteq X \times X$, $i = 0,1,2,\ldots,r-1$ be a suborbital. Then we form a graph Γ_i, by taking X as the set of vertices of Γ_i, and by including a directed edge from x to y (x, $y \in X$) if and only if $(x,y) \in O_i$.
The suborbital graph Γ_0 corresponding to the suborbit $\triangle_0$ is called the trivial suborbital graph. When the suborbits are selfpaired the corresponding suborbital graphs are undirected. If the suborbits are not selfpaired the corresponding suborbital graphs are directed. The trivial suborbital graph is selfpaired; it consists of a loop based at each vertex $x \in X$. We are mainly interested with the non - trivial suborbital graphs. If the suborbital graph Γ is paired with Γ^*, then Γ^* is just Γ with arrows reversed.
Let G act on a set X, then the character π of a permutation representation of G on X is defined by $\pi(g) = |Fix(g)|$, for all $g \in G$

Theorem 1.1.8. *(Cameron, 1975)*

Let G act transtively on a set X and let $g \in G$. Suppose π is the character of the permutation representation of G on X, then the number of selfpaired suborbits of G is given by $n_\pi = \frac{1}{|G|} \sum\limits_{g \in G} \pi(g^2)$.

Theorem 1.1.9. *(Sims, 1967)*
Let G be transitive on X. Then G is primitive if and only if each suborbital graph $\Gamma_i (i = 1,2,\ldots,r-1)$ is connected.

1.2 Literature Review

In this section a review of previous studies which are closely related to this work are discussed. Groups as mathematical structures form a major area of interest for mathematicians in abstract algebra. To be able to understand them better together with their properties, their suborbital graphs can be constructed. Wielandt (1964) wrote a little monograph on finite permutation groups. In this monograph a condition for imprimitivity of a group is given in terms of its subdegrees.

Higman (1964) introduced the rank of a group while working on finite permutation groups of rank 3. Also Higman (1970) gave a characterization of families of rank 3 permutation groups by the subdegrees. He proved that the symmetric group S_n on $X = \{1, 2, \ldots, n\}$ $n \geq 4$ acts as a rank 3 group on the set of $\begin{pmatrix} n \\ 2 \end{pmatrix}$ 2- elements subsets of X, with subdegrees $1, 2(n-2)$, $\begin{pmatrix} n-2 \\ 2 \end{pmatrix}$.

The idea of suborbital graphs of a permutation group G acting on a set X was introduced by Sims in 1967.

Tchuda (1986) computed the ranks and subdegrees of primitive permutation representations of $PSL(2,q)$.

Faradzev and Ivanov (1990) computed the subdegrees of primitive permutation representations of $PSL(2,q)$. If $G = PSL(2,q)$ acts on the cosets of its maximal subgroup H, then the rank is at least $\frac{|G|}{|H|^2}$ and if $q > 100$, the rank is greater than 5.

Kamuti (1992) devised a method for constructing some of the suborbital graphs of $PSL(2,q)$ and $PGL(2,q)$ acting on the cosets of their maximal dihedral subgroups of order $q-1$and $2(q-1)$ respectively. This method gave an alternative way of constructing the Coxeter graph which was first constructed by Coxeter (1986).

Lloyd and Jones (1998) published the paper $Reaction Graphs$ in which they showed that algebraic combinatorics and group theory are effective tools for studying such properties as connectivity and automorphisms in chemistry.

Akbas (2001) investigated the suborbital graphs for the modular group. He proved the conjecture by Jones, Singerman and Wicks (1991) that a suborbital graph for the modular group is a forest if and only if it contains no triangles.

Kamuti (2006) computed the ranks and subdegrees of primitive permutation representations of $PGL(2,q)$. It was shown in this paper that when $PGL(2,q)$ acts on the cosets of its maximal dihedral subgroup of order $2(q-1)$, the rank is $\frac{1}{2}(q+3)$ if q is odd and $\frac{1}{2}(q+2)$ if q is even.

Miyuki and Kano (2007) showed that a new visual cryptography scheme (VCS) with dihedral groups is possible. This is one of the many applications of the dihedral group.

Hamma S. and Audu M. S. (2010) investigated transitivity and primitivity of D_n acting on the vertices of a regular n - gon. In their research they considered dihedral groups of degree p (p prime), p^2 and $2^r (r \geq 2)$. When a dihedral group is of degree p, it was shown that the action was transitive and primitive; and when the degree is p^2 the action was shown to be transitive and imprimitive. Further it was shown that when a dihedral group is of degree $2^r(r \geq 2)$, the action is transitive and imprimitive. In this work they did not concinder all values of n.

Kamuti I. N., Inyangala E. B. and Rimberia J. K.(2012) investigated the action of Γ_∞ on $\mathbb{Z}$ and the corresponding suborbital graphs. It was shown that the action is transitive and imprimitive. They also constructed suborbital graphs corresponding to the action and gave the conditions necessary for the suborbital graphs to be connected or disconnected.

Nyaga L., Kamuti I. N., Mwathi C. and Akanga J.,(2012) showed that the action of S_n on $X^{(r)}$ is transitive and that the rank is $r+1$ if $n \geq 2r$. In the same work it was shown that the suborbits of S_n acting on $X^{(r)}$ are all selfpaired and that the subdegrees are

$$1, r\begin{pmatrix} n-r \\ r-1 \end{pmatrix}, \begin{pmatrix} r \\ 2 \end{pmatrix}\begin{pmatrix} n-r \\ r-2 \end{pmatrix}, \begin{pmatrix} r \\ 3 \end{pmatrix}\begin{pmatrix} n-r \\ r-3 \end{pmatrix}, \ldots \begin{pmatrix} r \\ r-1 \end{pmatrix}\begin{pmatrix} n-r \\ 1 \end{pmatrix}, \begin{pmatrix} n-r \\ r \end{pmatrix}.$$

1.3 Statement of the Problem

If a group acts on a set, the natural questions a group theorist may ask are:

- Is the action transitive
- Is the action primitive
- What are the mathematical structures and invariants associated with the action

In this work we try to answer the above questions with regard to the dihedral group D_n and the cyclic group C_n acting on the vertices of a regular $n-$ gon. To this end our research seeks to investigate transitivity, primitivity, ranks, subdegrees and suborbital graphs associated with the action. This will enable us fill the gap left by Hamma et. al (2010).

1.4 Justification

Graph theory has many application in chemistry and computer science. This is evident from the work of Lloyd and Jones (1998) where they showed that algebraic combinatorics is an effective tool for studying such properties as connectivity and automorphisms in chemistry.

Shirinivas *et al.* (2010) have discussed how graph theoretical ideas can be utilized in various computer science applications. These include research areas of computer science such as data mining, image segmentation, clustering, image capturing and networking . For example a data structure can be designed in the form of tree which in turn utilized vertices and edges. Similarly modeling of network topologies can be done using graph concepts. In the same way the most important concept of graph coloring is utilized in resource allocation, scheduling. Also, paths, walks and circuits in graph theory are used in tremendous applications say traveling salesman problem, database design concepts, resource networking. This leads to the development of new algorithms and new theorems that can be used in tremendous applications in computer science.

1.5 Objectives

1.5.1 General Objective

To construct the suborbital graphs corresponding to the action of cyclic group C_n and the dihedral group D_n acting on the vertices of a regular $n-$gon.

1.5.2 Specific Objectives

1. To determine transitivity and primitivity of D_n and C_n acting on the vertices of a regular $n-$ gon.

2. To determine the ranks and subdegrees of D_n and C_n acting on the vertices of a regular $n-$ gon.

3. To construct the suborbital graphs of D_n and C_n acting on the vertices of a regular $n-$ gon and to investigate their properties.

CHAPTER TWO
TRANSITIVITY, PRIMITIVITY AND SUBORBITS OF C_n

2.1 Introduction

This chapter investigates transitivity, primitivity and suborbits of the cyclic group C_n acting on the set of vertices of a regular $n-$ gon. Throughout the chapter we will be taking $G = C_n =< x >=< (12\ldots n) >$ and $X = \{1, 2, \ldots, n\}$; the set of vertices of a regular $n-$ gon.

2.2 Transitivity and Primitivity of G on X

Theorem 2.2.1. *Let $i \in X$, then $Stab_G(i) = \{e\}$.*

Proof. Clearly in G, it is only the identity element which fixes a point in X $\square$

Theorem 2.2.2. *G acts transitively on X.*

Proof. From Theorem 2.2.1, only the identity element which has a fixed point in X and in this case the number of points fixed by the identity is $|X| = n$. Hence by Cauchy - Frobenius lemma the number of orbits of G on X is $\frac{1}{|G|}\sum\limits_{g\in G}|fix(g)| = \frac{1}{n} \times n = 1$. Therefore from subsection 1.1.2, G acts transitively on X. $\square$

Theorem 2.2.3. *If $|X| = n$, where n is not a prime number, then G acts imprimitively on X.*

Proof. Since n is not prime, then there exists a positive integer k such that $1 < k < n$ and k divides n. Now $< x^{n/k} >$ is a proper subgroup of G of order k properly containing $Stab_G(i) = \{e\}$. Hence by Theorem 1.1.4 G acts imprimitively on X. $\square$

Example 2.2.4. Let $G = C_9 =< x >= \Big\langle (123456789) \Big\rangle$, then
$H =< x^3 >= \{1, (147)(258)(369), (174)(285)(396)\}$ and
$Stab_G(1) < H < G$. Hence G acts imprimitively on $X = \{1,2,\ldots,n\}$.

2.3 Suborbits and the Rank of G

Theorem 2.3.1. *Orbits of $Stab_G(1)$ on X are $\triangle_0 = \{1\}$, $\triangle_1 = \{2\}$, $\triangle_2 = \{3\}$, $\ldots$, $\triangle_i = \{i+1\}$, $\ldots$, $\triangle_{n-1} = \{n\}$. Thus the rank of G on X is n and the subdegrees are* $1, 1, 1, \ldots, 1$; *n ones*

Proof. From Theorem 2.2.1 $Stab_G(1) = \{e\}$, and therefore suborbits of G consist only of singleton elements. Hence the rank of G is n and the subdegrees are $1, 1, 1, \ldots, 1$; n ones $\square$

Theorem 2.3.2. *The number of selfpaired suborbits of G on X is 2 if n is even or 1 if n is odd.*

Proof. Let $g \in G$, then g^2 will have fixed points in X if either g is the identity or g is an element of order two. Also G contains an element of order two only when n is even. Therefore by Theorem 1.1.8 the number of selfpaired suborbits of G is $\frac{1}{n}(n+n) = \frac{2n}{n} = 2$ when n is even and $\frac{1}{n}(n) = \frac{n}{n} = 1$ when n is odd. $\square$

Example 2.3.3. Let $G = C_9 = \left\langle (123456789) \right\rangle$ and $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then

$$\begin{aligned} C_9 &= \{e, (123456789), (135792468), (147)(258)(369), (159483726), (162738495), \\ &\quad (174)(285)(396), (186429753), (198765432)\}, \end{aligned}$$

$Stab_G(1) = \{e\}$ and the suborbits of G are
$\triangle_0 = \{1\}, \triangle_1 = \{2\}, \triangle_2 = \{3\}, \triangle_3 = \{4\}, \triangle_4 = \{5\}, \triangle_5 = \{6\}, \triangle_6 = \{7\}, \triangle_7 = \{8\}, \triangle_8 = \{9\}$. Hence rank of G is 9 and the subdegrees are 1, 1, 1, 1, 1, 1, 1, 1, 1. By using the definition of $\triangle^*$ given in section 1.1.4 we obtain

$$\begin{aligned} \triangle_0^* &= \triangle_0 \\ \triangle_1^* &= \triangle_8 \\ \triangle_2^* &= \triangle_7 \\ \triangle_3^* &= \triangle_6 \\ \triangle_4^* &= \triangle_5 \end{aligned}$$

Hence the selfpaired suborbit is the trivial suborbit $\triangle_0 = \{1\}$

Example 2.3.4. Let $G = C_8 =< (12345678) >$ and $X = \{1,2,3,4,5,6,7,8\}$, then

$$\begin{aligned} C_8 &= \{e, (12345678), (1357)(2468), (14725836), (15)(26)(37)(48), (16385274), \\ &\quad (1753)(2864), (18765432)\}, \end{aligned}$$

$Stab_G(1) = \{e\}$ and the suborbits of G are

$\triangle_0 = \{1\}, \triangle_1 = \{2\}, \triangle_2 = \{3\}, \triangle_3 = \{4\}, \triangle_4 = \{5\}, \triangle_5 = \{6\}, \triangle_6 = \{7\}, \triangle_7 = \{8\}$. Hence the rank is 8 and the subdegrees are 1, 1, 1, 1, 1, 1, 1, 1.
By using the definition in section 1.1.4 we obtain

$$\begin{aligned}
\triangle_0^* &= \triangle_0 \\
\triangle_1^* &= \triangle_7 \\
\triangle_2^* &= \triangle_6 \\
\triangle_3^* &= \triangle_5 \\
\triangle_4^* &= \triangle_4
\end{aligned}$$

The two suborbits which are selfpaired are $\triangle_0$ and $\triangle_4$.

Theorem 2.3.5. *Let $G = C_n$ act on X, then the suborbit $\triangle_i$ of G is paired with $\triangle_{n-i}$.*

Proof. let $G =< x >$ and $i+1 \in \triangle_i$ (see Theorem 2.3.1). To get the suborbit paired with $\triangle_i$, first find $x^j \in G$ where $0 \leq j \leq n$ such that $x^j(i+1) = 1$. The value of j is gotten by solving the following equation $(j+i+1) mod n = 1$, which can be rewritten in this case as

$$\begin{aligned}
j+i+1 &= n+1 \\
j &= n-i
\end{aligned}$$

Secondly find where x^j takes 1 i.e $x^j 1$, which is $j+1 = n-i+1$. By the definition in section 1.1.4, the element $n-i+1$ exist in the suborbit which is paired with $\triangle_i$. If $i+1 \in \triangle_i$, then $n-i+1 \in \triangle_{n-i}$. Hence the suborbit $\triangle_i$ is paired with the suborbit $\triangle_{n-i}$, that is $\triangle_i^* = \triangle_{n-i}$. $\square$

Corollary 2.3.6. *$\triangle_0$ is the only selfpaired suborbit of G when n is odd and when n is even $\triangle_0$ and $\triangle_{\frac{n}{2}}$ are selfpaired suborbits.*

Proof. From Theorem 3.6, $\triangle_0^* = \triangle_{n-0} = \triangle_n = \triangle_0$ and $\triangle_{\frac{n}{2}}^* = \triangle_{n-\frac{n}{2}} = \triangle_{\frac{n}{2}}$. $\square$

CHAPTER THREE
TRANSITIVITY, PRIMITIVITY AND SUBORBITS OF D_n

3.1 Introduction

Stabilizer of a point, transitivity, primitivity, rank, suborbits and subdegrees of the dihedral group D_n acting on vertices of a regular $n-$ gon are discussed in this chapter. Throughout the chapter, G will denote the dihedral group D_n acting on $X = \{1, 2, \ldots, n\}$; the set of vertices of a regular $n-$ gon.

3.2 Transitivity and Primitivity of G acting on X

Theorem 3.2.1. *$Stab_G(1) = \{e, (1)(\frac{n}{2}+1)(2\ n)(3\ n-1)\ldots(i\ (n-i+2))\ldots(\frac{n}{2}\ \frac{n+4}{2})\}$, when n is even and when n is odd $Stab_G(1) = \{e, (1)(2\ n)(3\ n-1)\ldots(i\ (n-i+2))\ldots(\frac{n+1}{2}\ \frac{n+3}{2})\}$.*

Proof. Clearly when n is even there is a reflection y that fixes 1 and $\frac{n}{2}+1$ which is $(1)(\frac{n}{2}+1)(2\ n)(3\ n-1)\ldots(i\ (n-i+2))\ldots(\frac{n}{2}\ \frac{n+4}{2})$. Also when n is odd there is a reflection y that fixes 1 which is $(1)(2\ n)(3\ n-1)\ldots(i\ (n-i+2))\ldots(\frac{n+1}{2}\ \frac{n+3}{2})$. Hence the $Stab_G(1) = \{e, (1)(\frac{n}{2}+1)(2\ n)(3\ n-1)\ldots(i\ (n-i+2))\ldots(\frac{n}{2}\ \frac{n+4}{2})\}$ when n is even and $Stab_G(1) = \{e, (1)(2\ n)(3\ n-1)\ldots(i\ (n-i+2))\ldots(\frac{n+1}{2}\ \frac{n+3}{2})$ when n is odd. ☐

Theorem 3.2.2. *G acts transitively on X .*

Proof. The order of $G = 2n$ and $|Stab_G(1)| = 2$. Hence by Theorem 1.1.1 $|Orb_G(1)| = \frac{|G|}{|Stab_G(1)|} = \frac{2n}{2} = n$. Which implies that the action of G on X has one orbit. Hence the action is transitive according to subsection 1.1.2. ☐

Theorem 3.2.3. *G acts imprimitively on X if n is not prime.*

Proof. The dihedral group is generated by two elements, a rotation and a reflection, that is $G = \{x, y | x^n = y^2 = e\}$. The $Stab_G(1) = \{e, y\}$ according to Theorem 3.2.1, where y is a reflection that fixes $1 \in X$. Let k be such that $1 < k < n$ and k divides n, then the group $H =< x^{n/k}, y >$ is a proper subgroup of G of order k properly containing $Stab_G(1)$. Hence by Theorem 1.1.4 G acts imprimitively on X. ☐

Example 3.2.4. Consider $G = D_6 =< x,y >= \left\langle (123456), (26)(35) \right\rangle$
Then $H =< x^3, y >=< (14)(25)(36), (26)(35) >= \{e, (14)(25)(36), (26)(35), (14)(23)(56)\}$ satisfies the condition that $Stab_G(1) < H < G$. Therefore G acts imprimitively on X.

Now lets take a case when n is odd.

Example 3.2.5. Consider $D_9 =< x, y >= \left\langle (123456789), (29)(38)(47)(56) \right\rangle$,
Then $H =< x^3, y >= \left\langle (147)(258)(369), (29)(38)(47)(56) \right\rangle$
$= \{e, (147)(258)(369), (174)(285)(396), (29)(38)(47)(56), (17)(26)(35)(89), (14)(23)(59)(68)\}$.
Which satisfies the condition that $Stab_G(1) < H < G$. Therefore G acts imprimitively on X.

3.3 Ranks, Suborbits and Subdegrees of G

Theorem 3.3.1. *Orbits of $Stab_G(1)$ on X are $\triangle_0 = \{1\}, \triangle_1 = \{2, n\}, \ldots, \triangle_i = \{i+1, n-i+1\}, \ldots, \triangle_{\frac{n}{2}} = \{\frac{n}{2}+1\}$, when n even and when n is odd Orbits of $Stab_G(1)$ on X are $\triangle_0 = \{1\}, \triangle_1 = \{2, n\}, , \ldots, \triangle_i = \{i+1, n-i+1\}, \ldots, \triangle_{\frac{n-1}{2}} = \{\frac{n+1}{2}, \frac{n+3}{2}\}$.*

Proof. Clearly these are the cycles of the non identity element y in $Stab_G(1)$ in Theorem 3.2.1. $\square$

Corollary 3.3.2. *Subdegrees of G when n is even arranged in an increasing order of magnitude are $1, 1, 2, 2, \ldots, 2$; $\frac{n}{2}-1$ twos and when n is odd subdegrees of G are $1, 2, 2, \ldots, 2$; $\frac{n-1}{2}$twos. The rank of G when n is even is $\frac{n}{2}+1$ and when n is odd is $\frac{n+1}{2}$.*

Proof. Clearly from Theorem 3.3.1 the lengths of cycles of y arranged in an increasing order of magnitude are $1, 1, 2, 2, \ldots, 2 : \frac{n}{2}-1$ twos when n is even and $1, 2, 2, \ldots, 2$; $\frac{n-1}{2}$ twos when n is odd. The rank of G when n is even is computed using Theorem 1.1.5; which gives us $\frac{1}{2}[n+2] = \frac{n}{2}+1$, and when n is odd the rank is $\frac{1}{2}[n+1] = \frac{n}{2}+\frac{1}{2}$ $\square$

Theorem 3.3.3. *All the suborbits of G are selfpaired.*

Proof. Let $g \in G$, then g^2 will have fixed points in X if either g is the identity or g is of order two. In either case g^2 will fix all the points in X. If n is even, G contains $n+1$ elements of order 2. Therefore by Theorem 1.1.8 the number of selfpaired suborbits of G is $\frac{1}{2n}[n+n(n+1)] = \frac{n}{2}+1$. When n is odd, G contains n elements of order 2. Therefore by Theorem 1.1.8 the number of selfpaired suborbits in this case is $\frac{1}{2n}[n+n.n] = \frac{1}{2}+\frac{n}{2} = \frac{1+n}{2} = \frac{n+1}{2}$. Hence all suborbits of G are selfpaired. $\square$

Example 3.3.4. Consider $D_9 =< x, y >= \left\langle (123456789), (29)(38)(47)(56) \right\rangle$ and $X =$

$\{1,2,\ldots,9\}$, then

$$\begin{aligned} D_9 &= \{e,(123456789),(135792468),(147)(258)(369),(159483726),(162738495), \\ &\quad (174)(285)(396),(186429753),(198765432),(29)(38)(47)(56),(13)(49)(58)(67), \\ &\quad (15)(24)(69)(78),(17)(26)(53)(89),(19)(25)(34)(79),(12)(39)(48)(57), \\ &\quad (14)(23)(59)(68),(16)(25)(34)(79),(18)(27)(36)(45)\} \end{aligned}$$

$Stab_G(1) = \{e,(2\ 9)(3\ 8)(4\ 7)(5\ 7)\}$ and the suborbits of G are $\triangle_0 = \{1\}, \triangle_1 = \{2,9\}, \triangle_2 = \{3,8\}, \triangle_3 = \{4,7\}, \triangle_4 = \{5,6\}$. Hence the rank is equal to $\frac{1}{2}[9+1] = \frac{9}{2}+\frac{1}{2} = 5$ and the subdegrees (lengths of the suborbits) are $1,2,2,2,2$. The number selfpaired suborbits are equal to $n_\pi = \frac{1}{|G|}\sum\limits_{g\in G}\pi(g^2) = \frac{1}{18}[9+9+9+9+9+9+9+9+9+9] = \frac{90}{18} = 5 = \frac{n+1}{2}$.

Example 3.3.5. Let $G = D_6 =< x,y >= \left\langle (123456),(26)(35) \right\rangle$ and $X = \{1,2,\ldots,6\}$, then

$$\begin{aligned} D_6 &= \{e,(123456),(135)(246),(14)(25)(36),(153)(264), \\ &\quad (165432),(26)(35),(12)(36)(45),(13)(46),(14)(23)(56), \\ &\quad (15)(24),(16)(25)(34)\} \end{aligned}$$

$Stab_G(1) = \{e,(2\ 6)(3\ 5)\}$ and the suborbits of G are $\triangle_0 = \{1\}, \triangle_1 = \{2,6\}, \triangle_2 = \{3,5\}, \triangle_3 = \{4\}$. Hence the rank is equal to $\frac{1}{2}[6+2] = \frac{6}{2}+1 = 4$ and the subdegrees arranged in increasing order of magnitude are $1,1,2,2$. The number selfpaired suborbits are equal to $n_\pi = \frac{1}{|G|}\sum\limits_{g\in G}\pi(g^2) = \frac{1}{12}[6+6+6+6+6+6+6+6] = \frac{48}{12} = 4 = \frac{n}{2}+1$.

CHAPTER FOUR
SUBORBITAL GRAPHS OF C_n

4.1 Introduction

Construction of the suborbital graphs corresponding to C_n acting on the set of vertices of a regular $n-$ gon, and discussion of the properties of these graphs is done in this chapter. Through out this chapter G and X are defined as in chapter 2.

4.2 Suborbital Graphs for C_n acting on X

The suborbitals in this section have a one to one correspondence with the suborbits of G in chapter two. So $\triangle_i$ corresponds to O_i. Elements in X are assumed to be arranged cyclically and evenly spaced around a circle in anticlockwise direction. Any element $x^k \in G$ takes $i \in X$, k units around the circle in an anticlockwise direction.

Theorem 4.2.1. *Suppose* $(1, i)$ *is a representative of the non-trivial suborbital* O_{i-1} *of* G, *then* $(a, b) \in O_{i-1}$ *if and only if*

$$|b-a| = \begin{cases} i-1 & if\ b > a \\ n-(i-1) & if\ a > b \end{cases}.$$

.

Proof. Suppose $(a,b) \in O_{i-1}$, where $i > 1$, then there exists $x^j \in G$ such that $x^j(1,i) = (a,b)$. Now if $b > a$, then $a = 1+j$ and $b = i+j$. Thus $|b-a| = i-1$. Next if $a > b$, then $i+j > n$; and $a = 1+j$ and $b = i+j-n$ implies $a-b = n-(i-1)$. Therefore $|b-a| = n-(i-1)$.

Conversely, suppose that

$$|b-a| = \begin{cases} i-1 & if\ b > a \\ n-(i-1) & if\ a > b \end{cases}$$

We need to show that $(a,b) \in O_{i-1}$. In other words we need to show that there exists $x^k \in G$ such that $x^k(1,i) = (a,b)$. Now if $b > a$, $b-a = i-1$ implies $a = b-i+1$. Therefore

$$x^{a-1}(1,i) = (a, a-1+i) = (a,b).$$

On the other hand if $a > b$, then $a - b = n - (i-1)$ implies $a = n + b - i + 1$and

$$x^{a-1}(1,i) = (a, a+i-1) = (a, n+b) \equiv (a,b)(modn).$$

.

□

Example 4.2.2. Consider $G = C_9$ acting on $X = \{1,2,3,4,5,6,7,8,9\}$.
$O_1(1,2) = \{(1,2), (2,3), (3,4), (4,5), (5,6), (6,7), (7,8), (8,9), (9,1)\}$
$O_2(1,3) = \{(1,3), (2,4), (3,5), (4,6), (5,7), (6,8), (7,9), (8,1), (9,2)\}$
$O_3(1,4) = \{(1,4), (2,5), (3,6), (4,7), (5,8), (6,9), (7,1), (8,2), (9,3)\}$
$O_4(1,5) = \{(1,5), (2,6), (3,7), (4,8), (5,9), (6,1), (7,2), (8,3), (9,4)\}$
$O_5(1,6) = \{(1,6), (2,7), (3,8), (4,9), (5,1), (6,2), (7,3), (8,4), (9,5)\}$
$O_6(1,7) = \{(1,7), (2,8), (3,9), (4,1), (5,2), (6,3), (7,4), (8,5), (9,6)\}$
$O_7(1,8) = \{(1,8), (2,9), (3,1), (4,2), (5,3), (6,4), (7,5), (8,6), (9,7)\}$
$O_8(1,9) = \{(1,9), (2,1), (3,2), (4,3), (5,4), (6,5), (7,6), (8,7), (9,8)\}$
Corresponding suborbital graphs are as shown below.

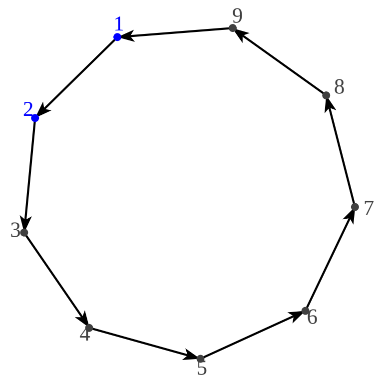

Figure 4.2.1: The suborbital graph Γ_1 corresponding to the action of C_9 on $X = \{1,2,\ldots,9\}$. The suborbital graph is connected and the girth is 9.

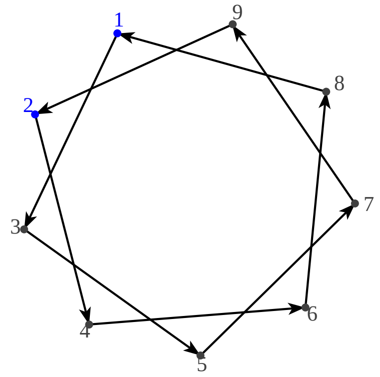

Figure 4.2.2: The suborbital graph Γ_2 corresponding to the action of C_9 on $X = \{1,2,\ldots,9\}$. The suborbital graph is connected and the girth is 9.

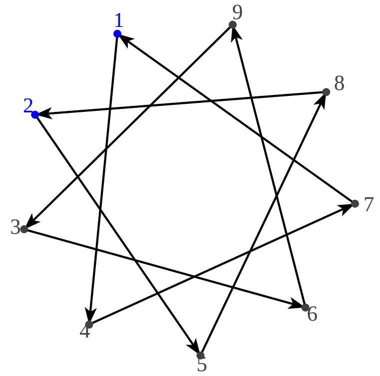

Figure 4.2.3: The suborbital graph Γ_3 corresponding to the action of C_9 on $X = \{1,2,\ldots,9\}$. The suborbital graph is disconnected with 3 connected components and the girth is 3.

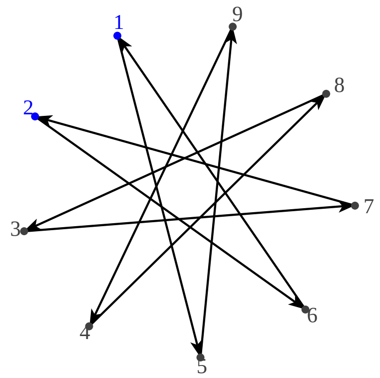

Figure 4.2.4: The suborbital graph Γ_4 corresponding to the action of C_9 on $X = \{1,2,\ldots,9\}$. The suborbital graph is connected and the girth is 9.

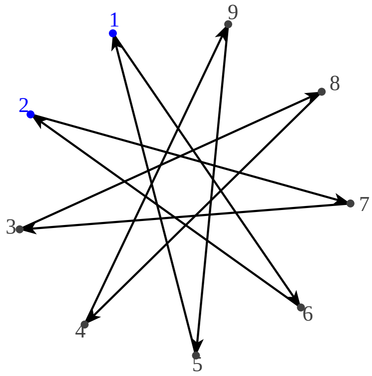

Figure 4.2.5: The suborbital graph Γ_5 corresponding to the action of C_9 on $X = \{1, 2, \ldots, 9\}$. The suborbital graph is connected and the girth is 9.

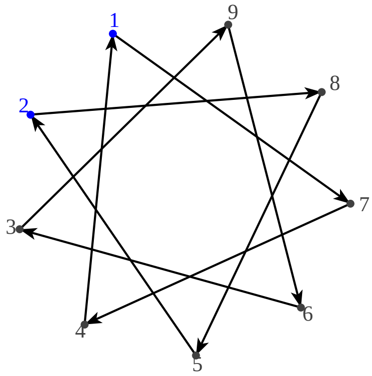

Figure 4.2.6: The suborbital graph Γ_6 corresponding to the action of C_9 on $X = \{1, 2, \ldots, 9\}$. The suborbital graph is disconnected with 3 connected components and the girth is 3.

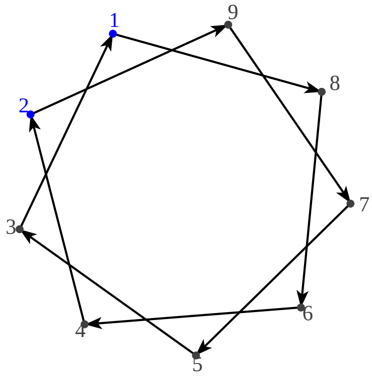

Figure 4.2.7: The suborbital graph Γ_7 corresponding to the action of C_9 on $X = \{1, 2, \ldots, 9\}$. The suborbital graph is connected and the girth is 9.

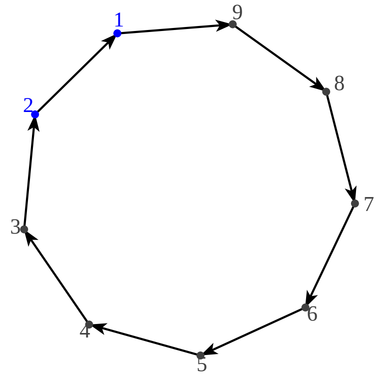

Figure 4.2.8: The suborbital graph Γ_8 corresponding to the action of C_9 on $X = \{1,2,\ldots,9\}$. The suborbital graph is connected and the girth is 9.

Example 4.2.3. Consider $G = C_6$ acting on $X = \{1,2,3,4,5,6\}$.
The suborbitals of G are obtained using Theorem 4.2.1

$$
\begin{aligned}
O_1(1,2) &= \{(1,2), (2,3), (3,4), (4,5), (5,6), (6,1)\} \\
O_2(1,3) &= \{(1,3), (2,4), (3,5), (4,6), (5,1), (6,2)\} \\
O_3(1,4) &= \{(1,4), (2,5), (3,6), (4,1), (5,2), (6,3)\} \\
O_4(1,5) &= \{(1,5), (2,6,), (3,1), (4,2), (5,3), (6,4)\} \\
O_5(1,6) &= \{(1,6), (2,1), (3,2), (4,3), (5,4), (6,5)\}
\end{aligned}
$$

The following are the non - trivial suborbital graphs corresponding to the action.

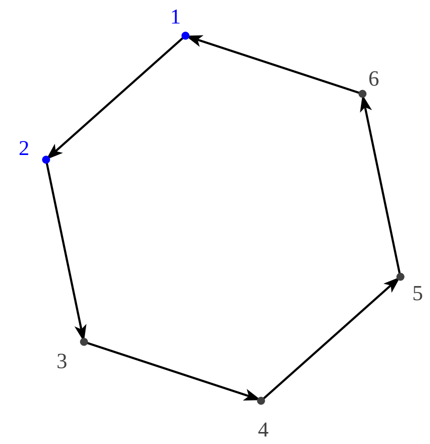

Figure 4.2.9: The suborbital graph Γ_1 corresponding to the action of C_6 on $X = \{1, 2, \ldots, 6\}$. The girth of the suborbital graph is 6, it is connected and directed.

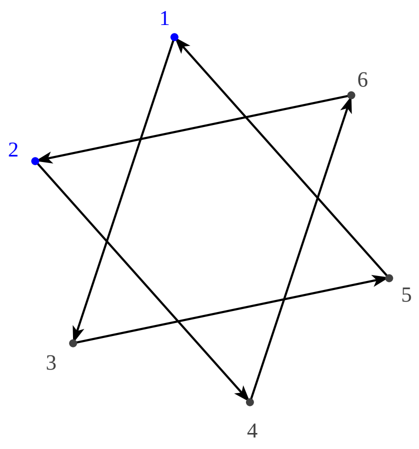

Figure 4.2.10: The suborbital graph Γ_2 corresponding to the action of C_6 on $X = \{1, 2, \ldots, 6\}$. The girth of the suborbital graph is 3, it is disconnected with 2 connected components.

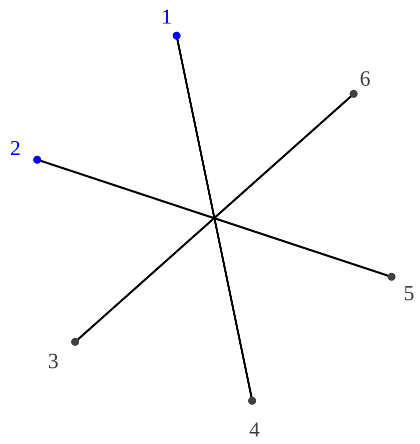

Figure 4.2.11: The suborbital graph Γ_3 corresponding to the action of C_6 on $X = \{1, 2, \dots, 6\}$. The suborbital graph is composed of 3 lines and it is undirected.

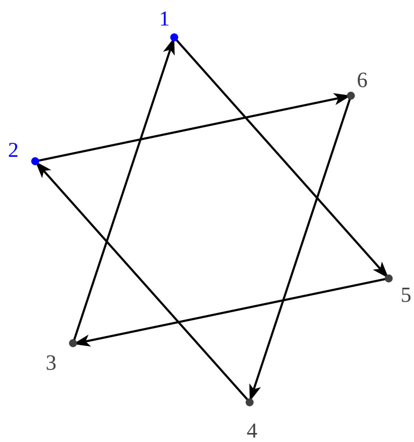

Figure 4.2.12: The suborbital graph Γ_4 corresponding to the action of C_6 on $X = \{1, 2, \dots, 6\}$. The girth of the suborbital graph is 3, it is disconnected with 2 connected components.

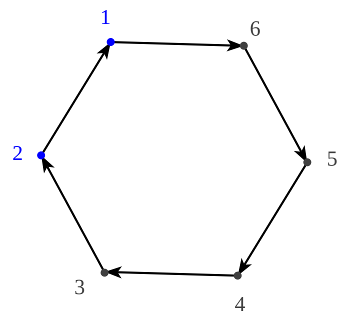

Figure 4.2.13: The suborbital graph Γ_5 corresponding to the action of C_6 on $X = \{1, 2, \ldots, 6\}$. The girth of the suborbital graph is 6, it is connected and directed.

Theorem 4.2.4. *Elements of O_{i-1} can be obtained by pairing each point of $x^{i-1} \in G$ to a point it is being mapped to.*

Proof. Consider $O_{i-1}(1,i) = \{(1,i),(2,i+1),\ldots,(k,k+i-1),\ldots,(n,i-1)\}$ and a rotation $x^{i-1} \in G$, where $1 < i \leq n$. In the rotation $x^{i-1} = \begin{pmatrix} 1 & 2 & \ldots & k & \ldots & n \\ (1+i-1) & (2+i-1) & \ldots & (k+i-1) & \ldots & (n+i-1) \end{pmatrix}$, if each point is paired with the point it is mapped to, we obtain

$$\begin{aligned} &\{(1,1+i-1),(2,2+i-1),\ldots,(k,k+i-1),\ldots,(n,n+i-1)\} \\ &= \{(1,i),(2,i+1),\ldots,(k,k+i-1),\ldots,(n,i-1)\} = O_{i-1}. \end{aligned}$$

□

From Theorem 4.2.4 we can deduce the following results.

Corollary 4.2.5. *There is a one to one correspondence between the cycles of x^{i-1} and the cycles of the suborbital graph Γ_{i-1}.*

Corollary 4.2.6. *The number of components of the suborbital graph Γ_{i-1} is equal to $gcd(n,i-1) = d$, and its girth is $r = \frac{n}{d}$, where $n,d,r,i \in \mathbb{Z}$ and $d \neq \frac{n}{2}$. When $d = \frac{n}{2}$, then the girth is zero.*

Proof. The number of disjoint cycles of $x^{i-1} \in G$ is equal to $gcd(n,i-1) = d$, and all the cycles are of equal length, which is $r = \frac{n}{d}$. From Corollary 4.2.5 we deduce that Γ_{i-1} has d components each of which is a cycle of length $r = \frac{n}{d}$ and therefore the girth of Γ_{i-1} is $\frac{n}{d}$ when $d \neq \frac{n}{2}$. When $d = \frac{n}{2}$, then $r = 2$, but since Γ_{i-1} does not have multiple edges, the girth of Γ_{i-1} must be zero in this case. ☐

Corollary 4.2.7. *The number of connected suborbital graphs is $\phi(n)$, where ϕ is the Euler's phi function.*

Proof. Since $\phi(n)$ is the number of i, $1 \leq i \leq n$ such that $gcd(n,i) = 1$, then from corollary 4.2.6 the number of suborbital graphs of G with exactly one connected component is $\phi(n)$. ☐

From subsection 1.1.4, Theorem 2.3.5 and Corollary 2.3.6 the following two results follow.

Theorem 4.2.8. *The suborbital graphs Γ_0 and $\Gamma_{\frac{n}{2}}$ are undirected when n is even and the other non - trivial suborbital graphs are directed .*

Theorem 4.2.9. *When n is odd only the trivial suborbital graph Γ_0 is undirected and the other non - trivial graphs are directed.*

From Theorems 1.1.3,1.1.9 and 2.2.3 the following result is trivial.

Theorem 4.2.10. *All the non - trivial suborbital graphs of G are connected if and only if n is prime*

CHAPTER FIVE
SUBORBITAL GRAPHS OF D_n

5.1 Introduction

Construction of the suborbital graphs corresponding to D_n acting on the set of vertices of a regular $n-$ gon and discussion of their properties is done in this chapter. Throughout this chapter G and X will be used as in chapter 3.

5.2 Suborbital Graphs for D_n acting on X

The suborbitals in this section have a one to one correspondence with the suborbits of G in chapter three. So $\triangle_i$ corresponds to O_i. Suppose that $(1,i)$ is a representative of the non - trivial suborbital graph O_{i-1} of G. Since G is acting on X and $C_n \subset G$, where C_n is composed of the rotations of G, then there is a connection between the suborbitals of C_n (in chapter 4) and the suborbitals of G in this chapter. In both cases same elements of G (to precise the rotations of G or $C_n \subset G$), act on the same set X hence will have the same results. Which implies that the first n elements of O_{i-1} of G corresponds to the suborbital $O_{i-1}$of C_n. By Theorem 3.3.3 all the suborbits of G are selfpaired. Therefore from subsection 1.1.4 if $(a,b) \in O_{i-1}$, then $(b,a) \in O_{i-1}$ also. That is paired suborbitals in C_n become one in G.

From Corollary 3.3.2, Theorem 3.3.3 and the discussion above, the following result is immediate.

Theorem 5.2.1. *(a) G has $\frac{n}{2}$ selfpaired non - trivial suborbitals O_{i-1}, $i = 2,3,\ldots,\frac{n}{2}+1$ when n is even, where a suborbital in G is the union of two paired suborbitals in C_n.*
(b) G has $\frac{n+1}{2}-1$ selfpaired non trivial suborbitals $O_{i-1}(1,i)$, $i = 2,3,\ldots,\frac{n+1}{2}$ when n is odd, where a suborbital in G is the union of two paired suborbitals in C_n.

Example 5.2.2. Consider $G = D_9$ acting on X

$$\begin{aligned} O_1(1,2) &= \{(1,2),(2,3),(3,4),(4,5),(5,6),(6,7),(7,8),(8,9),(9,1),\\ &\quad (1,9),(2,1),(3,2),(4,3),(5,4),(6,5),(7,6),(8,7),(9,8)\} \end{aligned}$$

$$\begin{aligned} O_2(1,3) &= \{(1,3),(2,4),(3,5),(4,6),(5,7),(6,8),(7,9),(8,1),(9,2),\\ &\quad (1,8),(3,1),(5,3),(7,5),(9,7),(2,9),(4,2),(6,4),(8,6)\} \end{aligned}$$

$$\begin{aligned} O_3(1,4) &= \{(1,4),(2,5),(3,6),(4,7),(5,8),(6,9),(7,1),(8,2),(9,3), \\ &\quad (1,7),(3,9),(5,2),(7,4),(9,6),(2,8),(4,1),(6,3),(8,5)\} \end{aligned}$$

$$\begin{aligned} O_4(1,5) &= \{(1,5),(2,6),(3,7),(4,8),(5,9),(6,1),(7,2),(8,3),(9,4), \\ &\quad (1,6),(3,8),(5,1),(7,3),(9,5),(2,7),(4,9),(6,2),(8,4)\} \end{aligned}$$

The following are the non - trivial suborbital graphs corresponding to the action.

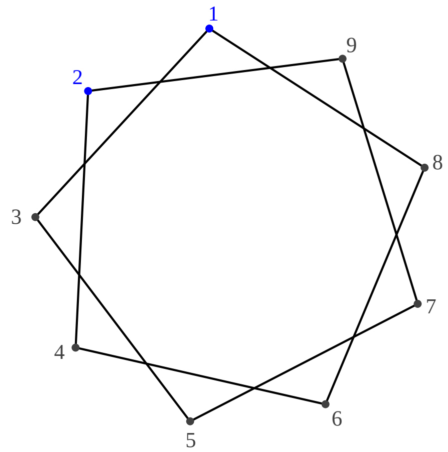

Figure 5.2.1: The suborbital graph Γ_1 corresponding to the action of D_9 on $X = \{1,2,\ldots,9\}$. The suborbital graph is undirected, connected and its girth is 9.

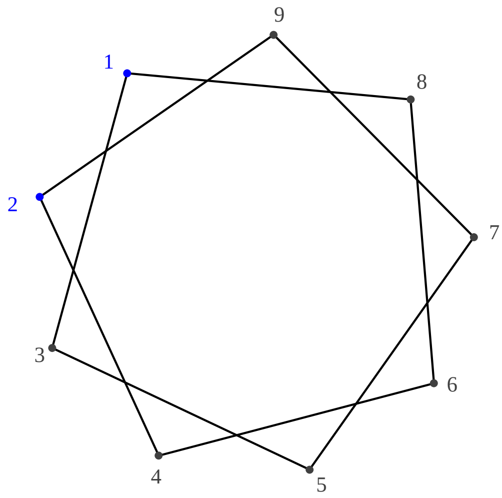

Figure 5.2.2: The suborbital graph Γ_2 corresponding to the action of D_9 on $X = \{1, 2, \ldots, 9\}$. The suborbital graph is undirected, connected and the girth is 9.

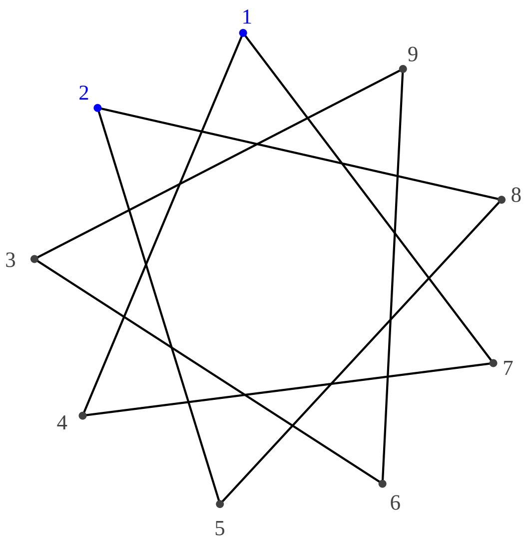

Figure 5.2.3: The suborbital graph Γ_3 corresponding to the action of D_9 on $X = \{1, 2, \ldots, 9\}$. The suborbital graph is undirected, disconnected and its girth is 3.

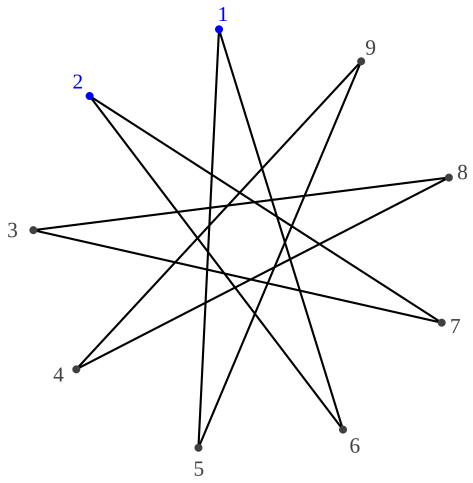

Figure 5.2.4: The suborbital graph Γ_4 corresponding to the action of D_9 on $X = \{1,2,\ldots,9\}$. The suborbital graph is undirected, connected and the girth is 9.

Example 5.2.3. Consider $G = D_6$ acting on X

$O_1(1,2) = \{(1,2), (2,3), (3,4), (4,5), (5,6), (6,1), (1,6), (2,1), (3,2), (4,3), (5,4), (6,5)\}$

$O_2(1,3) = \{(1,3), (2,4), (3,5), (4,6), (5,1), (6,2), (1,5), (2,6,), (3,1), (4,2), (5,3), (6,4)\}$

$O_3(1,4) = \{(1,4), (2,5), (3,6), (4,1), (5,2), (6,3)\}$

Following are the non - trivial suborbital graphs corresponding to the action.

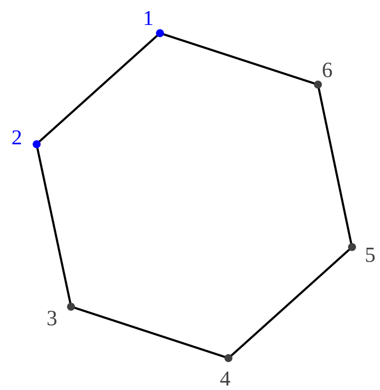

Figure 5.2.5: The suborbital graph Γ_1 corresponding to the action of D_6 on $X = \{1,2,\ldots,6\}$. The suborbital graph is undirected, connected and its girth is 6.

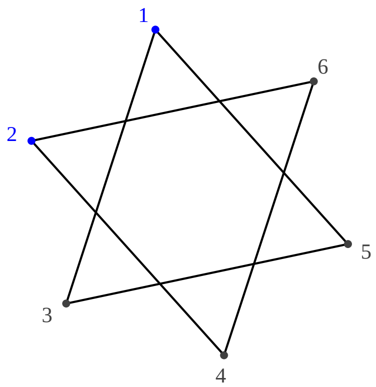

Figure 5.2.6: The suborbital graph Γ_2 corresponding to the action of D_6 on $X = \{1,2,\ldots,6\}$. The suborbital graph is undirected, disconnected with 2 connected components and its girth is 3.

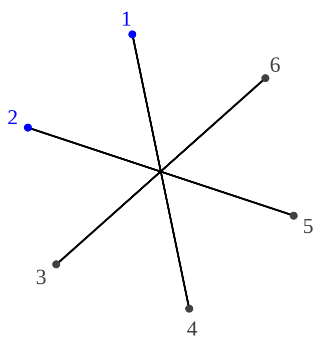

Figure 5.2.7: The suborbital graph Γ_3 corresponding to the action of D_6 on $X = \{1,2,\ldots,6\}$. The suborbital graph is disconnected with 3 lines and it is undirected.

From Theorem 5.2.1 and corollaries 4.2.5 and 4.2.6 we deduce the following two results.

Corollary 5.2.4. *The number of components of the suborbital graph* Γ_{i-1} *is equal to* $gcd(n, i-1) = d$ *and its girth is* $r = \frac{n}{d}$*, where* $d \neq \frac{n}{2}$*. When* $d = \frac{n}{2}$*, the girth is zero.*

Corollary 5.2.5. *The number of connected suborbital graphs of G is* $\frac{1}{2}\phi(n)$.

From Theorem 3.3.3 and subsection 1.1.4 we conclude the following

Theorem 5.2.6. *All the suborbital graphs of G are undirected.*

From Theorems 3.2.3 and 1.1.9 the following result are straight forward.

Theorem 5.2.7. *All the non - trivial suborbital graphs of G are connected if and only if n is prime*

CHAPTER SIX
CONCLUSIONS, RECOMMENDATIONS AND APPLICATIONS

6.1 Introduction

Some conclusions of this study, suggestions of areas for further research and applications are given in this chapter.

6.2 Conclusion

In chapter 2 it was proven that C_n acts transitively and imprimitively on X when n is not prime. It was also shown that the orbits of $Stab_G(1)$ on X were as follows $\triangle_0 = \{1\}, \triangle_1 = \{2\}, \triangle_2 = \{3\}, \ldots, \triangle_i = \{i+1\}, \ldots, \triangle_{n-1} = \{n\}$, the rank and subdegrees were also shown to be n and $1, 1, 1, \ldots, 1$; n ones respectively. It was further proven that the suborbit $\triangle_i$ is paired with $\triangle_{n-i}$.
In chapter 3 it was proven that D_n acts transitively and imprimitively on X when n is not prime. It was also shown that the orbits of $Stab_G(1)$ on X are $\triangle_0 = \{1\}, \triangle_1 = \{2,n\}, \ldots, \triangle_i = \{i+1,\, n-i+1\}, \ldots, \triangle_{\frac{n}{2}} = \{\frac{n}{2}+1\}$, when n is even and when n is odd Orbits of $Stab_G(1)$ on X were $\triangle_0 = \{1\}, \triangle_1 = \{2, n\},, \ldots, \triangle_i = \{i+1, n-i+1\}, \ldots, \triangle_{\frac{n-1}{2}} = \{\frac{n+1}{2}, \frac{n+3}{2}\}$. The rank of D_n was shown to be $\frac{n}{2}+1$ when n is even and $\frac{n+1}{2}$ when n is odd. The subdegrees of D_n when n is even arranged in an increasing order of magnitude were shown to be $1, 1, 2, 2, \ldots, 2$; $\frac{n}{2}-1$ twos and when n is odd the subdegrees were $1, 2, 2, \ldots, 2$; $\frac{n-1}{2}$ twos . Moreover it was proven that all the suborbits of D_n are selfpaired.
In chapter four a general construction of suborbital graphs of C_n was given. It was shown that $(a,b) \in O_{i-1}$ if and only if

$$|b-a| = \begin{cases} i-1 & if\ b>a \\ n-(i-1) & if\ a>b \end{cases}.$$

An alternative way of obtaining the elements of O_{i-1} was also derived in Theorem 4.2.4. The number of components of the non - trivial suborbital graph Γ_{i-1} was proven to be $d = gcd(n, n-1)$ and its girth was $r = \frac{n}{d}$ when $d \neq \frac{n}{2}$ and zero when $d = \frac{n}{2}$. Further the number of connected suborbital graphs corresponding to the action is shown to be $\phi(n)$. Finally it was proven that all the non trivial suborbital graphs of G are connected if and only if n is prime.
Chapter five dealt with investigation of the suborbital graphs of D_n acting on X. It was proven that the suborbital O_{i-1} of D_n is a union of two paired suborbitals of C_n. It was

also shown that the number of components of Γ_{i-1} is $d = gcd(n, n-1)$ and its girth is $r = \frac{n}{d}$ and zero when $d = \frac{n}{2}$. Further it was proven that the number of connected suborbital graphs of G was $\frac{1}{2}\phi(n)$. Finally it was proven that all the suborbital graphs of G were undirected and that they are connected if and only if n is prime.

6.3 Recommendations

One may investigate the action of the alternating group A_n acting on ordered and unordered $r-$ element subsets from the set $\{1, 2, \ldots, n\}$.

6.4 Applications

The major role of graph theory in computer applications is the development of graph algorithms. Numerous algorithms are used to solve problems that are modeled in the form of graphs . These algorithms are used to solve the graph theoretical concepts which in turn are used to solve the corresponding computer science application problems.

Some algorithms are:

1. Shortest path algorithm in a network.
2. Algorithms to determine connectedness.
3. Algorithms to find the cycles in a graph
4. Minimun spanning tree

Various computer languages are used to support the graph theory concepts. The main goal of such languages is to enable the user to formulate operations on graphs in a compact and natural manner.

1. GASP – Graph Algorithm Software Package.
2. FGRAAL – FORTRAN Extended Graph Algorithmic Language.

From the results obtained in this research new modules can be added to these already existing algorithms to make them more efficient in dealing with dihedral groups D_n and the cyclic groups C_n and their suborbital graphs.

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