NOMENCLATURE

$\mathbb{R}$	Field of real numbers
$\mathbb{C}$	Field of complex numbers
f,g	Invariants
$\mathbf{P}(\mathbb{R}^{\kappa}, \mathbb{R}^{\kappa})$	Vector space
Ax	Linear part of a system
N	Nilpotent matrix
$ker\ X$	Kernel of X
$\{X, Y, Z\}$	Triad of X, Y and Z.
lm X, lm Y	Image of X, Image of Y
L_A	Lie Operator
D_A	Differential Operator
$Sl(2)$	2×2 special linear Lie algebra
$\bigoplus$	Direct sum
$\boxtimes$	Box product
$\mathbb{R}[[x_1, ..., x_n]]$	Ring of power series in $x_1, ..., x_n$
w_f	Weigh of f

ABSTRACT

The concept of normal form was used to study the dynamics of non-linear systems. The problem of describing the normal forms for a system of differential equations at equilibrium with nilpotent linear part is solvable once the ring of invariants associated to the system is known.

This study was concerned with the description of the normal form for differential system with nilpotent linear part made up of n 3×3 Jordan blocks. The normal form of the systems with nilpotent linear part has the structure of a module of equivariants and is best described by giving its Stanley decomposition.

An algorithm based on the notion of transvectants from classical invariant theory was used to determine the Stanley decomposition for the ring of invariants for the coupled systems when the Stanley decompositions of the Jordan blocks of the linear part are known at each stage. The Stanley decomposition for the ring of invariants was then verified by developing a table function denoted by $T_{(3)^n}$, where $(3)^n$ is the dimension of the linear part. The normal form have been obtained by boosting the Stanley decomposition for the ring of invariants to Stanley Decomposition of the module of equivariants.

To put the normal form into practical use, asymptotic unfolding for a single block was included as an exposition to show the inclusion of arbitrarily parameters. The asymptotic unfolding was observed to exhibit all behavior which can be detected in perturbation of the original system up to a given degree, such as existence and stability of some bifurcations.

CHAPTER ONE

1.0 INTRODUCTION

Many nonlinear systems can be modeled by ordinary differential, or difference equations, and a central problem of dynamical systems theory is to obtain information on the long-time behavior of typical solutions. Because it is not possible, in general, to solve these equations explicitly, we identify particular solutions (such as equilibrium solutions or periodic solutions) and try to study the behavior of nearby solutions. Mathematically, equilibrium is a fixed point of a dynamical system, and the stability analysis is carried out by linearizing the system; that is, by replacing (close to the fixed point) the nonlinear equations by linear equations for the perturbations. The resulting linear system can be solved exactly, and the analysis of these solutions may give information about the behavior of the solutions of the nonlinear system around the fixed point. This method is extremely powerful when it works, and is the basis of many dynamical system analyses. In some cases, however, the behavior of the linear system may be entirely different from that of nonlinear system and no information can be obtained from the linear analysis.

Consider the normal form analysis of the zero fixed point of systems of two differential equations of the form

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + f_1(x_1, x_2)$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + f_2(x_1, x_2)$$

where the dots denote time derivatives and f_1 and f_2 are nonlinear analytic functions whose Taylor expansion about the origin contains a non linear term. Linear analysis of the fixed point at the origin is carried out by neglecting the nonlinear terms and considering the linear system. The dynamics of this system is governed by the eigenvalues λ_1, λ_2 of the matrix a_{ij}, which is assumed to be diagonalizable. If the eigenvalues have non vanishing real parts, then the dynamics of the original nonlinear system is qualitatively equivalent to the dynamics of the linear system and the fixed point is stable if both real parts are negative and unstable if one of the real parts is positive. However, if the eigenvalues are imaginary, the behavior of the nonlinear system cannot be inferred from analysis of the linear system, and the information on the stability of the origin is contained in the nonlinear terms.

For n dimensional system of differential equations, we can compute the linear eigenvalues $\lambda_1, ...\lambda_n$. If the real part of one of these eigenvalues vanishes, there is no guarantee that the dynamics of the linear system is equivalent to the dynamics of the nonlinear system close to the fixed point. This implies that there is crucial information contained in the nonlinear terms. Normal forms theory is a general method designed to extract this information.

1.1 The Normal Form Theory

Normal form is the simplest form into which any differential system can be transformed by change the coordinates. Normal form theory is a technique for transforming the ordinary differential equations describing nonlinear dynamical systems into standard forms. The starting point is a smooth system of differential equations with an equilibrium (rest point) at the origin, expanded as a power series

$$\dot{x} = Ax + a_1(x) + a_2(x) + ..., \tag{1.1.1}$$

where $x \in \mathbb{R}^n$ or $\mathbb{C}^n$. A is an $n \times n$ real or complex matrix and $a_j(x)$ is a homogeneous polynomial of degree $j + 1$ (for instance, $a_1(x)$ is quadratic). The expansion is taken to some finite order k and truncated there, or else is taken to infinity but is treated formally (the convergence or divergence of the series is ignored). The purpose is to obtain an approximation to the (unknown) solution of the original system, that will be valid over an extended range in time. The linear term Ax is assumed to be already in the desired normal form, usually the Jordan or a real canonical form. A transformation to new variables y is applied, having the form

$$x = y + u_1(y) + u_2(y) + ... \tag{1.1.2}$$

where u_j is homogeneous and of degree $j + 1$. This results in a new system

$$\dot{y} = Ay + b_1(y) + b_2(y) + ... \tag{1.1.3}$$

having the same general form as the original system. The goal is to make a careful choice of the u_j, so that the b_j are "simpler" in some sense than the a_j. "Simpler" may mean only that some terms have been eliminated, but in the best cases one hopes to achieve a system that has additional symmetries that were not present in the original system.

1.2 Literature Review

The idea of normal forms for nonlinear systems dates back as far as Poincare' (1880), who was among the first to bring forth the theory in a more definite form. Poincare' considered the problem of reducing a system of nonlinear differential equations to a system of linear ones. The formal solution of this problem entails finding near-identity coordinate transformations, which eliminate the analytic expressions of the nonlinear terms.

Among many historical references in the development of normal form theory, is Birkoff (1996) who shows that the early stages of the theory was confined to Hamiltonian systems and the normalizing transformations were canonical (now called symplectic). Bruno (1989) who treated in details the convergence and divergence of normalizing transformations.

Richard Cushman *et al.* (1988) using a method called "Co-variant of special equivariants" solved the problem of finding Stanley decomposition of $N_{22,\ldots,2}$. Their method begins by creating a scalar problem that is larger than the vector problem and their procedures are derived from classical invariant theory. Thus after obtaining the ring of invariants they repeated calculations of classical theory to obtain the ring of equivariants.

Sri Namachchivaya *et al.* (1994) studied a generalized Hopf bifurcation with non-semisimple 1:1 Resonance. The normal form for such a system contains only terms that belong to both the semisimple part of A and the normal form of the nilpotent part of A, which is a coupled Takens-Bogdanov system with $A = \begin{bmatrix} i\omega & 1 & & \\ & i\omega & & \\ & & i\omega & 1 \\ & & & i\omega \end{bmatrix}$. This example illustrates the physical significance of the study of normal forms for systems with a nilpotent linear part.

Murdock (2003) developed an efficient algorithm for the production of Stanley decomposition of the module of equivariants from the ring of invariants. The nonlinear terms are decomposed into essential and non essential terms and thus the normal form of nonlinear system can be computed.

Malonza (2004), using the technique of Groebner basis found in Adams *et al* (1994) computed the normal form of nilpotent systems consisting of 2×2 Jordan blocks by finding the Stanley decomposition for equivariants.

Murdock and Sanders (2006), developed an algorithm to determine the normal form of a vector field with a nilpotent linear part, when the form of the normal form is known for each Jordan block of the linear part taken separately. The algorithm is based on the notion of transvectant from classical invariant theory. Malonza (2010) using the algorithm for transvectants computed the Stanley decomposition for Taken-Bogdanov systems and obtained same results as using the technique of Groebner basis.

Our goal is to compute the normal forms for coupled $N_{33,\ldots,3}$ systems:

$$\dot{x} = Nx + h.o.t \tag{1.2.1}$$

where $x \in \mathbb{R}^{3n}$, $N = \begin{bmatrix} N_3 & & \\ & \ddots & \\ & & N_3 \end{bmatrix}$, $N_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $h.o.t$ are the higher order terms.

The normal form for such a system contains only terms that belong to the nilpotent part, which is a coupled $N_{33,\ldots,3}$ system.

Our results are mainly based on the work found in Murdock (2003) and Murdock and Sanders (2006), that is, the application of tranvectant's method for computing Stanley decompositions for the ring of invariants of nilpotent systems or what has come to be known as "box product". In chapter two, we put together some background information for understanding the content of this thesis. In chapter three, which forms the central part we use the method of transvectants (box product method) to compute Stanley decomposition for the ring of invariants of the coupled $N_{33,\ldots,3}$ systems when the Stanley decomposition of each Jordan block of the linear part is known separately. In chapter four we compute the normal form by boosting the known Stanley decomposition of the ring of invariants to the module of equivariants.

1.3 Statement of the Problem.

We shall compute the normal form of a system with a nilpotent linear part consisting of n 3×3 Jordan blocks. We use the transvectant algorithm found in Murdock and Sanders (2006) i.e. box product method to compute Stanley decomposition for the ring of invariants. To obtain the normal form we shall boost the ring of invariants to the module of equivariants which is best described by giving its Stanley decomposition.

1.4 General Research Objective

To compute the normal forms for coupled non-linear systems with nilpotent linear part $N_{33,\ldots,3}$.

1.5 Specific Objectives of the Study.

1. To compute the Stanley decomposition associated with the ring of invariants, $ker\ \mathcal{X}$, for the coupled systems with nilpotent linear part $N_{33,..,3}$.
2. To compute the module of equivariants, $ker\ \mathsf{X}$, which is the normal form for systems with nilpotent linear part $N_{33,..,3}$.

1.6 Hypothesis

There exist a generalization for computation of normal forms for coupled $N_{33,\ldots 3}$ systems

1.7 Applications

Normal form theory is one of the most powerful tools for the study of nonlinear differential equations for stability and bifurcations analysis. To analyse nonlinear three dimensional systems, we first transform the nonlinear dynamical systems into their normal forms.

The change in the qualitative character of a solution when a parameter is varied is known as a bifurcation. This occurs where a linear stability analysis yields an instability. Typically a new solution develops at this point. For parameter values near the bifurcation values the properties of the solutions are given by the method of normal forms and this general framework can be used to study and give a rigorous foundation to many physical and engineering problems. Other applications of normal forms include the analysis of chaotic behavior in nonlinear systems.

In scientific fields as diverse as fluid mechanics, electronics, chemistry and theoretical ecology, there is the application of bifurcation analysis; the analysis of a system of ordinary differential equations (ODE) under parameter variation. Performing a local bifurcation analysis is often a powerful way to analyse the properties of such systems, since it predicts the behaviour (system is in equilibrium, or there is cycling) that occurs where in parameter space. The normal form and its unfoldings in the parameter space explains the elementary dynamics exhibited by the system, and this model is still valid for certain perturbations in higher order terms (i.e. relations between polynomials of higher order).

Synthesizing dynamical systems by means of normal forms is practical with the appearance of algebraic symbolic algorithms, contributing to study the system dynamics in an analytic way. Although in majority of the nonlinear systems it not possible to find explicit solutions, except through the use of numerical methods, the technique of normal form is still very powerful for validating the results.

CHAPTER TWO

2.0 INVARIANTS AND BOX PRODUCTS OF STANLEY DECOMPOSITIONS

There are well-known procedures for putting a system of differential equations

$$\dot{x} = Ax + v(x) \tag{2.0.1}$$

(where v is a formal power series with quadratic terms) into normal form with respect to its linear part, A, which can be found in Cushman and Sanders (1990) and Murdock (2003). Our concern is to describe the Stanley decomposition of the module of equivariants, that is the set of all v such that $Ax + v(x)$ is in normal form. Our main result is a procedure that solves the description problem when A is a nilpotent matrix in Jordan form, N, with coupled 3×3 Jordan blocks, provided that the description problem is already solved for each Jordan block of A taken separately. Our method is based on adding one block at a time. This procedure will be illustrated with examples and then generalized.

A single N_3 system has the form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \textit{quadratic terms} + \textit{cubic terms} + \ldots \tag{2.0.2}$$

Let $N_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, then a coupled system $N_{33,\ldots,3}$ has the form

$$\dot{x} = Nx + h.o.t \tag{2.0.3}$$

where, $x \in \mathbb{R}^{3n}$ and $N = \begin{bmatrix} N_3 & & \\ & \ddots & \\ & & N_3 \end{bmatrix}$.

2.1 Invariant Ring and Stanley Decomposition

In differential geometry a vector field $a(x) = [a_1(x), ..., a_n(x)]$ is associated with a differential operator on smooth (scalar) functions defined by $\mathcal{D}_{a(x)} = a_1(x)\frac{\partial}{\partial x_1} + ... + a_n(x)\frac{\partial}{\partial x_n}$. Let $\mathcal{P}_j(\mathbb{R}^n, \mathbb{R}^m)$ denote the vector space of homogeneous polynomials of degree j on $\mathbb{R}^n$ with coefficients in $\mathbb{R}^m$, where $\mathbb{R}$ denotes the set of real numbers. Let $\mathcal{P}(\mathbb{R}^n, \mathbb{R}^m)$ be the vector space of all such polynomials of any degree and let $\mathcal{P}_*(\mathbb{R}^n, \mathbb{R}^m)$ be the vector space of formal power series. If $m = 1$, $\mathcal{P}_*(\mathbb{R}^n, \mathbb{R})$ becomes the ring of formal power series on $\mathbb{R}^n$, where $\mathbb{R}$ denotes the set of real numbers. For such smooth vectors fields, it is sufficient to work with polynomials. For any nilpotent matrix N, we define the Lie operator

$$L_N : \mathcal{P}_j(\mathbb{R}^n, \mathbb{R}) \to \mathcal{P}_j(\mathbb{R}^n, \mathbb{R}^n)$$

by

$$(L_N v)x = v'(x)Nx - Nv(x) \tag{2.1.1}$$

and the differential operator

$$\mathcal{D}_{Nx} : \mathcal{P}_j(\mathbb{R}^n, \mathbb{R}) \to \mathcal{P}_j(\mathbb{R}^n, \mathbb{R})$$

by

$$(\mathcal{D}_{Nx} f)(x) = f'(x)N(x) = (N(x).\bigtriangledown)f(x). \tag{2.1.2}$$

In addition,

$$L_N(fv) = (\mathcal{D}_N f)v + fL_N v. \tag{2.1.3}$$

Recall that if v is a vector field and f is a scalar field then $\mathcal{D}_{(x)}f$ is a scalar field called the derivation of f along (the flow) $v(x)$. We will write $\mathcal{D}_N$ for $\mathcal{D}_{Nx}$ to denote the derivation along the linear vector field Nx. A function f is called an *invariant* of Ax if $\frac{\partial}{\partial t}f(e^{At}x)\mid_{t=0}= 0$ or equivalently $\mathcal{D}_A f = 0$, that is, $f \in ker\mathcal{D}_A$. Since

$$\mathcal{D}_N(f+g) = \mathcal{D}_N f + \mathcal{D}_N g$$

and

$$\mathcal{D}_N fg = f\mathcal{D}_N g + g\mathcal{D}_N f;$$

it follows that if f and g are invariants, so are $f+g$ amd fg; that is *ker* $\mathcal{D}_N$ is both a vector space over $\mathbb{R}$ and also a subring of $\mathcal{P}(\mathbb{R}^n, \mathbb{R})$, known as the *ring of invariants.*

Similarly a vector field v is called an *equivariants* of Ax, if $\frac{\partial}{\partial t}(e^{-At}v(e^{At}x))\mid_{t=0}= 0$ or equivalently $L_A vx = 0$, that is, $v \in kerL_A$.

There are two normal form styles in common use for nilpotent systems, the *inner product normal form* and the *sl(2) normal form.* The inner product normal form is defined by $\mathcal{P}(\mathbb{R}^n, \mathbb{R}^n) = imL_N \oplus kerL_{N^*}$ where N^* is the conjugate transpose of N, a nilpotent matrix. To define the sl(2) normal form, one first sets $X = N$ and constructs matrices Y and Z such that

$$[X, Y] = Z, \qquad [Z, X] = 2X, \qquad [Z, Y] = -2Y. \tag{2.1.4}$$

An example of such an $sl(2)$ triad $\{X, Y, Z\}$ is

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Having obtained the triad $\{X, Y, Z\}$ we create two additional triads $\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\}$ and $\{\mathsf{X}, \mathsf{Y}, \mathsf{Z}\}$ defined as follows

$$\mathcal{X} = \mathcal{D}_Y, \qquad \mathcal{Y} = \mathcal{D}_X, \qquad \mathcal{Z} = \mathcal{D}_Z \tag{2.1.5}$$

$$\mathsf{X} = L_{\mathsf{Y}}, \qquad \mathsf{Y} = L_{\mathsf{X}}, \qquad \mathsf{Z} = L_{\mathsf{Z}} \tag{2.1.6}$$

The first of these is a triad of differential operators and the second is a triad of Lie operators. Both the operators $\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\}$ and $\{\mathsf{X}, \mathsf{Y}, \mathsf{Z}\}$ inherit the triad properties (2.1.4) that is,

$$[\mathcal{X}, \mathcal{Y}] = \mathcal{Z}, \qquad [\mathcal{Z}, \mathcal{X}] = 2\mathcal{X}, \qquad [\mathcal{Z}, \mathcal{Y}] = -2\mathcal{Y} \tag{2.1.7}$$

and

$$[\mathsf{X}, \mathsf{Y}] = \mathsf{Z}, \qquad [\mathsf{Z}, \mathsf{X}] = 2\mathsf{X}, \qquad [\mathsf{Z}, \mathsf{Y}] = -2\mathsf{Y}. \tag{2.1.8}$$

For example, with $X = N_{22}$ we have

$$\mathcal{X} = \mathcal{D}_Y = x_1 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_4}$$

$$\mathcal{Y} = \mathcal{D}_X = x_2 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_3}$$

$$\mathcal{Z} = \mathcal{D}_Z = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4}$$

The operators $\{\mathsf{X}, \mathsf{Y}, \mathsf{Z}\}$ map each $\mathcal{P}(\mathbb{R}^n, \mathbb{R}^n)$ into itself. It then follows from the representation theory $sl(2)$ found in Fulton and Harris (1991) that

$$\mathcal{P}(\mathbb{R}^n, \mathbb{R}^n) = im\mathsf{Y} \oplus ker\mathsf{X} = im\mathsf{X} \oplus ker\mathsf{Y}. \tag{2.1.9}$$

Clearly the *ker* $\mathcal{X}$ is a subring of $\mathcal{P}(\mathbb{R}^n, \mathbb{R})$, the ring of invariants and it follows from (2.1.3) that *ker* X is a module over this subring. This is the *sl(2) normal form module.*

The most effective way of describing the invariant ring associated with a nilpotent matrix N is by a device from commutative algebra called a Stanley decomposition, introduced for this purpose by Cushman and Sanders (1990). We write $\mathbb{R}[[x_1, ..., x_n]]$ for the ring of (scalar) power series in variables $x_1, ..., x_n$. A subalgebra $\Re$ of $\mathbb{R}[[x_1, ..., x_n]]$ is a subset that is both a subring and a vector subspace. The subalgebra is graded if

$$\Re = \bigoplus_{d=0}^{\infty} \Re_d,$$

where $\Re_d$ is the vector subspace of $\Re$ consisting of elements of degree d. To define a Stanley decomposition of a graded subalgebra, we begin with the definition of a Stanley term. A Stanley term is an expression of the form $\mathbb{R}[[f_1, ..., f_k]]\varphi$, where the elements $f_1, ..., f_k$ and φ are homogeneous polynomials and $f_1, ..., f_k$ not including φ are required to be algebraically independent. The Stanley term $\mathbb{R}[[f_1, ..., f_k]]\varphi$ denotes the set of all expressions of the form $F(f_1, ..., f_k)\varphi$ where F is a formal power series in k variables. When $\varphi = 1$, φ is omitted, and the Stanley term is a subalgebra, otherwise it is only a subspace. A Stanley decomposition is a finite direct sum of Stanley terms. A polynomial f is called *doubly homogeneous of type (d,w)* if every monomial in f has a degree d and weight w. A vector subspace V of *ker* $\mathcal{X}$ is doubly graded if

$$V = \bigoplus_{d=0}^{\infty} \bigoplus_{w=0}^{\infty} V_{dw},$$

where V_{dw}, is the vector subspace of V consisting of doubly homogeneous polynomials of degree d and weight w. A doubly graded Stanley decomposition of a doubly graded subalgebra $\mathfrak{R}$ of $ker\ \mathcal{X}$ is an expression of $\mathfrak{R}$ as a direct sum of vector subspaces of the form $\mathbb{R}[[f_1, ..., f_k]]\varphi$, where $f_1, ..., f_k, \varphi$ are doubly homogeneous polynomials and $f_1, ..., f_k$ are algebraically independent. All Stanley decompositions considered from here on will be of this kind and the words "doubly graded" will be omitted.

A standard monomial associated with a Stanley decomposition is an expression of the form $f_1^{m_1}, ..., f_k^{m_k}\varphi$, where $\mathbb{R}[[f_1, ..., f_n]]\varphi$ is a term in the Stanley decomposition. Notice that monomial here means a monomial in the basic invariants, which are polynomials in the original variables $x_1, ..., x_n$. Given a Stanley decomposition of $ker\ \mathcal{X}$, its standard monomials of a given degree (or of a given type) form a basis for the (finite-dimensional) vector space of invariants of that degree (or type).

The following lemma found in Murdock (2003) gives a method to check if all terms of Stanley decomposition have been found.

Lemma 2.1 *Let $\{X, Y, Z\}$ be a triad of $n \times n$ matrices, let $\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\}$ be the induced triad and suppose that $I_1, ..., I_n$ is a finite set of polynomials in ker $\mathcal{X}$. Let $\mathcal{R}$ be a subring of $\mathbb{R}[I_1, ..., I_n]$; suppose that the Stanley terms have been found, and that the Stanley decomposition and its associated table function $T(d, w)$ have been determined. Then $\mathcal{R} = ker\ \mathcal{X} \subset \mathcal{P}(\mathbb{R}^n, \mathbb{R}^n)$ if and only if*

$$\frac{\partial}{\partial w} wT \mid_{w=1} = \frac{1}{(1-d)^n}.$$

To generate the table function, we replace each term of the Stanley decomposition by a rational function P/Q in d and w (d for degree and w for weight) and construct as follows: for each basic invariants appearing inside the square brackets, the denominator will contain a factor $1 - d^p w^q$, where p and q are the degree and weight of the invariants; the numerator will be $d^p w^q$, where p and q are the degree and weight of the standard monomials of that term. When the rational function P/Q from each term of the Stanley decomposition are summed up we obtain the table function T given by $T = \sum_i P_i/Q_i$.

For example in the ring of invariants for N_{22}, $ker\ \mathcal{X}_{22} = \mathbb{R}[\alpha, \beta, \gamma]$, there are three invariants

$$\alpha = x_1$$

$$\beta = x_3$$

$$\gamma = x_1 x_4 - x_2 x_3$$

of degree 1, 1, 2 and weights 1, 1, 0 respectively. The table function is

$$T_4 = \frac{1}{(1-dw)^2(1-d^2)}.$$

Now

$$wT_4 = \frac{w}{(1-dw)^2(1-d^2)}.$$

It can be shown that

$$\frac{\partial}{\partial w} wT \mid_{w=1} = \frac{1}{(1-d)^4}.$$

2.2 Box Products of Stanley Decompositions

Let $V_k, k = 1, 2$ be $sl(2)$ representation spaces with triads $\{X_k, Y_k, Z_k\}$, then from Murdock (2002), $V_1 \otimes V_2$ is a representation space with triad $\{X, Y, Z\}$, where $X = X_1 \otimes I + I \otimes X_2$ (and similarly for Y and Z).

We define the box product of $ker\ X_1$ and $ker\ X_2$ by

$$(ker\ X_1) \boxtimes (ker\ X_2) = kerX.$$

Now, consider a system with the nilpotent linear part

$$N = \begin{bmatrix} \hat{N} & 0 \\ 0 & \tilde{N} \end{bmatrix},$$

where $\hat{N}$ and $\tilde{N}$ are nilpotent matrices of sizes $\hat{n} \times \hat{n}$ and $\tilde{n} \times \tilde{n}$ respectively ($\hat{n} + \tilde{n} = n$) in (upper) Jordan form, and each may consist of one or more Jordan blocks. Let

$\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\}$, $\{\hat{\mathcal{X}}, \hat{\mathcal{Y}}, \hat{\mathcal{Z}}\}$ and $\{\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}, \tilde{\mathcal{Z}}\}$ be the associated triads of operators. Notice that the first triads acts on $\mathbb{R}[[x_1, ..., x_n]]$, the second on $\mathbb{R}[[x_1, ..., x_{\hat{n}}]]$ and the third on $\mathbb{R}[[x_{\hat{n}+1}, ..., x_n]]$.

Suppose that $f = f(x_1, ..., x_{\hat{n}}) \in ker\ \hat{\mathcal{X}}$ and $g = g(x_{\hat{n}+1}, ..., x_n) \in ker\ \tilde{\mathcal{X}}$ are weight invariants of weights w_f and w_g, and i is an integer in the range $0 \leq i \leq min(w_f, w_g)$. The basis elements of $ker\ \mathcal{X}$ containing f and g are of the form $(f, g)^{(i)}$ known as transvectants. Then we define external transvectant of f and g of order i to be the polynomial $(f, g)^i \in \mathbb{R}[[x_1, ..., x_n]]$ given by

$$(f, g)^i = \sum_{j=0}^{i} (-1)^j W_{f,g}^{i,j} (\hat{\mathcal{Y}}^j f)(\tilde{\mathcal{Y}}^{i-j} g) \tag{2.2.1}$$

where

$$W_{f,g}^{i,j} = \binom{i}{j} \frac{(w_f - j)!}{(w_f - i)!} . \frac{(w_g - i + j)!}{(w_g - i)!}.$$

We say that a transvectant $(f, g)^i$ is well defined if $0 \leq i \leq min(w_f, w_g)$. Notice that the zeroth transvectant is always well-defined and reduces to the product:$(f, g)^0 = fg$. Given Stanley decompositions for $ker\ \hat{\mathcal{X}}$ and $ker\ \tilde{\mathcal{X}}$, the following theorem provides a basis for $ker\ X$ in each degree which is a first step toward obtaining a Stanley decomposition for $ker\ \mathcal{X}$.

Theorem 2.2 *Each well-defined transvectant $(f, g)^i$ of $f \in ker\ \hat{\mathcal{X}}$ and $g \in ker\ \tilde{\mathcal{X}}$ belongs to ker $\mathcal{X}$. If f and g are doubly homogeneous polynomials of types (d_f, w_f) and (d_g, w_g) respectively, $(f, g)^i$ is a doubly homogeneous polynomial of type $(d_f + d_g, w_f + w_g - 2i)$. Suppose that Stanley decompositions for ker $\hat{\mathcal{X}}$ and ker $\tilde{\mathcal{X}}$ are given, then a basis for the (finite-dimensional) subspace $(ker\ \mathcal{X})_d$ of homogeneous polynomials in ker $\mathcal{X}$ with degree d is given by the set of all well-defined transvectants $(f, g)^i$ where f is a standard monomial of the Stanley decomposition for ker $\hat{\mathcal{X}}$ and g is a standard monomial of the Stanley decomposition for ker $\tilde{\mathcal{X}}$ and $d_f + d_g = d$.*

The proof of this theorem is given in section 6 of Murdock and Sanders (2006). The bases given by Theorem 2.2 are sufficient to determine *ker* $\mathfrak{X}$ one degree at a time, but to find all of *ker* $\mathfrak{X}$ in this way would require finding infinitely many transvectants. A Stanley decomposition for *ker* $\mathfrak{X}$ must be based on a finite number of basic invariants. To construct such a decomposition, we must find an alternative basis for each (*ker* $\mathfrak{X}$) that uses only a finite number of transvectants overall. (We do not count zeroth transvectants, which are simply products. A Stanley decomposition can produce an infinite number of products). Such alternatives bases can be found by the following replacement theorem found in Murdock and Sanders (2006).

Theorem 2.3 *Any transvectant* $(f,g)^i$ *in the basis given by Theorem 2.2 can be replaced by a product* $(f_1,g_1)^{i_1}...(f_j,g_j)^{i_j}$ *of transvectants, provided that* $f_1...f_j = f, g_1...g_j = g$ *and* $i_1+...+i_j = i$.

The following corollary of the Replacement Theorem will play a crucial role in our calculations.

Corollary 2.4 *If* $w_h = w_k = r$ *so that* $(h,k)^{(r)}$ *has weight zero, then whenever* $(fh,gk)^{(i+r)}$ *is well defined, it may be replaced by* $(f,g)^{(i)}(h,k)^{(r)}$.

Proof. Clearly $(fh,gk)^{(i+r)}$ and $(f,g)^{(i)}(h,k)^{(r)}$ have the same stripped form and total transvectant order. It is only necessary to observe that $(f,g)^{(i)}$ is well-defined. But $w_{fh} = w_f + w_h = w_f + r \geq i + r$, so $w_f \geq i$ and similarly $w_g \geq i$. □

The following lemma is now trivial, but essential to our method.

Lemma 2.5 *Box distributes over direct sums of admissible subspaces: If* $\hat{V} \subset ker\hat{X}, \tilde{V}_1 \subset ker\tilde{X}$ *and* $\tilde{V}_2 \subset ker\tilde{X}$ *are admissible subspaces, with* $\tilde{V}_1 \cap \tilde{V}_2 = 0$, *then* $\tilde{V}_1 \oplus \tilde{V}_2$ *is admissible and* $\hat{V} \boxtimes (\tilde{V}_1 \oplus \tilde{V}_2) = (\hat{V} \boxtimes \tilde{V}_1) \oplus (\hat{V} \boxtimes \tilde{V}_2)$, *and similarly for* $(\tilde{V}_1 \oplus \tilde{V}_2) \boxtimes \hat{V}$.

We complete this section by the following theorem which is Theorem (9) in Murdock and Sanders (2006), that outlines the procedure for computing *ker* $\mathfrak{X}$.

Theorem 2.6 *A Stanley decomposition of ker* $\mathfrak{X} = ker\ \hat{\mathfrak{X}} \boxtimes ker\ \tilde{\mathfrak{X}}$ *is computable in a finite number of steps given decomposition of ker* $\hat{\mathfrak{X}}$ *and ker* $\tilde{\mathfrak{X}}$.

Proof. The proof is given in Murdock and Sanders (2006), but we will briefly outline the ideas used in the proof because of their importance to our calculations. By Lemma 2.5, we can compute $ker\ \mathfrak{X}$ if we can compute any box product of the form $\mathbb{R}[[f_1,...,f_k]]\varphi \boxtimes \mathbb{R}[[g_1,...,g_l]]]\psi$, where each factor is a Stanley term from the given decompositions of $ker\ \hat{\mathfrak{X}}$ and $ker\ \tilde{\mathfrak{X}}$.

Let p be the number of elements of weight > 0 in $f_1,...,f_k$ and q the number of such elements in $g_1,...,g_l$. We proceed by double induction on p and q.

Suppose $p = q = 0$, then the box product is spanned by transvectants of the form $(f_1^{m_1},...,f_k^{m_k}\varphi, g_1^{n_1},...,g_l^{n_l}\psi)^{(i)}$, which is well-defined if and only if $0 \le i \le r$, where $r = min(w_\varphi, w_\psi)$. The f and g factors add no weight, and cannot support any higher transvectants. By Theorem 2.3 each transvectant may be replaced by $f_1^{m_1}...f_k^{m_k}, g_1^{n_1}...g_l^{n_l}(\varphi,\psi)^i$ which remains well-defined. Therefore

$$\mathbb{R}[[f_1,...,f_k]]\varphi \boxtimes \mathbb{R}[[g_1,...,g_l]]\psi \cong \bigoplus_{i=0}^{r} \mathbb{R}[[f_1,...,f_k,g_1,...,g_l]](\varphi,\psi)^i. \tag{2.2.2}$$

Now we make induction hypothesis that all cases with $p = 0$ are computable up through the case $q-1$, and we discuss case q. Choose one of the q elements of $g_1,...,g_l$ having positive weight; we assume the chosen element is g_1. Then we may expand

$$\mathbb{R}[[g_1,...,g_l]]\psi = \left(\bigoplus_{\nu=0}^{t-1} \mathbb{R}[g_2,...,g_l]]g_1^\nu\psi\right) \oplus \mathbb{R}[[g_1,...,g_l]]g_1^t\psi,$$

where t is the smallest integer such that $w_{g_1^t\varphi} > w_\psi$. This decomposition corresponds to classifying monomials according to the power of g_1 that occurs, with all powers greater than or equal to t assigned to the last term. Now take the box product of $\mathbb{R}[[f_1^{m_1},...,f_k^{m_k}]]\varphi$ times this expression, and distribute the product according to Lemma 2.5. All of the terms except the last are computable by the induction hypothesis. We claim the last term is computable by the formula

$$\mathbb{R}[[f_1,...,f_k]]\varphi \boxtimes \mathbb{R}[[g_1,...,g_l]]g_1^t\psi \cong \bigoplus_{i=0}^{w_\varphi} \mathbb{R}[[f_1,...,f_k,g_1,...,g_l]](\varphi, g_1^t\psi)^{(i)}. \tag{2.2.3}$$

This is because w_φ is an absolute limit to the order of transvectants in this box product that will be well-defined, and any such transvectant $(f_1^{m_1}...f_k^{m_k}\varphi, g_1^{n_1}...g_l^{n_l}...g_l^t\psi)^i$ can be replaced by $f_1^{m_1}...f_k^{m_k}\varphi, g_1^{n_1}...g_l^{n_l}(g_1^t\varphi, \psi)^{(i)}$.

Now we make the induction hypothesis that cases $(p-1, q)$, $(p, q-1)$, and $(p-1, q-1)$ can be handled, and we treat the case (p, q). Choose one of the p functions in $f_1, ..., f_k$ having positive weight; we assume the chosen element is f_1. Similarly, choose a function of positive weight from $g_1, ..., g_l$ and suppose it is g_1. Let s and t be the smallest integers such that $s.w_{f_1} = t.w_{g_1}$

Expand

$$\mathbb{R}[[f_1, ..., f_k]]\varphi = (\bigoplus_{\mu=0}^{s-1} \mathbb{R}[[f_2, ..., f_k]]f_1^\mu\varphi) \oplus \mathbb{R}[[f_1, ..., f_k]]f_1^s\varphi$$

and

$$\mathbb{R}[[g_1, ..., g_l]]\psi = (\bigoplus_{\nu=0}^{t-1} \mathbb{R}[[g_2, ..., g_l]]g_1^\nu\psi) \oplus \mathbb{R}[[g_1, ..., g_l]]g_1^t\psi.$$

Taking the box product of these last two expansions and distributing the product there are four kinds of terms. Terms that are missing both f_1 and g_1 in square brackets are of type $(p-1, q-1)$. Terms that are missing f_1 in square brackets, but not g_1 are of type $(p-1, q)$ and there are likewise terms of type $(p, q-1)$. All of these can be handled by the induction hypothesis. Finally, there is the term

$$\mathbb{R}[[f_1, ..., f_k]]f_1^s\varphi \boxtimes \mathbb{R}[[g_1, ..., g_l]]g_1^t\psi. \tag{2.2.4}$$

There is no upper limit to the transvectant order that can occur here, since in general there remain terms of positive weight in the square brackets. However, setting $r = s.w_{f1} = t.w_{g1}$ it can be shown that this box product is equivalent to

$$\mathbb{R}[[f_1, ..., f_k]]f_1^s\varphi \boxtimes \mathbb{R}[[g_1, ..., g_l]]g_1^t\psi \cong$$

$$\bigoplus_{i=0}^{r-1}\mathbb{R}[[f_1,...,f_k,g_1,...,g_l]](f_1^s\varphi,g_1^t\psi)^{(i)}\oplus\Big(\mathbb{R}[[f_1,...,f_k]]\varphi\boxtimes\mathbb{R}[[g_1,...g_l]]\psi\Big)(f_1^s,g_1^t)^{(r)} \tag{2.2.5}$$

The final term is quite different from any other considered so far, since it involves a box product of subspaces as the coefficient of $(f_1^s,g_1^t)^{(r)}$. At this point we have reduced the calculation of $\mathbb{R}[[f_1,...,f_k]]\varphi\boxtimes\mathbb{R}[[g_1,...,g_l]]\psi$ in the case (p,q) to a number of terms computable by the induction hypothesis plus one special term that lead in circles since it involves the very same box product that we are trying to calculate. Thus our result has the form

$$\mathfrak{R}=\delta\oplus\mathfrak{R}\theta$$

where $\theta=(f_1^s,g_1^t)^{(r)}$ has weight zero. But this implies $\mathfrak{R}=\delta\oplus(\delta\oplus\mathfrak{R}\theta)\delta=\delta\oplus\delta\theta\oplus\mathfrak{R}\theta^2+...$ which reduces to $\mathfrak{R}=\delta[[\theta]]$.

This simply means that we erase the term $(\mathbb{R}[[f_1,...,f_k]]\varphi\boxtimes\mathbb{R}[[g_1,...,g_l]]\psi)(f_1^s,g_1^t)^{(r)}$ from our computation, and instead insert $\theta=(f_1^s,g_1^t)^{(r)}$ into the square brackets in all the coefficient rings that have already been computed. This does not affect the induction, because the new elements added have weight zero, and the induction is on the numbers p and q of elements of positive weight. □

2.3 Boosting Ring of Invariants to the Module of Equivariants.

In this section we describe the procedure for obtaining a Stanley decomposition of the module of equivariants (or normal form space) $ker\ \mathsf{X}$ when the Stanley decomposition of the ring of invariants $ker\ \mathcal{X}$ is already known.

The module of all formal power series vector fields on $\mathbb{R}^n$ can be viewed as the tensor product $\mathbb{R}[[x_1,...,x_n]]\otimes\mathbb{R}^n$ as in Murdock (2002). In fact the tensor product can be identified with the ordinary product (of a field times a constant vector) since (just as in the case of a tensor product of two polynomial spaces with nonoverlapping variables, used in section 2.2) the ordinary product satisfies the same algebraic rules as a tensor product. Specifically, every formal power series vector field can be written as

$$f_1(x)e_1 + ... + f_n(x)e_n = \begin{bmatrix} f_1(x) \\ . \\ . \\ . \\ f_n(x) \end{bmatrix}$$

where the e_i are the standard basis vectors of $\mathbb{R}^n$.

It is known that the Lie derivative $\mathsf{X} = \mathsf{L}_{X^*}$ can be expressed as the tensor product of $\mathcal{X}$ and $-X^*$, that is $\mathsf{X} = \mathcal{X} \otimes I + I \otimes (-X^*)$. Under the identification of $\otimes$ with ordinary product, this means $\mathsf{X}(fv) = (\mathcal{X}f)v + f(-X^*v)$, where $f \in \mathbb{R}[[x_1, ..., x_n]]$ and $v \in \mathbb{R}^n$ in agreement with the following calculation (in which $v' = 0$ because v is constant).

$$\begin{aligned} \mathsf{X}(fv) &= \mathsf{L}_{X^*}(fv) \\ &= (\mathcal{D}_{X^*}f)v + f(\mathsf{L}_{X^*}v) \\ &= (\mathcal{D}_{X^*}f)v + f(v'X^*x - X^*v) \\ &= (\mathcal{D}_{X^*}f)v + f(-X^*v). \end{aligned}$$

This kind of calculation also shows that $sl(2)$ representation (on vector fields) with triad $\{\mathsf{X}, \mathsf{Y}, \mathsf{Z}\}$ is the tensor product of the representation (on scalar fields) with triad $\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\}$ and the representation (on $\mathbb{R}^n$) with triad $\{-X^*, -Y^*, -Z\}$.

It follows that a basis for the normal form space, *ker* X i.e the set of all vectors that spans *ker* X, is given by the well defined transvectants $(f, v)^{(i)}$ as f ranges over a basis for *ker* $\mathcal{X} \subset \mathbb{R}[[x_1, ..., x_n]]$ and v ranges over a basis for *ker* $X^* \subset \mathbb{R}^n$. The first of these bases is given by the standard monomials of a Stanley decomposition for $ker\ \mathcal{X}$. The second is given by the standard basis vectors $e_r \in \mathbb{R}$ such that r is the index of the bottom row of a Jordan block in X^* (or equivalently, in X). It is useful to note that the weight of such an e_r is one less than the size of the block. The definition (2.2.1) of transvectant in this case becomes

$$(f, e_r)^i = \sum_{j=0}^{i} (-1)^j W^{i,j}_{f,e_r} (\mathcal{Y}^{\mathbf{j}}\mathbf{f})((\text{-}\,Y^*)^{\mathbf{i-j}} e_r)$$

$$= (f, g)^i = (-1)^i \sum_{j=0}^{i} W^{i,j}_{f,g} (\mathcal{Y}^{\mathbf{j}}\mathbf{f})((Y^*)^{\mathbf{i-j}} g). \qquad (2.3.1)$$

The computational procedures are the same as those used in previous section of rings of invariants in section (2.2), except that infinite iterations never arise. As an example, we illustrate how to compute Stanley decomposition for the normal form of vector fields with linear part N_{22}.

We begin with $ker\ \mathcal{X}_{22} = \mathbb{R}[[\alpha_1, \alpha_2, \beta_{12}]]$ from Malonza (2010). Since β_{12} has weight zero, it is convenient to suppress it from the calculation by setting $\mathcal{R} = \mathbb{R}[[\beta_{12}]]$ since we do not expand on invariants of weight zero and we write

$$\text{ker } \mathcal{X}_{22} = \mathcal{R}[[\alpha_1, \alpha_2]]. \qquad (2.3.2)$$

Expanding the Stanley decomposition:

$$\begin{aligned} \text{ker } \mathcal{X}_{22} &= \mathcal{R}[[\alpha_2]] \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\alpha_1 \\ &= \mathcal{R} \oplus \mathcal{R}[[\alpha_2]]\alpha_2 \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\alpha_1 \qquad (2.3.3) \end{aligned}$$

The basis element of N_{22} are e_2 and e_4, the bottom rows of each Jordan block. Therefore:

$$ker\ \mathsf{X}_{22} = ker\ \mathcal{X}_{22} \boxtimes (\mathbb{R}e_2 \oplus \mathbb{R}e_4)$$

Distributing the box product over (2.3.2) gives:

$$\begin{aligned} \ker\ \mathcal{X}_{22} &= (\mathcal{R} \oplus \mathcal{R}[[\alpha_2]]\alpha_2 \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\alpha_1) \boxtimes (\mathbb{R}e_2 \oplus \mathbb{R}e_4) \\ &= (\mathcal{R} \oplus \mathcal{R}[[\alpha_2]]\alpha_2 \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\alpha_1) \boxtimes \mathbb{R}e_2 \oplus \\ &\quad (\mathcal{R} \oplus \mathcal{R}[[\alpha_2]]\alpha_2 \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\alpha_1) \boxtimes \mathbb{R}e_4 \end{aligned} \tag{2.3.4}$$

We have two cases to consider.

case 1: $(\mathcal{R} \oplus \mathcal{R}[[\alpha_2]]\alpha_2 \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\alpha_1) \boxtimes \mathbb{R}e_2$

1. If $f \in \mathcal{R}$ the only transvectant that can be formed is $(f, e_2)^{(0)} = fe_2$. So,

$$\mathcal{R} \boxtimes \mathbb{R}e_2 = \mathcal{R}e_2.$$

2. If $f \in \mathcal{R}[[\alpha_2]]\alpha_2$ then $f = g\alpha_2$ with $g \in \mathcal{R}[[\alpha_2]]$ (having weight one). Then $(f, e_2)^{(i)}$ can be formed for $i = 0, 1$ and can be replaced by $g(\alpha_2, e_2)^{(i)}$. Therefore

$$\mathcal{R}[[\alpha_2]]\alpha_2 \boxtimes \mathbb{R}e_2 = \mathcal{R}[[\alpha_2]]\alpha_2 e_2 \oplus \mathcal{R}[[\alpha_2]](\alpha_2, e_2)^{(1)}.$$

3. If $f \in \mathcal{R}[[\alpha_1, \alpha_2]]\alpha_1$ then $f = g\alpha_1$ with $g \in \mathcal{R}[[\alpha_1, \alpha_2]]$ (having weight one). Tranvectants $(f, e_2)^{(i)}$ can be formed for $i = 0, 1$ and can be replaced by $g(\alpha_1, e_2)^{(i)}$. Thus,

$$\mathcal{R}[[\alpha_1, \alpha_2]]\alpha_1 \boxtimes \mathbb{R}e_2 = \mathcal{R}[[\alpha_1, \alpha_2]]\alpha_1 e_2 \oplus \mathcal{R}[[\alpha_1, \alpha_2]](\alpha_1, e_2)^{(1)}.$$

Case 2: Similarly for the basis e_4 we have:

1. $\mathcal{R} \boxtimes \mathbb{R}e_4 = \mathcal{R}e_4$.
2. $\mathcal{R}[[\alpha_2]]\alpha_2 \boxtimes \mathbb{R}e_4 = \mathcal{R}[[\alpha_2]]\alpha_2 e_4 \oplus \mathcal{R}[[\alpha_2]](\alpha_2, e_4)^{(1)}$.
3. $\mathcal{R}[[\alpha_1, \alpha_2]]\alpha_1 \boxtimes \mathbb{R}e_4 = \mathcal{R}[[\alpha_1, \alpha_2]]\alpha_1 e_4 \oplus \mathcal{R}[[\alpha_1, \alpha_2]](\alpha_1, e_4)^{(1)}$.

Adding the terms in both cases and inserting β_{12} we have:

$$\begin{aligned}\ker\ \mathsf{X}_{22} &= \mathcal{R}[[\alpha_1,\alpha_2,\beta_{12}]]e_2 \oplus \mathcal{R}[[\alpha_1,\alpha_2,\beta_{12}]](\alpha_1,e_2)^{(1)} \oplus \mathcal{R}[[\alpha_2,\beta_{12}]](\alpha_2,e_2)^{(1)} \oplus \\ &\quad \mathcal{R}[[\alpha_1,\alpha_2,\beta_{12}]]e_4 \oplus \mathcal{R}[[\alpha_1,\alpha_2,\beta_{12}]](\alpha_1,e_4)^{(1)} \oplus \mathcal{R}[[\alpha_2,\beta_{12}]](\alpha_2,e_4)^{(1)}.\end{aligned}$$

It is necessary to compute the transvectants that appear in the Stanley decomposition. These are of the form $(f,e_2)^{(1)}$ and $(f,e_4)^{(1)}$ where $f=\alpha_1,\alpha_2$.

From the definition of transvectant,

$$(f,g)^i = (-1)^i \sum_{j=0}^{i} W_{f,g}^{i,j}(\mathcal{Y}^{\mathbf{j}}\mathbf{f})((Y^*)^{\mathbf{i-j}}g)$$

where $W_{f,g}^{i,j} = \binom{i}{j}\frac{(w_f-j)!}{(w_f-i)!}.\frac{(w_g-i+j)!}{(w_g-i)!}$.

With $f=\alpha_1$ and $g=e_2$, for $j=0, i=1$, then $W_{f,g}^{1,0} = \binom{1}{0}\frac{(1-0)!}{(1-1)!}.\frac{(1-1+0)!}{(1-1)!} = 1$ and for $j=1, i=1$, then $W_{f,g}^{1,1} = \binom{1}{1}\frac{(1-1)!}{(1-1)!}.\frac{(1-1+1)!}{(1-1)!} = 1$.

Thus,

$$\begin{aligned}(f,g)^1 &= -fY^*g - \mathcal{Y}fg \\ &= -f\begin{bmatrix}1\\0\\0\\0\end{bmatrix} - \mathcal{Y}f\begin{bmatrix}0\\1\\0\\0\end{bmatrix} \\ &= -\begin{bmatrix}f\\ \mathcal{Y}f\\0\\0\end{bmatrix} = \begin{bmatrix}\mathcal{X}\mathcal{Y}f\\ \mathcal{Y}f\\0\\0\end{bmatrix}\end{aligned} \tag{2.3.5}$$

Therefore:

$$(\alpha_1, e_2)^1 \;=\; \begin{bmatrix} \mathcal{X}\mathcal{Y}\alpha_1 \\ \mathcal{Y}\alpha_1 \\ 0 \\ 0 \end{bmatrix} \qquad (\alpha_1, e_4)^1 = \begin{bmatrix} 0 \\ 0 \\ \mathcal{X}\mathcal{Y}\alpha_1 \\ \mathcal{Y}\alpha_1 \end{bmatrix}$$

$$(\alpha_2, e_2)^1 \;=\; \begin{bmatrix} \mathcal{X}\mathcal{Y}\alpha_2 \\ \mathcal{Y}\alpha_2 \\ 0 \\ 0 \end{bmatrix} \qquad (\alpha_2, e_4)^1 = \begin{bmatrix} 0 \\ 0 \\ \mathcal{X}\mathcal{Y}\alpha_2 \\ \mathcal{Y}\alpha_2 \end{bmatrix}$$

The normal form is then:

$$\begin{aligned} ker\ \mathsf{X}_{22} \;=\; & \mathcal{R}[[\alpha_1, \alpha_2, \beta_{12}]] \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \oplus \mathcal{R}[[\alpha_1, \alpha_2, \beta_{12}]] \begin{bmatrix} \mathcal{X}\mathcal{Y}\alpha_1 \\ \mathcal{Y}\alpha_1 \\ 0 \\ 0 \end{bmatrix} \oplus \mathcal{R}[[\alpha_2, \beta_{12}]] \begin{bmatrix} \mathcal{X}\mathcal{Y}\alpha_2 \\ \mathcal{Y}\alpha_2 \\ 0 \\ 0 \end{bmatrix} \oplus \\ & \mathcal{R}[[\alpha_1, \alpha_2, \beta_{12}]] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \oplus \mathcal{R}[[\alpha_1, \alpha_2, \beta_{12}]] \begin{bmatrix} 0 \\ 0 \\ \mathcal{X}\mathcal{Y}\alpha_1 \\ \mathcal{Y}\alpha_1 \end{bmatrix} \oplus \mathcal{R}[[\alpha_2, \beta_{12}]] \begin{bmatrix} 0 \\ 0 \\ \mathcal{X}\mathcal{Y}\alpha_2 \\ \mathcal{Y}\alpha_2 \end{bmatrix} \end{aligned}$$

which coincides with the one in Malonza (2004) that was computed by the Groebner basis method.

CHAPTER THREE

3.0 STANLEY DECOMPOSITION OF RING OF INVARIANTS WITH LINEAR PART $N_{33,\dots,3}$

In this chapter we shall apply the algorithm developed by Murdock (2006) to obtain the Stanley decomposition of the ring of invariants, $ker\ \mathcal{X}$ for systems with nilpotent linear part $N_{33,\dots,3}$. Before generalizing the result of writing down the Stanley decomposition we give various examples as motivation.

3.1 The Stanley Decomposition for System with Linear Part N_{33}

The ring of invariants of N_3 in $R[x, y, z]$ is $ker\ \mathcal{X}_3$. Let $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

The $sl(2)$ triad will be as follows;

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \qquad Z = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Having obtained the triad $\{X, Y, Z\}$, we create additional triad $\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\}$ as

$$\mathcal{X} = \mathcal{D}_Y = 2x\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial z}$$

$$\mathcal{Y} = \mathcal{D}_X = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}$$

$$\mathcal{Z} = \mathcal{D}_Z = 2x\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial z}.$$

The differential operators $\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\}$ map each vector space of homogeneous scalar polynomials into itself, with $\mathcal{X}$ and $\mathcal{Y}$ being nilpotent and $\mathcal{Z}$ semisimple. The eigenvectors of $\mathcal{Z}$ (called weight vectors) are the monomials x^m and the associated eigenvalues(called weights) are $< m, \mu >$ where $\mu = (\mu_1, ..., \mu_n)$ are the eigenvalues of $\mathcal{Z}$ that is $\mathcal{Z}(x^m) =< m, \mu > x^m$.

The basic invariants of N_3 can be shown to be $\alpha = x, \beta = y^2 - 2xz$. Here α is of degree 1 weight 2 and β is of degree 2 weight 0. Every element of $ker\ \mathcal{X}_3$ can be written uniquely as a formal series $f[\alpha, \beta]$ in x, y, z. We describe this by the Stanley decomposition $ker\ \mathcal{X}_3 = \mathbb{R}[[\alpha, \beta]]$.

From the Stanley decomposition of system with linear part N_3, $ker\ \mathcal{X}_3 = \mathbb{R}[[\alpha_1, \beta_1]]$, we have by Theorem 2.6 that $ker\ \mathcal{X}_{33} = ker\ \mathcal{X}_3 \boxtimes ker\ \bar{\mathcal{X}}_3$, where $ker\ \bar{\mathcal{X}}_3 = [[\alpha_2, \beta_2]]$ corresponds to the second Jordan block in N_{33}. Expanding we have

$$ker\ \mathcal{X}_3 = \mathbb{R}[[\beta_1]] \oplus \mathbb{R}[[\alpha_1, \beta_1]]\alpha_1$$

$$ker\ \bar{\mathcal{X}}_3 = \mathbb{R}[[\beta_2]] \oplus \mathbb{R}[[\alpha_2, \beta_2]]\alpha_2$$

Note that β_1 and β_2 are terms of weight zero and we do not expand along terms of weight zero, so they are suppressed and inserted in every square brackets of the box product we compute i.e.

$$ker\ \mathcal{X}_{33} = [\mathbb{R} \oplus \mathbb{R}[[\alpha_1]]\alpha_1] \boxtimes [\mathbb{R} \oplus \mathbb{R}[[\alpha_2]]\alpha_2]$$

Distributing the box product by considering all well defined transvectants $(f, g)^{(i)}$ gives three kinds of terms.

1. Two terms that are immediately computed in final form: $\mathbb{R} \oplus \mathbb{R}[[\alpha_1]]\alpha_1$
2. One box product: $\mathbb{R} \boxtimes \mathbb{R}[[\alpha_2]]\alpha_2 = \mathbb{R}[[\alpha_2]]\alpha_2$
3. One box product $\mathbb{R}[[\alpha_1]]\alpha_1 \boxtimes \mathbb{R}[[\alpha_2]]\alpha_2$. This will recycle to $\mathbb{R}[[\alpha_1]] \boxtimes \mathbb{R}[[\alpha_2]]$.

Indeed:

$$\mathbb{R}[[\alpha_1]]\alpha_1 \boxtimes \mathbb{R}[[\alpha_2]]\alpha_2 = \mathbb{R}[[\alpha_1, \alpha_2]]\alpha_1\alpha_2 \oplus \mathbb{R}[[\alpha_1, \alpha_2]](\alpha_1, \alpha_2)^{(1)} \oplus [\mathbb{R}[[\alpha_1]] \boxtimes \mathbb{R}[[\alpha_2]]](\alpha_1, \alpha_2)^{(2)}.$$

Let $(\alpha_1, \alpha_2)^{(1)} = \gamma_{1,2}$ and $(\alpha_1, \alpha_2)^{(2)} = \xi_{1,2}$. According to recycling rule in theorem 2.6 the last term will be deleted and $\xi_{1,2}$ which has weight zero will be inserted to all square brackets along side the suppressed invariants. Collecting and recombining all the terms, whenever possible we have:

$$ker\ \mathcal{X}_{33} = \mathbb{R}[[\alpha_1, \alpha_2, \beta_1, \beta_2, \xi_{1,2}]] \oplus \mathbb{R}[[\alpha_1, \alpha_2, \beta_1, \beta_2, \xi_{1,2}]]\gamma_{1,2}.$$

The same Stanley decomposition can be obtained from the lattice diagram below:

$$\begin{tikzcd}
\alpha_2 & & & \\
 & \gamma_{1,2} \arrow[dl] & & \\
\alpha_1 \arrow[uu] & \xi_{1,2} \arrow[l] \arrow[u] & \beta_2 \arrow[l] & \beta_1 \arrow[l]
\end{tikzcd}$$

We define the paths from β_1 to α_2 to be any moves in the direction of the arrows.The Stanley decomposition is then given by the two paths $(\beta_1 \to \beta_2 \to \xi_{1,2} \to \alpha_1 \to \alpha_2)$ with no corner and $(\beta_1 \to \beta_2 \to \xi_{1,2} \to \alpha_1 \to \alpha_2)$ with $\gamma_{1,2}$ as a corner.

To verify that all basic invariants have been found we consider the table function of the Stanley decomposition, $ker\ \mathcal{X}_{33}$ which in this case is given by

$$T_6 = \frac{1}{(1 - dw^2)^2(1 - d^2)^3} + \frac{d^2w^2}{(1 - dw^2)^2(1 - d^2)^3}.$$

Multiplying the table function by w, differentiating with respect to w and putting $w = 1$, it can be shown that

$$\frac{\partial}{\partial w} wT_6 \mid_{w=1} = \frac{1}{(1-d)^6}.$$

Hence by lemma 2.1 all the tranvectants has been found.

3.2 The Stanley Decomposition for System with Linear Part N_{333}

When $n = 3$ the Stanley decomposition of the ring of invariants is given by

$$\begin{aligned} ker\mathcal{X}_{333} &= ker\mathcal{X}_{33} \boxtimes ker\mathcal{X}_3 \\ &= [\mathbb{R}[[\alpha_1, \alpha_2, \beta_1, \beta_2, \xi_{1,2}]] \oplus \mathbb{R}[[\alpha_1, \alpha_2, \beta_1, \beta_2, \xi_{1,2}]]\gamma_{1,2}] \boxtimes \mathbb{R}[[\alpha_3, \beta_3]]. \end{aligned}$$

Suppressing $\beta_1, \beta_2, \beta_3$ and $\xi_{1,2}$ since we do not expand along terms of weight zero and noting that they will be inserted in all square brackets of the box product, then we have that

$$ker\ \mathcal{X}_{333} = [\mathbb{R}[[\alpha_1, \alpha_2]] \oplus \mathbb{R}[[\alpha_1, \alpha_2]]\gamma_{1,2}] \boxtimes \mathbb{R}[[\alpha_3]]$$

There are two cases to consider.

Case 1: $\mathbb{R}[[\alpha_1, \alpha_2]] \boxtimes \mathbb{R}[[\alpha_3]]$. Expanding the terms of Stanley decomposition we have:

$$\mathbb{R}[[\alpha_1, \alpha_2]] \boxtimes \mathbb{R}[[\alpha_3]] = [\mathbb{R}[[\alpha_2]] \oplus \mathbb{R}[[\alpha_1, \alpha_2]]\alpha_1] \boxtimes [\mathbb{R} \oplus \mathbb{R}[[\alpha_3]]\alpha_3].$$

Distributing the box product gives three kinds of terms:

1. Two terms that are immediately computed to final form: $\mathbb{R}[[\alpha_2]] \oplus \mathbb{R}[[\alpha_1, \alpha_2]]\alpha_1$.

2. One box product that by theorem 2.6 must be computed by further expansions:

$$\begin{aligned}
\mathbb{R}[[\alpha_2]] \boxtimes \mathbb{R}[[\alpha_3]]\alpha_3 &= [\mathbb{R} \oplus \mathbb{R}[[\alpha_2]]\alpha_2] \boxtimes [\mathbb{R}\alpha_3 \oplus \mathbb{R}[[\alpha_3]]\alpha_3^2] \\
&= \mathbb{R} \oplus \mathbb{R}[[\alpha_3]]\alpha_3^2 \oplus \mathbb{R}[[\alpha_2]]\alpha_2\alpha_3 \oplus \mathbb{R}[[\alpha_2]](\alpha_2, \alpha_3)^{(1)} \oplus \\
&\quad \mathbb{R}[[\alpha_2]](\alpha_2, \alpha_3)^{(2)} \oplus \mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2\alpha_3^2 \oplus \mathbb{R}[[\alpha_2, \alpha_3]](\alpha_2, \alpha_3^2)^{(1)} \oplus \\
&\quad [\mathbb{R}[[\alpha_2]] \boxtimes \mathbb{R}[[\alpha_3]]\alpha_3](\alpha_2, \alpha_3)^{(2)}
\end{aligned}$$

Let $(\alpha_2, \alpha_3)^{(1)} = \gamma_{2,3}$ and $(\alpha_2, \alpha_3)^{(2)} = \xi_{2,3}$. Recombining terms we obtain,

$$\mathbb{R}[[\alpha_2]] \boxtimes \mathbb{R}[[\alpha_3]]\alpha_3 = \mathbb{R}[[\alpha_2, \alpha_3, \xi_{2,3}]]\alpha_3 \oplus \mathbb{R}[[\alpha_2, \alpha_3, \xi_{2,3}]]\gamma_{2,3} \oplus \mathbb{R}[[\alpha_3, \xi_{2,3}]]\xi_{2,3}.$$

3. One box product $\mathbb{R}[[\alpha_1, \alpha_2]]\alpha_1 \boxtimes \mathbb{R}[[\alpha_3]]\alpha_3$ that will recycle to $\mathbb{R}[[\alpha_1, \alpha_2]] \boxtimes \mathbb{R}[[\alpha_3]]$. Indeed:

$$\begin{aligned}
\mathbb{R}[[\alpha_1, \alpha_2]]\alpha_1 \boxtimes \mathbb{R}[[\alpha_3]]\alpha_3 &= \mathbb{R}[[\alpha_1, \alpha_2, \alpha_3]]\alpha_1\alpha_3 \oplus \mathbb{R}[[\alpha_1, \alpha_2, \alpha_3]](\alpha_1, \alpha_3)^{(1)} \\
&\quad \oplus [\mathbb{R}[[\alpha_1, \alpha_2]] \boxtimes \mathbb{R}[[\alpha_3]]](\alpha_1, \alpha_3)^{(2)}.
\end{aligned}$$

Let $(\alpha_1, \alpha_3)^{(1)} = \gamma_{1,3}$ and $(\alpha_1, \alpha_3)^{(2)} = \xi_{1,3}$. According to recycling rule we delete the last term and insert $\xi_{2,3}$ to all square brackets of case 1. After recombining terms where possible we obtain:

$$\begin{aligned}
\mathbb{R}[[\alpha_1, \alpha_2]] \boxtimes \mathbb{R}[[\alpha_3]] &= \mathbb{R}[[\alpha_1, \alpha_2, \alpha_3, \xi_{1,3}]]\alpha_1 \oplus \mathbb{R}[[\alpha_1, \alpha_2, \alpha_3, \xi_{1,3}]]\gamma_{1,3} \\
&\quad \oplus \mathbb{R}[[\alpha_2, \alpha_3, \xi_{1,3}, \xi_{2,3}]] \oplus \mathbb{R}[[\alpha_2, \alpha_3, \xi_{1,3}, \xi_{2,3}]]\gamma_{2,3}.
\end{aligned}$$

Case 2: $\mathbb{R}[[\alpha_1, \alpha_2]]\gamma_{1,2} \boxtimes [[\alpha_3]]$. Expanding

$$\mathbb{R}[[\alpha_1, \alpha_2]]\gamma_{1,2} \boxtimes \mathbb{R}[[\alpha_3]] = [\mathbb{R}[[\alpha_2]]\gamma_{1,2} \oplus \mathbb{R}[[\alpha_1, \alpha_2]]\alpha_1\gamma_{1,2}] \boxtimes [\mathbb{R} \oplus \mathbb{R}[[\alpha_3]]\alpha_3]$$

Distributing the box products we have three kinds of terms.

1. Two terms that are completely computed: $\mathbb{R}[[\alpha_2]]\gamma_{1,2} \oplus \mathbb{R}[[\alpha_1, \alpha_2]]\alpha_1\gamma_{1,2}$

2. One box product that by theorem 2.6 that must be computed by further expansions:

$\mathbb{R}[[\alpha_2]]\gamma_{1,2} \boxtimes \mathbb{R}[[\alpha_3]]\alpha_3 = [\mathbb{R}\gamma_{1,2} \oplus \mathbb{R}[[\alpha_2]]\alpha_2\gamma_{1,2}] \boxtimes [\mathbb{R}\alpha_3 \oplus \mathbb{R}[[\alpha_3]]\alpha_3^2]$. There are four box products to be computed by further expansions namely:

$a)\mathbb{R}\gamma_{1,2} \boxtimes \mathbb{R}\alpha_3 = \mathbb{R}\alpha_3\gamma_{1,2} \oplus \mathbb{R}(\alpha_3, \gamma_{1,2})^{(1)} \oplus \mathbb{R}(\alpha_3, \gamma_{1,2})^{(2)}$.

$$b)\mathbb{R}\gamma_{1,2} \boxtimes \mathbb{R}[[\alpha_3]]\alpha_3^2 = \mathbb{R}[[\alpha_3]]\alpha_3^2\gamma_{1,2} \oplus \mathbb{R}[[\alpha_3]](\alpha_3^2, \gamma_{1,2})^{(1)} \oplus \mathbb{R}[[\alpha_3]](\alpha_3^2, \gamma_{1,2})^{(2)}$$
$$= \mathbb{R}[[\alpha_3]]\alpha_3^2\gamma_{1,2} \oplus \mathbb{R}[[\alpha_3]]\alpha_3(\alpha_3, \gamma_{1,2})^{(1)} \oplus \mathbb{R}[[\alpha_3]]\alpha_3(\alpha_3, \gamma_{1,2})^{(2)}.$$

$$c)\mathbb{R}[[\alpha_2]]\alpha_2\gamma_{1,2}\boxtimes\mathbb{R}\alpha_3 = \mathbb{R}[[\alpha_2]]\alpha_2\gamma_{1,2}\alpha_3\oplus\mathbb{R}[[\alpha_2]](\alpha_2\gamma_{1,2}, \alpha_3)^{(1)}\oplus\mathbb{R}[[\alpha_2]](\alpha_2\gamma_{1,2}, \alpha_3)^{(2)}.$$
$$= \mathbb{R}[[\alpha_2]]\alpha_2\gamma_{1,2}\alpha_3\oplus\mathbb{R}[[\alpha_2]]\gamma_{1,2}(\alpha_2, \alpha_3)^{(1)}\oplus\mathbb{R}[[\alpha_2]]\gamma_{1,2}(\alpha_2, \alpha_3)^{(2)}.$$

$$d)\mathbb{R}[[\alpha_2]]\alpha_2\gamma_{1,2} \boxtimes \mathbb{R}[[\alpha_3]]\alpha_3^2 = \mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2\gamma_{1,2}, \alpha_3^2 \oplus \mathbb{R}[[\alpha_2, \alpha_3]](\alpha_2\gamma_{1,2}, \alpha_3^2)^{(1)}\oplus$$
$$[\mathbb{R}[[\alpha_2]]\gamma_{1,2} \boxtimes \mathbb{R}[[\alpha_3]]\alpha_3](\alpha_2, \alpha_3)^{(2)}.$$

According to recycling rule we delete the last term and insert $\xi_{2,3}$ in all square brackets of calculation 2 to get the final product as:

$$\begin{aligned}
\mathbb{R}[[\alpha_2]]\gamma_{1,2} \boxtimes \mathbb{R}[[\alpha_3]]\alpha_3 &= \mathbb{R}[[\alpha_2, \alpha_3, \xi_{2,3}]]\alpha_3\gamma_{1,2} \oplus \mathbb{R}[[\alpha_3, \xi_{2,3}]](\alpha_3, \gamma_{1,2})^{(1)} \oplus \\
&\quad \mathbb{R}[[\alpha_3, \xi_{2,3}]](\alpha_3, \gamma_{1,2})^{(2)} \oplus \mathbb{R}[[\alpha_2, \alpha_3, \xi_{2,3}]]\gamma_{12}(\alpha_2, \alpha_3)^{(1)} \oplus \\
&\quad \mathbb{R}[[\alpha_3, \xi_{2,3}]]\gamma_{1,2}(\alpha_2, \alpha_3)^{(2)} \\
&= \mathbb{R}[[\alpha_2, \alpha_3, \xi_{2,3}]]\alpha_3\gamma_{1,2} \oplus \mathbb{R}[[\alpha_3, \xi_{2,3}]](\alpha_3, \gamma_{1,2})^{(1)} \oplus \\
&\quad \mathbb{R}[[\alpha_3, \xi_{2,3}]](\alpha_3, \gamma_{1,2})^{(2)} \oplus \mathbb{R}[[\alpha_2, \alpha_3, \xi_{2,3}]]\gamma_{1,2}\gamma_{2,3} \oplus \\
&\quad \mathbb{R}[[\alpha_3, \xi_{2,3}]]\gamma_{1,2}\xi_{2,3}
\end{aligned}$$

3. One box product : $\mathbb{R}[[\alpha_1, \alpha_2]]\alpha_2\gamma_{1,2} \boxtimes \mathbb{R}[[\alpha_3]]\alpha_3$ that recycles to $\mathbb{R}[[\alpha_1, \alpha_2]]\gamma_{1,2} \boxtimes \mathbb{R}[[\alpha_3]]$. Indeed:

$$
\begin{aligned}
\mathbb{R}[[\alpha_1,\alpha_2]]\alpha_1\gamma_{1,2}\boxtimes\mathbb{R}[[\alpha_3]]\alpha_3 &= \mathbb{R}[[\alpha_1,\alpha_2,\alpha_3]]\alpha_1\gamma_{1,2},\alpha_3\oplus\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3]](\alpha_1\gamma_{1,2},\alpha_3)^{(1)}\oplus\\
&\quad [\mathbb{R}[[\alpha_1,\alpha_2]]\gamma_{1,2}\boxtimes\mathbb{R}[[\alpha_3]]](\alpha_1,\alpha_3)^{(2)}\\
&= \mathbb{R}[[\alpha_1,\alpha_2,\alpha_3]]\alpha_1\gamma_{1,2},\alpha_3\oplus\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3]]\gamma_{1,2}\gamma_{1,3}\oplus\\
&\quad [\mathbb{R}[[\alpha_1,\alpha_2]]\gamma_{1,2}\boxtimes\mathbb{R}[[\alpha_3]]](\alpha_1,\alpha_3)^{(2)}
\end{aligned}
$$

Deleting the last term, inserting $\xi_{1,3}$ in all square brackets of case 2 and recombining terms where possible we obtain:

$$
\begin{aligned}
\mathbb{R}[[[\alpha_1,\alpha_2]]\gamma_{1,2}]\boxtimes\mathbb{R}[[\alpha_3]] &= \mathbb{R}[[\alpha_2,\alpha_3,\xi_{1,3},\xi_{2,3}]]\gamma_{1,2}\oplus\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\xi_{1,3}]]\alpha_1\gamma_{1,2}\\
&\quad \mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\xi_{13}]]\gamma_{1,2}\gamma_{1,3}\oplus\mathbb{R}[[\alpha_2,\alpha_3,\xi_{1,3},\xi_{2,3}]]\gamma_{1,2}\gamma_{2,3}\\
&\quad \mathbb{R}[[\alpha_3,\xi_{13},\xi_{23}]](\alpha_3,\gamma_{1,2})^{(1)}\oplus\mathbb{R}[[\alpha_3,\xi_{1,3},\xi_{2,3}]](\alpha_3,\gamma_{1,2})^{(2)}
\end{aligned}
$$

Adding both cases, rearranging and inserting suppressed transvectants the Stanley decomposition of a system with linear part N_{333} is

$$
\begin{aligned}
ker\ \mathfrak{X}_{333} &= \mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3}]]\oplus\\
&\quad \mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3}]]\gamma_{1,2}\oplus\\
&\quad \mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3}]]\gamma_{1,3}\oplus\\
&\quad \mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3}]]\gamma_{1,2}\gamma_{1,3}\oplus\\
&\quad \mathbb{R}[[\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3}]]\gamma_{2,3}\oplus\\
&\quad \mathbb{R}[[\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3}]]\xi_{2,3}\oplus\\
&\quad \mathbb{R}[[\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3}]]\gamma_{1,2}\gamma_{2,3}\oplus\\
&\quad \mathbb{R}[[\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3}]]\gamma_{1,2}\xi_{2,3}\oplus\\
&\quad \mathbb{R}[[\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3}]](\gamma_{1,2},\alpha_3)^{(1)}\oplus\\
&\quad \mathbb{R}[[\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3}]](\gamma_{1,2},\alpha_3)^{(2)}.
\end{aligned}
$$

To verify that all basic invariants have been found we consider the table function of the Stanley decomposition, *ker* $\mathcal{X}_{333}$ which in this case is given by

$$T_9 = \frac{dw^2}{(1-dw^2)^3(1-d^2)^5} + \frac{d^3w^4}{(1-dw^2)^3(1-d^2)^5} + \frac{d^2w^2}{(1-dw^2)^3(1-d^2)^5}$$

$$+\frac{d^4w^4}{(1-dw^2)^3(1-d^2)^5} + \frac{1}{(1-dw^2)^2(1-d^2)^6} + \frac{d^2w^2}{(1-dw^2)^2(1-d^2)^6}$$

$$+\frac{d^4w^4}{(1-dw^2)^2(1-d^2)^6} + \frac{d^3w^2}{(1-dw^2)(1-d^2)^6} + \frac{d}{(1-dw^2)(1-d^2)^6}.$$

Multiplying the table function by w, differentiating with respect to w and putting $w = 1$, it can be shown that

$$\frac{\partial}{\partial w} wT_9 \mid_{w=1} = \frac{1}{(1-d)^9}.$$

Hence by lemma 2.1 all the tranvectants has been found.

The following observations are made for system with linear part N_{333};

a) the first term of Stanley decomposition has no product outside the square bracket.

b) there are new transvectants $\gamma_{k,l}$ and $\xi_{k,l}$ where $1 \leq k < l \leq 3$

c) the transvectants $\gamma_{k,l}$ never appears inside the square brackets.

d) the transvectants $\xi_{k,l}$ appears inside as well as outside the square brackets.

If the new transvectants $\alpha_{k,l}$ where $1 \leq k < l \leq 3$ together with α_3 are added to the lattice diagram for N_{33}, then it is evident that the same Stanley decomposition can be obtained from the following lattice diagram:

$$
\begin{tikzcd}
\alpha_3 & & & & & \\
 & \gamma_{2,3} \arrow[dl] & & & & \\
\alpha_2 \arrow[uu] & \xi_{2,3} \arrow[l] \arrow[u] & & & & \\
 & \gamma_{1,3} \arrow[dl] & \gamma_{1,2} \arrow[dl] & & & \\
\alpha_1 \arrow[uu] & \xi_{1,3} \arrow[l] \arrow[u] & \xi_{1,2} \arrow[l] \arrow[u] & \beta_3 \arrow[l] & \beta_2 \arrow[l] & \beta_1 \arrow[l]
\end{tikzcd}
$$

- We define paths from β_1 to α_3 to be any moves in the direction of the arrows, that is to be made up of moves left, moves diagonal or moves up. Every monotone path i.e. paths in the given single direction takes the form

$$
\begin{tikzcd}
{} & & & \\
{} \arrow[u] & * \arrow[l] \arrow[dl] & & \\
{} & {} \arrow[u] & * \arrow[l] \arrow[dl] & \\
 & {} & {} \arrow[u] & {} \arrow[l]
\end{tikzcd}
$$

  Each of the vertex marked * will be called a corner of a maximal monotone path as they represent the maximum points along the paths.

- Each square brackets of the Stanley decomposition contains all invariants in a path except $\gamma_{k,l}$ where $1 \leq k < l \leq n$ and there is no monotone path from $\gamma_{k,l}$ to $\gamma_{k,l+1}$ as these are corners. The product of tranvectants outside the square bracket is the products of the invariants at the corners.

- A Stanley decomposition of the ring of invariants, $ker\ \mathcal{X}_{333}$ is then given by the sum of the terms of T_1 and T_2, i.e. $ker\ \mathcal{X}_{333} = \bigoplus_j T_1 \oplus \bigoplus_j T_2$ where

T_1=$\bigoplus_j \mathbb{R}$[[invariants on the j^{th} path]](product of corners on the j^{th} path) exiting at α_k and ending at α_3 where $k = 1, 2$.

T_2=$\bigoplus_j \mathbb{R}$[[invariants on the j^{th} path]](product of corners on the j^{th} path, $\alpha_3)^{(i)}$ exiting at α_2 through $\gamma_{2,3}$ and ending at α_3 where $i = 1, 2$.

For T_1

$(\beta_1 \to \beta_2 \to \beta_3 \to \xi_{1,2} \to \xi_{1,3} \to \alpha_1 \to \alpha_2 \to \alpha_3)$ with no corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \xi_{1,2} \to \xi_{1,3} \to \alpha_1 \to \alpha_2 \to \alpha_3)$, with $\gamma_{1,2}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \xi_{1,2} \to \xi_{1,3} \to \alpha_1 \to \alpha_2 \to \alpha_3)$, with $\gamma_{1,3}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \xi_{1,2} \to \xi_{1,3} \to \alpha_1 \to \alpha_2 \to \alpha_3)$ with $\gamma_{1,2}$ and $\gamma_{1,3}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \alpha_2 \to \alpha_3)$, with $\gamma_{2,3}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \alpha_2 \to \alpha_3)$, with $\xi_{2,3}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \alpha_2 \to \alpha_3)$,with $\gamma_{1,2}$ and $\gamma_{2,3}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \alpha_2 \to \alpha_3)$, with $\gamma_{1,2}$ and $\xi_{2,3}$ as corners

For T_2

$(\beta_1 \to \beta_2 \to \beta_3 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \alpha_3)$, with $(\gamma_{1,2}, \alpha_3)^{(1)}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \alpha_3)$, with $(\gamma_{1,2}, \alpha_3)^{(2)}$ as a corner

3.3 The Stanley Decomposition for System with Linear Part N_{3333}

From the example above we know that the Stanley decomposition of $ker\ \mathfrak{X}_{333} =$ $\mathbb{R}[[\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \xi_{1,2}, \xi_{1,3}]]\oplus$

$\mathbb{R}[[\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \xi_{1,2}, \xi_{1,3}]]\gamma_{1,2} \oplus \mathbb{R}[[\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \xi_{1,2}, \xi_{1,3}]]\gamma_{1,3}\oplus$

$\mathbb{R}[[\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \xi_{1,2}, \xi_{1,3}]]\gamma_{1,2}\gamma_{1,3} \oplus \mathbb{R}[[\alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}]]\gamma_{2,3}\oplus$

$\mathbb{R}[[\alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}]]\xi_{2,3} \oplus \mathbb{R}[[\alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}]]\gamma_{1,2}\gamma_{2,3}\oplus$

$\mathbb{R}[[\alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}]]\gamma_{1,2}\xi_{2,3} \oplus \mathbb{R}[[\alpha_3, \beta_1, \beta_2, \beta_3, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}]](\gamma_{1,2}, \alpha_3)^{(1)}$

$\oplus \mathbb{R}[[\alpha_3, \beta_1, \beta_2, \beta_3, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}]](\gamma_{1,2}, \alpha_3)^{(2)}$ and also $ker X_3 = \mathbb{R}[[\alpha_4, \beta_4]]$

Suppressing the transvectants $\beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}$ and $\xi_{2,3}$ since we do not expand along terms of weight zero and inserting them in all square brackets computed, then

$$\begin{aligned}
ker\ \mathcal{X}_{3333} &= ker\ \mathcal{X}_{333} \boxtimes ker\ \mathcal{X}_3 \\
&= [\mathbb{R}[[\alpha_1, \alpha_2, \alpha_3]] \oplus \mathbb{R}[[\alpha_1, \alpha_2, \alpha_3]]\gamma_{1,2} \oplus \mathbb{R}[[\alpha_1, \alpha_2, \alpha_3]]\gamma_{1,3} \oplus \\
&\quad \mathbb{R}[[\alpha_1, \alpha_2, \alpha_3]]\gamma_{1,2}\gamma_{1,3} \oplus \mathbb{R}[[\alpha_2, \alpha_3]]\gamma_{2,3} \oplus \mathbb{R}[[\alpha_2, \alpha_3]]\xi_{2,3} \oplus \\
&\quad \mathbb{R}[[\alpha_2, \alpha_3]]\gamma_{12}\gamma_{2,3} \oplus \mathbb{R}[[\alpha_2, \alpha_3]]\gamma_{1,2}\xi_{2,3} \oplus \mathbb{R}[[\alpha_3]](\gamma_{1,2}, \alpha_3)^{(1)} \\
&\quad \oplus \mathbb{R}[[\alpha_3,]](\gamma_{1,2}, \alpha_3)^{(2)}] \boxtimes \mathbb{R}[[\alpha_4]]
\end{aligned}$$

There are ten cases to consider.

Case 1:

$$\mathbb{R}[[\alpha_1, \alpha_2, \alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]] = [\mathbb{R}[[\alpha_2, \alpha_3]] \oplus \mathbb{R}[[\alpha_1, \alpha_2, \alpha_3]]\alpha_1] \boxtimes [\mathbb{R} \oplus \mathbb{R}[[\alpha_4]]\alpha_4].$$

There are three kinds of terms after distributing the box products:

1. Terms that are computed to final form: $\mathbb{R}[[\alpha_2, \alpha_3]] \oplus \mathbb{R}[[\alpha_1, \alpha_2, \alpha_3]]\alpha_1$

2. One box product to be computed by further expansions:

$\mathbb{R}[[\alpha_2, \alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4 = [\mathbb{R}[[\alpha_3]] \oplus \mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2] \boxtimes [\mathbb{R}\alpha_4 \oplus \mathbb{R}[[\alpha_4]]\alpha_4^2]$. There are four box products to be computed by further expansions namely:

$$\begin{aligned}
a)\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}\alpha_4 &= [\mathbb{R} \oplus \mathbb{R}[[\alpha_3]]\alpha_3] \boxtimes \mathbb{R}\alpha_4 \\
&= \mathbb{R}\alpha_4 \oplus \mathbb{R}[[\alpha_3]]\alpha_3\alpha_4 \oplus \mathbb{R}[[\alpha_3]](\alpha_3, \alpha_4)^{(1)} \oplus \mathbb{R}[[\alpha_3]](\alpha_3, \alpha_4)^{(2)} \\
&= \mathbb{R}[[\alpha_3]]\alpha_4 \oplus \mathbb{R}[[\alpha_3]](\alpha_3, \alpha_4)^{(1)} \oplus \mathbb{R}[[\alpha_3]](\alpha_3, \alpha_4)^{(2)}
\end{aligned}$$

Let $(\alpha_3, \alpha_4)^{(1)} = \gamma_{3,4}$ and $(\alpha_3, \alpha_4)^{(2)} = \xi_{3,4}$. Then

$$\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}\alpha_4 \;=\; \mathbb{R}[[\alpha_3]]\alpha_4 \oplus \mathbb{R}[[\alpha_3]]\gamma_{3,4} \oplus \mathbb{R}[[\alpha_3]]\xi_{3,4}$$

$$\begin{aligned} b)\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4^2 &= [\mathbb{R} \oplus \mathbb{R}[[\alpha_3]]\alpha_3] \boxtimes \mathbb{R}[\alpha_4^2 \oplus \mathbb{R}[[\alpha_4]]\alpha_4^3] \\ &= \mathbb{R}\alpha_4^2 \oplus \mathbb{R}[[\alpha_4]]\alpha_4^3 \oplus \mathbb{R}[[\alpha_3]]\alpha_3\alpha_4^2 \oplus \mathbb{R}[[\alpha_3]](\alpha_3, \alpha_4^2)^{(1)} \oplus \\ &\quad \mathbb{R}[[\alpha_3]](\alpha_3, \alpha_4^2)^{(2)}\mathbb{R}[[\alpha_3, \alpha_4]]\alpha_3\alpha_4^3 \oplus \mathbb{R}[[\alpha_3, \alpha_4]](\alpha_3, \alpha_4^3)^{(1)} \oplus \\ &\quad [\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4^2](\alpha_3, \alpha_4)^{(2)} \\ &= \mathbb{R}\alpha_4^2 \oplus \mathbb{R}[[\alpha_4]]\alpha_4^3 \oplus \mathbb{R}[[\alpha_3]]\alpha_3\alpha_4^2 \oplus \mathbb{R}[[\alpha_3]]\alpha_4(\alpha_3, \alpha_4)^{(1)} \oplus \\ &\quad \mathbb{R}[[\alpha_3]]\alpha_4(\alpha_3, \alpha_4)^{(2)} \oplus \mathbb{R}[[\alpha_3, \alpha_4]]\alpha_3, \alpha_4^3 \oplus \\ &\quad \mathbb{R}[[\alpha_3, \alpha_4]]\alpha_4^2(\alpha_3\alpha_4)^{(1)} \oplus [\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4^2](\alpha_3, \alpha_4)^{(2)} \end{aligned}$$

Deleting the last term, recombining terms and inserting $\xi_{3,4}$ in all square brackets in (b) alone, we obtain:

$$\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4^2 \;=\; \mathbb{R}[[\alpha_3, \alpha_4, \xi_{3,4}]]\alpha_4^2 \oplus \mathbb{R}[[\alpha_3, \alpha_4, \xi_{3,4}]]\alpha_4\gamma_{3,4} \oplus \mathbb{R}[[\alpha_3, \xi_{3,4}]]\alpha_4\xi_{3,4}.$$

c) $\mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2 \boxtimes \mathbb{R}\alpha_4 = \mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2\alpha_4 \oplus \mathbb{R}[[\alpha_2, \alpha_3]](\alpha_2, \alpha_4)^{(1)} \oplus \mathbb{R}[[\alpha_2, \alpha_3]](\alpha_2, \alpha_4)^{(2)}$.
Let $(\alpha_2, \alpha_4)^{(1)} = \gamma_{2,4}$ and $(\alpha_2, \alpha_4)^{(2)} = \xi_{2,4}$. Then,
$\mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2 \boxtimes \mathbb{R}\alpha_4 = \mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2\alpha_4 \oplus \mathbb{R}[[\alpha_2, \alpha_3]]\gamma_{2,4} \oplus \mathbb{R}[[\alpha_2, \alpha_3]]\xi_{2,4}$.

$$\begin{aligned} d)\mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2 \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4^2 &= \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4]]\alpha_2\alpha_4^2 \oplus \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4]](\alpha_2, \alpha_4^2)^{(1)} \oplus \\ &\quad [\mathbb{R}[[\alpha_2, \alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]](\alpha_2, \alpha_4)^{(2)} \end{aligned}$$

Deleting the last term according to recycling rule and inserting ξ_{24} to all square brackets in calculation 2 we get:

$$\begin{aligned}\mathbb{R}[[\alpha_2,\alpha_3]]\boxtimes\mathbb{R}[[\alpha_4]]\alpha_4 &= \mathbb{R}[[\alpha_3,\xi_{2,4}]]\alpha_4\oplus\mathbb{R}[[\alpha_3,\xi_{2,4}]]\gamma_{3,4}\oplus\mathbb{R}[[\alpha_3,\xi_{2,4}]]\xi_{3,4}\oplus\\ &\quad \mathbb{R}[[\alpha_3,\alpha_4,\xi_{2,4},\xi_{3,4}]]\alpha_4^2\oplus\mathbb{R}[[\alpha_3,\alpha_4,\xi_2,\xi_{3,4}]]\alpha_4\gamma_{3,4}\oplus\\ &\quad \mathbb{R}[[\alpha_3,\xi_{2,4},\xi_{3,4}]]\alpha_4\xi_{3,4}\oplus\mathbb{R}[[\alpha_2,\alpha_3,\alpha_4,\xi_{2,4}]]\alpha_2\alpha_4\oplus\\ &\quad \mathbb{R}[[\alpha_2,\alpha_3,\alpha_4,\xi_{2,4}]]\gamma_{2,4}\oplus\mathbb{R}[[\alpha_2,\alpha_3,\xi_{2,4}]]\xi_{2,4}.\end{aligned}$$

3. One box product $\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3]]\alpha_1\boxtimes\mathbb{R}[[\alpha_4]]\alpha_4$ which recycles to $\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3]]\boxtimes\mathbb{R}[[\alpha_4]]$. Indeed:

$$\begin{aligned}\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3]]\alpha_1\boxtimes\mathbb{R}[[\alpha_4]]\alpha_4 &= \mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4]]\alpha_1\alpha_4\oplus\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4]]\\ &\quad (\alpha_1,\alpha_4)^{(1)}\oplus[\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3]]\boxtimes\mathbb{R}[[\alpha_4]](\alpha_1,\alpha_4)^{(2)}.\end{aligned}$$

Let $(\alpha_1,\alpha_4)^{(1)}=\gamma_{1,4}$ and $(\alpha_1,\alpha_4)^{(2)}=\xi_{1,4}$. Deleting the last term and inserting $\xi_{1,4}$ along with the suppressed transvectants in all square brackets of case 1 we have:

$$\begin{aligned}\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3]]\times\mathbb{R}[[\alpha_4]] &= \mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4}]]\oplus\\ &\quad \mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4}]]\gamma_{1,4}\oplus\\ &\quad \mathbb{R}[[\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4}]]\xi_{2,4}\oplus\\ &\quad \mathbb{R}[[\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{3,4}\oplus\\ &\quad \mathbb{R}[[\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\xi_{3,4}\oplus\\ &\quad \mathbb{R}[[\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4}]]\gamma_{2,4}.\end{aligned}$$

Case 2:

$$\begin{aligned}\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3]]\gamma_{12}\boxtimes\mathbb{R}[[\alpha_4]] &= [\mathbb{R}[[\alpha_2,\alpha_3]]\gamma_{12}\oplus\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3]]\alpha_1\gamma_{12}]\\ &\quad \boxtimes[\mathbb{R}\oplus\mathbb{R}[[\alpha_4]]\alpha_4]\end{aligned}$$

Distributing the box products there are three kinds of terms, namely:

1. Terms that are completely computed to final form:

$\mathbb{R}[[\alpha_2,\alpha_3]]\gamma_{12}\oplus\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3]]\alpha_1\gamma_{1,2}$

2.One box product computed by further expansion as:

$\mathbb{R}[[\alpha_2,\alpha_3]]\gamma_{1,2}\boxtimes\mathbb{R}[[\alpha_4]]\alpha_4=[\mathbb{R}[[\alpha_3]]\gamma_{1,2}\oplus\mathbb{R}[[\alpha_2,\alpha_3]]\alpha_2\gamma_{1,2}]\boxtimes[\mathbb{R}\alpha_4\oplus\mathbb{R}[[\alpha_4]]\alpha_4^2]$

There are four box products to be computed by further expansions namely:

$$\begin{aligned}a)\mathbb{R}[[\alpha_3]]\gamma_{1,2}\boxtimes\mathbb{R}\alpha_4 &= [\mathbb{R}\gamma_{1,2}\oplus\mathbb{R}[[\alpha_3]]\alpha_3\gamma_{1,2}]\boxtimes\mathbb{R}\alpha_4\\ &= \mathbb{R}\alpha_4\gamma_{1,2}\oplus\mathbb{R}(\alpha_4,\gamma_{1,2})^{(1)}\oplus\mathbb{R}(\alpha_4,\gamma_{1,2})^{(2)}\oplus\mathbb{R}[[\alpha_3]]\alpha_3\gamma_{1,2}\alpha_4\\ &\quad \oplus\mathbb{R}[[\alpha_3]](\alpha_3\gamma_{1,2},\alpha_4)^{(1)}\oplus\mathbb{R}[[\alpha_3]](\alpha_3\gamma_{1,2},\alpha_4)^{(2)}\\ &= \mathbb{R}[[\alpha_3]]\alpha_4\gamma_{1,2}\oplus\mathbb{R}(\alpha_4,\gamma_{1,2})^{(1)}\oplus\mathbb{R}(\alpha_4,\gamma_{1,2})^{(2)}\oplus\\ &\quad \mathbb{R}[[\alpha_3]]\gamma_{1,2}\gamma_{3,4}\oplus\mathbb{R}[\alpha_3]\gamma_{1,2}\xi_{3,4}.\end{aligned}$$

$$\begin{aligned}b)\mathbb{R}[[\alpha_3]]\gamma_{1,2}\boxtimes\mathbb{R}[[\alpha_4]]\alpha_4^2 &= [\mathbb{R}\gamma_{1,2}\oplus\mathbb{R}[[\alpha_3]]\alpha_3\gamma_{1,2}]\boxtimes[\mathbb{R}\alpha_4^2\oplus\mathbb{R}[[\alpha_4]]\alpha_4^3]\\ &= \mathbb{R}\alpha_4^2\gamma_{1,2}\oplus\mathbb{R}(\alpha_4^2,\gamma_{1,2})^{(1)}\mathbb{R}(\alpha_4^2,\gamma_{1,2})^{(2)}\oplus\mathbb{R}[[\alpha_4]]\alpha_4^3\gamma_{1,2}\\ &\quad \oplus\mathbb{R}[[\alpha_4]](\alpha_4^3,\gamma_{1,2})^{(1)}\oplus\mathbb{R}[[\alpha_4]](\alpha_4^3,\gamma_{1,2})^{(2)}\oplus\\ &\quad \mathbb{R}[[\alpha_3]]\alpha_3\gamma_{1,2}\alpha_4^2\oplus\mathbb{R}[[\alpha_3]](\alpha_3\gamma_{1,2},\alpha_4^2)^{(1)}\oplus\\ &\quad \mathbb{R}[[\alpha_3]](\alpha_3\gamma_{1,2},\alpha_4^2)^{(2)}\oplus\mathbb{R}[[\alpha_3,\alpha_4]]\alpha_3\gamma_{1,2}\alpha_4^3\oplus\\ &\quad \mathbb{R}[[\alpha_3,\alpha_4]](\alpha_3\gamma_{1,2},\alpha_4^3)^{(1)}\oplus\mathbb{R}[[[\alpha_3]]\gamma_{1,2}\boxtimes\\ &\quad \oplus\mathbb{R}[[\alpha_4]]\alpha_4^2](\alpha_3,\alpha_4)^{(2)}.\end{aligned}$$

According to recycling rule we delete the last term and insert $\xi_{3,4}$ in all square brackets in (b). We recombine the terms to obtain:

$$\begin{aligned}\mathbb{R}[[\alpha_3]]\gamma_{1,2} \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4^2 &= \mathbb{R}[[\alpha_3, \alpha_4, \xi_{3,4}]]\alpha_4^2\gamma_{1,2} \oplus \mathbb{R}[[\alpha_3, \alpha_4, \xi_{3,4}]](\alpha_4^2, \gamma_{1,2})^{(1)} \oplus \\ &\quad \mathbb{R}[[\alpha_4, \xi_{3,4}]](\alpha_4^2, \gamma_{1,2})^{(2)} \oplus \mathbb{R}[[\alpha_3, \xi_{3,4}]](\alpha_3\gamma_{1,2}, \alpha_4^2)^{(2)}.\end{aligned}$$

$$\begin{aligned}c)\mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2\gamma_{1,2} \boxtimes \mathbb{R}\alpha_4 &= \mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2\gamma_{1,2}, \alpha_4 \oplus \mathbb{R}[[\alpha_2, \alpha_3]](\alpha_2\gamma_{1,2}, \alpha_4)^{(1)} \oplus \\ &\quad \mathbb{R}[[\alpha_2, \alpha_3]](\alpha_2\gamma_{1,2}, \alpha_4)^{(2)} \\ &= \mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2\gamma_{1,2}, \alpha_4 \oplus \mathbb{R}[[\alpha_2, \alpha_3]]\gamma_{1,2}(\alpha_2, \alpha_4)^{(1)} \oplus \\ &\quad \mathbb{R}[[\alpha_2, \alpha_3]]\gamma_{1,2}(\alpha_2, \alpha_4)^{(2)}.\end{aligned}$$

d) One box product that recycles:

$$\begin{aligned}\mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2\gamma_{1,2} \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4^2 &= \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4]]\alpha_2\gamma_{1,2}\alpha_4^2, \oplus \\ &\quad \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4]](\alpha_2\gamma_{1,2}, \alpha_4^2)^{(1)} \oplus \\ &\quad [\mathbb{R}[[\alpha_2, \alpha_3]]\gamma_{1,2} \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4](\alpha_2, \alpha_4)^2.\end{aligned}$$

Deleting the last term and inserting $\xi_{2,4}$ in all square brackets of calculation 2, we recombine terms to obtain

$\mathbb{R}[[\alpha_2, \alpha_3]]\gamma_{1,2} \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4$

$$\begin{aligned}&= \mathbb{R}[[\alpha_3, \alpha_4, \xi_{2,4}, \xi_{3,4}]]\alpha_4\gamma_{1,2}\xi_{3,4} \oplus \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4, \xi_{2,4}]]\alpha_4\gamma_{1,2} \oplus \\ &\quad \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4, \xi_{2,4}]]\gamma_{1,2}\gamma_{2,4} \oplus \mathbb{R}[[\alpha_2, \alpha_3, \xi_{2,4}]]\gamma_{1,2}\xi_{2,4} \oplus \\ &\quad \mathbb{R}[[\alpha_4, \xi_{2,4}, \xi_{3,4}]]\alpha_4(\alpha_4, \gamma_{1,2})^{(1)} \oplus \mathbb{R}[[\alpha_4, \xi_{2,4}, \xi_{3,4}]]\alpha_4(\alpha_4, \gamma_{12})^{(2)} \oplus \\ &\quad \mathbb{R}[[\alpha_3, \alpha_4, \xi_{2,4}, \xi_{3,4}]]\alpha_4\gamma_{1,2}\gamma_{3,4} \oplus \mathbb{R}[[\xi_{2,4}]](\alpha_4, \gamma_{1,2})^{(1)} \oplus \\ &\quad \mathbb{R}[[\xi_{2,4}]](\alpha_4, \gamma_{1,2})^{(2)} \oplus \mathbb{R}[[\alpha_3, \xi_{2,4}]]\gamma_{1,3}\gamma_{3,4} \oplus \mathbb{R}[[\alpha_3, \xi_{2,4}]]\gamma_{1,3}\xi_{3,4}\end{aligned}$$

3. One box product that recycles:

$\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3]]\alpha_1\gamma_{1,2} \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4$

$$\begin{aligned} = & \ \mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4]]\alpha_1\alpha_4\gamma_{1,2} \oplus \mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4]](\alpha_1\gamma_{1,2},\alpha_4)^{(1)} \oplus \\ & [\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3]]\gamma_{1,2} \boxtimes \mathbb{R}[\alpha_4]](\alpha_1,\alpha_4)^{(2)} \end{aligned}$$

According to the recycling rule we delete the last term and insert ξ_{14} in all square brackets of case 2, we obtain:

$\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3]]\gamma_{1,2} \boxtimes \mathbb{R}[[\alpha_4]] =$

$$\begin{aligned} & \mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4]]\alpha_1\gamma_{1,2} \oplus \mathbb{R}[[\alpha_2,\alpha_3,\alpha_4,\xi_{2,4}]]\gamma_{1,2}\oplus \\ & \mathbb{R}[[\alpha_3,\alpha_4,\xi_{2,4},\xi_{3,4}]]\gamma_{1,2}\xi_{3,4} \oplus \mathbb{R}[\alpha_4,\xi_{2,4},\xi_{3,4}](\alpha_4,\gamma_{1,2})^{(1)}\oplus \\ & \mathbb{R}[[\alpha_4,\xi_{2,4},\xi_{3,4}]](\alpha_4,\gamma_{1,2})^{(2)} \oplus \mathbb{R}[[\alpha_3,\alpha_4,\xi_{2,4},\xi_{3,4}]]\gamma_{1,2}\gamma_{3,4}\oplus \\ & \mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4]]\gamma_{1,2}\gamma_{1,4} \oplus \mathbb{R}[[\alpha_2,\alpha_3,\alpha_4,\xi_{2,4}]]\gamma_{1,2}\gamma_{2,4} \end{aligned}$$

Adding the suppressed transvectants

$$\begin{aligned} \mathbb{R}[[\alpha_1,\alpha_2,\alpha_3]]\gamma_{1,2} \boxtimes \mathbb{R}[[\alpha_4]] = & \ \mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4}]]\gamma_{1,2} \oplus \\ & \mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4}]]\gamma_{1,2}\gamma_{1,4} \oplus \\ & \mathbb{R}[[\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4}]]\gamma_{1,2}\gamma_{2,4} \oplus \\ & \mathbb{R}[[\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4}]]\gamma_{1,2}\xi_{2,4} \oplus \\ & \mathbb{R}[[\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{1,2}\gamma_{3,4} \oplus \\ & \mathbb{R}[[\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{1,2}\xi_{3,4} \oplus \\ & \mathbb{R}[[\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{12},\xi_{1,3},\xi_{1,4},\xi_{2,4},\xi_{3,4}]](\alpha_4,\gamma_{1,2})^1 \oplus \\ & \mathbb{R}[[\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4},\xi_{3,4}]](\alpha_4.\gamma_{1,2})^2 \end{aligned}$$

Note that case 3 and 4 are similar to case 2 with $\gamma_{1,2}$ replaced by $\gamma_{1,3}$ and $\gamma_{1,2}\gamma_{1,3}$ respectively.

Hence, case 3 :

$$\begin{aligned}
\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3]]\gamma_{1,3}\boxtimes\mathbb{R}[[\alpha_4]] \quad = \quad & \mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4}]]\gamma_{1,3}\oplus \\
& \mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4}]]\gamma_{1,3}\gamma_{1,4}\oplus \\
& \mathbb{R}[[\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4}]]\gamma_{1,3}\gamma_{2,4}\oplus \\
& \mathbb{R}[[\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4}]]\gamma_{1,3}\xi_{2,4}\oplus \\
& \mathbb{R}[[\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{12},\xi_{1,3},\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{1,3}\gamma_{3,4}\oplus \\
& \mathbb{R}[[\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{1,3}\xi_{3,4}\oplus \\
& \mathbb{R}[[\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4},\xi_{3,4}]](\alpha_4,\gamma_{1,3})^1\oplus \\
& \mathbb{R}[[\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4},\xi_{3,4}]](\alpha_4,\gamma_{1,3})^2
\end{aligned}$$

Case 4:

$\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3]]\gamma_{1,2}\gamma_{1,3} \boxtimes \mathbb{R}[[\alpha_4]] =$

$$\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4}]]\gamma_{1,2}\gamma_{1,3}\oplus$$
$$\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4}]]\gamma_{1,2}\gamma_{1,3}\gamma_{1,4}\oplus$$
$$\mathbb{R}[[\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4}]]\gamma_{1,2}\gamma_{1,3}\gamma_{2,4}\oplus$$
$$\mathbb{R}[[\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4}]]\gamma_{1,2}\gamma_{1,3}\xi_{2,4}\oplus$$
$$\mathbb{R}[[\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{1,2}\gamma_{1,3}\gamma_{3,4}\oplus$$
$$\mathbb{R}[[\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{1,2}\gamma_{1,3}\xi_{3,4}\oplus$$
$$\mathbb{R}[[\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4},\xi_{3,4}]](\alpha_4,\gamma_{1,2}\gamma_{1,3})^{(1)}\oplus$$
$$\mathbb{R}[[\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{1,4},\xi_{2,4},\xi_{3,4}]](\alpha_4,\gamma_{1,2}\gamma_{1,3})^{(2)}.$$

Case 5.

$$\mathbb{R}[[\alpha_2,\alpha_3]]\gamma_{2,3} \boxtimes \mathbb{R}[[\alpha_4]] \quad = \quad [\mathbb{R}[[\alpha_3]]\gamma_{2,3} \oplus \mathbb{R}[[\alpha_2,\alpha_3]]\alpha_2\gamma_{2,3}] \boxtimes [\mathbb{R} \oplus \mathbb{R}[[\alpha_4]]\alpha_4]$$

Distributing the products gives three kinds of terms.

1. Two terms that are completely computed: $\mathbb{R}[[\alpha_3]]\gamma_{2,3} \oplus \mathbb{R}[[\alpha_2,\alpha_3]]\alpha_2\gamma_{2,3}$.

2. One box product that is computed by further expansion:

$$\mathbb{R}[[\alpha_3]]\gamma_{2,3} \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4 \quad = \quad [\mathbb{R}\gamma_{2,3} \oplus \mathbb{R}[[\alpha_3]]\alpha_3\gamma_{2,3} \boxtimes [\mathbb{R}\alpha_4 \oplus \mathbb{R}[[\alpha_4]]\alpha_4^2]$$

There are four box products namely:

a) $\mathbb{R}\gamma_{2,3} \boxtimes \mathbb{R}\alpha_4 = \mathbb{R}\alpha_4\gamma_{2,3} \oplus \mathbb{R}(\alpha_4,\gamma_{2,3})^{(1)} \oplus \mathbb{R}(\alpha_4,\gamma_{2,3})^{(2)}$.

b)$\mathbb{R}\gamma_{2,3} \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4^2 = \mathbb{R}[[\alpha_4]]\alpha_4^2\gamma_{2,3} \oplus \mathbb{R}[[\alpha_4]]\alpha_4(\alpha_4,\gamma_{2,3})^{(1)}\mathbb{R}[[\alpha_4]]\alpha_4(\alpha_4,\gamma_{2,3})^{(2)}$

c)$\mathbb{R}[[\alpha_3]]\alpha_3\gamma_{2,3}\boxtimes\mathbb{R}\alpha_4 = \mathbb{R}[[\alpha_3]]\alpha_3\alpha_4\gamma_{2,3}\oplus\mathbb{R}[[\alpha_3]]\gamma_{2,3}(\alpha_3,\alpha_4)^{(1)}\mathbb{R}[[\alpha_3]]\gamma_{2,3}(\alpha_3,\alpha_4)^{(2)}$

d)$\mathbb{R}[[\alpha_3]]\alpha_3\gamma_{2,3}\boxtimes\mathbb{R}[[\alpha_4]]\alpha_4^2 = \mathbb{R}[[\alpha_3,\alpha_4]]\alpha_3\alpha_4^2\gamma_{2,3}\oplus\mathbb{R}[[\alpha_3,\alpha_4]](\alpha_3\gamma_{2,3},\alpha_4^2)^{(1)}\oplus\mathbb{R}[[[\alpha_3]]\gamma_{2,3}\boxtimes \mathbb{R}[[\alpha_4]]\alpha_4](\alpha_3,\alpha_4)^{(2)}$

Deleting the last term and inserting $\xi_{3,4}$ in all square brackets of calculation 2 and recombining terms we get:

$$\begin{aligned}\mathbb{R}[[\alpha_3]]\gamma_{2,3} \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4 \quad &= \quad \mathbb{R}[[\alpha_3, \alpha_4, \xi_{3,4}]]\alpha_4\gamma_{2,3} \oplus \mathbb{R}[[\alpha_4, \xi_{3,4}]](\alpha_4, \gamma_{2,3})^{(1)} \oplus \\ &\quad\quad \mathbb{R}[[\alpha_4, \xi_{34}]](\alpha_4, \gamma_{23})^{(2)} \oplus \mathbb{R}[[\alpha_3, \alpha_4, \xi_{34}]]\gamma_{23}\gamma_{34} \oplus \\ &\quad\quad \mathbb{R}[[\alpha_3, \xi_{34}]]\gamma_{23}\xi_{34}\end{aligned}$$

3. One box product that recycles:

$$\begin{aligned}\mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2\gamma_{2,3} \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4 \quad &= \quad \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4]]\alpha_2\alpha_4\gamma_{2,3} \oplus \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4]]\gamma_{2,3}(\alpha_2, \alpha_4)^{(1)} \\ &\quad\quad \oplus\mathbb{R}[[[\alpha_2\alpha_3]]\gamma_{2,3} \boxtimes \mathbb{R}[[\alpha_4]]](\alpha_2, \alpha_4)^{(2)}\end{aligned}$$

Deleting the last term and inserting $\xi_{2,4}$ in all square brackets of case 5, we recombine terms and add suppressed tranvectants to conclude that:

$$\begin{aligned}\mathbb{R}[[\alpha_2, \alpha_3]]\gamma_{2,3} \boxtimes \mathbb{R}[[\alpha_4]] \quad &= \quad \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}]]\gamma_{2,3} \oplus \\ &\quad\quad \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}]]\gamma_{2,3}\gamma_{2,4} \oplus \\ &\quad\quad \mathbb{R}[[\alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]]\gamma_{2,3}\gamma_{3,4} \oplus \\ &\quad\quad \mathbb{R}[[\alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]]\gamma_{2,3}\xi_{3,4} \oplus \\ &\quad\quad \mathbb{R}[[\alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]](\alpha_4, \gamma_{2,3})^{(1)} \oplus \\ &\quad\quad \mathbb{R}[[\alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]](\alpha_4, \gamma_{2,3})^{(2)}\end{aligned}$$

Case 6:

$\mathbb{R}[[\alpha_2, \alpha_3]]\xi_{2,3} \boxtimes \mathbb{R}[[\alpha_4]]$

We compute in this case with $\xi_{2,3}$ appearing outside each square bracket. Expanding we have

$\mathbb{R}[[\alpha_2,\alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]] = [\mathbb{R}[[\alpha_3]] \oplus \mathbb{R}[[\alpha_2,\alpha_3]]\alpha_2] \boxtimes [\mathbb{R} \oplus \mathbb{R}[[\alpha_4]]\alpha_4]$

Distributing the box product gives three kinds of terms:

1. Two terms that are computed completely to final form; $\mathbb{R}[[\alpha_3]] \oplus \mathbb{R}[[\alpha_2,\alpha_3]]\alpha_2$.

2. One box product that must be computed by further expansion:

$$\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4 \quad = \quad [\mathbb{R} \oplus \mathbb{R}[[\alpha_3]]\alpha_3] \boxtimes [\mathbb{R}\alpha_4 \oplus \mathbb{R}[[\alpha_4]]\alpha_4^2]$$

We have three kinds of box products:

a)Terms computed completely to final form: $\mathbb{R}\alpha_4 \oplus \mathbb{R}[[\alpha_4]]\alpha_4^2$.

b) One box product: $\mathbb{R}[[\alpha_3]]\alpha_3 \boxtimes \mathbb{R}\alpha_4 = \mathbb{R}[[\alpha_3]]\alpha_3\alpha_4 \oplus \mathbb{R}[[\alpha_3]](\alpha_3,\alpha_4)^{(1)}\mathbb{R}[[\alpha_3]](\alpha_3,\alpha_4)^{(2)}$

c) One box product that recycles:

$\mathbb{R}[[\alpha_3]]\alpha_3 \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4^2 = \mathbb{R}[[\alpha_3,\alpha_4]]\alpha_3\alpha_4^2 \oplus \mathbb{R}[[\alpha_3,\alpha_4]]\alpha_4(\alpha_3,\alpha_4)^{(1)} \oplus [\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4](\alpha_3,\alpha_4)^{(2)}$.

The last term will be deleted and $\xi_{3,4}$ will be inserted in all square brackets from calculation 2. After recombining terms whenever possible we get:

$$\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4 \quad = \quad \mathbb{R}[[\alpha_3,\alpha_4,\xi_{3,4}]]\alpha_4 \oplus \mathbb{R}[[\alpha_3,\alpha_4,\xi_{3,4}]]\gamma_{34} \oplus \mathbb{R}[[\alpha_3]]\xi_{3,4}$$

3. One box product that recycles:

$$\begin{aligned}\mathbb{R}[[\alpha_2,\alpha_3]]\alpha_2 \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4 \quad &= \quad \mathbb{R}[[\alpha_2,\alpha_3,\alpha_4]]\alpha_2\alpha_4 \oplus \mathbb{R}[[\alpha_3,\alpha_4]]\gamma_{2,4} \oplus \\ &\quad\ [\mathbb{R}[[\alpha_2,\alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]](\alpha_2,\alpha_4^{(2)}\end{aligned}$$

According to recycling rule the last term here is deleted and ξ_{24} which has weight zero will be inserted in all square brackets of case 6 alongside suppressed transvectants. Finally after recombining terms where possible we obtain:

$\mathbb{R}[[\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}]]\xi_{2,3} \boxtimes \mathbb{R}[[\alpha_4]]$

$$\begin{aligned} = \; & \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}]]\xi_{2,3} \oplus \\ & \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}]]\gamma_{2,4}, \xi_{2,3} \oplus \\ & \mathbb{R}[[\alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]]\gamma_{3,4}, \xi_{2,3} \oplus \\ & \mathbb{R}[[\alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]]\xi_{2,3}, \xi_{3,4} \end{aligned}$$

Cases 7 and 8 are similar to case 5 with $\gamma_{2,3}$ replaced by $\gamma_{1,2}\gamma_{2,3}$ and $\gamma_{1,2}\xi_{2,3}$ respectively. Case 7:$\mathbb{R}[[\alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}]]\gamma_{1,2}\gamma_{2,3} \boxtimes \mathbb{R}[[\alpha_4]]$

$$\begin{aligned} = \; & \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}]]\gamma_{1,2}\gamma_{2,3} \oplus \\ & \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}]]\gamma_{1,2}\gamma_{2,3}\gamma_{2,4} \oplus \\ & \mathbb{R}[[\alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]]\gamma_{1,2}\gamma_{2,3}\gamma_{3,4} \oplus \\ & \mathbb{R}[[\alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]]\gamma_{1,2}\gamma_{2,3}\xi_{3,4} \oplus \\ & \mathbb{R}[[\alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]](\alpha_4, \gamma_{1,2}\gamma_{2,3})^{(1)} \oplus \\ & \mathbb{R}[[\alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]](\alpha_4, \gamma_{1,2}\gamma_{2,3})^{(2)} \end{aligned}$$

Case 8: $\mathbb{R}[[\alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}]]\gamma_{1,2}\xi_{2,3} \boxtimes \mathbb{R}[[\alpha_4]]$

$$\begin{aligned} = \; & \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}]]\gamma_{1,2}\xi_{2,3} \oplus \\ & \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}]]\gamma_{1,2}\gamma_{2,4}\xi_{2,3} \oplus \\ & \mathbb{R}[[\alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{13}, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]]\gamma_{1,2}\gamma_{3,4}\xi_{2,3} \oplus \\ & \mathbb{R}[[\alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]]\gamma_{1,2}\xi_{2,3}\xi_{3,4} \oplus \\ & \mathbb{R}[[\alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]](\alpha_4, \gamma_{1,2}\xi_{2,3})^{(1)} \oplus \\ & \mathbb{R}[[\alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]](\alpha_4, \gamma_{1,2}\xi_{2,3})^{(2)} \end{aligned}$$

Case 9:

$\mathbb{R}[[\alpha_3]](\alpha_3,\gamma_{1,2})^{(1)} \boxtimes \mathbb{R}[[\alpha_4]]$. Expanding, we have that:

$$\begin{aligned}\mathbb{R}[[\alpha_3]](\alpha_3,\gamma_{1,2})^{(1)} \boxtimes \mathbb{R}[[\alpha_4]] &= [\mathbb{R}(\alpha_3,\gamma_{1,2})^{(1)} \oplus \mathbb{R}[[\alpha_3]]\alpha_3(\alpha_3,\gamma_{1,2})^{(1)}] \\ &\quad \boxtimes[\mathbb{R} \oplus \mathbb{R}[[\alpha_4]]\alpha_4]\end{aligned}$$

Distributing the box products we get three kinds of terms namely:

1. Terms computed completely to final form: $\mathbb{R}(\alpha_3,\gamma_{1,2})^{(1)} \oplus \mathbb{R}[[\alpha_3]]\alpha_3(\alpha_3,\gamma_{1,2})^{(1)}$.
2. One box product:

$$\begin{aligned}\mathbb{R}(\alpha_3,\gamma_{1,2})^{(1)} \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4 &= \mathbb{R}[[\alpha_4]]\alpha_4(\alpha_3,\gamma_{1,2})^{(1)} \oplus \mathbb{R}[[\alpha_4]](\alpha_4,(\alpha_3,\gamma_{1,2})^{(1)})^{(1)} \oplus \\ &\quad \mathbb{R}[[\alpha_4]](\alpha_4,(\alpha_3,\gamma_{1,2})^{(1)})^{(2)}\end{aligned}$$

3. One box product that recycles:

$$\begin{aligned}\mathbb{R}[[\alpha_3]]\alpha_3(\alpha_3,\gamma_{1,2})^{(1)} \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4 &= \mathbb{R}[[\alpha_3,\alpha_4]]\alpha_3(\alpha_3,\gamma_{12})^{(1)}\alpha_4 \oplus \\ &\quad \mathbb{R}[[\alpha_3,\alpha_4]](\alpha_3(\alpha_3,\gamma_{1,2})^{(1)},\alpha_4)^{(1)} \oplus \\ &\quad [\mathbb{R}[[\alpha_3]]\alpha_3(\alpha_3,\gamma_{1,2})^{(1)} \boxtimes \mathbb{R}[[\alpha_4]]](\alpha_3,\alpha_4)^{(2)}\end{aligned}$$

We delete the last term and insert $\xi_{3,4}$ in every square bracket of case 9. Recombining terms where possible we state the final result for this case as:

$\mathbb{R}[[\alpha_3]](\alpha_3,\gamma_{1,2})^{(1)} \boxtimes \mathbb{R}[[\alpha_4]]$

$$\begin{aligned}&= \mathbb{R}[[\alpha_3,\alpha_4,\xi_{3,4}]](\alpha_3,\gamma_{1,2})^{(1)} \oplus \mathbb{R}[[\alpha_3,\alpha_4,\xi_{3,4}]](\alpha_4,\alpha_3(\alpha_3,\gamma_{1,2})^{(1)})^{(1)} \oplus \\ &\quad \mathbb{R}[[\alpha_4,\xi_{3,4}]](\alpha_4,(\alpha_3,\gamma_{1,2})^{(1)})^{(1)} \oplus \mathbb{R}[[\alpha_4,\xi_{3,4}]](\alpha_4,(\alpha_3,\gamma_{1,2})^{(1)})^{(2)}\end{aligned}$$

Inserting the suppressed transvectants we conclude that:

$\mathbb{R}[[\alpha_3, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}]](\alpha_3\gamma_{1,2})^{(1)} \boxtimes \mathbb{R}[[\alpha_4]]$

$$\begin{aligned}
&= \mathbb{R}[[\alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{3,4}]](\alpha_3, \gamma_{1,2})^{(1)} \oplus \\
&\quad \mathbb{R}[[\alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{3,4}]](\alpha_4, \alpha_3(\alpha_3, \gamma_{1,2})^{(1)})^{(1)} \\
&\quad \mathbb{R}[[\alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{3,4}]](\alpha_4, (\alpha_3, \gamma_{1,2})^{(1)})^{(1)} \oplus \\
&\quad \mathbb{R}[[\alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}, \xi_{3,4}]](\alpha_4, (\alpha_3, \gamma_{1,2})^{(1)})^{(2)}
\end{aligned}$$

Lastly,

Case 10: $\mathbb{R}[[\alpha_3]](\alpha_3, \gamma_{1,2})^{(2)} \boxtimes \mathbb{R}[[\alpha_4]]$.

We compute with $(\alpha_3, \gamma_{1,2})^{(2)}$ appearing outside each square bracket. Expanding, we have that:

$$\begin{aligned}
\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]] &= [\mathbb{R} \oplus \mathbb{R}[[\alpha_3]]\alpha_3] \boxtimes [\mathbb{R} \oplus \mathbb{R}[[\alpha_4]]\alpha_4] \\
&= \mathbb{R} \oplus \mathbb{R}[[\alpha_4]]\alpha_4 \oplus \mathbb{R}[[\alpha_3]]\alpha_3 \oplus \\
&\quad \mathbb{R}[[\alpha_3, \alpha_4]]\alpha_3\alpha_4 \oplus \mathbb{R}[[\alpha_3, \alpha_4]](\alpha_3, \alpha_4)^{(1)} \oplus \\
&\quad [\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]](\alpha_3, \alpha_4)^{(2)}
\end{aligned}$$

Recombining terms and adding $(\alpha_3, \gamma_{1,2})^{(2)}$ to all square brackets we get:

$$\begin{aligned}
\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]] &= \mathbb{R}[[\alpha_3, \alpha_4, \xi_{3,4}]](\alpha_3, \gamma_{1,2})^{(2)} \oplus \\
&\quad \mathbb{R}[[\alpha_3, \alpha_4, \xi_{3,4}]](\alpha_3, \alpha_4(\alpha_3, \gamma_{1,2})^{(2)})^{(1)}
\end{aligned}$$

Inserting the suppressed transvectants we have;

$$\mathbb{R}[[\alpha_3,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{2,3}]](\alpha_3\gamma_{1,2})^{(2)} \boxtimes \mathbb{R}[[\alpha_4]]$$

$$\begin{aligned} &= \mathbb{R}[[\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{2,3},\xi_{3,4}]](\alpha_3,\gamma_{1,2})^{(2)} \oplus \\ &\quad \mathbb{R}[[\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3},\xi_{2,3},\xi_{3,4}]](\alpha_4,\alpha_3(\alpha_3,\gamma_{1,2})^{(2)})^{(1)} \end{aligned}$$

We now state the Stanley decomposition, $ker\ \mathfrak{X}_{3333}$. Let $\mathfrak{R} = \mathbb{R}[[\beta_1,\beta_2,\beta_3,\beta_4,\xi_{1,2},\xi_{1,3}]]$. Then:

$$\begin{aligned} ker\ \mathfrak{X}_{3333} = \ & \mathfrak{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\xi_{1,4}]] \oplus \mathfrak{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\xi_{1,4}]]\gamma_{1,2} \oplus \\ & \mathfrak{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\xi_{1,4}]]\gamma_{1,3} \oplus \mathfrak{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\xi_{1,4}]]\gamma_{1,4} \oplus \\ & \mathfrak{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\xi_{1,4}]]\gamma_{1,2}\gamma_{1,3} \oplus \mathfrak{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\xi_{1,4}]]\gamma_{1,2}\gamma_{1,4} \oplus \\ & \mathfrak{R}[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\xi_{1,4}]]\gamma_{1,3}\gamma_{1,4} \oplus \mathfrak{R}_1[[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\xi_{1,4}]]\gamma_{1,2}\gamma_{1,3}\gamma_{1,4} \oplus \\ & \mathfrak{R}[[\alpha_2,\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4}]]\gamma_{2,4} \oplus \mathfrak{R}[[\alpha_2,\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4}]]\gamma_{1,2}\gamma_{2,4} \oplus \\ & \mathfrak{R}[[\alpha_2,\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4}]]\gamma_{1,3}\gamma_{2,4} \oplus \mathfrak{R}[[\alpha_2,\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4}]]\gamma_{1,2}\gamma_{1,3}\gamma_{2,4} \oplus \\ & \mathfrak{R}[[\alpha_2,\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4}]]\xi_{2,4} \oplus \mathfrak{R}[[\alpha_2,\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4}]]\gamma_{1,2}\xi_{2,4} \oplus \\ & \mathfrak{R}[[\alpha_2,\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4}]]\gamma_{1,3}\xi_{2,4} \oplus \mathfrak{R}[[\alpha_2,\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4}]]\gamma_{1,2}\gamma_{1,3}\xi_{2,4} \oplus \\ & \mathfrak{R}[[\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{3,4} \oplus \mathfrak{R}[[\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{1,2}\gamma_{3,4} \oplus \\ & \mathfrak{R}[[\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{1,3}\gamma_{3,4} \oplus \mathfrak{R}[[\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\xi_{3,4} \oplus \\ & \mathfrak{R}[[\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{1,2}\xi_{3,4} \oplus \mathfrak{R}[[\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{1,3}\xi_{3,4} \oplus \\ & \mathfrak{R}[[\alpha_3,\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]]\gamma_{1,2}\gamma_{1,3}\xi_{3,4} \oplus \mathfrak{R}[[\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]](\gamma_{1,2},\alpha_4)^{(1)} \oplus \\ & \mathfrak{R}[[\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]](\gamma_{1,2},\alpha_4)^{(2)} \oplus \mathfrak{R}[[\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]](\gamma_{1,3},\alpha_4)^{(1)} \oplus \\ & \mathfrak{R}[[\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]](\gamma_{1,3},\alpha_4)^{(2)} \oplus \mathfrak{R}[[\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]](\gamma_{1,2}\gamma_{1,3},\alpha_4)^{(1)} \oplus \\ & \mathfrak{R}[[\alpha_4,\xi_{1,4},\xi_{2,4},\xi_{3,4}]](\gamma_{1,2}\gamma_{1,3},\alpha_4)^{(2)} \oplus \mathfrak{R}[[\alpha_2,\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4}]]\gamma_{2,3} \oplus \\ & \mathfrak{R}[[\alpha_2,\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4}]]\gamma_{1,2}\gamma_{2,3} \oplus \mathfrak{R}[[\alpha_2,\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4}]]\gamma_{1,3}\gamma_{2,3} \oplus \\ & \mathfrak{R}[[\alpha_2,\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4}]]\gamma_{1,2}\gamma_{1,3}\gamma_{2,3} \oplus \mathfrak{R}[[\alpha_2,\alpha_3,\alpha_4,\xi_{2,3},\xi_{2,4}]]\xi_{2,3} \oplus \end{aligned}$$

$\mathfrak{R}[[\alpha_2, \alpha_3, \alpha_4, \xi_{2,3}, \xi_{2,4}]]\gamma_{1,2}\xi_{2,3} \oplus \mathfrak{R}[[\alpha_2, \alpha_3, \alpha_4, \xi_{2,3}, \xi_{2,4}]]\gamma_{1,3}\xi_{2,3} \oplus$

$\mathfrak{R}[[\alpha_2, \alpha_3, \alpha_4, \xi_{2,3}, \xi_{2,4}]]\gamma_{1,2}\gamma_{1,3}\xi_{2,3} \oplus \mathfrak{R}[[\alpha_3, \alpha_4, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]]\gamma_{2,3}\gamma_{3,4} \oplus$

$\mathfrak{R}[[\alpha_3, \alpha_4, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]]\gamma_{1,2}\gamma_{2,3}\gamma_{3,4} \oplus \mathfrak{R}[[\alpha_3, \alpha_4, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]]\gamma_{2,3}\xi_{3,4} \oplus$

$\mathfrak{R}[[\alpha_3, \alpha_4, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]]\gamma_{1,2}\gamma_{2,3}\xi_{3,4} \oplus \mathfrak{R}[[\alpha_3, \alpha_4, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]]\gamma_{3,4}\xi_{2,3} \oplus$

$\mathfrak{R}[[\alpha_3, \alpha_4, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]]\gamma_{1,2}\gamma_{3,4}\xi_{2,3} \oplus \mathfrak{R}[[\alpha_3, \alpha_4, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]]\xi_{2,3}\xi_{3,4} \oplus$

$\mathfrak{R}[[\alpha_3, \alpha_4, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]]\gamma_{1,2}\xi_{2,3}\xi_{3,4} \oplus \mathfrak{R}[[\alpha_4, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]](\gamma_{2,3}, \alpha_4)^{(1)} \oplus$

$\mathfrak{R}[[\alpha_4, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]](\gamma_{2,3}, \alpha_4)^{(2)} \oplus \mathfrak{R}[[\alpha_4, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]](\gamma_{1,2}\gamma_{2,3}, \alpha_4)^{(1)} \oplus$

$\mathfrak{R}[[\alpha_4, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]](\gamma_{1,2}\gamma_{2,3}, \alpha_4)^{(2)} \oplus \mathfrak{R}[[\alpha_4, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]](\gamma_{1,2}\xi_{2,3}, \alpha_4)^{(1)} \oplus$

$\mathfrak{R}[[\alpha_4, \xi_{2,3}, \xi_{2,4}, \xi_{3,4}]](\gamma_{1,2}\xi_{2,3}, \alpha_4)^{(2)} \oplus \mathfrak{R}[[\alpha_3, \alpha_4, \xi_{2,3}, \xi_{3,4}]](\gamma_{1,2}, \alpha_3)^{(1)} \oplus$

$\mathfrak{R}[[\alpha_3, \alpha_4, \xi_{2,3}, \xi_{3,4}]](\gamma_{1,2}, \alpha_3)^{(2)} \oplus \mathfrak{R}[[\alpha_3, \alpha_4, \xi_{2,3}, \xi_{3,4}]](\gamma_{1,2}\gamma_{3,4}, \alpha_3)^{(1)} \oplus$

$\mathfrak{R}[[\alpha_3, \alpha_4, \xi_{2,3}, \xi_{3,4}]](\gamma_{1,2}\gamma_{3,4}, \alpha_3)^{(2)} \oplus \mathfrak{R}[[\alpha_4, \xi_{2,3}, \xi_{3,4}]](\gamma_{1,2}, \alpha_3)^{(1)}, \alpha_4)^{(1)} \oplus$

$\mathfrak{R}[[\alpha_4, \xi_{2,3}, \xi_{3,4}]]((\gamma_{1,2}, \alpha_3)^{(1)}, \alpha_4)^{(2)}.$

The new transvectants created are $\gamma_{1,4}, \gamma_{2,4}, \gamma_{3,4}, \xi_{1,4}, \xi_{2,4}, \xi_{3,4}$. We illustrate how to get the Stanley decomposition from the lattice diagram. By adding the new transvectants and α_4 to the lattice diagram for N_{333} we obtained the lattice diagram below:

$$\begin{tikzcd}
\alpha_4 & & & & & & & \\
& \gamma_{3,4} \arrow[dl] & & & & & & \\
\alpha_3 \arrow[uu] & \xi_{3,4} \arrow[l] \arrow[u] & & & & & & \\
& \gamma_{2,4} \arrow[u] \arrow[dl] & \gamma_{2,3} \arrow[dl] & & & & & \\
\alpha_2 \arrow[uu] & \xi_{2,4} \arrow[l] \arrow[u] & \xi_{2,3} \arrow[l] \arrow[u] & & & & & \\
& \gamma_{1,4} \arrow[u] \arrow[dl] & \gamma_{1,3} \arrow[u] \arrow[dl] & \gamma_{1,2} \arrow[dl] & & & & \\
\alpha_1 \arrow[uu] & \xi_{1,4} \arrow[l] \arrow[u] & \xi_{1,3} \arrow[l] \arrow[u] & \xi_{1,2} \arrow[l] \arrow[u] & \beta_4 \arrow[l] & \beta_3 \arrow[l] & \beta_2 \arrow[l] & \beta_1 \arrow[l]
\end{tikzcd}$$

The Stanley decomposition of the ring of invariants, $ker\ \mathcal{X}_{3333}$ is then given by the sum of the terms $\bigoplus_j T_1$ and $\bigoplus_j T_2$ with the following maximal monotone paths:

T_1=$\bigoplus_j \mathbb{R}$[[variables on the j^{th} path]](products of corners on the j^{th} path). Exit at α_k and end at α_4.

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \alpha_1 \to \alpha_2 \to \alpha_3 \to \alpha_4)$ with no corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \alpha_1 \to \alpha_2 \to \alpha_3 \to \alpha_4)$, with γ_{12} as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \alpha_1 \to \alpha_2 \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,3}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \alpha_1 \to \alpha_2 \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,4}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \alpha_1 \to \alpha_2 \to \alpha_3 \to \alpha_4)$ with $\gamma_{1,2}$ and

$\gamma_{1,3}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \alpha_1 \to \alpha_2 \to \alpha_3 \to \alpha_4)$ with $\gamma_{1,2}$ and $\gamma_{1,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \alpha_1 \to \alpha_2 \to \alpha_3 \to \alpha_4)$ with $\gamma_{1,3}$ and $\gamma_{1,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \alpha_1 \to \alpha_2 \to \alpha_3 \to \alpha_4)$ with $\gamma_{1,2}$,$\gamma_{1,3}$ and $\gamma_{1,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \alpha_2 \to \alpha_3 \to \alpha_4)$, with $\gamma_{2,4}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \alpha_2 \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,2}$ and γ_{24} as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \alpha_2 \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,3}$ and $\gamma_{2,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \alpha_2 \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,2}$, $\gamma_{1,3}$ and $\gamma_{2,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \alpha_2 \to \alpha_3 \to \alpha_4)$, with $\xi_{2,4}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \alpha_2 \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,2}$ and $\xi_{2,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \alpha_2 \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,3}$ and $\xi_{2,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \alpha_2 \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,2}$, $\gamma_{1,3}$ and $\xi_{2,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \alpha_2 \to \alpha_3 \to \alpha_4)$, with $\gamma_{2,3}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \alpha_2 \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,2}$ and $\gamma_{2,3}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \alpha_2 \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,3}$ and $\gamma_{2,3}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \alpha_2 \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,2}$, $\gamma_{2,3}$ and $\gamma_{2,3}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \alpha_2 \to \alpha_3 \to \alpha_4)$, with $\xi_{2,3}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \alpha_2 \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,2}$ and $\xi_{2,3}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \alpha_2 \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,3}$ and $\xi_{2,3}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \alpha_2 \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,2}$, $\gamma_{2,3}$ and $\xi_{2,3}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $\gamma_{3,4}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,2}$ and $\gamma_{3,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,3}$ and $\gamma_{3,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,2}$, $\gamma_{1,3}$ and $\gamma_{3,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $\xi_{3,4}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,2}$ and $\xi_{3,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,3}$ and $\xi_{3,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,2}$, $\gamma_{2,3}$ and $\gamma_{3,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $\gamma_{2,3}$ and $\gamma_{3,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,2}$, $\gamma_{2,3}$ and $\gamma_{3,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $\gamma_{2,3}$ and $\xi_{3,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,2}$, $\gamma_{2,3}$ and $\xi_{3,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $\xi_{2,3}$ and $\gamma_{3,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,2}$, $\xi_{2,3}$ and $\gamma_{3,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $\xi_{2,3}$ and $\xi_{3,4}$ as corners

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $\gamma_{1,2}$, $\xi_{2,3}$ and $\xi_{3,4}$ as corners.

T_2=$\bigoplus_j \mathbb{R}$[[invariants on the j^{th} path]](products of corners on the j^{th} path, $\alpha_m)^{(i)}$. Exit at α_m through $\gamma_{m-1,m}$ and end at α_4 where $i = 1, 2$ and $m = 3, 4$.

m=3

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $(\gamma_{1,2}, \alpha_3)^{(1)}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $(\gamma_{1,2}, \alpha_3)^{(2)}$ as a corner

m=4

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_4)$, with $(\gamma_{1,2}, \alpha_4)^{(1)}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_4)$, with $(\gamma_{1,2}, \alpha_4)^{(2)}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_4)$, with $(\gamma_{1,3}, \alpha_4)^{(1)}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_4)$, with $(\gamma_{1,3}, \alpha_4)^{(2)}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_4)$, with $(\gamma_{1,2}\gamma_{1,3}, \alpha_4)^{(1)}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{1,4} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_4)$, with $(\gamma_{1,2}\gamma_{1,3}, \alpha_4)^{(2)}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_4)$, with $(\gamma_{2,3}, \alpha_4)^{(1)}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_4)$, with $(\gamma_{2,3}, \alpha_4)^{(2)}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_4)$, with $(\gamma_{1,2}\gamma_{2,3}, \alpha_4)^{(1)}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_4)$, with $(\gamma_{1,2}\gamma_{2,3}, \alpha_4)^{(2)}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_4)$, with $(\gamma_{1,2}\xi_{2,3}, \alpha_4)^{(1)}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{2,4} \to \xi_{3,4} \to \alpha_4)$, with $(\gamma_{1,2}\xi_{2,3}, \alpha_4)^{(2)}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $((\gamma_{1,2}\gamma_{3,4}, \alpha_3)^{(1)})^{(1)}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{3,4} \to \alpha_3 \to \alpha_4)$, with $((\gamma_{1,2}\gamma_{3,4}, \alpha_3)^{(2)})^{(1)}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{3,4} \to \alpha_4)$, with $((\gamma_{1,2}, \alpha_3)^{(1)}, \alpha_4)^{(1)}$ as a corner

$(\beta_1 \to \beta_2 \to \beta_3 \to \beta_4 \to \xi_{1,2} \to \xi_{1,3} \to \xi_{2,3} \to \xi_{3,4} \to \alpha_4)$, with $((\gamma_{1,2}, \alpha_3)^{(1)}, \alpha_4)^{(2)}$ as a corner

The Stanley decomposition, $\mathfrak{X}_{333}$ of the system with linear part N_{3333} is given by $\bigoplus_j T_1 + \bigoplus_j T_2$ which is the same as in 3.

To verify that all invariants have been found consider the table function of $ker\ \mathfrak{X}_{3333}$ shown below:

$$T_{12} = \frac{w + 3d^2w^3 + 3d^4w^5 + d^6w^7}{(1-dw^2)^4(1-d^2)^7} + \frac{2d^2w^3 + 4d^4w^5 + 2d^6w^7 + 2d^2w + 4d^4w^3 + 2d^6w^5}{(1-dw^2)^3(1-d^2)^8}$$

$$+ \frac{d^2w^3 + 3d^4w^5 + 2d^6w^7 + d^2w + 4d^4w^3 + 3d^6w^5 + d^4w + d^6w^3}{(1-dw^2)^2(1-d^2)^9}$$

$$+ \frac{3d^3w^3 + 3d^3w + 2d^5w^5 + 3d^5w^3 + d^5w}{(1-dw^2)(1-d^2)^9} + \frac{d^3w^3 + d^3w + d^5w^5 + d^5w^3}{(1-dw^2)^2(1-d^2)^8}$$

$$+ \frac{d^4w^3 + d^4w}{(1-dw^2)(1-d^2)^8}.$$

Multiplying the table function by w, differentiating with respect to w and putting $w = 1$, it can be shown that

$$\frac{\partial}{\partial w} wT_{12} \mid_{w=1} = \frac{1}{(1-d)^{12}}.$$

3.4 The Stanley Decomposition for Coupled $N_{33,...,3}$ System

In this section we generalize to any coupled $N_{33,...,3}$ system. The following are the general observations made from the above examples:

a) for each product there are three kinds of terms to be considered.

b) for every additional n, there are new transvectants $(\alpha_k, \alpha_l)^{(i)}$ where $i = 1, 2$ and $1 \leq k < l \leq n$.

c) the lattice diagram of $N_{(3)^n}$ is obtained by adding these new transvectants together with α_n to the lattice diagram for $N_{(3)^{n-1}}$.

d) the invariants β_k, where $k = 1, ..., n$ appear inside the square brackets.

e) the first term of the Stanley decomposition has no products of transvectants outside the square brackets.

f) the transvectants $\gamma_{k,l}$ never appears inside the square brackets.

g) the transvectants $\xi_{k,l}$ appears inside as well as outside the square brackets.

h) the Stanley decomposition for a system with linear part $N_{(3)^n}$ is given by the sum of terms of T_1 and T_2 where

$T_1 = \bigoplus_j \mathbb{R}$[[invariants on the jth path]](products of corners on the jth path). Exit at α_k and ending at α_n where $k = 1, 2, ..., n-1$.

$T_2 = \bigoplus \mathbb{R}$[[invariants on the jth path]]$(\textit{product of corners on the jth path}, \alpha_m)^{(i)}$. Exit at α_m through $\gamma_{m-1,m}$ and ending at α_n, where $i = 1, 2$ and $m = 3, 4, ..., n$.

We now state the main theorem in this thesis.

Theorem 3.1 *The Stanley decomposition of the ring of invariants, ker $\mathfrak{X}_{(3)^n}$ is given by the sum of terms of $\bigoplus_j T_1$ and $\bigoplus_j T_2$ from the lattice diagram:*

$$
\begin{tikzcd}
\alpha_n & & & & & & & \\
\alpha_{n-1} \arrow[u] & \gamma_{n-1,n} \arrow[dl] & & & & & & \\
\alpha_{n-1} \arrow[u] & \xi_{n-1,n} \arrow[l] \arrow[u] & & & & & & \\
\vdots \arrow[u] & \vdots \arrow[u] & & & & & & \\
 & \gamma_{i,n} \arrow[u] \arrow[dl] & \cdots \arrow[dl] & \gamma_{i,j} \arrow[dl] & & & & \\
\alpha_i \arrow[uu] & \xi_{i,n} \arrow[l] \arrow[u] & \cdots \arrow[l] \arrow[u] & \xi_{i,j} \arrow[l] \arrow[u] & & & & \\
\vdots \arrow[u] & \vdots \arrow[l] \arrow[u] & \vdots \arrow[l] \arrow[u] & \vdots \arrow[l] \arrow[u] & & & & \\
 & \gamma_{1,n} \arrow[u] \arrow[dl] & \cdots \arrow[u] \arrow[dl] & \cdots \arrow[u] \arrow[dl] & \gamma_{1,2} \arrow[dl] & & & \\
\alpha_1 \arrow[uu] & \xi_{1,n} \arrow[l] \arrow[u] & \cdots \arrow[l] \arrow[u] & \cdots \arrow[l] \arrow[u] & \xi_{1,2} \arrow[l] \arrow[u] & \beta_n \arrow[l] & \cdots \arrow[l] & \beta_1 \arrow[l]
\end{tikzcd}
$$

where j will range over all possible number of paths for ker $\mathcal{X}_{(3)^n}$ in the lattice diagram.

$T_1 = \bigoplus_j \mathbb{R}[[$*invariants on the jth path*$]]($*products of corners on the jth path*$)$, *exiting at α_k and ending at α_n* where $k = 1, 2, .., n-1$ and

$T_2 = \bigoplus_j \mathbb{R}[[$*invariants on the jth path*$]]($ *product of corners on the jth path, α_m*$)^{(i)}$, *exiting at α_m through $\gamma_{m-1,m}$ and ending at α_n, where $i = 1, 2$ and $m = 3, 4, ..., n$.*

Proof. By Theorem 2.6,

$$ker\ \mathcal{X}_{(3)^n} = ker\ \mathcal{X}_{(3)^{n-1}} \boxtimes ker\ \mathcal{X}_{3.}$$

We prove by induction on n. It is true for $n = 3$ and $n = 4$, by the above examples. We suppose that it is true for $k = n - 1$ and show that it hold for

$k = n$. We have that:

$$ker\ \mathfrak{X}_{(3)^n} = \big(\bigoplus_j T_1 \oplus \bigoplus_j T_2\big) \boxtimes \mathbb{R}[[\alpha_n, \beta_n]]$$

where j will range over all possible paths of the lattice diagram of $ker\ \mathfrak{X}_{(3)^{n-1}}$. We distribute the box product over the direct sums of $ker\ \mathfrak{X}_{(3)^{n-1}}$. Suppressing all transvectants of the form $\beta_1, ...\beta_n$ and $\xi_{k,l}$ for $1 \leq k < l \leq n-1$ since they are of weight zero to be inserted in every square brackets computed depending on the terms they are found and noticing that the corners multiply each square bracket. we have: $\mathbb{R}[[\alpha_i, ..., \alpha_{n-1}]]\varphi \boxtimes \mathbb{R}[[\alpha_n]]$.

Expanding the Stanley decompositions:

$\mathbb{R}[[\alpha_k, ..., \alpha_{n-1}]]\varphi \boxtimes \mathbb{R}[[\alpha_n]] = \big(\mathbb{R}[[\alpha_{k+1}, ..., \alpha_{n-1}]]\varphi \oplus \mathbb{R}[[\alpha_k, \alpha_{k+1}, ..., \alpha_{n-1}]]\alpha_k\varphi\big) \boxtimes$
$\big(\mathbb{R} \oplus \mathbb{R}[[\alpha_n]]\big)$

Distributing the box product gives three kinds of terms.

1. Two terms that are computed to final form: $\mathbb{R}[[\alpha_{k+1}, ..., \alpha_{n-1}]]\varphi \oplus \mathbb{R}[[\alpha_k, \alpha_{k+1}, ..., \alpha_{n-1}]]\alpha_k\varphi$

2. One box product: $\mathbb{R}[[\alpha_{k+1}, ..., \alpha_{n-1}]]\varphi \boxtimes \mathbb{R}[[\alpha_n]]\alpha_n$, that must be computed by further expansions according to theorem 2.6. Expanding, we have that:

$\mathbb{R}[[\alpha_{k+1}, ..., \alpha_{n-1}]]\varphi \boxtimes \mathbb{R}[[\alpha_n]]\alpha_n = (\mathbb{R}[[\alpha_{k+2}, ..., \alpha_{n-1}]]\varphi \oplus \mathbb{R}[[\alpha_{k+1}, ..., \alpha_{n-1}]]\alpha_{k+1}, \varphi) \boxtimes$
$(\mathbb{R}\alpha_n \oplus \mathbb{R}[[\alpha_n]]\alpha_n^2)$

Distributing the box product,there are four box products namely:

a) $\mathbb{R}[[\alpha_{k+1}, ..., \alpha_{n-1}]]\varphi \boxtimes \mathbb{R}\alpha_n$

$= \mathbb{R}[[\alpha_{k+1}, ..., \alpha_{n-1}]]\varphi\alpha_n \oplus \mathbb{R}[[\alpha_{k+1}, ..., \alpha_{n-1}]](\varphi, \alpha_n)^{(1)} \oplus \mathbb{R}[[\alpha_{k+1}, ..., \alpha_{n-1}]](\varphi, \alpha_n)^{(2)}$

b)$(\mathbb{R}[[\alpha_{k+2}, ..., \alpha_{n-1}]]\varphi \boxtimes \mathbb{R}[[\alpha_n]]\alpha_n^2$

$= (\mathbb{R}[[\alpha_{k+3}, ..., \alpha_{n-1}]]\varphi \oplus \mathbb{R}[[\alpha_{k+2}, ..., \alpha_{n-1}]]\alpha_{k+2}\varphi) \boxtimes (\mathbb{R}\alpha_n^2 \oplus \mathbb{R}[[\alpha_n]]\alpha_n^3)$.

Distributing the box product, all term are computed explicitly except the last product that recycles to: $\mathbb{R}[[\alpha_{k+2}, ..., \alpha_{n-1}]]\varphi \boxtimes \mathbb{R}[[\alpha_n]]\alpha_n^2$. Indeed:

$\mathbb{R}[[\alpha_{k+2},...,\alpha_{n-1}]]\alpha_{k+2}\varphi \boxtimes \mathbb{R}[[\alpha_n]]\alpha_n^3 = \mathbb{R}[[\alpha_{k+2},...,\alpha_{n-1}]]\alpha_{k+2}\varphi\alpha_n^3$

$\oplus\mathbb{R}[[\alpha_{k+2},...,\alpha_{n-1}]](\alpha_{k+2}\varphi,\alpha_n^3)^{(1)} \oplus (\mathbb{R}[[\alpha_{k+2},...,\alpha_{n-1}]]\varphi \boxtimes \mathbb{R}[[\alpha_n]]\alpha_n^2)(\alpha_{k+2},\alpha_n)^{(2)}$

We delete the last term and insert $(\alpha_{k+2},\alpha_n)^{(2)} = \xi_{k+2,n}$ in all square brackets in (b).

c) $\mathbb{R}[[\alpha_{k+1},...,\alpha_{n-1}]]\alpha_{k+1}\varphi \boxtimes \mathbb{R}\alpha_n = \mathbb{R}[[\alpha_{k+1},...,\alpha_{n-1}]]\alpha_{k+1}\varphi\alpha_n\oplus$

$\mathbb{R}[[\alpha_{k+1},...,\alpha_{n-1}]](\alpha_{k+1}\varphi,\alpha_n)^{(1)} \oplus \mathbb{R}[[\alpha_{k+1},...,\alpha_{n-1}]](\alpha_{k+1}\varphi,\alpha_n)^{(2)}$

d) One box product: $\mathbb{R}[[\alpha_{k+1},...,\alpha_{n-1}]]\alpha_{k+1}\varphi \boxtimes \mathbb{R}[[\alpha_n]]\alpha_n^2$ which recycles to the same box product we are trying to compute.

$\mathbb{R}[[\alpha_{k+1},...,\alpha_{n-1}]]\alpha_{k+1}\varphi \boxtimes \mathbb{R}[[\alpha_n]]\alpha_n^2 = \mathbb{R}[[\alpha_{k+1},...,\alpha_{n-1},\alpha_n]]\alpha_{k+1}\varphi\alpha_n^2\oplus$

$\mathbb{R}[[\alpha_{k+1},...,\alpha_{n-1},\alpha_n]](\alpha_{k+1}\varphi,\alpha_n^2)^{(1)}\oplus[\mathbb{R}[[\alpha_{k+1},...,\alpha_{n-1}]]\varphi\boxtimes\mathbb{R}[[\alpha_n]]\alpha_n](\alpha_{k+1},\alpha_n)^{(2)}$

By recycling rule from theorem 2.6 we delete the last term and insert $(\alpha_{k+1},\alpha_n)^{(2)} = \xi_{k+1,n}$ in all square brackets resulting from this calculation 2.

3. One box product that recycles: $\mathbb{R}[[\alpha_k,\alpha_{k+1},...,\alpha_{n-1}]]\alpha_k\varphi \boxtimes \mathbb{R}[[\alpha_n]]\alpha_n$

$= \mathbb{R}[[\alpha_k,\alpha_{k+1},...,\alpha_{n-1},\alpha_n]]\alpha_k,\alpha_n,\varphi \oplus \mathbb{R}[[\alpha_k,\alpha_{k+1},...,\alpha_{n-1},\alpha_n]](\alpha_k,\alpha_n,\varphi)^{(1)}\oplus$

$[\mathbb{R}[[\alpha_k,\alpha_{k+1},...,\alpha_{n-1}]]\varphi) \boxtimes \mathbb{R}[[\alpha_n]]](\alpha_k,\alpha_n)^{(2)}$

We delete the last term and insert $(\alpha_k,\alpha_n)^{(2)} = \xi_{k,n}$ in all square brackets together with other suppressed tranvectants. Recombining terms whenever possible, we finally find the Stanley decomposition, $ker\ X_{(3)^n}$.

Equivalently, finding all the additional transvectants for $ker\ \mathfrak{X}_{(3)^n}$ of the form $\gamma_{k,l}$ and $\xi_{k,n}$ where $1 \leq k < n$ and adding these together with α_n to the lattice diagram of $ker\ \mathfrak{X}_{(3)^{n-1}}$, we obtain the lattice diagram for $ker\ \mathfrak{X}_{(3)^n}$ as shown above. We also obtain that the sum of the jth paths of the form T_1 and T_2 gives the Stanley decomposition, $ker\ \mathfrak{X}_{3^n}$ as required.

CHAPTER FOUR

4.0 NORMAL FORMS FOR COUPLED $N_{33,\ldots,3}$ SYSTEMS

In this chapter we shall compute the Stanley decomposition of the module of equivariants (Normal form space), *ker* X, from Stanley decomposition of the ring of invariants, *ker* $\mathcal{X}$. We shall compute the normal forms of systems with linear part having two and three blocks, that is N_{33} and N_{333} as examples before generalizing.

4.1 Normal Form for System with Linear Part N_{33}

From chapter three, we know that the Stanley decomposition of the ring of invariants with linear part N_{33} is given by: *ker* $\mathcal{X}_{33} = \mathbb{R}[[\alpha_1, \alpha_2, \beta_1, \beta_2, \xi_{1,2}]] \oplus \mathbb{R}[[\alpha_1, \alpha_2, \beta_1, \beta_2, \xi_{1,2}]]\gamma_{1,2}$. Since β_1, β_2 and $\xi_{1,2}$ have weight zero, it is convenient to suppress them since we do not expand along terms of weight zero by setting $\mathcal{R} = \mathbb{R}[[\beta_1, \beta_2, \xi_{1,2}]]$ and write

$$ker\ \mathcal{X}_{33} = \mathcal{R}[[\alpha_1, \alpha_2]] \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\gamma_{1,2} \qquad (4.1.1)$$

Expanding the Stanley decomposition:

$$\begin{aligned} ker\ \mathcal{X}_{33} &= \mathcal{R}[[\alpha_2]] \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\alpha_1 \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\gamma_{1,2} \\ &= \mathcal{R} \oplus \mathcal{R}[[\alpha_2]]\alpha_2 \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\alpha_1 \oplus \mathbb{R}[[\alpha_1, \alpha_2]]\gamma_{1,2} \end{aligned}$$

Since we have two blocks the basis elements are e_3 and e_6. We need to compute the box product of the ring *ker* $\mathcal{X}_{33}$ with $\mathbb{R}e_3 \oplus \mathbb{R}e_6$.

Therefore *ker* X_{33}=(*ker* $\mathcal{X}_{33}$) $\boxtimes$ $(\mathbb{R}e_3 \oplus \mathbb{R}e_6)$. Distributing the box product, there are two cases to consider.

Case 1: $\mathcal{R}[[\alpha_1, \alpha_2]] \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\gamma_{1,2} \boxtimes \mathbb{R}e_3$

a) $\mathcal{R}[[\alpha_1, \alpha_2]] \boxtimes \mathbb{R}e_3 = [[\mathcal{R} \oplus \mathcal{R}[[\alpha_2]]\alpha_2 \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\alpha_1]] \boxtimes \mathbb{R}e_3$

$= \mathbb{R}e_3 \oplus \mathcal{R}[[\alpha_2]]\alpha_2 e_3 \oplus \mathcal{R}[[\alpha_2]](\alpha_2, e_3)^{(1)} \oplus \mathcal{R}[[\alpha_2]](\alpha_2, e_3)^{(2)} \oplus$

$\mathcal{R}[[\alpha_1, \alpha_2]]\alpha_1 e_3 \oplus \mathcal{R}[[\alpha_1, \alpha_2]](\alpha_1, e_3)^{(1)} \oplus \mathcal{R}[[\alpha_1, \alpha_2]](\alpha_1, e_3)^{(2)}$

b)$\mathcal{R}[[\alpha_1, \alpha_2]]\gamma_{1,2} \boxtimes \mathbb{R}e_3 = \mathcal{R}[[\alpha_1, \alpha_2]]\gamma_{1,2}e_3 \oplus \mathcal{R}[[\alpha_1, \alpha_2]](\gamma_{1,2}, e_3)^{(1)} \oplus \mathcal{R}[[\alpha_1, \alpha_2]](\gamma_{1,2}, e_3)^{(2)}$

Recombining terms in a and b gives

$[\mathcal{R} \oplus \mathcal{R}[[\alpha_2]]\alpha_2 \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\alpha_1 \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\gamma_{1,2}] \boxtimes \mathbb{R}e_3 =$

$\mathcal{R}[[\alpha_1, \alpha_2]]e_3 \oplus \mathcal{R}[[\alpha_2]](\alpha_2, e_3)^{(1)} \oplus \mathcal{R}[[\alpha_2]](\alpha_2, e_3)^{(2)} \oplus \mathcal{R}[[\alpha_1, \alpha_2]](\alpha_1, e_3)^{(1)} \oplus$

$\mathcal{R}[[\alpha_1, \alpha_2]](\alpha_1, e_3)^{(2)} \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\gamma_{1,2}e_3 \oplus \mathcal{R}[[\alpha_1, \alpha_2]](\gamma_{1,2}, e_3)^{(1)} \oplus \mathcal{R}[[\alpha_1, \alpha_2]](\gamma_{1,2}, e_3)^{(2)}$.

Case 2: Similarly, we have

$[\mathcal{R} \oplus \mathcal{R}[[\alpha_2]]\alpha_2 \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\alpha_1 \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\gamma_{1,2}] \boxtimes \mathbb{R}e_6 =$

$\mathcal{R}[[\alpha_1, \alpha_2]]e_6 \oplus \mathcal{R}[[\alpha_2]](\alpha_2, e_6)^{(1)} \oplus \mathcal{R}[[\alpha_2]](\alpha_2, e_6)^{(2)} \oplus \mathcal{R}[[\alpha_1, \alpha_2]](\alpha_1, e_6)^{(1)} \oplus$

$\mathcal{R}[[\alpha_1, \alpha_2]](\alpha_1, e_6)^{(2)} \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\gamma_{1,2}e_6 \oplus \mathcal{R}[[\alpha_1, \alpha_2]](\gamma_{1,2}, e_6)^{(1)} \oplus \mathcal{R}[[\alpha_1, \alpha_2]](\gamma_{1,2}, e_6)^{(2)}$

Adding the terms in cases 1 and 2 we obtain:

$$\begin{aligned}
ker\ \mathsf{X}_{33} \;=\; & \mathcal{R}[[\alpha_1, \alpha_2]]e_3 \oplus \mathcal{R}[[\alpha_1, \alpha_2]](\alpha_1, e_3)^{(1)} \oplus \mathcal{R}[[\alpha_1, \alpha_2]](\alpha_1, e_3)^{(2)} \oplus \\
& \mathcal{R}[[\alpha_2]](\alpha_2, e_3)^{(1)} \oplus \mathcal{R}[[\alpha_2]](\alpha_2, e_3)^{(2)} \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\gamma_{1,2}e_3 \oplus \\
& \mathcal{R}[[\alpha_1, \alpha_2]](\gamma_{1,2}, e_3)^{(1)} \oplus \mathcal{R}[[\alpha_1, \alpha_2]](\gamma_{1,2}, e_3)^{(2)} \oplus \mathcal{R}[[\alpha_1, \alpha_2]]e_6 \oplus \\
& \mathcal{R}[[\alpha_1, \alpha_2]](\alpha_1, e_6)^{(1)} \oplus \mathcal{R}[[\alpha_1, \alpha_2]](\alpha_1, e_6)^{(2)} \oplus \mathcal{R}[[\alpha_2]](\alpha_2, e_6)^{(1)} \oplus \\
& \mathcal{R}[[\alpha_2]](\alpha_2, e_6)^{(2)} \oplus \mathcal{R}[[\alpha_1, \alpha_2]]\gamma_{1,2}e_6 \oplus \mathcal{R}[[\alpha_1, \alpha_2]](\gamma_{1,2}, e_6)^{(1)} \oplus \\
& \mathcal{R}[[\alpha_1, \alpha_2]](\gamma_{1,2}, e_6)^{(2)}.
\end{aligned}$$

To complete the calculation, it is necessary to compute the transvectants that appear from the cases of the normal form.. These are $(f, e_3)^{(i)}$ and $(f, e_6)^{(i)}$ for $i = 0, 1, 2$.

From the definition of transvectant,

$$(f,e_3)^i = (-1)^i \sum_{j=0}^{i} W^{i,j}_{f,e_3} (\mathcal{Y}^{\mathbf{j}}\mathbf{f})((Y^{*})^{\mathbf{i-j}}e_3).$$

$W^{i,j}_{f,e_3} = \binom{i}{j}\frac{(w_f-j)!}{(w_f-i)!}.\frac{(w_{e_3}-i+j)!}{(w_{e_3}-i)!}$

We have:

$w_f = 2$ and $w_{e_3} = w_{e_6} = 2$, therefore

$$(f,e_3)^{(0)} = \begin{bmatrix} 0 \\ 0 \\ f \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

To compute $(f,e_3)^{(1)}$,

$j=0, i=1$, $W^{1,0}_{f,e_3} = \binom{1}{0}\frac{(2-0)!}{(2-1)!}.\frac{(2-1+0)!}{(2-1)!} = 2$

$j=1, i=1$, $W^{1,1}_{f,e_3} = \binom{1}{1}\frac{(2-1)!}{(2-1)!}.\frac{(2-1+1)!}{(2-1)!} = 2$

Thus,

$$(f,e_3)^{(1)} = -2f(Y^{*})e_3 - 2\mathcal{Y}fe_3 \tag{4.1.2}$$

$$(f, e_3)^{(1)} = -2f \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 2\mathcal{Y}f \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.1.3}$$

$$= -2 \begin{bmatrix} 0 \\ 2f \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 0 \\ \mathcal{Y}f \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.1.4}$$

$$(f, e_3)^{(1)} = -2 \begin{bmatrix} 0 \\ 2f \\ \mathcal{Y}f \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.1.5}$$

It can be shown that $w_f f = \mathcal{X}\mathcal{Y}f$. Therefore,

$$(f, e_3)^{(1)} = -2 \begin{bmatrix} 0 \\ \mathcal{X}\mathcal{Y}f \\ \mathcal{Y}f \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.1.6}$$

To compute $(f, e_3)^{(2)}$,

$j = 0, i = 2$, $W^{2,0}_{f,e_3} = \binom{2}{0}\frac{(2-0)!}{(2-2)!}.\frac{(2-2+0)!}{(2-2)!} = 2$

$j = 1, i = 2$, $W^{2,1}_{f,e_3} = \binom{2}{1}\frac{(2-1)!}{(2-2)!}.\frac{(2-2+1)!}{(2-2)!} = 2$

$j = 2, i = 2$, $W^{2,2}_{f,e_3} = \binom{2}{2}\frac{(2-2)!}{(2-2)!}.\frac{(2-2+2)!}{(2-2)!} = 2$

$$\begin{aligned} (f, e_3)^{(2)} &= 2f(Y^*)^2 e_3 + 2\mathcal{Y}fY^* e_3 + 2\mathcal{Y}^2 f e_3 \\ &= 2 \begin{bmatrix} \mathcal{X}^2\mathcal{Y}^2 f \\ \mathcal{X}\mathcal{Y}^2 f \\ \mathcal{Y}^2 f \\ 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned} \tag{4.1.7}$$

We ignore the nonzero constants -2 and 2 because we are concerned with computing basis elements. For the basis e_6 we have:

$$(f,e_6)^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ f \end{bmatrix}, \qquad (f,e_6)^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mathcal{X}\mathcal{Y}f \\ \mathcal{Y}f \end{bmatrix}, \qquad (f,e_6)^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathcal{X}^2\mathcal{Y}^2 f \\ \mathcal{X}\mathcal{Y}^2 f \\ \mathcal{Y}^2 f \end{bmatrix}.$$

Therefore the normal form for the system with linear part N_{33} is :

$$\begin{aligned} ker\ \mathsf{X}_{33} \;=\; & \mathcal{R}[[\alpha_1,\alpha_2,\beta_1,\beta_2,\xi_{1,2}]] \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \oplus \mathcal{R}[[\alpha_1,\alpha_2,\beta_1,\beta_2,\xi_{1,2}]] \begin{bmatrix} 0 \\ \mathcal{X}\mathcal{Y}\alpha_1 \\ \mathcal{Y}\alpha_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \oplus \\ & \mathcal{R}[[\alpha_1,\alpha_2,\beta_1,\beta_2,\xi_{1,2}]] \begin{bmatrix} \mathcal{X}^2\mathcal{Y}^2\alpha_1 \\ \mathcal{X}\mathcal{Y}^2\alpha_1 \\ \mathcal{Y}^2\alpha_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \oplus \mathcal{R}[[\alpha_2,\beta_1,\beta_2,\xi_{1,2}]] \begin{bmatrix} 0 \\ \mathcal{X}\mathcal{Y}\alpha_2 \\ \mathcal{Y}\alpha_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \oplus \\ & \mathcal{R}[[\alpha_2,\beta_1,\beta_2,\xi_{1,2}]] \begin{bmatrix} \mathcal{X}^2\mathcal{Y}^2\alpha_2 \\ \mathcal{X}\mathcal{Y}^2\alpha_2 \\ \mathcal{Y}^2\alpha_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \oplus \mathcal{R}[[\alpha_1,\alpha_2,\beta_1,\beta_2,\xi_{1,2}]] \begin{bmatrix} 0 \\ 0 \\ \gamma_{12} \\ 0 \\ 0 \\ 0 \end{bmatrix} \oplus \end{aligned}$$

$$\mathcal{R}[[\alpha_1,\alpha_2,\beta_1,\beta_2,\xi_{1,2}]]\begin{bmatrix}0\\ \mathcal{X}\mathcal{Y}\gamma_{1,2}\\ \mathcal{Y}^2\gamma_{1,2}\\ 0\\ 0\\ 0\end{bmatrix}\oplus\mathcal{R}[[\alpha_1,\alpha_2,\beta_1,\beta_2,\xi_{1,2}]]\begin{bmatrix}\mathcal{X}^2\mathcal{Y}^2\gamma_{1,2}\\ \mathcal{X}\mathcal{Y}^2\gamma_{1,2}\\ \mathcal{Y}^2\gamma_{1,2}\\ 0\\ 0\\ 0\end{bmatrix}\oplus$$

$$\mathcal{R}[[\alpha_1,\alpha_2,\beta_1,\beta_2,\xi_{1,2}]]\begin{bmatrix}0\\ 0\\ 0\\ 0\\ 0\\ 1\end{bmatrix}\oplus\mathcal{R}[[\alpha_1,\alpha_2,\beta_1,\beta_2,\xi_{1,2}]]\begin{bmatrix}0\\ 0\\ 0\\ 0\\ \mathcal{X}\mathcal{Y}\alpha_1\\ \mathcal{Y}\alpha_1\end{bmatrix}\oplus$$

$$\mathcal{R}[[\alpha_1,\alpha_2,\beta_1,\beta_2,\xi_{1,2}]]\begin{bmatrix}0\\ 0\\ 0\\ \mathcal{X}^2\mathcal{Y}^2\alpha_1\\ \mathcal{X}\mathcal{Y}^2\alpha_1\\ \mathcal{Y}^2\alpha_1\end{bmatrix}\oplus\mathcal{R}[[\alpha_2,\beta_1,\beta_2,\xi_{1,2}]]\begin{bmatrix}0\\ 0\\ 0\\ 0\\ \mathcal{X}\mathcal{Y}\alpha_2\\ \mathcal{Y}\alpha_2\end{bmatrix}\oplus$$

$$\mathcal{R}[[\alpha_2,\beta_1,\beta_2,\xi_{1,2}]]\begin{bmatrix}0\\ 0\\ 0\\ \mathcal{X}^2\mathcal{Y}^2\alpha_2\\ \mathcal{X}\mathcal{Y}^2\alpha_2\\ \mathcal{Y}^2\alpha_2\end{bmatrix}\oplus\mathcal{R}[[\alpha_1,\alpha_2,\beta_1,\beta_2,\xi_{1,2}]]\begin{bmatrix}0\\ 0\\ 0\\ 0\\ 0\\ \gamma_{1,2}\end{bmatrix}\oplus$$

$$\mathcal{R}[[\alpha_1,\alpha_2,\beta_1,\beta_2,\xi_{1,2}]]\begin{bmatrix}0\\0\\0\\0\\\mathcal{X}\mathcal{Y}\gamma_{1,2}\\\mathcal{Y}^2\gamma_{1,2}\end{bmatrix}\oplus\mathcal{R}[[\alpha_1,\alpha_2,\beta_1,\beta_2,\xi_{1,2}]]\begin{bmatrix}0\\0\\0\\\mathcal{X}^2\mathcal{Y}^2\gamma_{1,2}\\\mathcal{X}\mathcal{Y}^2\gamma_{1,2}\\\mathcal{Y}^2\gamma_{1,2}\end{bmatrix}\oplus$$

This results can then summarized as

$$ker\ \mathsf{X}_{(3)^2}=\bigoplus_{r=1}^{r=2}\left[SD\big(ker\ \mathcal{X}_{(3)^2}\big)\left(\bigoplus_{i=0}^{2}(f,e_{3r})^{(i)}\right)\right].$$

The vector fields $(f,e_{3r})^{(i)}$ are the basis of the normal form of $ker\ \mathsf{X}_{(3)^2}$ and f are the standard monomials of the ring of invariants, $ker\ \mathcal{X}_{(3)^2}$.

4.2 Normal Form for System with Linear Part N_{333}

From section 3.3, the Stanley decomposition of ring of invariants of a system with linear part N_{333} is given by:

$ker\ \mathcal{X}_{333}=$

$\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3}]]\oplus\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3}]]\gamma_{1,2}\oplus$

$\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3}]]\gamma_{1,3}\oplus\mathbb{R}[[\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3}]]\gamma_{1,2}\gamma_{1,3}$

$\oplus\mathbb{R}[[\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3}]]\gamma_{2,3}\oplus\mathbb{R}[[\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3}]]\xi_{2,3}\oplus$

$\mathbb{R}[[\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3}]]\gamma_{1,2}\gamma_{2,3}\oplus\mathbb{R}[[\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3}]]$

$\gamma_{1,2}\xi_{2,3}\oplus\mathbb{R}[[\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3}]](\gamma_{1,2},\alpha_3)^{(1)}\oplus$

$\mathbb{R}[[\alpha_3,\beta_1,\beta_2,\beta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3}]](\gamma_{1,2},\alpha_3)^{(2)}.$

We have three blocks and therefore the basis elements for N_{333} are e_3, e_6 and e_9. We need to compute the box product of the invariants ring, $ker\ \mathcal{X}_{333}$ with $\mathbb{R}e_3 \oplus \mathbb{R}e_6 \oplus \mathbb{R}e_9$. Thus $ker\ \mathsf{X}_{333} = ker\mathcal{X}_{333} \boxtimes [\mathbb{R}e_3 \oplus \mathbb{R}e_6 \oplus \mathbb{R}e_9]$. Let $\mathcal{R} = \mathbb{R}[[\beta_1, \beta_2, \beta_3, \xi_{1,2}, \xi_{1,3}]]$, then

$ker\ \mathsf{X}_{333} =$

$$\begin{aligned}
&[\mathcal{R}[[\alpha_1, \alpha_2, \alpha_3]] \oplus \mathcal{R}[[\alpha_1, \alpha_2, \alpha_3]]\gamma_{1,2} \oplus \mathcal{R}[[\alpha_1, \alpha_2, \alpha_3]]\gamma_{1,3} \oplus \\
&\mathcal{R}[[\alpha_1, \alpha_2, \alpha_3]]\gamma_{1,2}\gamma_{1,3} \oplus \mathcal{R}[[\alpha_2, \alpha_3, \xi_{2,3}]]\gamma_{2,3} \oplus \mathcal{R}[[\alpha_2, \alpha_3, \xi_{2,3}]]\xi_{2,3} \oplus \\
&\mathcal{R}[[\alpha_2, \alpha_3, \xi_{2,3}]]\gamma_{1,2}\gamma_{2,3} \oplus \mathcal{R}[[\alpha_2, \alpha_3, \xi_{2,3}]]\gamma_{1,2}\xi_{2,3} \oplus \mathcal{R}[[\alpha_3, \xi_{2,3}]](\alpha_3, \gamma_{1,2})^{(1)} \\
&\oplus \mathcal{R}[[\alpha_3, \xi_{2,3}]](\alpha_3, \gamma_{1,2})^{(2)}] \boxtimes [\mathbb{R}e_3 \oplus \mathbb{R}e_6 \oplus \mathbb{R}e_9].
\end{aligned}$$

There are three cases to consider. Computing and simplifying the cases we obtain the normal form of N_{333} as

$$\begin{aligned}
ker\ \mathsf{X}_{333} \quad = \quad & \mathcal{R}[[\alpha_1, \alpha_2, \alpha_3]]e_{3r} \oplus \mathcal{R}[[\alpha_1, \alpha_2, \alpha_3]](\alpha_1, e_{3r})^{(i)} \oplus \\
& \mathcal{R}[[\alpha_2, \alpha_3]](\alpha_2, e_{3r})^{(i)} \oplus \mathcal{R}[[\alpha_3]](\alpha_3, e_{3r})^{(i)} \oplus \\
& \mathcal{R}[[\alpha_1, \alpha_2, \alpha_3]]\gamma_{1,2}e_{3r} \oplus \mathcal{R}[[\alpha_1, \alpha_2, \alpha_3]](\gamma_{1,2}, e_{3r})^{(i)} \oplus \\
& \mathcal{R}[[\alpha_1, \alpha_2, \alpha_3]]\gamma_{1,3}e_{3r} \oplus \mathcal{R}[[\alpha_1, \alpha_2, \alpha_3]](\gamma_{1,3}, e_{3r})^{(i)} \oplus \\
& \mathcal{R}[[\alpha_1, \alpha_2, \alpha_3]]\gamma_{1,2}\gamma_{1,3}e_{3r} \oplus \mathcal{R}[[\alpha_1, \alpha_2, \alpha_3]](\gamma_{1,2}\gamma_{1,3}, e_{3r})^{(i)} \oplus \\
& \mathcal{R}[[\alpha_2, \alpha_3, \xi_{2,3}]]\gamma_{2,3}e_{3r} \oplus \mathcal{R}[[\alpha_3, \xi_{2,3}]](\gamma_{2,3}, e_{3r})^{(i)} \oplus \\
& \mathcal{R}[[\alpha_2, \alpha_3, \xi_{2,3}]]\xi_{2,3}e_{3r} \oplus \mathcal{R}[[\alpha_3, \xi_{2,3}]](\alpha_3\xi_{2,3}, e_{3r})^{(i)} \oplus \\
& \mathcal{R}[[\alpha_2, \alpha_3, \xi_{2,3}]](\alpha_2\xi_{2,3}, e_{3r})^{(i)} \oplus \mathcal{R}[[\alpha_2, \alpha_3, \xi_{2,3}]]\gamma_{1,2}\gamma_{2,3}e_{3r} \oplus \\
& \mathcal{R}[[\alpha_3, \xi_{2,3}]](\gamma_{1,2}\gamma_{2,3}, e_{3r})^{(i)} \oplus \mathcal{R}[[\alpha_2, \alpha_3, \xi_{2,3}]]\gamma_{1,2}\xi_{2,3}e_{3r} \oplus \\
& \mathcal{R}[[\alpha_2, \alpha_3, \xi_{2,3}]](\gamma_{1,2}\xi_{2,3}, e_{3r})^{(i)} \oplus \mathcal{R}[[\alpha_3, \xi_{2,3}]](\gamma_{1,2}, \alpha_3)^{(1)}e_{3r} \oplus \\
& \mathcal{R}[[\alpha_3, \xi_{2,3}]]\Big((\gamma_{1,2}, \alpha_3)^{(1)}, e_{3r}\Big)^{(i)} \oplus \mathcal{R}[[\alpha_3, \xi_{2,3}]](\gamma_{1,2}, \alpha_3)^{(2)}e_{3r} \oplus \\
& \mathcal{R}[[\alpha_3, \xi_{2,3}]]\Big(\alpha_3(\gamma_{1,2}, \alpha_3)^{(2)}, e_{3r}\Big)^{(i)}
\end{aligned}$$

where $i = 1, 2$ and $r = 1, 2, 3$ such that $e_{3(1)} = e_3$, $e_{3(2)} = e_6$ and $e_{3(3)} = e_9$.

We compute the transvectants that appear from the cases of the normal form. These are $(f, e_3)^{(i)}$, $(f, e_6)^{(i)}$ and $(f, e_9)^{(i)}$ for $i = 0, 1, 2$ where f are the monomials from the Stanley decomposition of the ring of invariants i.e.

$$(f, e_3)^{(0)} = \begin{bmatrix} 0 \\ 0 \\ f \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad (f, e_3)^{(1)} = \begin{bmatrix} 0 \\ \mathcal{X}\mathcal{Y}f \\ \mathcal{Y}f \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad (f, e_3)^{(2)} = \begin{bmatrix} \mathcal{X}^2\mathcal{Y}^2 f \\ \mathcal{X}\mathcal{Y}^2 f \\ \mathcal{Y}^2 f \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$(f, e_6)^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ f \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad (f, e_6)^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mathcal{X}\mathcal{Y}f \\ \mathcal{Y}f \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad (f, e_6)^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathcal{X}^2\mathcal{Y}^2 f \\ \mathcal{X}\mathcal{Y}^2 f \\ \mathcal{Y}^2 f \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$(f,e_9)^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ f \end{bmatrix}, \qquad (f,e_9)^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \mathcal{X}\mathcal{Y}f \\ \mathcal{Y}f \end{bmatrix}, \qquad (f,e_9)^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \mathcal{X}^2\mathcal{Y}^2 f \\ \mathcal{X}\mathcal{Y}^2 f \\ \mathcal{Y}^2 f \end{bmatrix}.$$

The Stanley decomposition for the normal form of N_{333} can then summarized as

$$ker\ \mathsf{X}_{(3)^3} = \bigoplus_{r=1}^{r=3} \left[SD\big(ker\ \mathcal{X}_{(3)^3}\big) \left(\bigoplus_{i=0}^{2} (f, e_{3r})^{(i)} \right) \right].$$

The vector fields $(f, e_{3r})^{(i)}$ are the basis of the normal form of $ker\ \mathsf{X}_{(3)^3}$ and f are the standard monomials of the ring of invariants, $ker\ \mathcal{X}_{(3)^3}$.

In general from the above examples the normal form of the coupled $N_{33,\ldots,3}$ system is obtained by computing the box product

$$ker\ \mathsf{X}_{(3)^n} = ker\ \mathcal{X}_{(3)^n} \boxtimes [\mathbb{R}e_3 \oplus \ldots \oplus \mathbb{R}_{3n}].$$

The basis of the normal form of $ker\ \mathsf{X}_{(3)^n}$ are transvectants of the form: $(f, e_{3r})^{(i)}$ where f is the standard monomials of Stanley decomposition of the ring of invariants, $ker\ \mathcal{X}_{(3)^n}$, where $i = 0, 1, 2$ and $r = 1, 2, \ldots, n$. The Stanley decomposition for the normal form for coupled system $N_{33,\ldots 3}$ is given by

$$ker\ \mathsf{X}_{(3)^n} = \bigoplus_{r=1}^{r=n} \left[SD\big(ker\ \mathcal{X}_{(3)^n}\big) \left(\bigoplus_{i=0}^{2} (f, e_{3r})^{(i)} \right) \right]$$

where

1. n is the number of Jordan blocks

2. $SD(ker\ \mathcal{X}_{(3)^n})$ is the Stanley decomposition of the ring of invariants of $ker\ \mathcal{X}_{(3)^n}$.

3. The transvectants $(f, e_{3r})^{(i)}$ are the basis of the normal form of $ker\ \mathsf{X}_{(3)^n}$.

CHAPTER FIVE

5.0 UNFOLDING AND CONCLUSION

In this chapter we briefly discuss the unfolding of a single N_3 dynamical system in $sl(2)$ normal form i.e. show the inclusion of perturbation parameters into the normal form. There are two reasons for doing unfolding to a dynamical system, namely imperfection in modeling and bifurcation theory as illustrated by Murdock (2003). We will also give a conclusion and suggestion of future research work.

5.1 Unfolding of Dynamical Systems

To unfold a system of vector fields is to add parameters to the system with the intention of studying behavior of all possible systems close to the original one. On normalizing the perturbed system up to a given degree, the arbitrarily parameters that remain are called the unfolding parameters, and the number of such parameters is called the co-dimension of the unfolding. There exists a natural notion of unfolding known as asymptotic unfolding (for vector fields having a rest point at the origin), under which all systems have unfolding of finite co-dimensions. Such unfolding have been computed for many years in applied contexts, and are treated, for example in Murdock (2003) and Murdock and Sanders (2006). The asymptotically unfolded system can be used to study the behavior which can be detected in the perturbation of the original system, such as stability and bifurcation analysis.

The first order unfolding.

The goal here is to find a simple form representing a system close to the given system $\dot{x} = Ax + Q(x) + ..$, where A is a given matrix and Q is a given quadratic part.

All perturbations of this system can be obtained by adding $\epsilon(p + B(x + ...) + ...)$ where $p \in \mathbb{R}^n$ is an arbitrary constant vector, B is an arbitrary $n \times n$ matrix, ϵ is a small parameter, and the second set of dots represent terms of higher order in ϵ. Defining the equivalence relation $\cong$ to mean congruence modulo cubic in x, quadratic terms in ϵ and ϵ times quadratic terms in x, the system to be normalized appears as

$$\dot{x} \cong Ax + Q(x) + \epsilon(p + Bx). \tag{5.1.1}$$

A is assumed to be in Jordan form and Q is in normal form with respect to A (in our case Q will be in $sl(2)$ form). Our aim is to simplify p and B in (5.1.1) as much as possible with the intention of reducing these $n + n^2$ quantities to a much smaller number of unfolding parameters $\mu_1, ..., \mu_k$. The result is called *the first unfolding of* $\dot{x} = Ax + Q(x) + ...$

According to Murdock (2002) and (2003) , if Q were in simplified normal form the normalization is achieved in three stages:

1. A coordinate shift $x = y + \epsilon k$ called a *primary shift*, that simplifies p.

2. A linear coordinate transformation $y = z + \epsilon Bz$ to simplify B.

3. A coordinate shift $z = w + \epsilon h$, called a *secondary shift*, that has no effect on p but simplifies B further.

However, this is not possible when Q is in $sl(2)$ normal form, the difficulty being that the secondary shift does not preserve the normal form achieved for the linear part B. The problem can be resolved by carrying out the normalization of the perturbed linear part and the secondary shift simultaneously rather than successively, an idea proposed by Murdock (2003). This normalization can be achieved by using coordinate transformation of the form

$$x = \bar{x} + \epsilon(h + T\bar{x}). \tag{5.1.2}$$

The result of applying this transformation to (5.1.1) and then dropping the bar from$\bar{x}$ is

$$\dot{x} \cong Ax + Q(x) + \epsilon(p + Ah) + \epsilon(B + C(h) + ad_A T)x \tag{5.1.3}$$

where $C(h)x = Q'(x)h$ and $ad_A T = AT - TA$. Since $Q(x)$ is homogeneous quadratic, $Q'(x)$ is linear in both x and h and may be written as the product of a matrix $C(h)$ that depends linearly on h and x.

5.2 Unfolding for System with Linear Part N_3

To find the $sl(2)$ normal form, we first find the ring of invariants $ker\ \mathcal{X}$ where $\mathcal{X} = 2x\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial z}$ using x, y, z. By inspection $\alpha = x$ and $\beta = y^2 - xz$, and this generates the entire ring; that is

$$ker\ \mathcal{X} = \mathbb{R}[\alpha, \beta] \tag{5.2.1}$$

To check this, we note that the weight of α is two and β is of weight zero, so the table function of $\mathbb{R}[\alpha, \beta]$ is

$$T = \frac{1}{(1 - dw^2)(1 - d^2)}$$

Hence

$$\frac{\partial}{\partial w}(wT_3)\mid_{w=1} = \frac{1}{(1-d)^3},$$

which implies (5.2.1).

The next step is to compute $ker\ \mathsf{X}$ as a module over $ker\ \mathcal{X}_3$. N_3 contains one Jordan block of size 3 whose differential operators are

$$\mathcal{X} = 2x\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial z}$$

$$\mathcal{Y} = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}.$$

The basis element is e_3 of weight 2, therefore the normal form is

$$\begin{aligned} ker\ \mathsf{X}_3 &= ker\ \mathcal{X}_3 \boxtimes \mathbb{R}e_3 \\ &= \mathbb{R}[[\alpha,\beta]]e_3 \oplus \mathbb{R}[[\alpha,\beta]](\alpha,e_3)^{(1)} \oplus \mathbb{R}[[\alpha,\beta]](\alpha,e_3)^{(2)} \end{aligned}$$

Thus

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + f_1(\alpha,\beta) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + f_2(\alpha,\beta) \begin{bmatrix} 0 \\ \mathcal{X}\mathcal{Y}\alpha \\ \mathcal{Y}\alpha \end{bmatrix} + f_3(\alpha,\beta) \begin{bmatrix} \mathcal{X}^2\mathcal{Y}^2\alpha \\ \mathcal{X}\mathcal{Y}^2\alpha \\ \mathcal{Y}^2\alpha \end{bmatrix}$$

We compute;

$$\mathcal{Y}\alpha = \mathcal{Y}x = y, \qquad \tfrac{1}{2}\mathcal{X}\mathcal{Y}\alpha = x$$

$$\mathcal{Y}^2\alpha = z, \qquad \tfrac{1}{2}\mathcal{X}\mathcal{Y}^2\alpha = y, \qquad \tfrac{1}{4}\mathcal{X}^2\mathcal{Y}^2\alpha = x$$

Therefore,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + f_1(\alpha,\beta) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + f_2(\alpha,\beta) \begin{bmatrix} 0 \\ x \\ y \end{bmatrix} + f_3(\alpha,\beta) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The differential equations in $sl(2)$ normal form upto quadratic terms are:

$$\dot{x} = y + h(x, y^2 - xz)x$$

$$
\begin{aligned}
&= y + (\gamma_1 x + \gamma_2 x^2)x \\
&= y + \gamma_1 x^2 + ... \\
\dot{y} &= z + g(x, y^2 - xz)x + h(x, y^2 - xz)y \\
&= z + g(x)x + h(x)y \\
&= y + (\beta_1 x + \beta_2 x^2 + ...)x + (\gamma_1 x + \gamma_2 x^2 + ...)y \\
&= z + \beta_1 x^2 + \gamma_1 xy + ... \\
\dot{z} &= f(x, y^2 - xz)x + g(x, y^2 - xz)y + h(x, y^2 - xz)z \\
&= f(x)x + g(x)y + h(x)z \\
&= (\alpha_1 x + \alpha_2 x^2)x + ... + (\beta_1 x + \beta_2 x^2 + ...)y + (\gamma_1 x + \gamma_2 x^2 + ...)z \\
&= \alpha_1 x^2 + \beta_1 xy + \gamma_1 xz + ...
\end{aligned}
$$

The system can now be written as:

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \alpha x^2 \\ \beta x^2 + \alpha xy \\ \gamma x^2 + \beta xy + \alpha xz \end{bmatrix}. \tag{5.2.2}
$$

Perturbing the system (5.2.2), we have:

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \alpha x^2 \\ \beta x^2 + \alpha xy \\ \gamma x^2 + \beta xy + \alpha xz \end{bmatrix} + \epsilon \begin{bmatrix} p \\ q \\ r \end{bmatrix} + \epsilon \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{5.2.3}
$$

Our goal is to reduce the number of arbitrary parameters $p, q, r, a, b, c, d, e, f, g, h, i$

from 12 to 3 by applying the transformation of the form (5.1.2):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cong \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} + \begin{bmatrix} \alpha\bar{x}^2 \\ \beta\bar{x}^2 + \alpha\bar{x}\bar{y} \\ \gamma\bar{x}^2 + \beta\bar{x}\bar{y} + \alpha\bar{x}\bar{z} \end{bmatrix} + \epsilon \begin{bmatrix} h \\ k \\ l \end{bmatrix} + \epsilon \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix}. \tag{5.2.4}$$

Computing,

$$ad_A T = \begin{bmatrix} d_1 & e_1 - a_1 & f_1 - b_1 \\ g_1 & h_1 - d_1 & i_1 - e_1 \\ 0 & -g_1 & -h_1 \end{bmatrix} \tag{5.2.5}$$

and

$$C(h) = \begin{bmatrix} 2\alpha h & 0 & 0 \\ 2\beta h + \alpha k & \alpha h & 0 \\ 2\gamma h + \beta k + \alpha l & \beta h & \alpha h \end{bmatrix} \tag{5.2.6}$$

From equation (5.1.3) we obtain $\dot{\bar{x}}$, $\dot{\bar{y}}$ and $\dot{\bar{z}}$ as

$$\begin{aligned} \begin{bmatrix} \dot{\bar{x}} \\ \dot{\bar{y}} \\ \dot{\bar{z}} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} + \begin{bmatrix} \alpha\bar{x}^2 \\ \beta\bar{x}^2 + \alpha\bar{x}\bar{y} \\ \gamma\bar{x}^2 + \beta\bar{x}\bar{y} + \alpha\bar{x}\bar{z} \end{bmatrix} + \epsilon \begin{bmatrix} p + k \\ q + l \\ r \end{bmatrix} + \\ &\epsilon \begin{bmatrix} a + d_1 + 2\alpha h & b + e_1 - a_1 & c + f_1 - b_1 \\ d + g_1 + 2\beta h + \alpha k & e + h_1 + \alpha h - d_1 & f + i_1 - e_1 \\ g + 2\gamma h + \beta k + \alpha l & h + \beta h - g_1 & i + \alpha l - h_1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} \end{aligned} \tag{5.2.7}$$

We drop the bar from $\bar{x}$ and choose $h, k, l, r, a, b, c, d, e, f, g, h, i$ so that the system has the form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \cong \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \alpha x^2 \\ \beta x^2 + \alpha xy \\ \gamma x^2 + \beta xy + \alpha xz \end{bmatrix} + \epsilon \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} + \epsilon \begin{bmatrix} \bar{a} & 0 & 0 \\ \bar{b} & \bar{a} & 0 \\ \bar{c} & \bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{5.2.8}$$

That is, we want

$0 = p + k$

$0 = q + l$

$0 = r$

$\bar{a} = a + d_1 + 2\gamma h$

$0 = b + e_1 - a_1$

$\bar{b}d = 2\beta h + \gamma k$

$\bar{a} = e + h_1 + \gamma h - d_1$

$0 = c + f_1 - b_1$

$0 = f + i_1 - e_1$

$\bar{c} = g + 2\alpha h + \beta k + \gamma l$

$\bar{b} = h + \beta h - g_1$

$\bar{a} = i + \gamma l - h_1$

So we choose $k = -p,\ l = -q,\ b = a_1 - e_1,\ c = b_1 - f_1$ and $f = e_1 - i_1$.

To make the diagonal elements equal so that our system is of the form (5.2.6) , then

$\bar{a} = a + d_1 + 2\gamma h ....(i)$

$\bar{a} = e + h_1 + \gamma h - d_1 ......(ii)$

$\bar{a} = i + \gamma l - h_1 ....(iii)$

Solving, $h_1 = 2i - a - e - 3\gamma h - 2\gamma l$

$\bar{b} = d + 2\beta h + \gamma k$

$\bar{b} = h + \beta h - g_1$

$g_1 = h - d - \beta h \alpha k$

If $\alpha \neq 0$ a generic condition, we can eliminate $\bar{c}$ by choosing

$$\bar{c} = g + 2\alpha h + \beta k + \gamma l = 0$$

$$g = -(2\alpha h + \beta k + \gamma l) = \beta p + \gamma q - 2\alpha h$$

$$h = \frac{\beta p + \gamma q - g}{2\alpha}$$

Modifying $\bar{a}$ and $\bar{b}$ the resulting system can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \epsilon r \end{bmatrix} + \begin{bmatrix} \epsilon a & 1 & 0 \\ \epsilon b & \epsilon a & 1 \\ 0 & \epsilon b & \epsilon a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \alpha x^2 \\ \beta x^2 + \alpha xy \\ \gamma x^2 + \beta xy + \alpha xz \end{bmatrix}. \quad (5.2.9)$$

Taking $u_1 = \epsilon r,\ u_1 = \epsilon b$ and $u_3 = \epsilon a$ gives

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u_1 \end{bmatrix} + \begin{bmatrix} u_3 & 1 & 0 \\ u_2 & u_3 & 1 \\ 0 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \alpha x^2 \\ \beta x^2 + \alpha xy \\ \gamma x^2 + \beta xy + \alpha xz \end{bmatrix}. \quad (5.2.10)$$

If $\beta \neq 0$ we can eliminate $\bar{b}$ by choosing

$\bar{b} = d + 2\beta h + \gamma k$

$\bar{b} = h + \beta h$

$h = \frac{d+2h}{\beta-1}$ and $k = \frac{dh(1-\beta)}{\gamma}$

Modifying $\bar{a}$ and $\bar{c}$ the resulting system can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \epsilon r \end{bmatrix} + \begin{bmatrix} \epsilon a & 1 & 0 \\ 0 & \epsilon a & 1 \\ \epsilon c & 0 & \epsilon a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \alpha x^2 \\ \beta x^2 + \alpha xy \\ \gamma x^2 + \beta xy + \alpha xz \end{bmatrix}. \tag{5.2.11}$$

Taking $u_1 = \epsilon r,\ u_2 = \epsilon a$ and $u_3 = \epsilon c$ gives

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u_1 \end{bmatrix} + \begin{bmatrix} u_2 & 1 & 0 \\ 0 & u_2 & 1 \\ u_3 & 0 & u_2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \alpha x^2 \\ \beta x^2 + \alpha xy \\ \gamma x^2 + \beta xy + \alpha xz \end{bmatrix}. \tag{5.2.12}$$

Similarly if $\gamma \neq 0$, we eliminate $\bar{a}$ by choosing;

$\bar{a} = a + d_1 + 2\gamma h$ and $\bar{a} = e + h_1 + \gamma h$

then

$$h = \frac{a + e + d_1 + h_1 - 2i - 2\gamma l - 2g_1}{3\gamma}$$

$$l = \frac{-(a + e + d_1 + h_1 + 3\gamma h - 2i + 2g_1}{2\gamma}$$

Modifying $\bar{b}$ and $\bar{c}$ we get

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u_1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ u_2 & 0 & 1 \\ u_3 & u_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \alpha x^2 \\ \beta x^2 + \alpha xy \\ \gamma x^2 + \beta xy + \alpha xz \end{bmatrix}. \tag{5.2.13}$$

The systems (5.2.10), (5.2.11) and (5.2.13) are equivalent to system (5.2.2) each having four perturbation parameters. It remains to be shown that the results obtained above for the truncated differential equations accurately reflect the behavior of the full system. This leads to the study of the behavior of the unfolded system by scaling methods. Other future problems includes:

1. Unfolding using $sl(2)$ normal form for the general case $N_{33,\ldots,3}$.

2. Possible reduction of co-dimension numbers of truncated systems.

3. Computation of the normal form of Hopf-bifurcation with perturbation parameters.

4. Bifurcations analysis of systems with nilpotent linear part $N_{33,\ldots,3}$.

5. Normal form Hamiltonian systems with $N_{22,\ldots,2}$ for their linear part.

6. Normal forms for $N_{44,\ldots,4}$ systems.

5.3 Conclusion

The use of Stanley decomposition to solve the descriptive problem for normal forms with nilpotent case begins by obtaining the invariants and using the table function to verify the result and then by repeating the same type of construction for the equivariants, using the triad X, Y, Z i.e. the problem begins by the creation of a scalar problem that is larger than the vector problem.

The approach discussed in this thesis is a method that begins by studying a scalar problem (the problem of invariants) that is smaller than the vector problem (problem of equivariants). When the scalar problem has been solved, our approach makes it unnecessary to repeat the calculations of classical invariants at the level of equivariants. Instead an algorithm is used that boosts Stanley decomposition of the ring of invariants into a Stanley decomposition of the module of equivariants.

In this thesis the normal form for coupled systems $N_{33,\ldots,3}$ has been computed. The Stanley decomposition of the ring of invariants of systems with linear part $N_{33,\ldots,3}$ has been computed explicitly and verified by its table function. An alternative method of obtaining the Stanley decomposition of the ring of invariants by use of lattice diagram has been discussed in chapter three. The boosting of the ring of invariants to the module of equivariants has also been computed explicitly in chapter four by giving specific cases and then generalizing leading to the normal forms for the systems.

Normal forms are important for determining bifurcations of a system. However, in general, a physical system or an engineering problem always involves some system parameters. To put the normal form into practical use, asymptotic unfolding for a single block is included as an exposition to show the inclusion of arbitrarily parameters. The asymptotic unfolding exhibit all behavior which can be detected in the perturbation of the original system up to a given degree, such as existence and stability of some bifurcations.

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