DEVELOPMENT OF EULER BERNOULLI SIMULATION MODEL FOR ANALYSIS OF THIN CURVED PLATES

PHILIP OSIEMO MOGIRE

MASTERS OF SCIENCE

(Civil Engineering)

JOMO KENYATTA UNIVERSITY OF

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Development of Euler Bernoulli Simulation Model for Analysis of Thin Curved Plates

Philip Osiemo Mogire

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DECLARATION

This thesis is my original work and has not been presented for a degree in any other University.

Signature: ……………………. …   Date ………………………

Philip Osiemo Mogire

This thesis has been submitted for examination with our approval as University supervisors.

Signature: ……………………. …   Date ………………………

DR. (Eng.) G.N.Mang'uriu

JKUAT, Kenya

Signature: ……………………. …   Date ………………………

Prof. (Eng.) Walter Oyawa

JKUAT, Kenya

DEDICATION

To my entire family

for their support and patience.

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ABSTRACT

Analysis of curved plate elements requires a high computational effort to obtain reliable solutions for design purposes. Available commercial programs are expensive and they need thorough knowledge for effective use. There is therefore need to code cheaper and accessible programs. This is also in line with vision 2030 of the government of Kenya, of using sustainable methods of production to drive the economy to the middle class level. To address this issue, a uniquemodel is formulated based on the Euler-Bernoulli beam model. This model is applicable to thin elements which include plate and membrane elements.

This thesispresents a finite element theory to calculate the master stiffness of a curved plate. The master stiffness takes into account the stiffness, the geometry and the loading of the element. The determinant of the master stiffness is established from which the buckling load which is unknown in the matrix is evaluated by the principal of bifurcation.The curved element is divided into 2,3,6,9 or 12 elements; this demonstrates the computational effort to a reliable solution.

Numerical analysis is carried out by abstracting the procedural development of the theory and programming it to run on a Visual Basic platform. The results obtained show that curved plates resist a higher load when it is directed towards the center of the arcand plates with large curvatures resist higher loads than those with smaller curvatures. A comparison made between the result obtained in this research and those of other methods show that there is a good agreement between the proposed and the existing methods. Thus the proposed method is suitable for analysis of curved plates.The research is useful for the study of curved plate elements as it manipulates the given plate and loading parameters to give optimal output.

CHAPTER ONE

INTRODUCTION

1.1 Background

Curved plate structures have been used in many civil, mechanical, and aerospace engineering applications such as curved roofs, curved aircraft body structures,turbo machineryblade, tire dynamic, and ship body parts. It can also be used as a simplified model of a shell structure. Key aspects of a curved plate structure relate to the curved form as well as the plate form (Nam-II & Chan-Ki, 2013).A plate may be described geometrically as a smooth 2D reference plane surrounded by material with thickness *h* to form a 3D body with one dimension much smaller than the other two.In general, a point in the plate can be represented mathematically by its Cartesian coordinates (x_i, Yi) where $x_{i,}$ $y_i$are two orthogonal lines in the reference plane. (Cem, & Eric, 2003)

A plate structure may be as simple as the flat web of a stiffener or as complex as an integrally stiffened plate supported by heavy frames and rings. In the behavior of these plate structures under in plane compression, a critical point exists where an infinitesimal increase in load can cause the plate surface to buckle. The load at this critical point defines the buckling strength of the plate. Increases in load beyond the load at the initiation of buckling increases the buckling deformations until collapse occurs. Thus, the load at collapse defines the post buckling or crippling strength of the plate. The behavior of plate structures in this regard differs markedly from the behavior of columns and many thin curved shell structures for which the buckling load corresponds closely to the collapse load.

A fundamental problem in plate design is to size plate elements so that plate buckling will not induce detrimental deformations at any expected service load up to limit load (the maximum load expected in service), and that the design ultimate load (the limit load multiplied by the ultimate design factor) will not exceed the plate crippling strength. Buckling of a plate structure can cause an unacceptable degradation. It can trigger general buckling of a larger structure because of a redistribution of loads; it can also affect the response to the structure sufficiently to cause failure from excessive displacement or fatigue.

This study presents an analytical criterion for determining the buckling load of a curved plate considering three load cases and recommends analysis of other loading cases and other plate structures. It is concerned with curved plates, but is also applicable for plates with shallow curvature. The analysis of the plate assemblies assumes a sinusoidal response along the plate length. The analysis uses stiffness matrices that result from the exact solution to the differential equations that describe the behavior of the plates.

Currently, most programs approximate a curved plate by subdividing it into a series of flat-plate segments that are joined along their longitudinal edges to form the complete curved-plate structure (Chang-Yong Lee et al, 2013;Chernuka, M.W.et al, 2005). This procedure is analogous to the discretization approach used in finite element analysis. The program uses exact stiffness's for the flat-plate segments and enforces continuity of displacements and rotations at the segment connections. Thus, the analyst must ensure that an adequate number of flat-plate segments are used in the analysis.

In this study, the capability to analyze curved-plate segments with an adequate number of divisions will be developed. The study will describe the methodology that will be used to accomplish this enhancement and will present results utilizing this new capability. The procedure will involve deriving a numerical method of appropriate differential equations of equilibrium for the analysis of fully anisotropic curved plates

1.2 Problem statement

Existing programs for analysis of curved plates are expensive and need a thorough knowledge of their usage. Further, analyzing curved elements manually is time consuming and the results outputs have low reliability due to the operations involved. Results are subsequently costly in terms of production and usage hence the need to program the operations.

As we enter into the evolution of smart structures, there is demand for techniques of analysis that will lead to design of cost effective structures. Less time should be spent by the engineers to do the analysis with high computational efficiency. Curved plates are nonlinear; they are best divided into smaller linear elements whose length approximates the length of the curve for analysis. Use of existing program modules for their analysis demands a clear understanding of the basis of their formulation with special regard to assumptions and limitations of the theories applied. When an analysis consumes time to obtain results the cost is transferred to the end users, and if results are not of high reliability the inherent errors results to under design or overdesign of elements.

1.3 Objectives

1.3.1 Main objective

To develop an Euler Bernoulli simulation model for analysis of thin curved plates

1.3.2 Specific objectives

a) To develop a finite element technique based on the Euler Bernoulli model for analysis of thin curved plate structural elements.

b) To develop a numerical procedure of analysis of thin curved plate structural elements.

c) To develop and validate a finite element computer simulation program based on Euler Bernoulli model for analysis of thin curved plate structural elements.

1.4 Limitations and scope of study

There are limited resources to undertake a study of a physical model. This includes finance, lab and equipment to undertake the study. The study will cover analytical formulation of the curved-plate non-linear equilibrium equations. The analytical formulation will be implemented through a computer based program. A convergence study using a segmented-plate approach in will be performed for a simple example problem to obtain baseline results for use in future comparisons. Results comparing the computational effort required by the new analysis to that of the analysis from classical plate equations will be presented.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

This chapter reviews existing literature on development of analysis of curved elements. It reviews modeling of curved elements by finite element method and existing theories for analysis. It also reviews the analytical and numerical methods of analysis. It further outlines development of programming languages, models for program development and available programs for analysis of curved elements.

2.2 Analysis of curved plates

There is a growing interest in analysis of curved plates. Chernuka et al (2005) analyzed curved plates using finite elements. He derived edge conditions for nodal points on these boundaries. He found the error inherent in representing the shape of a curved boundary by a series of straight segments to be the limiting factor on accuracy, while the effect of approximations in the actual boundary conditions as negligible. To overcome the error in shape representation, the high precision element was modified to include one curved edge. Caramalian (2005) using the Cowper triangle analyzed a two-nodded curved element with 12 degrees-of-freedom, which has one straight side and one curved side. Independent expansions were assumed in each domain and explicit shape functions were derived in all cases the curved triangle produced perfect results.

Application of the convex composite boundary element results in less numerical effort for a comparable error in circular and elliptical domain test problems than the application of straight-edged elements when compared to the curved isoperimetric elements. For domains with concave boundaries, the application of straight-edged, concave composite boundary and curved isoparametric elements give comparable accuracies and numerical efforts because of a fortuitous cancellation of error that occurs with straight-edged elements (Silva et al, 2005).

Eziefula, U.G. et al (2014) developed a technique for analysis of the plastic buckling analysis of a thin, rectangular, isotropic plate under uniform in-plane compression in the longitudinal direction. This technique is applicable to analysis of plastic buckling of thin isotropic plates with C-SS-SS-SS boundary conditions.

Raviprakash et al (2010) used a probabilistic approach in his analysis of plate structures. In his method, he kept the variance of imperfections of all the models at assumed manufacturing tolerance of 1.71 mm and maintained the maximum amplitude of imperfections within ±8 mm1024 random geometrical imperfect plate models were generated by the linear combination of the first 10 eigen value mode shapes using 2k factorial design. Murat, A.et al (2011) in his analysis of a clamped curved plate, used nonlinear algebraic equations obtained by the finite difference method and solved them by the Newton-Raphson method. The boundary conditions at the support and at the center of the plate were satisfied exactly.

2.3 Modeling of plates by finite element

Development of the finite element method began in earnest in the middle to late 1950s for airframe and structural analysis and gathered momentum at the University of Tuttgart through the work of John Argyris and at Berkeley through the work of Ray W. Clough in the 1960s for use in civil engineering. By late 1950s, the key concepts of stiffness matrix and element assembly existed essentially in the form used today and NASA issued a request for proposals for the development of the finite element software NASTRAN in 1965. The method was provided with a rigorous texematical foundation in 1973 with the publication of Strang and Fix's An Analysis of The Finite Element Method, and has since been generalized into a branch of applied texematics for numerical modeling of physical systems in a wide variety of engineering disciplines, e.g., electromagnetism and fluid dynamics.

The finite element method became even more popular with the advancement of microcomputers and development of various efficient programming languages.

Energy derivations are commonly used to form the stiffness for a variety of element types. The most common elements are the membrane (planar), plate, shell and solid elements. Each of these elements has a given set of nodes and displacements associated with those nodes. The common forms of these elements are given in figure 2.1.

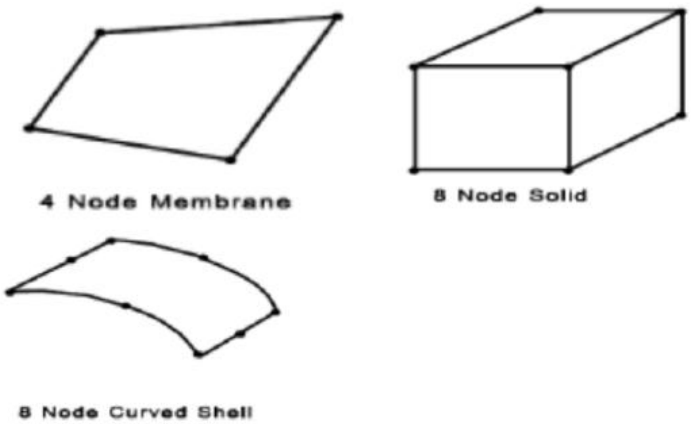


Figure 2.1 Common finite elements

These elements have additional restrictions on their behavior that depend on their derivation. However, the result is always a stiffness matrix that can then be treated like any other stiffness matrix and may be rotated and transformed as desired. When combining these elements, the same concerns about boundary conditions and matching DOF at the nodes must be accounted for. Additional concerns are also generated since the shape function assumption can affect the accuracy of the results. The standard beam element can be derived in a similar fashion using the cubic beam functions given on Consistent Geometric Stiffness. Element derivation has become increasingly complex. Techniques that include nonlinearities while still reducing the number of unknowns in the element have become very demanding. However the use of these sophisticated elements is identical to their simpler counterparts.

The three elements most commonly used by structural engineers are the membrane, plate/shell and solid elements. The membrane element is a two dimensional flat extensional element. The common versions are triangular and rectangular elements (figure2.2) (Manjunath & Bandayopadhyay, 2007). The triangular elements vary from three to six nodes. The rectangular elements vary from four to nine nodes. There are two in-plane displacement DOF's at each node of the element. The elements can be used to model two dimensional elasticity problems, plane strain and plane stress. It can reproduce the two normal and one shear stress in the plane of the element.

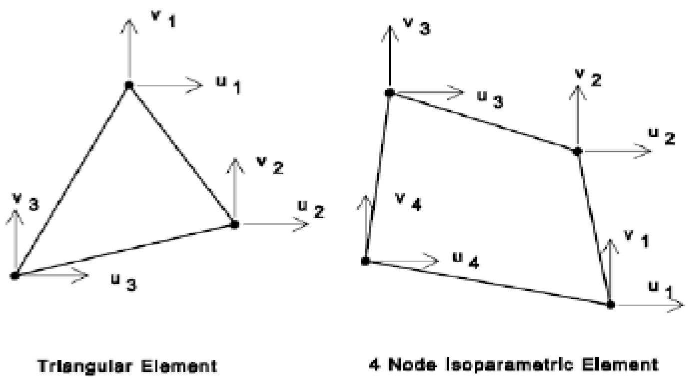


Figure 2.2 Local DOF for planar Elements (Manjunath & Bandayopadhyay, 2007).

2.4 Plate Bending Elements

The flat plate element is a two dimensional element that acts like a flat plate. It is found in triangular and rectangular versions. There are two out of plane rotations and the normal displacement as DOF. These elements model plate-bending behavior in two dimensions. The element can model the two normal moments and the cross moment in the plane of the element. Some versions will also give the transverse shear as a result. The three node triangular version models constant moment. The higher node elements can model linear variation of moment across the element. This element has no rotational stiffness normal to the plane and no in plane stiffness. Superimposing the membrane and plate elements on top of one another creates flat shell elements. Loading on plate elements can consist of any combination of forces normal to the plate and out of plane moments. Loading on shell elements can consist of the combination of plate and membrane loadings.

There has been considerable interest in the development of plate bending elements ever since their use became popular for representing the bending behavior of the shell elements. Many plate bending elements have been developed. A review of all plate bending elements as a part of the study on the effectiveness of plate bending elements has been presented (Hrabok & Hrudey, 1984).Triangular plate bending elements was developed by dividing the main triangle into three sub triangles (Clough & Tocher, 1965).

Conforming and nonconforming plate bending elements were also developed (Bazeley et al, 1966). The developed triangular plate bending elements were by use of shape functions based on the area coordinates. The nonconforming plate bending element does not pass the patch test for some mesh patterns, and the confirming element is costly to use because of the high order numerical integration scheme required to determine the stiffness matrix of the element.

Several effective triangular plate bending elements for the analysis of plates and shells were developed (Batoz et al, 1980). These elements had two rotational and one translational degree of freedom at each node for a total of 9 degrees of freedom. Three of plate bending elements were developed, namely;

(1) The DKT element based on Discrete Kirchoff Theory assumptions,

(2) The HSM element based on the Hybrid Stress Method, to overcome the problems in the development of pure displacement based models, and

(3) The SRI element based on a Selective Reduced Integration scheme that includes transverse shear deformation.

The results obtained for these elements were compared and was seen that the DKT and HSM elements are more effective than the SRI element. It was also found that the DKT element gives better results than the HSM element as the DKT element requires less storage compared to the HSM element.

Quadrilateral plate bending elements are popular in analyzing slab structures and are used in formulating shell elements for the analysis of regular shaped shell structures. Earlier attempts to develop quadrilateral plate bending elements involved combining four triangular plate bending elements (Batoz & Tahar, 1982).However their formulation was very complicated.A four node quadrilateral shell element using isoparametric shape functions was developed (McNeal, 1978), which gives very good results for plate bending.

Earlier attempts to develop plate bending elements were reviewed and it was concluded that these elements were useful for thick plates (Batoz &Tahar, 1982), but when applied to the thin plates they do not give very good results. A four node quadrilateral element based on the Discrete Kirchoff theory was developed (Batoz & Tahar, 1992). The basis of the formulation of this element was the Discrete Kirchoff Triangular (DKT) element developed earlier (Batoz*et al.*, 1980). The quadrilateral plate bending element (DKQ) formulated by Batoz and Tahar (1982) and the triangular plate bending element (DKT) formulated by Batoz*et al.* 1980, are based on the discrete Kirchoff assumptions in which the transverse shear strain is neglected. They considered transverse shear strain to be present in the element in the initial development and then removed the transverse shear strain terms by applying discrete Kirchoff constraints. Several tests on these elements were conducted by Batoz and Tahar (1982). Based on their study, they suggested that the convergence rates in displacements and stresses for DKQ element is not good as for the QUAD4 element (McNeal, 1978) and LORA by (Robinson & Haggenmacher, 2005).

2.5 Models for analysis of curved elements

The selection criteria for beam theories are generally given by means of some deterministic rule involving beam dimensions.

2.5.1. Linear elastic beam model

a) Euler-Bernoulli Model

This model often referred to as the classical beam model accounts for the bending moment effects on stresses and deformations. The effect of transverse shear forces on beam deformation is neglected. Its fundamental assumption is that cross sections remain plane and normal to the deformed longitudinal axis before and after bending. This assumption is valid if length to thickness ratio is large and for small deflection of beam. However, if length to thickness ratio is small, the plane section will not remain normal to the neutral axis after bending and the total rotation θ will be due to the bending stress alone. This rotation occurs about a neutral axis that passes through the centroid of the cross section of the beam as shown in figure 2.3.

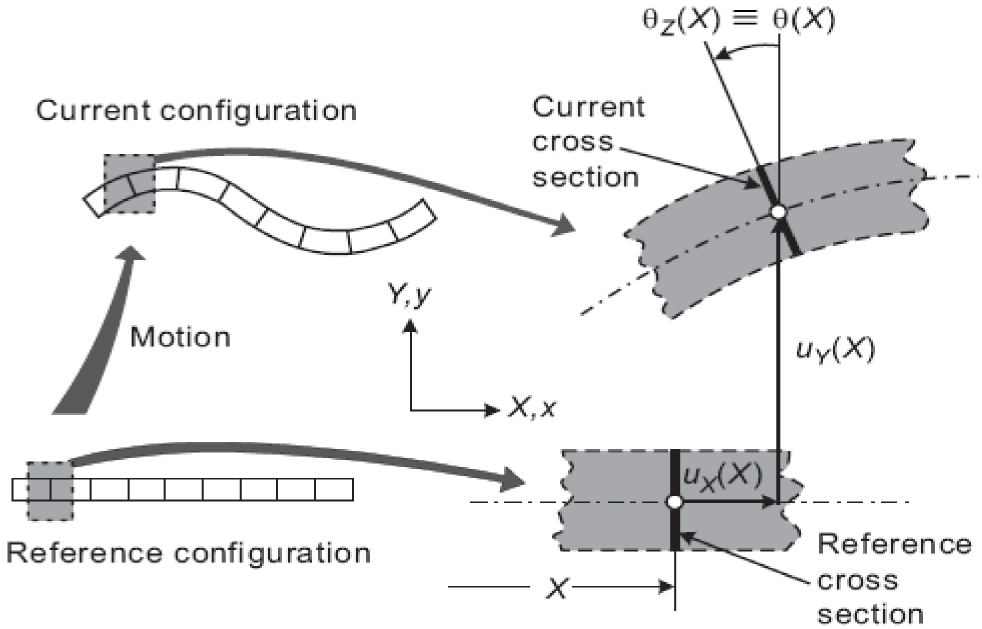


Figure 2.3 Euler-Bernoulli beam model (Carrera et al, 2011)

Shear forces and axial displacement are neglected in this theory. Euler-Bernoulli theory, slightly inaccurate results may be obtained. Timoshenko Beam Theoryis used to overcome the drawbacks of the Euler-Bernoulli beam theory by considering the effect of shear and axial displacements.

b) Timoshenko model

This model corrects the classical beam model with first ordershear deformation effects (Archiniega & Reddy, 2006;Marcelo & Pachero, 2006; Hibbit et al, 2003). In this model, the cross sections of the beam remain plane and rotate about the same neutral axis as the Euler-Bernoulli model, but do not remain normal to the deformed longitudinal axis. The deviation from normality is produced by a transverse shear that is assumed to be constant over the cross section. Thus, the TimoshenkoBeam model is superior to Euler-Bernoulli model in predicting precisely the beam response. The total slope of the beam in this model consists of two parts, one due to bending θ, and the other due to shearβ.

c) Gao and Russell beam model

It is known that the classical Euler–Bernoulli beam theory is valid only for long span (equivalently, thin) beams. In 1921, S. P. Timoshenko proposed a dynamic beam theory with two generalized displacements; i.e., the deflection $w(x)$ and the transverse shear deformation $v(x)$. In 1996, Gao and Russell proposed an extended beam model (figure 2.4), which allows the shear deformation to vary in y-direction, $v(x,y)$.

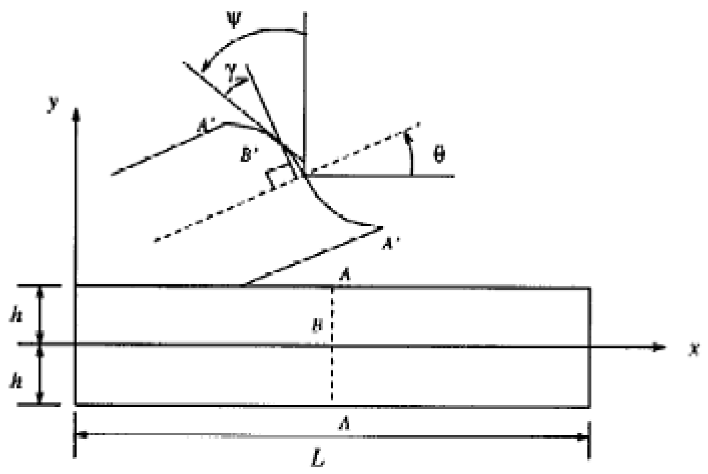


Figure 2.4 Extended beam model

i) Static beam model

$$\int_{-h}^{h} \left(Ey^2 \frac{\partial^4 w}{\partial x^4} - Ey \frac{\partial^3 v}{\partial x^3} \right) dy = p(x),$$

$$Ey \frac{\partial^3 w}{\partial x^3} - E \frac{\partial^2 v}{\partial x^2} - G \frac{\partial^2 v}{\partial y^2} = 0. \quad (2.1)$$

ii) Dynamic beam model

$$EI \frac{\partial^4 w}{\partial x^4} - \int_{-h}^{h} Ey \frac{\partial^3 v}{\partial x^3} dy - p(x) = \bar{\rho} \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x} \left(I_\rho \frac{\partial^3 w}{\partial t^2 \partial x} \right) + \int_{-h}^{h} y \frac{\partial}{\partial x} \left(\rho \frac{\partial^2 v}{\partial t^2} \right) dy, \quad (2.2)$$

$$Ey \frac{\partial^3 w}{\partial x^3} - E \frac{\partial^2 v}{\partial x^2} - G \frac{\partial^2 v}{\partial y^2} = \rho \frac{\partial}{\partial t} \left(y \frac{\partial^2 w}{\partial t \partial x} - \frac{\partial v}{\partial t} \right), \quad \forall (x, y, t) \in \Omega \times [0, T], \quad (2.3)$$

$$w(x, t) = 0, \qquad \frac{\partial w}{\partial x}(x, t) = 0, \qquad v(x, y, t) = 0 \qquad \text{at} \quad x = 0, L, \quad (2.4)$$

$$\frac{\partial v}{\partial y}(x, y, t) = 0 \qquad \text{at} \quad y = \pm h.$$

(2.5)

2.5.2 Elasto-perfectly plastic beam model:

The dual problem of this model is equivalent to a quadratic minimization problem subjected to linear and nonlinear inequality constraints given below (Larry et al, 2003);

2.6 Development of numerical analysis of curved elements

In order to obtain the values of deflections and stresses of curved elements, various analyses techniques have been developed. The energy methods like Castigliano's theorem(Nam-II & Chan-Ki II, 2008) can be used,Castigliano's theorem is only useful in solving simple and decoupled problems, and the formulation has to be done separately for each problem. Rayleigh-Ritz method may be applied to evaluate solutions for the curved elements, but the accuracy of the results obtained by this method depends on the displacement functions chosen and a large number of terms have to be used in the displacement functions to get the good results for complicated problems. Moreover, this method has to be formulated separately for each problem since the chosen displacement functions chosen depend on the boundary conditions of curved beam. Thus for the analysis of curved elements, the numerical methods have applied finite element method mostly because of its versatility and applicability. This work will develop a new numerical procedure for analysis of thin curved plates based on the finite element method.

2.7 Development of finite element program for analysis of curved elements

2.7.1 Introduction

The current target application area is analysis of thin curved plates of engineering structures such as water tanks, pressure vessels and roof structures. Although there are hundreds of engineering analysis and optimization programs having been written in the past 20years(Pachero, 2006; Hibbit et al, 2003; Larry et al, 2003).

During the past two decades,computers have been providing approximately 25% more power per dollar per year (Goodman et al, 2000; Balling, 2003).Advances in computer hardware and software have allowed for exploration of many new ideas,and have been a key catalyst in what has led to the maturing of computing as a discipline.In the 1970’s computers were viewed as a machines for research engineers and scientists-compared to today’s standards,computer memory was very expensive and the central processing units were very slow.

Early versions of computer analysis and finite element were developed with the goal of optimizing numerical and/or instructional considerations alone.These programs offered a restricted,but well implemented,set of numerical procedures for static structural analyses,and linear/nonlinear time history calculations.And even though these early computer programs were not particularly easy to use,practicing engineers gradually adopted them because they allowed for the analysis of new structural systems in ways that were previously intractable.

During the last twenty years,the use of computers in engineering has matured to the point where importance is now placed on ease of use,and a wide array of services being available to the engineering profession as a whole.Computer programs written for engineering computations are expected to be fast,accurate,flexible,reliable and easy to use.Whereas an engineer in the 1990’s might have been satisfied by a computer program that provided numerical solutions to a very specific engineering problem,the same engineer today might require engineering analysis,support for design code,optimization,interactive computer graphics and network connectivity.Many of the latter features are not a bottleneck for getting the job done.Rather features such as interactive computer graphics makes the job of describing a problem and interpreting results easier-the pathway from ease of use to productivity gains is well defined.Computers are now are an indispensable tool for computing and communications.

The aforementioned hardware advances with appropriate software developments is reflected in the economic costs of project development. In the 1970's software consumed 25% while hardware consumed 75% of total costs of development of data intensive systems.Currently development and maintenance of software consumes more than 80% of the total project costs (Goodman et al, 2000; Balling R.J, 2003).This change in economics is the combined result of falling hardware costs and increased software development budgets needed to implement systems that are more complex than they used to be.

2.7.2 Finite element modeling

Finite element methods are now widely used to solve structural, fluid, and multi-physics problems numerically(Zienkiewitz & Taylor,2005).The Euler –Bernoulli beam model applies since only thin elements are considered here (shear deformations are neglected) (Memarzader et al,2010).Two methods of analysis of curved elements have been considered: the eigenvalue buckling analysis and nonlinear buckling analysis.

The eigenvalue analysis predicts the theoretical buckling strength of an ideal linear elastic structure. This is analogous to the classical plate equation approach to elastic buckling analysis (Bathe,2009).However, imperfections and nonlinearities prevent most real-world structures from achieving their theoretical elastic buckling strength.

Nonlinear buckling analysis takes into account of imperfections and nonlinearities of real-world structures. In this method the load is increased until the solution fails to converge, indicating that the structure cannot support the applied load (or that numerical difficulties prevent solution) (Mohammad et al,1988). If the structure does not lose its ability to support additional load when it buckles, a nonlinear analysis can be used to track post-buckling behavior.

2.7.3 Basic element shapes

For the discretization of problems involving curved geometries, finite elements with curved sides are useful (Pedro et al, 2011). The ability to model curved boundaries has been made possible by the addition of midsized nodes. Finite elements with straight sides are known as linear elements, whereas those with curved sides are called higher order elements.

2.7.4 Size of Elements

The size of chosen elements influences the convergence of the solution directly. If the size of the elements is small, the final solution is expected to be more accurate.

2.7.5 Number of Elements

The number of elements is related to the accuracy required and the number of degrees of freedom involved (Dalvi et al, 2012). Although an increase in the number of elements generally means more accurate results for any given problem, there will be a certain number of elements beyond which the accuracy cannot be improved by any significant amount, as shown graphically in figure 2.6.

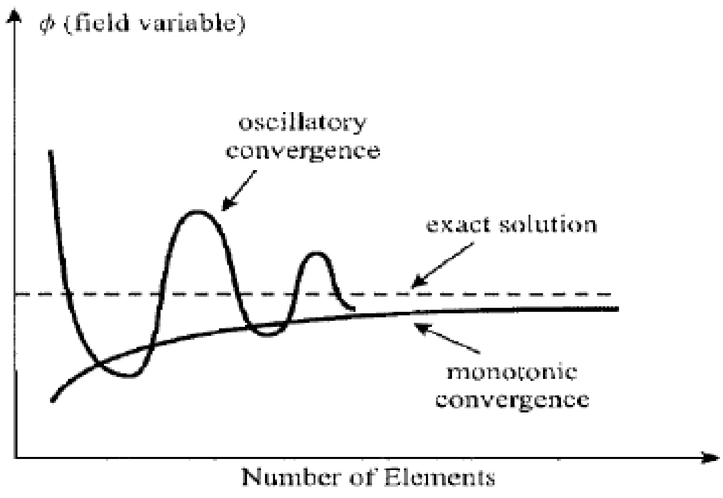


Figure 2.5 Relationship between the number of elements and accuracy(Dalvi M.V. et al, 2012)

2.7.6 The modeling procedure

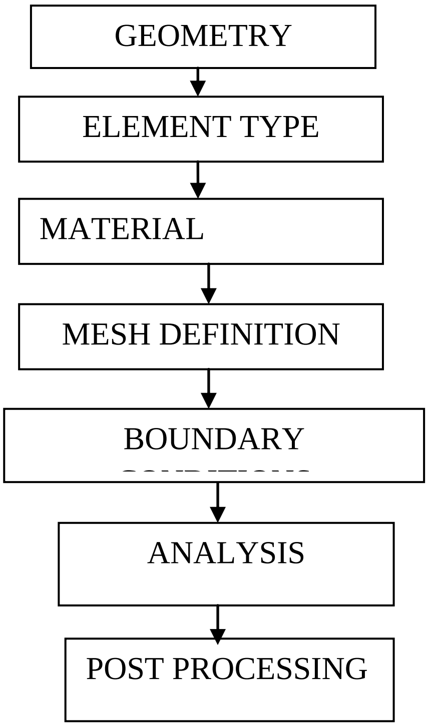


Figure 2.6 The finite element modeling procedure.

Finite element analysis is used in problems where analytical solution is not easily obtained and mathematical expressions required for solution are not simple because of complex geometries loadings material properties.

a) Basic Finite Element Equation.

The fundamental FEA equation greatly simplifies problem formulation and solution (Pravin & Kachare, 2012).

$$[F] = [K] * [d] \quad (2.7)$$

where [F] is the known vector of nodal loads, [K] is the known stiffness matrix, and [d] is the unknown vector of nodal displacements

This matrix equation describes the behavior of Finite element analysis models. It contains a very large number of linear algebraic equations, varying from several thousand to several million depending on the model size. The stiffness matrix [K] depends on the geometry, material properties, and restraints. Under the linear analysis assumption that the model stiffness never changes, those equations are assembled and solved just once, with no need to update anything while the model is deforming. Thus linear analysis follows a straight path from problem formulation to completion. It produces results in a matter of seconds or minutes, even for very large models.

b) Principal Steps of Finite element analysis

i) Pre-processing:

The user constructs a model of the part to be analyzed in which the geometry is divided into a number of discrete sub regions, or elements, connected at discrete points called nodes Certain of these nodes will have fixed displacements, and others will have prescribed loads. These models can be extremely time consuming to prepare, and commercial codes vie with one another to have the most user-friendly graphical pre-processor" to assist in this rather tedious chore. Some of these pre-processors can overlay a mesh on a pre-existing CAD, so that finite element analysis can be done conveniently as part of the computerized drafting-and-design process.

The dataset prepared by the preprocessor is used as input to the finite element code itself, which constructs and solves a system of linear or nonlinear algebraic equations .$K_{ij}u_j = f_i$ where u and f are the displacements and externally applied forces at the nodal points. The formation of the K matrix is dependent on the type of problem being attacked, and this module will outline the approach for truss and linear elastic stress analyses. Commercial codes may have very large element libraries, with elements appropriate to a wide range of problem types. One of FEA's principal advantages is that many problem types can be addressed with the same code, merely by specifying the appropriate element types from the library.

ii) Post processing:

In the earlier days of finite element analysis, the user would pore through reams of numbers generated by the code, listing displacements and stresses at discrete positions within the model. It is easy to miss important trends and hot spots this way, and modern codes use graphical displays to assist in visualizing the results. A typical postprocessor display overlay colored contours representing stress levels on the model, showing a full field picture similar to that of photoelastic or moiré experimental results. The operation of a specific code is usually detailed in the documentation accompanying the software, and vendors of the more expensive codes will often over workshops or training sessions as well to help users learn the intricacies of code operation. One problem users may have even after this training is that the code tends to be a black box" whose inner workings are not understood. In this module we will outline the principles underlying most current Finite element stress analysis codes, limiting the discussion to linear elastic analysis for now. Understanding this theory helps dissipate the black-box syndrome, and also serves to summarize the analytical foundations of solid mechanics.

2.8 Programming Language

Computer programming languages are built around two approaches; namely procedural programming and Object oriented programming.

In procedural programming, the program is prepared by a series of steps or routines that follow the data provided. The programming languages such as FORTRAN, C BASIC are procedural programming languages.The main drawback of the procedural programming languages is that they are not structured and the flow of the program largely depends on conditional statements that induce more chances of errors. These languages are good for small programs, but procedural programs are difficult to maintain when they become larger.

In the object oriented programming approach, the program is organized around its data in the form of objects (Schildt, 2001). The object oriented programming languages are built on the concept of abstraction. Large complex procedures can be subdivided in to small procedures by abstraction. Each of these sub procedures represents different objects with their own separate identity (Tabarrok et al, 1988; Kakuchi, 2014). The series of process steps can be achieved by passing information to the objects without being affected by the complexity of the whole procedure. The three unique aspects of the object oriented programming languages are; encapsulation, inheritance and polymorphism.

Encapsulation is the most important aspect of the object oriented programming. In object oriented programming languages, classes perform the task of encapsulation. Class defines the structure and behavior of the process that will be shared by a set of objects (Schildt, 2001) such as variables and methods. The variables or methods are declared by access specifiers such as public, private or protected. A variable or method declared as public can be accessed from outside the class in the program. Variables or methods that are defined as private cannot be accessed from outside the class and hence the privacy of the data is maintained. Variables or methods declared protected are only accessible to the superclass and the subclass where the properties of the superclass are inherited.

Many programs contain objects that are dependent on each other and inherit certain properties from one object to another. In object oriented programming, the classes are divided in the superclass and subclass. The subclass inherits all of the properties of the superclass except those declared as private. Any subclass that inherits properties from its subclass may have additional properties that give it an individual identity, which is not common to the other objects or subclasses that inherit the same properties from the super class.

Polymorphism is another valuable feature of object oriented programming. Polymorphism allows the programmer to use the same interface to perform multiple tasks. The same class may contain multiple methods that are related to different activities and each method will perform a different task when it is called by the object. The call to one method will not affect the contents or activity of another method in the same class.Several object oriented programming languages have been developed in recent years. These include C++, C#, and Java. Java is one of the more popular object oriented programming languages because it has several unique features.

Java is an object oriented programming language developed by Sun Microsystems in 1991. Java is based on the popular programming language C++. It has many of the same features of C++. Over the years, Java has become very popular because of some of its unique features. Java is platform independent, which means code developed in Java can be used on a variety of different computers without making any changes. Another advantage is that, Java programs can be embedded within HTML pages (where they are called applets) and can be easily transmitted over the internet. All this needs is a Java compatible web browser to run these applets. Another reason for the popularity of Java is its robustness. Java provides automatic protections for memory loss and run-time errors that occur during the program execution.

Java has a special garbage collection class that dynamically allocates memory and hence reverts memory loss. In other programming languages, this is done manually by the programmer and any mistake in allocating or de-allocating memory may result in failure of the program. There is also an exception handling class for handling runtime errors in Java. With the use of this class it is possible to catch many common runtime errors which would otherwise result in program failure.

Until recently most finite element analysis program were written in FORTRAN. Although FORTRAN is an efficient language for developing scientific applications, it is not well suited for writing large complex programs. With the advancements in finite element analysis, many different types of elements are being developed and elements are constantly being modified to improve their behavior. It is very difficult to maintain the codes for the finite element analysis that were developed using procedural programming languages because of its complexity. An object oriented programming language such as Java or matlab is better suited for the development of large complex programs for finite element analysis, because of the many advantages discussed above. These advantages make it very attractive to implement object oriented programming techniques for the development of the finite element analysis codes.

The Matlab programming language is useful in illustrating how to program the finite element method due to the fact it allows one to very quickly code numerical methods and has a vast Predefined mathematical library. This is also due to the fact that matrix, vector and many linear algebra tools are already defined and the developer can focus entirely onthe implementation of the algorithm not defining these data structures. The extensive mathematics and graphics functions further free the developer from the drudgery of developing these functions themselves or finding equivalent pre-existing libraries. A simple two dimensional finite element program in Matlab need only be a few hundred lines of code whereas in Fortran or C++ one might need a few thousand lines.

Although the Matlab programming language is very complete with respect to its mathematical functions there are a few finite element specific tasks that are helpful to develop as separate functions. As usual there is a trade on to this ease of development. Since Matlab is an interpretive language; each line of code is interpreted by the Matlab command line interpreter and executed sequentially at run time, the run times can be much greater than that of compiled programming languages like Fortran or C++. It should be noted that the built-in Matlab functions are already compiled and are extremely efficient and should be used as much as possible. Keeping this slow down due to the interpretive nature of Matlab in mind, one programming constrain that should be avoided at all costs is the for loop, especially nested for loops since these can make a Matlab programs run time orders of magnitude longer than may be needed. Often for loops can be eliminated using Matlab'svectorized addressing.

A typical finite element program consists of the following sections; preprocessing section, processing section and post-processing section

In the preprocessing section the data and structures that define the particular problem statement are defined. These include the finite element discretization, material properties, solution parameters etc. The processing section is where the finite element objects i.e. Stiffness matrices, force vectors etc. are computed, boundary conditions are enforced and the system is solved. Thepost-processing section is where the results from the processing section are analyzed. Here stresses may be calculated and data may be visualized.

2.9 Available programs for analysis of curved elements.

i) Prokon Structural Analysis and Design

This is a suite of over thirty years for structural analysis, design and detailing .The first Prokon programs were developed in 1989, and today Prokon is used worldwide in over eighty countries. The suite is modular in nature, but its true power lies in the tight integration between analysis, design and detailing programs.This software does not have a module for analysis of curved plates. Only thicker elements are considered. There is need to redefine the module that has buckling before usage.

ii) STAAD PRO

STAAD.Pro is a general purpose structural analysis and design program with applications primarily in the building industry - commercial buildings, bridges and highway structures, industrial structures, chemical plant structures, dams, retaining walls, turbine foundations, culverts and other embedded structures, etc. The program hence consists of the following facilities to enable this task.

Using STAAD.Pro for analysis of curved plates, the module to use is referred to as STAAD. beava. This module will generate loads on plates. It is first required to model the element into a plate mesh model and then model some fictitious beam entities (with negligible section properties) from node to node within the plate mesh to deliver the load. This process is time consuming, needs a thorough knowledge of modeling and is bound to computational errors.

iii) ANSYS

This is researchsoftware developed by AnsysInc. It was developed into many different software modules; amongst these is the Ansys workbench platform. It is the framework upon which advanced engineering simulations are built. Ansys workbench combines the strength of core problem solvers with project management tools necessary to manage project workflow. In Ansys, Workbench Analysis is built as systems which can be combined together into a project. Ansys workbench is laborious as you have to model the element whose knowledge must be inherent before analysis.

2.10 Models for Program development

2.10.1 Introduction

The process of building computer software and information systems has always been dictated by different development methodologies. A software development methodology refers to the framework that is used to plan, manage, and control the process of developing an information system (Ian Sommerville, 2011). A software development methodology is known as SDLC short for Software Development Life Cycle and is majorly used in several engineering and industrial fields such as systems engineering, software engineering, mechanical engineering, computer science, computational sciences, and applied engineering (Richard. et al,1986). In effect, SDLC has been studied and investigated by many researchers and practitioners all over the world, and numerous models have been proposed, each with its own acknowledged strengths and weaknesses. The Waterfall, spiral, incremental, rational unified process (RUP), rapid application development (RAD), agile software development, and rapid prototyping are few to mention as successful SDLC models. In a way or another, all SDLC models suggested so far share basic properties. They all consist of a sequence of phases or steps that must be followed and completed by system designers and developers in order to attain some results and deliver a final product.

2.10.2 SDLC Waterfall Model

In this model (figure2.7), the software development activity is divided into different phases and each phase consists of a series of tasks and has different objectives.Waterfall model is the pioneer of the SDLC processes,it was the first model which was widely used in the software industry. It is divided into phases and output of one phase becomes input of the next phase. It is mandatory for a phase to be completed before the next phase starts. In short, there is no overlapping in

In waterfall model, development of one phase starts only when the previous phase is complete. Because of this nature, each phase of waterfall model is quite precise well defined. Since the phase's falls from higher level to lower level, like a water fall, it's named as waterfall model.

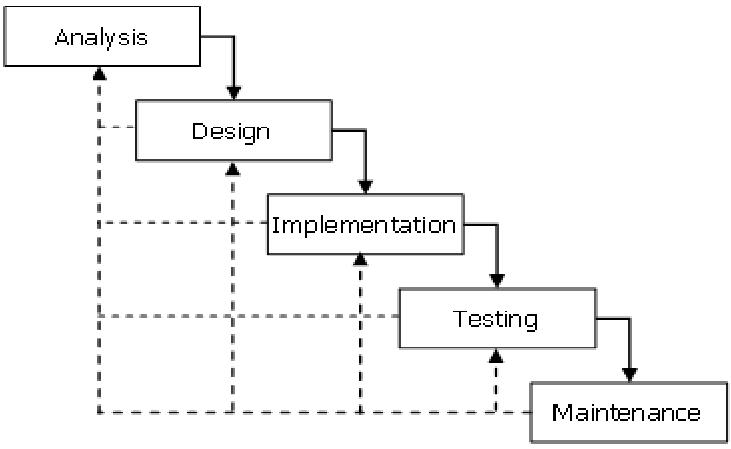


Figure 2.7 The water fall model (Youssef Bassil, 2012)

CHAPTER THREE

METHODOLOGY

3.1 Introduction

This chapter describes the formulation of ananalytical model of analysis based on the Euler-Bernoulli beam model. The formulation combines other research works into a new formulation based on the finite element method.The chapter also describes the numerical procedure for analysis of curved elements. This model is based on the analytical finite element formulation for analysis. Lastly, an approach in the development of the code of analysis base of curved elements based on the analytical formulation of finite element is developed.

3.2Developing analytical formulation for analysis of thin curved plates

The work done during a virtual displacement of the element is equated to zero to obtain the equilibrium equations for the element (Jerrel, 1970). The bifurcation concept of elastic stability is used to postulate that two possible sets of, displacements which satisfy the equilibrium equations may exist under the same magnitude of external load if the magnitude is such that a structure is unstable. Each of these sets of displacements is substituted in turn into the equilibrium equations. The resulting sets of equations are combined to obtain a relationship between the nodal forces and the nodal displacements during buckling. When placed in matrix form this relationship becomes (Jerrel, 1970).

$$\{Q^i\} = [[K] + [G] + [L]]\,\{q^i\} = \{0\} \tag{3.1}$$

Where Q^i is the column matrix of nodal forces,

q^i is the column matrix of nodaldis placements,K is the usual beam element stiffness matrix,G is the geometric stiffness matrix and L is the load behavior stiffness matrix.

[G] and [L] both contain the magnitude of external load as a factor. Thus an element stiffness can be derived which is a function of the applied load including the applied load behavior.

A number of such elements may be combined to represent a particular structure, so that equation (3.1) applies to the entire structure and [K] + [G] + [L] is the master stiffness matrix. Boundary conditions may be applied to reduce the size of the master stiffness matrix. For instability to exist the determinant of the master stiffness matrix must vanish. Hence, an eigenvalue problem is formulated where the eigenvalues are the magnitudes of the applied load at which the structure is unstable.

3.2.1 Development of element stiffness matrix

a)Description of Element.

Consider a beam column element as shown in Figure 3.1 which is subjected to nodal forces and moments and a uniformly distributed load, p, applied along its length. It is desired to determine as tiffness matrix for the element which may be used to calculate the stability of structures made-up of such elements. The stiffness matrix is to account for the fact that the components of the nodal forces or distributed load, or both, may be a function of the element displacements. Three cases are to be considered: the loads remain normal to the deformed element; parallel to their original direction; and, directed towards a fixed point.

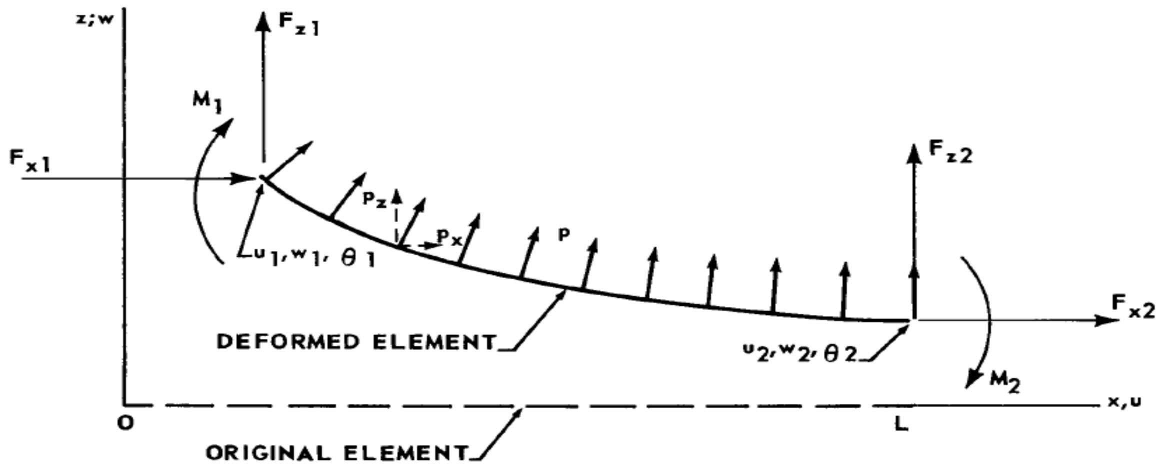


Figure 3.1 Element modeling

b) Displacement Functions.

If it is assumed that the lateral displacement of the beam element of Figure 3.1 above may be represented by

$$W=\alpha_1+\alpha_2\, x+\alpha_3\, x^2+\alpha_4\, x^3 \tag{3.2}$$

And the longitudinal displacement is given by

$$U=\alpha_5+\alpha_6\, x \tag{3.3}$$

then the six unknown constants, may be determined in terms of the nodal displacement from the element boundary conditions

$$\left.\begin{array}{lll} W & = w_1 & \text{at } x=0 \\ w_{,x} & = -\theta_1 & \text{at } x=0 \\ W & = w_2 & \text{at } x=L \\ w_{,x} & = -\theta_2 & \text{at } x=L \\ u & = u_1 & \text{at } x=0 \\ u & = u_2 & \text{at } x=L \end{array}\right\} \tag{3.4}$$

The resulting displacement functions are:

$$W= w_1\left(1-3\frac{x^2}{L^2}+2\frac{x^3}{L^3}\right)+w_1\left(3\frac{x^2}{L^2}-2\frac{x^3}{L^3}\right)+\theta_1\left(-x+2\frac{x^2}{L}-\frac{x^3}{L^2}\right)+\theta_2\left(\frac{x^2}{L}-\frac{x^3}{L^2}\right)$$

$$= w_1\, \emptyset_1(x) + w_2\, \emptyset_2(x) + \theta_1\emptyset_3(x) + \theta_1\emptyset_4(x)$$

$$= w_i\emptyset_i(x) \tag{3.5}$$

$$U = u_1\left(1-\frac{x}{L}\right)+u_2\left(\frac{x}{L}\right)$$

$$= u_i\, \gamma_i(x) \tag{3.6}$$

These displacement functions have been used by for the beam column element. However, it should be mentioned that an inconsistency arises when the cubic function is used for the lateral displacement of the present element which has a distributed load. This is illustrated by taking the third derivative of w, which should be the equation for the shear load on the element, and observing that a constant results. But the element has a distributed load, and the shear should obviously be a linear function, not a constant. Since the change in shear over the length of an element becomes negligible in the limit as the element becomes smaller and smaller, this inconsistency is not unacceptable and it will be seen that adequate results are obtained. The correct fourth order function could not be used for w because there is no other boundary condition available for evaluating another constant in equation (3.2). This will be discussed further in the conclusions.

c) Development of equilibrium equations

The element equilibrium equations will now be developed from the principle of virtual displacements

$$\delta U - \delta V = 0 \tag{3.7}$$

Where δU is the change in strain energy and δV is the work done by the external forces during a virtual displacement.

For the uniaxial state of stress assumed to exist in a beam the change in strain energy is given by

$$\delta U = \int_v \delta_x \delta \in_x \, dv \tag{3.8}$$

Here, $\in_x$ is the strain of the beam at any point of the cross section and is given by

$$\in_x = \in_{xx} + zk_{xx} \tag{3.9}$$

Where $\in_{xx}$ is the strain of the beam mid-surface, z is the distance of the point in the beam from the mid-surface, and

$$k_{xx} = w_{,xx} \tag{3.10}$$

is the beam curvature. The non-linear strain – displacement relationship

$$\in_{xx} = u_{,x} + \frac{1}{2} w_{,x}^2 + \frac{1}{2} u_{,x}^2 \tag{3.11}$$

is used for the mid-surface strain. The $u_{,x}^2$ term in this expression is not known to have been used in previous derivation of non-linear beam element stiffness matrices.

Substituting equations (3.9),(3.10) and (3.11) into equation (3.8)

$$\delta U = \int_A \int_L E \in_x \delta \in_x dxdA = \int_A \int_L E(\in_{xx} + zk_{xx})E(\delta \in_{xx} + z\delta k_{xx})dxdA \tag{3.12}$$

$$= \int_A \int_L [E \in_{xx} \delta \in_{xx} + EZ \in_{xx} \delta k_{xx} + Ezk_{xx} \delta \in_{xx} + Ez^2 k_{xx} \delta \in_{xx}]$$

$$= \int_A zdA = 0, \int_A z^2 dA = I$$

$$= \int_L [AE\epsilon_{xx}\ \delta\ \epsilon_{xx} + EI\epsilon_{xx}\ \delta k_{xx}]dx \tag{3.13}$$

Where

$$\delta_{xx} = \delta u_{,x} + \delta\left(\frac{1}{2} w_{,x}^2\right) + \delta\left(\frac{1}{2} u_{,x}^2\right) = \delta u_{,x} + w_{,x}\ \delta w_{,x} + u_{,x}\ \delta u_{,x}$$

$$\delta k_{,xx} = \delta w_{,xx}$$

$$\delta u = \int_L \left[EA\left(u_{,x} + \frac{1}{2} w_{,x}^2 + \frac{1}{2} U_{,x}^2\right)\left(\delta u_{,x} + w_{,x}\ \delta w_{,x} + u_{,x}\ \delta u_{,x}\right) + EIw_{,xx}\ \delta w_{,x}\right] dx$$

$$\delta u = \int_L [EA(u_{,x} + \frac{1}{2} w_{,x}^2 + \frac{3}{2} u_{,x}^2)\ \delta u_{,x} + EA(u_{,x}\ w_{,x})\delta w_{,x} + EIw_{,xx}\ \delta w_{,xx})]\ dx$$

+higher order terms. (3.14)

The virtual work of the external forces acting on the element is given by:

$$\delta v = \bar{F}xi^{\delta u}i + \bar{F}zi^{\delta w}i + \bar{m}i^{\delta\theta}i + \int_L [p_z \delta w + p_z \delta u]dx \tag{3.15}$$

where the force components depend upon the load behavior. Three load behavior cases will be considered.

Case i. Loads remain normal to the deformed element

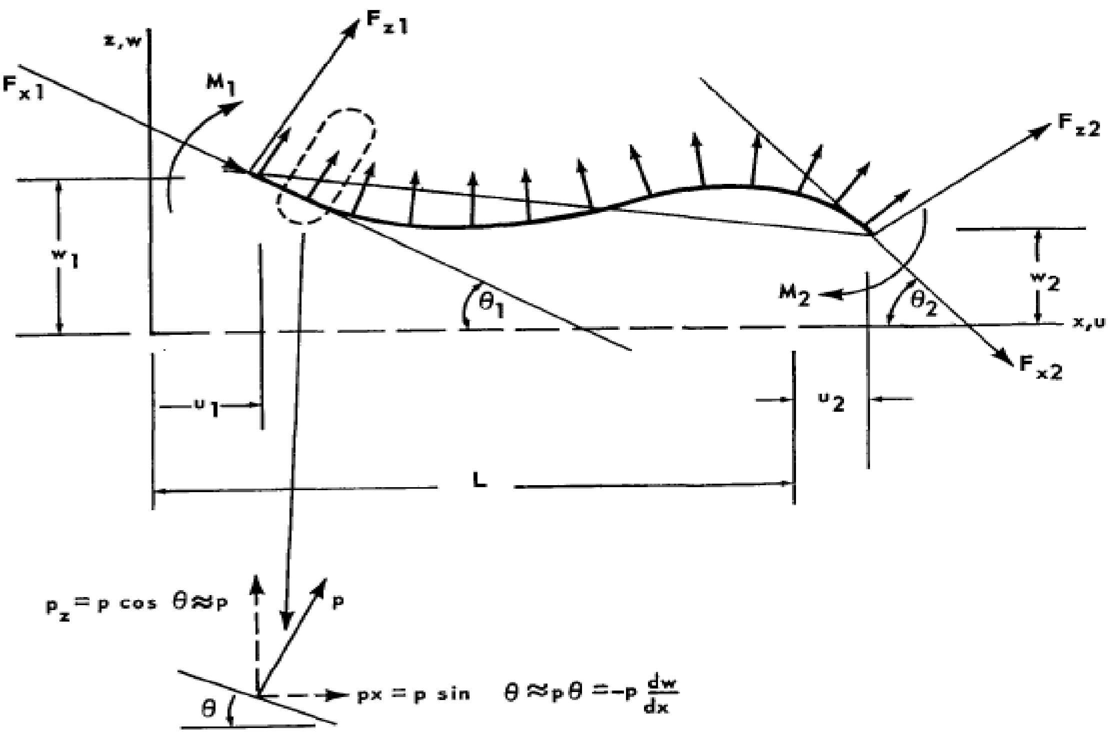


Figure 3.2 Loads normal to the deformed element

$$\left.\begin{aligned}
P_z &= p \\
P_x &= p w_{,z} \\
\bar{F}_{Xi} &= \bar{F}_{Xi} + F_{Zi}\,\theta_{\underline{i}} \\
\bar{F}_{zi} &= F_{zi} + F_{xi}\,\theta_{\underline{i}} \\
\bar{M}_i &= M_i
\end{aligned}\right\} \qquad (3.16)$$

Then

$$\delta v = \left(F_{Xi} + F_{Zi}\theta_{\underline{i}}\right)\delta u_i + \left(F_{Xi} + F_{Zi}\theta_{\underline{i}}\delta w_i\right) + m_i\,\delta\theta_i + \int_L \left[p\delta w - p w_X \delta u_{,x}\right]dx \qquad (3.17)$$

Case ii. Loads remain parallel to their original direction

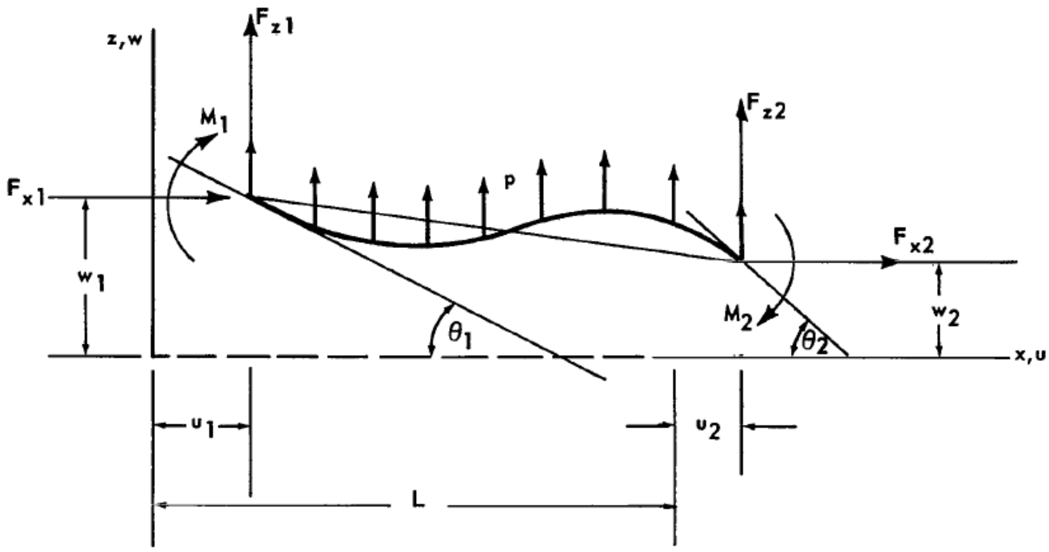


Figure 3.3 Loads parallel to their original direction

$$\left.\begin{aligned}
P_z &= p \\
P_x &= 0 \\
\bar{F}_{Xi} &= F_{Xi} \\
\bar{F}_{zi} &= F_{zi} \\
\bar{M}_i &= M_i
\end{aligned}\right\} \qquad (3.18)$$

$$\delta v = F_{Xi}\,\delta u_i + F_{zi}\,\delta w_i + M_i\,\delta\theta_i + \int_L p\delta w dx \qquad (3.19)$$

Case iii Loads remain directed towards a fixed point

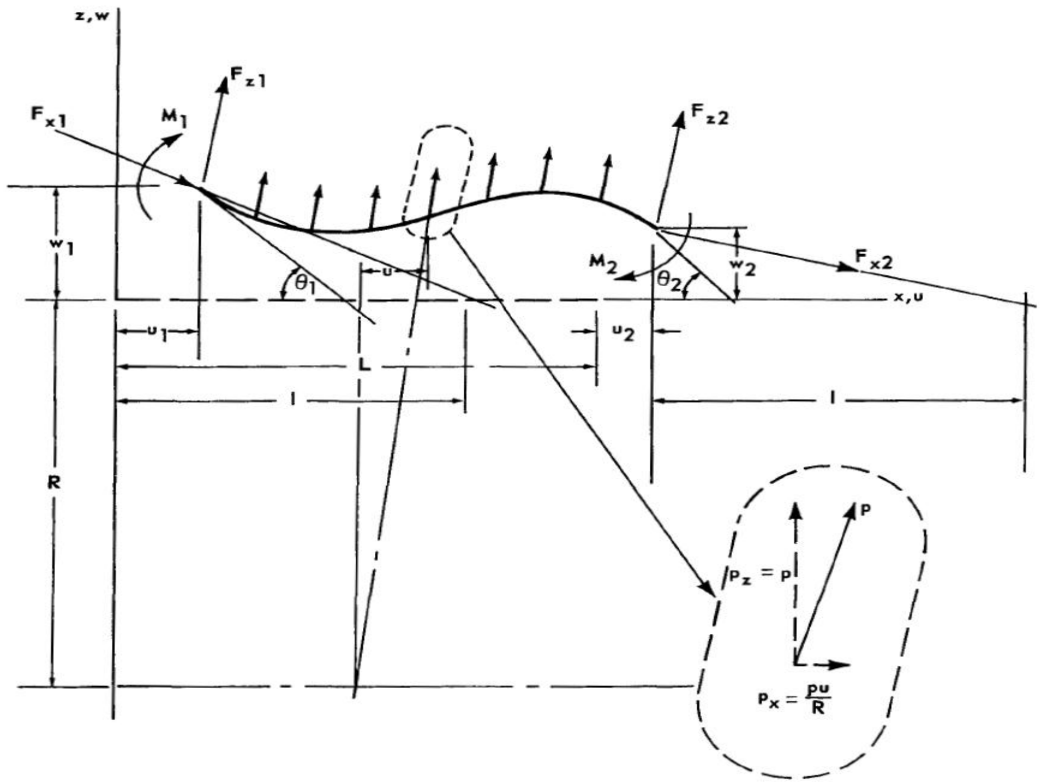


Figure 3.4 Loads directed towards a fixed point

$$\left.\begin{aligned}
P_z &= p\\
P_x &= \frac{pu}{R}\\
\bar{F}_{Xi} &= F_{Xi} + F_{zi}\frac{ui}{R}\\
\bar{F}_{zi} &= F_{zi} - F_{xi}\frac{wi}{1}\\
\bar{M}_i &= M_i
\end{aligned}\right\} \qquad (3.20)$$

$$\delta v = \left(F_{Xi} + F_{zi}\frac{ui}{R}\right)\delta u_i + \left(F_{zi} - F_{xi}\frac{wi}{1}\right)\delta w_i + m_i\,\delta\theta_i + \int_L \left[\,p\delta w + \frac{Pu}{R}\,\delta u\right]dx \qquad (3.21)$$

Substituting equation (3.14),(3.17),(3.19), and (3.20) for the three load cases into equation (3.7), we obtain the following;

Case i

$$\int_L \left[EA\left(u_{,x} + \frac{1}{2} w_{,x}^2 + \frac{3}{2} u_{,x}^2\right) \delta u_{,x} + EA\left(u_{,x}\, w_{,x}\right)\delta w_{,x} + EIw_{,xx}\, \delta w_{,x}\right] dx =$$

$$\left(F_{Xi} + F_{Zi}\,\theta_i\right)\delta u_{,i} + \left(F_{zi} + F_{zi}\,\theta_i\right)\delta w_{,i} + M_i\,\delta\theta_i + \left[\int_L p\delta w - pw_{,x}\, \delta u\right] dx \qquad (3.22)$$

Case ii

$$\int_L \left[EA\left(u_{,x} + \frac{1}{2} w_{,x}^2 + \frac{3}{2} u_{,x}^2\right) \delta u_{,x} + EA\left(u_{,x}\, w_{,x}\right)\delta w_{,x} + EIw_{,xx}\, \delta w_{,xx}\right] dx =$$

$$\left(F_{Xi}\,\delta u_i + F_{Zi}\,\delta w_i + M_i\,\delta\theta_i + \int_L p\delta w\, dx \qquad (3.23)$$

Case iii

$$\int_L \left[EA\left(u_{,x} + \frac{1}{2} w_{,x}^2 + \frac{3}{2} u_{,x}^2\right) \delta u_{,x} + EA\left(u_{,x}\, w_{,x}\right)\delta w_{,x} + EIw_{,xx}\, \delta w_{,xx}\right] dx$$

$$=\left(F_{Xi} + F_{zi}\,\frac{ui}{R}\right)\delta u_i + \left(F_{zi} - F_{xi}\,\frac{wi}{1}\right)\delta w_i + M_i\,\delta\theta_i + \int_L \left[p\delta w + \frac{Pu}{R}\,\delta u\right] d \qquad (3.24)$$

From the displacement functions, equations (3.5) and (3.6)

$$u_{,x} = u_{,i}\, \gamma_{i,x}$$

$$w_{,x} = w_i\, \emptyset_{i,x}$$

$$w_{,xx} = w_i\, \emptyset_{i,xx} \qquad (3.25)$$

And the above variation can be written,

$$\delta u = \gamma_i\, \delta u_i$$

$$\delta u_{,x} = \gamma_{i,x}\, \delta u_i$$

$$\delta w = \emptyset_i\, \delta w_i$$

$$\delta w_{,x} = \emptyset_{i,x}\, \delta w_i$$

$$\delta w_{,xx} = \emptyset_{i,xx}\, \delta w_i \qquad (3.26)$$

Upon substituting equation (3.25) and (3.26) in to equation (3.2) through (3.24) and recalling that $\theta_i = w_{1+2}$, following equations are obtained

Case i

$$\int_L \{EA[u_j\gamma_{j,}\tfrac{1}{2}w_k w_j \emptyset_{k,x}\emptyset_{j,x} + \tfrac{3}{2}u_k u_j \gamma_{k,x}\gamma_j)\gamma_{i,x}\delta u_i + (u_j\gamma_{j,x}w_k\emptyset_{k,x})\emptyset_{i,x}\delta w_i]$$

$$+EIw_j\emptyset_{i,xx}\emptyset_{j,xx}\delta w_i\}dx + (F_{Xi} + F_{zi}w_{\underline{1}+2})\delta u_i + (F_{zi} + F_{xi}w_{\underline{1}+2})\delta w_i$$

$$+M_i\delta w_{1+2}\int_L[p\emptyset_i\delta w_i - pw_j\emptyset_{j,x}\gamma_i\delta u_i]dx=0 \qquad (3.27)$$

Case ii

$$\int_L \{EA[u_j\gamma_{j,x} + \tfrac{1}{2}w_k w_j \emptyset_{k,x}\emptyset_{k,x}\emptyset_j + \tfrac{3}{2}u_k u_j \gamma_{k,x}\gamma_{,xj})\gamma_{i,x}\delta u_i + (u_j\gamma_{j,x}w_k\emptyset_{k,x})\emptyset_{i,x}\delta w_i]$$

$$+EIw_j\emptyset_{i,xx}\emptyset_{j,xx}\delta w_i\}dx + F_{xi}\delta u_i - F_{zi}\delta w_i - M_i\delta w_{1+} - p\emptyset_i\delta w_i\ dx = 0 \qquad (3.28)$$

Case iii

$$\int_L \{EA[(u_j\gamma_{j,x} + \tfrac{1}{2}w_k w_j \emptyset_{k,x}\emptyset_{k,x}\emptyset_{j,x} + \tfrac{3}{2}u_k u_j \gamma_{k,x}\gamma_j)\gamma_{i,x}\delta u_i + (u_j\gamma_{j,x}w_k\emptyset_{k,x})\emptyset_{i,x}\delta w_i]$$

$$+EIw_j\emptyset_{i,xx}\emptyset_{j,xx}\delta w_i\}dx - (F_{Xi} + F_{zi}\tfrac{\underline{ui}}{R})\delta u_i - (F_{zi} - F_{xi}\tfrac{\underline{wi}}{1})\delta w_i - M_i\delta w_{1+2}$$

$$\int_L (p\emptyset_i\delta w_i + \tfrac{P}{R}u_j\gamma_{j,}\gamma_{i,\delta u_i})dx=0 \qquad (3.29)$$

For independent virtual displacement, δu_i and δw_i, the equilibrium equations are obtained from Equations (3.27) through (3.29)

Case i

$$\int_L [EA\ (u_j\gamma_{j,x}\gamma_{i,x} + \tfrac{1}{2}w_k w_j \emptyset_{k,x}\emptyset_{j,x}\gamma_{i,x} + \tfrac{3}{2}u_k u_j \gamma_{k,x}\gamma_{j,x}\gamma_{i,x}) + pw_j\emptyset_{j,x}\gamma_i]dx - Q_{xi} -$$

$$F_{zi}w_{\underline{1}+2} = 0 \qquad (3.30)$$

$$\int_L [EAu_j\gamma_{j,x}w_k\emptyset_{k,x}\emptyset_{i,x} + EAw_j\emptyset_{i,xx}\emptyset_{j,xx} - p(\emptyset_i)]dx - Q_{zi} + F_{xi}w_{\underline{1}+2} = 0 \qquad (3.31)$$

Case ii

$$\int_L [EA\ (u_j\gamma_{j,x}\gamma_{i,x} + \tfrac{1}{2}w_k w_j \emptyset_{k,x}\emptyset_{j,x}\gamma_{i,x} + \tfrac{3}{2}u_k u_j \gamma_{k,x}\gamma_{j,x}\gamma_{i,x})]dx - Q_{xi} = 0 \qquad (3.32)$$

$$\int_L [EAu_j\gamma_{j,x}w_k\emptyset_{k,x}\emptyset_{i,x} + EIw_j\emptyset_{i,xx}\emptyset_{j,xx} - p\emptyset_i]dx - Q_{Zi} = 0 \qquad (3.33)$$

Case iii

$$\int_L [EA(u_j\gamma_{j,x}\gamma_{i,x} + \tfrac{1}{2}w_k w_j \emptyset_{k,x}\emptyset_{j,x}\gamma_{i,x} + \tfrac{3}{2}u_k u_j \gamma_{k,x}\gamma_{j,x}\gamma_{i,x})$$

$$-\tfrac{P}{R}u_j\gamma_j\gamma_i]dx - Q_{xi} - F_{zi}\tfrac{\underline{ui}}{R} = 0 \qquad (3.34)$$

$$\int_L (EAu_j\gamma_{j,x}w_k\emptyset_{k,x}\emptyset_{i,x} + EIw_j\emptyset_{i,xx}\emptyset_{j,xx} - p\emptyset_i)dx - Q_{xi} - F_{xi}\tfrac{\underline{wi}}{1} = 0 \qquad (3.35)$$

Equations (30) through (35) are the final equilibrium equations for the beam column element.

d) Bifurcation theory of instability:

Let a solution of the equilibrium equations be$u_1^0 w_1^0$. According to the bifurcation concept of instability, at the instability load magnitude there is another set of displacement, arbitrarily close to the first set, which also satisfies the equations of equilibrium. We denote this second set of nodal displacement by $u_i^0 + u_i^1, w_i^0 + w_i^2$ Upon substituting this new solution into the equilibrium equation, the results for Case 1

$$\int_L \{EA[(u_j^0+u_j^1)\gamma_{j,x}\gamma_{i,x}+\frac{1}{2}(w_k^0 + w_k^1)(w_j^0 + w_j^1)\emptyset_{k,x}\emptyset_{j,x}\gamma_{i,x}+\frac{3}{2}(u_k^0 + u_k^1)$$

$$u_j^0+u_j^1)\gamma_{k,x}\gamma_{j,x}\gamma_{i,x}]+p(w_j^0 + w_j^1)\emptyset_{j,x}\gamma_i\} - Q_{xi}\text{-}F_{zi}(w_{\underline{i}+2}^0 + w_{\underline{i}+2}^1)=0 \quad (3.36)$$

$$[EA(u_j^0+u_j^1)(w_k^0 + w_k^1)\gamma_{j,x}\emptyset_{k,x}\emptyset_{i,x}+EI(w_j^0 + w_j^1)\emptyset_{i,xx}\emptyset_{j,xx}p\emptyset_i]dx - Q_{zi}+$$

$$F_{xi}(w_{\underline{i}+2}^0 + w_{\underline{i}+2}^1) \quad (3.37)$$

Similar results are obtained for cases ii and iii

If these equations are expanded, if the q_i^0 state terms (which themselves satisfy the equations) are canceled, and if only linear terms in the arbitrarily small q_i^1 state are retained, the following sets of equations result

Case i

$$\int_L [EA\ (u_j^1\gamma_{j,x}\gamma_{i,x} + w_j^1 w_k^0\emptyset_{k,x}\emptyset_{j,x}\gamma_{i,x} + \frac{3}{2}u_j^1\gamma_{k,x}\gamma_{j,x}\gamma_{i,x}) + pww_j^1\emptyset_{j,x}\gamma_{i,}]dx - F_{zi}\theta_{\underline{i}}^0 = 0$$

(3.38)

$$\int_L [EA\ (u_j^1 w_k^0\emptyset_{i,x}\emptyset_{k,x}\gamma_{j,x} + w_j^1 u_k^0\emptyset_{i,x}\emptyset_{j,x}\gamma_{k,x})+EIw_j^I\emptyset_{i,XX}\emptyset_{j,xx}] + f_{xi}\,\theta_{\underline{i}}^1 = 0 \quad (3.39)$$

Case ii

$$\int_L [EA\ (u_j^1\gamma_{j,x}\gamma_{i,x} + w_j^1 w_k^0\emptyset_{k,x}\emptyset_{j,x}\gamma_{i,x} + \frac{3}{2}u_j^1\gamma_{k,x}\gamma_{j,x}\gamma_{i,x})\ ]dx = 0 \quad (3.40)$$

$$\int_l [ea\left(u_j^1 w_k^0\emptyset_{i,x}\emptyset_{k,x}\gamma_{j,x} + w_j^1 u_k^0\emptyset_{i,x}\emptyset_{j,x}\gamma_{k,x}\right) + eiw_j^i\emptyset_{i,xx}\emptyset_{j,xx}]dx=0 \quad (3.41)$$

Case iii

$$\int_L [EA\ (u_j^1\gamma_{j,x}\gamma_{i,x} + w_j^1 w_k^0\emptyset_{k,x}\emptyset_{j,x}\gamma_{i,x} + \frac{3}{2}u_j^1\gamma_{k,x}\gamma_{j,x}\gamma_{i,x})] - \frac{P}{R}u_j^1\gamma_j\gamma_i]\ dx\ \text{-}F_{zi}\frac{ui}{R} = 0$$

(3.42)

$$\int_L [EA\left(u_j^1 w_k^0 \emptyset_{i,x} \emptyset_{k,x} \gamma_{j,x} + w_j^1 u_k^0 \emptyset_{i,x} \emptyset_{j,x} \gamma_{k,x}\right) + EI w_j^1 \emptyset_{i,xx} \emptyset_{j,xx}]dx + F_{xi} \frac{w_i^1}{1} = 0 \quad (3.43)$$

These equations may be expressed in matrix notation for all three cases as follows

$$\big[[K]+[G]+[L_P]+[L_F]\big]\{q^1\} = \{0\} \quad (3.44)$$

Or

$[k^1]\ \{q^1\} = \{0\}$

Where

$[k^1] = [K] + [G] + [L_P] + [L_F]$

$$[k]\ [q^1] = \left(\begin{array}{c:c} \int_L EA\gamma_{j,x}\gamma_{i,x}dx & 0 \\ \hdashline 0 & \\ \int_L EA\gamma_{j,x}\gamma_{i,x} & \end{array}\right) \left(\begin{array}{c} u_J^1 \\ \hdashline w_J^1 \end{array}\right) \quad (3.45)$$

$$[k]\ [q^1] = \left(\begin{array}{c:c} \int_L 3EA\, u_k^0 \gamma_{k,x}\gamma_{i,x}dx & \int_L 3EA\, w_k^0 \emptyset_{k,x} \emptyset_{j,x} \gamma_{i,x}dx \\ \hdashline \int_L 3EA\, w_k^0 \emptyset_{i,x} \emptyset_{k,x} \gamma_{j,x}dx & \int_L 3EA\, u_k^0 \emptyset_{i,x} \emptyset_{j,x} \gamma_{k,x}dx \end{array}\right) \left(\begin{array}{c} u_J^1 \\ \hdashline w_J^1 \end{array}\right)$$

(3.46)

For all cases

The L matrices are different for each case

Case i

$$[L_P]\{q^1\} = \left(\begin{array}{c:c} 0 & \int_L p\varnothing_{j,}\gamma_{i,x}dx \\ \hdashline 0 & 0 \end{array}\right)\left[\begin{array}{c} u_J^1 \\ \hdashline w_J^1 \end{array}\right] \tag{3.47}$$

Case ii

$$[L_F]\{q^1\} = \begin{pmatrix} -F_{zi}\theta_{\underline{i}}^1 \\ -F_{xi}\theta_{\underline{i}}^1 \end{pmatrix} \tag{3.48}$$

$$[L_P] = [L_F] = [0] \tag{3.49}$$

Case iii

$$[L_P]\{q^1\} = \left(\begin{array}{c:c} -\int_L \frac{P}{R}\gamma_{j,}\gamma_i dx & 0 \\ \hdashline 0 & 0 \end{array}\right)\left[\begin{array}{c} u_J^1 \\ \hdashline w_J^1 \end{array}\right] \tag{3.50}$$

$$[L_F]\{q^1\} = \begin{pmatrix} F_{zi}\dfrac{u_{\underline{i}}^1}{R} \\ F_{zi}\dfrac{w_{\underline{i}}^1}{1} \end{pmatrix} \tag{3.51}$$

d) Conventional stiffness matrix K

Equation (3.45) represents the convectional beam column element stiffness matrix, which is suitable for use when there is no interest in nonlinear effects or instability. The elements of this matrix may be derived from the general expression, equation (3.45). For example

$\gamma_1=(1-\frac{x}{l}),\emptyset_1 = (1\text{-}3\frac{x^2}{L^2}+\frac{x^3}{L^3})$

$\gamma_{1,x} =-\frac{1}{L},\ \emptyset_{1,x}= -6\frac{x}{L^2}+6\frac{x^2}{L^3},\ \emptyset_{1,xx} = -\frac{6}{L^2}+6\frac{12x}{L^3}$

Therefore

$K_{11} = \int_0^l EA_{\gamma i,x}\gamma_{i,x}dx= \int_0^l EA\ (\frac{1}{L^2})\ dx =\frac{AE}{L}$

$K_{33} = \int_0^l EA\emptyset_{i,xx_{,xx}}\emptyset_{i,xx}dx= \int_0^l EA\ (-\frac{6}{L^2}+\frac{12x}{L^3})dx =\frac{12EI}{L^3}$

Other elements of K are obtained similarly to yield the stiffness matrix K from the matrix equation 3.52 .

$$
\begin{matrix} u_1 & u_2 & w_1 & w_2 & \theta_1 & \theta_2 \end{matrix}
$$

$$
[K] = \begin{bmatrix}
\frac{AE}{L} & -\frac{AE}{L} & 0 & 0 & 0 & 0 \\
-\frac{AE}{L} & \frac{AE}{L} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{12EI}{L^3} & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & -\frac{6EI}{L^2} \\
0 & 0 & -\frac{12EI}{-L^3} & \frac{12EI}{L^3} & \frac{6E}{L^2} & \frac{6EI}{L^2} \\
0 & 0 & -\frac{6EI}{L^2} & \frac{6EI}{L^2} & \frac{4EI}{L} & \frac{2EI}{L} \\
0 & 0 & -\frac{6EI}{L^2} & \frac{6EI}{L^2} & \frac{2EI}{L} & \frac{4EI}{L}
\end{bmatrix}
\begin{pmatrix} F_X^1 \\ F_X^2 \\ F_Z^1 \\ F_Z^2 \\ M_1 \\ M_2 \end{pmatrix}
\qquad (3.52)
$$

e) Geometric Stiffness Matrix G.

Equation 3.46; is the general form of the geometric stiffness matrix which accounts for the effect of loads existing in the element on the stiffness of the element. For example, it is well known that the axial load in a beam column has an appreciable effect on the lateral stiffness. The geometric stiffness matrix is an adjustment to the conventional stiffness matrix to account for such effects. Such matrices have also been referred to in the literature as stability coefficient matrices and incremental stiffness matrices. Elements of the G matrix will now be derived

Evaluation of $\int_0^L \emptyset_{k,x}\emptyset_{j,x}\,\gamma_{i,x}dx$

For i = 1

$\gamma_{i,x} = \gamma_{i,x} = \frac{d}{dx}(1-\frac{x}{L})$

$= -\int_0^L \emptyset_{k,x}\emptyset_{j,x}\,\gamma_{i,x}dx$

$= -\frac{1}{L}\int_0^L \emptyset_{k,x}\emptyset_{j,x}\;dx$

When k=1, j=1

$\emptyset_{1,x}\frac{d}{dx}(1\text{-}3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}) = -\frac{6x}{L^2} + \frac{6x}{L^3}\text{-}\frac{1}{L}\int_0^l(-\frac{6x}{L^2} + \frac{6x}{L^3})^2\;dx$

$= -\frac{1}{L}\int_0^l(\frac{36x^2}{L^4} - \frac{72x^3}{L^5} + \frac{36x^4}{L^6})\;dx$

$= -\frac{1}{L}(\frac{12x^3}{L^4} - \frac{18x^4}{L^5} + \frac{36}{5}\frac{x^5}{L^6})_0^L = -\frac{1}{L}[\frac{12x^3}{L^4} - \frac{18x^4}{L^5} + \frac{36}{5}\frac{x^5}{L^6}]$

$= \frac{12}{L^2} + \frac{18}{L^2} - \frac{36}{5L^2} = (-\frac{60}{5} + \frac{90}{5} - \frac{36}{5})\frac{1}{L^2} =- \frac{6}{5L^2}$

For i=2

$\gamma_{2,x} = -\frac{d}{dx}, (\frac{x}{L}) = \frac{1}{L}$

$\int_0^L \emptyset_{k,x} \emptyset_{j,x}\, \gamma_{2,x} dx = -\int_0^L \emptyset_{k,x} \emptyset_{j,x}\, \gamma_{1,x} dx$

Other values of j and are evaluated similarly to yield

$$\int_0^L \emptyset_{k,x} \emptyset_{j,x}\, \gamma_{1,x} dx = \begin{pmatrix} w_1 & w_2 & \theta_1 & \theta_2 \\ -\frac{6}{5L^2} & \frac{6}{5L^2} & \frac{1}{10L} & \frac{1}{10L} \\ \frac{6}{5L^2} & -\frac{6}{5L^2} & -\frac{1}{10L} & -\frac{1}{10L} \\ \frac{1}{10L} & -\frac{1}{10L} & -\frac{4}{30} & -\frac{4}{30} \\ \frac{1}{10L} & -\frac{1}{10L} & \frac{1}{30} & -\frac{4}{30} \end{pmatrix} \cdot \left[\emptyset_k \emptyset_j\, \gamma_1\right] \quad (3.53)$$

$$\int_0^L \emptyset_{k,x} \emptyset_{j,x}\, \gamma_{2,x} dx = \begin{pmatrix} w_1 & w_2 & \theta_1 & \theta_2 \\ \frac{6}{5L^2} & -\frac{6}{5L^2} & -\frac{1}{10L} & -\frac{1}{10L} \\ -\frac{6}{5L^2} & \frac{6}{5L^2} & \frac{1}{10L} & \frac{1}{10L} \\ -\frac{1}{10L} & \frac{1}{10L} & \frac{4}{30} & \frac{1}{30} \\ -\frac{1}{10L} & \frac{1}{10L} & \frac{1}{30} & \frac{4}{30} \end{pmatrix} \left[\emptyset_k \emptyset_j\, \gamma_2\right] \quad (3.54)$$

Evaluation of $\int_0^L \emptyset_{k,x} \emptyset_{j,x}\, \gamma_{K,x} dx$

For i=1

$$\emptyset_{i,x} = \emptyset_{i,x} = \frac{6x}{L^2} + \frac{6x^2}{L^3}$$

j=1, k=1

$$\int_0^L \emptyset_{i,x} \emptyset_{1,x}\, \gamma_{i,x}\, dx = -\frac{6}{5L^2}$$

When j=1, k=2

$$\int_0^L \left(\frac{6x}{L^2} + \frac{6x^2}{L^3}\right)\left(-\frac{6}{5L^2} + + \frac{6x^2}{L^3}\right)\left(-\frac{1}{L}\right) = \frac{6}{5L^2}$$

$$\int_0^L \emptyset_{1,x} \emptyset_{k,x}\, \gamma_{j,x}\, dx = \begin{pmatrix} -\frac{6}{5L^2} & \frac{6}{5L^2} \\ \frac{6}{5L^2} & -\frac{6}{5L^2} \\ \frac{1}{10L} & -\frac{1}{10L} \\ \frac{1}{10L} & -\frac{1}{10L} \end{pmatrix} \left[\emptyset_1 \emptyset_k\, \gamma_j\right] \tag{3.55}$$

$$\int_0^L \emptyset_{2,x} \emptyset_{k,x}\, \gamma_{j,x}\, dx = \begin{pmatrix} \frac{6}{5L^2} & -\frac{6}{5L^2} \\ -\frac{6}{5L^2} & \frac{6}{5L^2} \\ -\frac{1}{10L} & \frac{1}{10L} \\ -\frac{1}{10L} & \frac{1}{10L} \end{pmatrix} \left[\emptyset_2 \emptyset_k\, \gamma_j\right] \tag{3.56}$$

$$\int_0^L \emptyset_{3,x}\emptyset_{k,x}\,\gamma_{j,x}dx = \begin{pmatrix} \frac{1}{10L} & -\frac{1}{10L} \\ -\frac{1}{10L} & \frac{1}{10L} \\ -\frac{4}{30} & \frac{4}{30} \\ \frac{1}{30} & -\frac{1}{30} \end{pmatrix} \left[\emptyset_3\emptyset_k\,\gamma_j\right] \tag{3.57}$$

$$\int_0^L \emptyset_{4,x}\emptyset_{k,x}\,\gamma_{j,x}dx = \begin{pmatrix} \frac{1}{10L} & -\frac{1}{10L} \\ -\frac{1}{10L} & \frac{1}{10L} \\ \frac{1}{30} & -\frac{1}{30} \\ -\frac{4}{30} & \frac{4}{30} \end{pmatrix} \left[\emptyset_4\emptyset_k\,\gamma_j\right] \tag{3.58}$$

Evaluation of $\int_0^L \gamma_{k,x}\gamma_{j,x}\,\gamma_{i,x}dx$

$\gamma_{1,x} = -\frac{1}{L}, \gamma_{2,x} = \frac{1}{L}$

Then,

$$\int_0^L \gamma_{k,x}\gamma_{j,x}\,\gamma_{1,x}dx = \begin{pmatrix} -\frac{1}{L^2} & \frac{1}{L^2} \\ \frac{1}{L^2} & -\frac{1}{L^2} \end{pmatrix} [\gamma_k\gamma_j\gamma_1] \tag{3.59}$$

$$\int_0^L \gamma_{k,x}\gamma_{j,x}\,\gamma_{2,x}dx = \begin{pmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{pmatrix} [\gamma_k \gamma_j \gamma_2] \tag{3.60}$$

Evaluation of $\int_0^L \emptyset_{i,x}\emptyset_{j,x}\,\gamma_{k,x}dx$

$$\int_0^L \emptyset_{i,x}\emptyset_{j,x}\,\gamma_{k,x}dx = \left(\emptyset_i \emptyset_j \gamma_k \right) = \left(\emptyset_i \emptyset_k \gamma_j \right) \tag{3.61}$$

The above matrices are now multiplied by the q_k^0 state displacement as indicated in equation (3.46)

$$w_k^0 \int_0^L \emptyset_{k,x}\emptyset_{j,x}\,\gamma_{1,x}dx = [w_1^0 \;\; w_2^0 \;\; \theta_1^0 \;\; \theta_2^0] \begin{pmatrix} -\frac{6}{5L^2} & \frac{6}{5L^2} & \frac{1}{10L} & \frac{1}{10L} \\ \frac{6}{5L^2} & -\frac{6}{5L^2} & -\frac{1}{10L} & -\frac{1}{10L} \\ \frac{1}{10L} & -\frac{1}{10L} & -\frac{4}{30} & \frac{1}{30} \\ \frac{1}{10L} & -\frac{1}{10L} & \frac{1}{30} & -\frac{4}{30} \end{pmatrix}$$

$$= [w^0]\;[\emptyset_k \emptyset_j \gamma_1] \tag{3.62}$$

$$w_k^0\int_0^L \emptyset_{k,x}\emptyset_{j,x}\,\gamma_{2,x}dx = [w_1^0 \;\; w_2^0 \;\; \theta_1^0 \;\; \theta_2^0]\begin{pmatrix} \frac{6}{5L^2} & -\frac{6}{5L^2} & -\frac{1}{10L} & -\frac{1}{10L} \\ -\frac{6}{5L^2} & \frac{6}{5L^2} & \frac{1}{10L} & \frac{1}{10L} \\ -\frac{1}{10L} & \frac{1}{10L} & \frac{4}{30} & -\frac{1}{30} \\ -\frac{1}{10L} & \frac{1}{10L} & -\frac{1}{30} & \frac{4}{30} \end{pmatrix}$$

$$= [w^0]\,[\emptyset_k\emptyset_j\,\gamma_2] \qquad (3.63)$$

$$w_k^0\int_0^L \emptyset_{1,x}\emptyset_{k,x}\,\gamma_{j,x}dx = [w_1^0 \;\; w_2^0 \;\; \theta_1^0 \;\; \theta_2^0]\begin{pmatrix} -\frac{6}{5L^2} & \frac{6}{5L^2} \\ \frac{6}{5L^2} & -\frac{6}{5L^2} \\ \frac{1}{10L} & -\frac{1}{10L} \\ \frac{1}{10L} & -\frac{1}{10L} \end{pmatrix} = [w^0]\,[\emptyset_1\emptyset_k\,\gamma_j] \qquad (3.64)$$

$$w_k^0\int_0^L \emptyset_{2,x}\emptyset_{k,x}\,\gamma_{j,x}dx = [w^0]\,[\emptyset_2\emptyset_k\,\gamma_j] \qquad (3.65)$$

$$w_k^0\int_0^L \emptyset_{3,x}\emptyset_{k,x}\,\gamma_{j,x}dx = [W^0]\,[\emptyset_3\emptyset_k\,\gamma_j] \qquad (3.66)$$

$$w_k^0\int_0^L \emptyset_{4,x}\emptyset_{k,x}\,\gamma_{j,x}dx = [w^0]\,[\emptyset_4\emptyset_k\,\gamma_j] \qquad (3.67)$$

$$u_k^0\int_0^L \emptyset_{1,x}\emptyset_{j,x}\,\gamma_{k,x}dx = [u_1^0 \; u_2^0]\begin{pmatrix} -\frac{6}{5L^2} & \frac{6}{5L^2} & \frac{1}{10L} & \frac{1}{10L} \\ \frac{6}{5L^2} & -\frac{6}{5L^2} & -\frac{1}{10L} & -\frac{1}{10L} \end{pmatrix}$$

$$= [u^0]\ [\emptyset_1 \emptyset_j\, \gamma_k] \quad (3.68)$$

$$u_k^0 \int_0^L \emptyset_{2,x} \emptyset_{j,x}\, \gamma_{k,x} dx = [u^0]\ [\emptyset_2 \emptyset_j\, \gamma_k] \quad (3.69)$$

$$u_k^0 \int_0^L \emptyset_{3,x} \emptyset_{j,x}\, \gamma_{k,x} dx = [u^0]\ [\emptyset_3 \emptyset_j\, \gamma_k] \quad (3.70)$$

$$u_k^0 \int_0^L \emptyset_{4,x} \emptyset_{j,x}\, \gamma_{k,x} dx = [u^0]\ [\emptyset_4 \emptyset_j\, \gamma_k] \quad (3.71)$$

$$u_k^0 \int_0^L \gamma_{k,x} \gamma_{j,x}\, \gamma_{1,x} dx = [u_1^0\ u_2^0] \begin{pmatrix} -\frac{1}{L^2} & \frac{1}{L^2} \\ \frac{1}{L^2} & -\frac{1}{L^2} \end{pmatrix} = [u^0]\ [\gamma_k \gamma_j\, \gamma_1] \quad (3.72)$$

$$u_k^0 \int_0^L \gamma_{k,x} \gamma_{j,x}\, \gamma_{2,x} dx = [u^0]\ [\gamma_k \gamma_j\, \gamma_2] \quad (3.73)$$

$$[G] = \begin{pmatrix} u_1^1 & u_2^1 & w_1^1 \quad w_2^1 \quad \theta_1^1 \quad \theta_2^1 \\ 3EA[u^0]\,[\gamma_k \gamma_j \gamma_1] & & EA[w^0]\,[\emptyset_k \emptyset_j \gamma_1] \\ 3EA[u^0]\,[\gamma_k \gamma_j \gamma_2] & & EA[w^0]\,[\emptyset_k \emptyset_j \gamma_2] \\ EA[w^0]\,[\emptyset_1 \emptyset_k \gamma_j] & & EA[u^0]\,[\emptyset_1 \emptyset_j \gamma_k] \\ EA[w^0]\,[\emptyset_2 \emptyset_k \gamma_j] & & EA[u^0]\,[\emptyset_2 \emptyset_j \gamma_k] \\ EA[w^0]\,[\emptyset_3 \emptyset_k \gamma_j] & & EA[u^0]\,[\emptyset_3 \emptyset_j \gamma_k] \\ EA[w^0]\,[\emptyset_3 \emptyset_k \gamma_j] & & EA[u^0]\,[\emptyset_2 \emptyset_j \gamma_k] \end{pmatrix} \quad (3.74)$$

d) Load behavior matrix L_p

The effect of applied load behavior on the element stiffness is obtained by adjusting the $\underline{k}$ matrix with the $\underline{L}$ matrix, where

$$[L] = [L_P] + [L_F] \tag{3.75}$$

The $\underline{L}$ matrix will now be derived from equations (3.47) through (3.50) for each of the three load behavior cases.

Case i

$$\int_0^L pw_j^i \emptyset_{j,x}\gamma_1 dx = \frac{1}{2}pw_1 + \frac{1}{2}pw_2 + \frac{1}{2}pw\theta_1 - \frac{1}{12}pw\theta_2$$

$$+ w_2^1\left(\frac{6x}{L^2} + \frac{6x^2}{L^3}\right) + \theta_2^1\left(\frac{2x}{L} + 3\frac{x^2}{L^2}\right)\Big]\, dx$$

$$= -\frac{1}{2}pw_1^1 + \frac{1}{2}pw_2^1 - \frac{1}{12}pL\theta_1^1 + \frac{1}{12}pL\theta_2^1$$

Then

$$[L_P] = P\begin{pmatrix} 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{L}{12} & \frac{L}{12} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{L}{12} & -\frac{L}{12} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.76}$$

$\underline{L}_F$ May be written directly from equation (3.48) in view of equations (3.38) And (3.39)

$$[L_F] = P\begin{pmatrix} 0 & 0 & 0 & 0 & -\overrightarrow{F_{Z1}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\overrightarrow{F_{Z2}} \\ 0 & 0 & 0 & 0 & -\overrightarrow{F_{X1}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\overrightarrow{F_{X2}} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.77}$$

Where $\overrightarrow{-F}$'S are proportionality constant between the indicated forces and p. the magnitude of the applied forces are assumed to remain in a constant ration to each other and to the lateral load, p, during loading of the element. This assumption is made here and in the stability criterion section for illustrates purposes. The assumption is easily altered to study buckling under pressure with specified nodal forces or buckling under nodal forces with specified pressure.

Case ii

$$[L_P] = [L_F] = [0] \tag{3.78}$$

$$\int_L \frac{P}{R} u_j^1 \gamma_j \gamma_1 dx = \int_L \frac{P}{R} \left[u_1^1\left(1 - \frac{x}{L}\right) + u_2^1 \left(\frac{x}{L}\right)\right]\left[1 - \frac{x}{L}\right] dx$$

$$= \frac{P}{R}\int_L \left[\left(1 - \frac{2x}{L} + \frac{x^2}{L^2}\right) u_1^1 + \left(\frac{x}{L} - \frac{x^2}{L^2}\right) u_2^1\right] dx$$

$$= \frac{P}{R}\left[\left(x - \frac{x^2}{L}\right) + \frac{x^3}{3L^2}\right) u_1^1 + \left(\frac{x^2}{2L} - \frac{x^3}{3L^2}\right) u_2^1\right]_0^L$$

$$= \frac{P}{R}\left[\left(L - L + \frac{L}{3}\right) u_1^1 + \left(\frac{L}{2} - \frac{L}{2}\right) u_2^1\right]_0^L$$

$$= \frac{P}{R}\left[\frac{L}{3} u_1^1 + \frac{L}{6} u_2^1\right]$$

$$\int_L \frac{P}{R} u_j^1 \gamma_j \gamma_2 dx = \frac{PL}{R}\left[\frac{1}{6} u_1^1 + \frac{1}{3} u_2^1\right]$$

$$[L_P] = P \begin{pmatrix} -\frac{L}{3R} & -\frac{L}{6R} & 0 & 0 & 0 & 0 \\ -\frac{L}{6R} & -\frac{L}{3R} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.79}$$

L_F be written directly from equation (3.51)

$$[L_F]= \begin{pmatrix} -\frac{\vec{F}_{Z_1}}{R} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\vec{F}_{Z_2}}{R} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\vec{F}_{X_1}}{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\vec{F}_{Z_2}}{R} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.80)$$

e) Transformation Matrix

The developments thus far have been in a coordinate system which was oriented so that the x-axis coincides with the element centerline. To combine several elements for solution of a particular problem, it is necessary to obtain the stiffness matrices of the elements in a common or global coordinate system. Then to determine the transformation matrix for the curved beam element, wefirst consider a transformation matrix in two dimensions of straight beam element,of length L. If the lateral and normal displacements are n and u, and if theangle that the element makes with the X direction is a, as shown in figure (3.5),then the relationship between the displacements n; u; q and X; Y; q is known. It is noted that the unknown vector does not involve the rotation angle; the essential boundary condition can be imposed with the penalty function method (Sun & Liew, 2008).

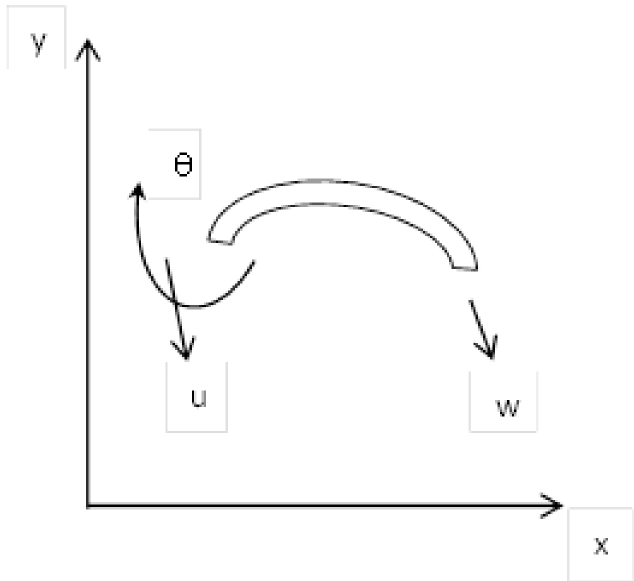


Figure 3.5 Curved beam element related to local and global displacements

$$
\begin{Bmatrix} u_1 \\ U_1 \\ \theta_1 \\ w_2 \\ W_2 \\ \theta_2 \end{Bmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} X_1 \\ Y_1 \\ \theta_1 \\ X_2 \end{Bmatrix} \quad (3.81)
$$

$d_s = \varphi d_g$

Where

ds = the local straight beam displacement vector

d_g = global coordinate displacement vector

φ = transformation matrix

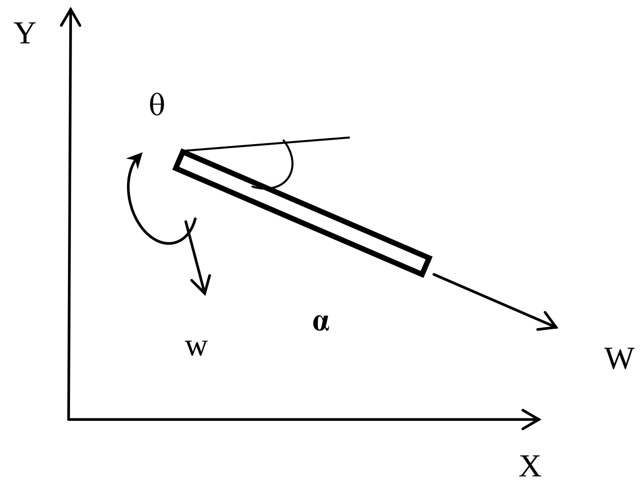


Figure 3.6: Straight beam element displacement

Consider the relationship of the displacement of the curved element w*;* u*;* qwith the displacements of the straight elements n *;* u*;* q , then as indicatedin figure 3.6.

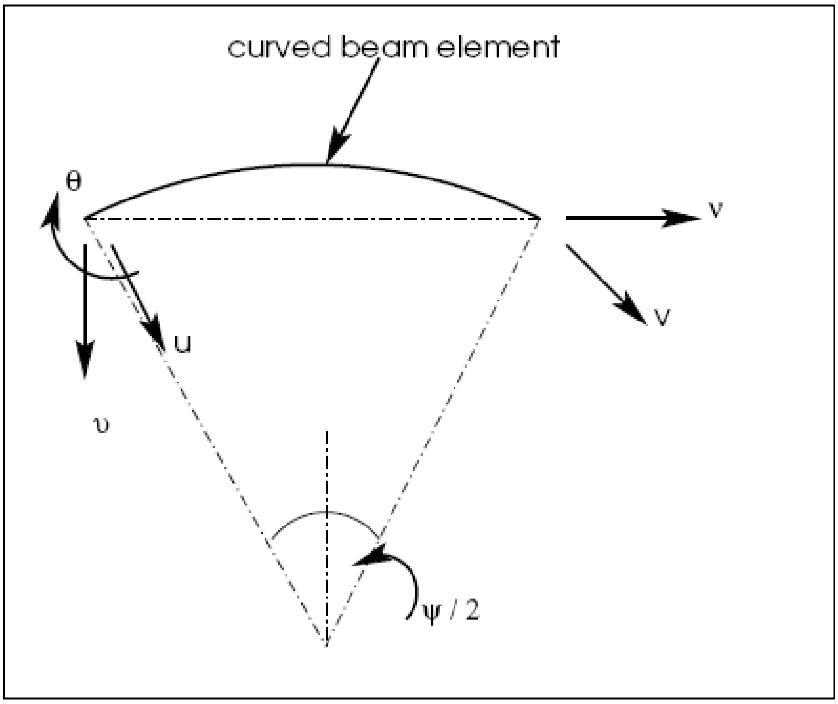


Figure 3.7 Relation of curved beam element in local and global coordinates system

$$\begin{Bmatrix} u_1 \\ U_1 \\ \theta_1 \\ w_2 \\ W_2 \\ \theta_2 \end{Bmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 & 0 & 0 & 0 \\ -\sin\psi & \cos\psi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\psi & \sin\psi & 0 \\ 0 & 0 & 0 & -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} X_1 \\ Y_1 \\ \theta_1 \\ X_2 \end{Bmatrix} \quad (3.82)$$

$d_c = \text{ɤ} d_g$

d_c = the displacement vector for a curved beam in the local coordinate system

d_g = global coordinate displacement vector

ɤ = the transformation matrix above.

The relation of the nodal displacement of thecurved beam in the local systemdcand those in the global system dgisfor the transformation matrixglobal coordinate displacement vector

$d_c = \text{ɤ}\varphi d_g$ (Kakuchi,2014)

LetT= $\text{ɤ}\varphi$ (3.83)

Since for the nodal forces we also have to transform from local to global coordinatesystem, by identifying the global resultant force component when a force isapplied in local coordinates, thus the force vector is transformed as below

$F_C = TF_g$

It follows;

$Fc = K^{(e)} d_c$

$TF_g = K^{(e)} Td_g$

$F_g = K^{(e)} Td_g$

$$K_g = T^T K^{(e)} T \tag{3.84}$$

3.2.2 Master Stiffness Matrix

The stiffness matrix for complete structure is obtained by combining the element stiffness matrices in the global coordinate system.

3.2.3 Stability Criterion

Elements with stiffness matrices of the type given by equation (3.44) may be assembled into a master stiffness matrix to represent a structure subjected to the critical (buckling) magnitude of applied loads. Thus,

$$[[\bar{k}] + p\,[\bar{k}_0]]\{\bar{q}_0\} = \{0\} \tag{3.85}$$

Where

$$p\,[\bar{k}_0] = [\bar{G}] + [\bar{L}_P] + [\bar{L}_F]$$

the null matrix of applied external loads indicates that the structural stiffness has vanished under the critical load magnitude (a physical interpretation of instability).The$\underline{\bar{K}}$, $\underline{\bar{G}}$, and $\underline{\bar{L}_P}$ matrices in equation 3.85 are obtained by assembly of element matrices as described in the preceding section. The $\underline{\bar{L}_F}$ matrix is more conveniently obtained by direct application of equations (3.48) and (3.51) to each node of the assembled structure in the global coordinate system.A nontrivial solution of equations will exist only when the determinant of the matrix of $\underline{\bar{K}} + p\,\underline{\bar{K}_0}$vanishes;

$$\text{Det}\,(\underline{\bar{K}} + p\bar{k}_0) = 0 \tag{3.86}$$

This is an eigenvalue problem where the magnitudes of applied load, p, at which instability will occur, are the eigenvalues.This formulation combines the works of Thomas,1970and Kikuchi, 2014; into a new formulation, hereunder used to formulate a code for analysis.

3.3 Developing numerical procedure for analyzing thin curved plates

The following is the numerical procedure for analysis following the analytical formulation.

i) Identify the principal theory from equation 3.1

ii) Divide the curved plate into appropriate elements and calculate the arc length (L) and the internal angle (ψ) and angle (α) of each element relative to the x axis.

iii) Enter the following specific member properties (variables).

a) Area in m^2

b) Moment of inertia I in m^4

c) Length L in m

d) Young's modulus E in N/m^2

e) (ψ) and (α) in degrees

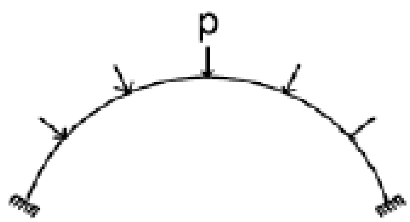
iv) For each element calculate:

a) The stiffness matrix K from the matrix from equation 3.52
b) The geometric matrix from the matrix from equation 3.74

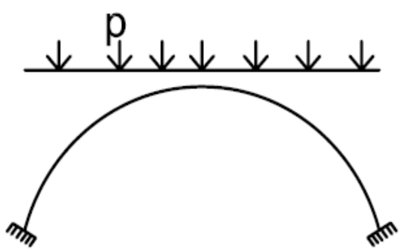
c) Select the load case and calculate the load matrix L

$[L] = [L_P] + [L_F]$

Load case i-Loads remain normal to the element

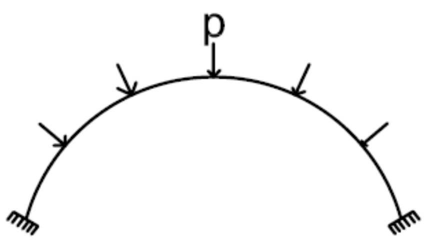


Load case ii-Loads remain parallel to the original direction



$L_p = L_F = [0]$

Load case iii-Loads remain directed towards a fixed point



d) Rearranging all the matrices in the following order from equation 3.87

$\underline{K}$, $\underline{G}$, and $\underline{L}_F$

$$[\underline{K}^1] = \begin{array}{c} \begin{matrix} U_1 & w_1 & \theta_1 & u_2 & w_2 & \theta_1 \end{matrix} \\ \left(\begin{array}{c:c} K^1_{11} & K^1_{12} \\ \hdashline K^1_{21} & K^1_{22} \end{array} \right) \end{array} \begin{Bmatrix} F_{X1} \\ F_{Z1} \\ M_1 \\ F_{X2} \\ F_{Z2} \\ M_2 \end{Bmatrix}$$

(3.87)

v) Matrix transformation

Rotate all element matrices to the global coordinate system by the operation given in equation 3.83.

vi) Calculate the master stiffness from the summation of all the matrices of the system, e.g,$K_1+K_2+K_3=k$, $G_1+G_2+G_3=G$, $L1+L_2+L_3=L$ for three elements.

viii) Calculate the determinant of the resulting 6x6 matrix from bifurcation theory

$$|\bar{K} + p\,\bar{K}_0| = 0 \tag{3.88}$$

where

$$\rho[\bar{K}_0] = [\bar{G}] + [\bar{L}_\rho] + [\bar{L}_F] = 0.$$

The buckling load p is obtained from the solution determinant in equation 3.88.

3.4 Development of finite element program for analyzing thin curved plates

The methodology approach used in this development is waterfall approach phases(Sun & Liew, 2008). The project was segmented into a hierarchy of the numerical procedures developed in section 3.3 above. This phase also involves developing the problem case for the project. A problem case provides the information that a user needs to decide whether to proceed with the program.

Other subsequent procedures are as follows:

a) Initiation / Software system requirements

This phase involved a macro level study of the project requirements. This phase also involved understanding and defining the solutions to the problem requirements and cost-benefit justification of these alternatives.

b) Analysis

This involved carrying out detailed study of the model procedure and arriving at the exact steps. The phase involved freezing the requirements before the design phase begins.

c) Design

Involved translating the identified requirements into a logical structure that can be implemented in a programming logic.

d) Construction

Involves development and integration and testing all the modules developed in the previous phase as a complete system.

e) Testing

Involves integrating and testing all the modules developed in the previous phase as a complete system. Model used in testing is as follows

1. Test planning
2. Test development – Creating the testing environment
3. Test execution – Writing the test cases/Creating & Executing the Test Script
4. Result analysis – Analysis result and Bug report
5. Bug tracking - Analyze Bugs and application errors

f) Implementation and deployment

Involves converting the new system design into operation. This involved implementing the software system and operating the software system and its functionalities (figure 3.8).

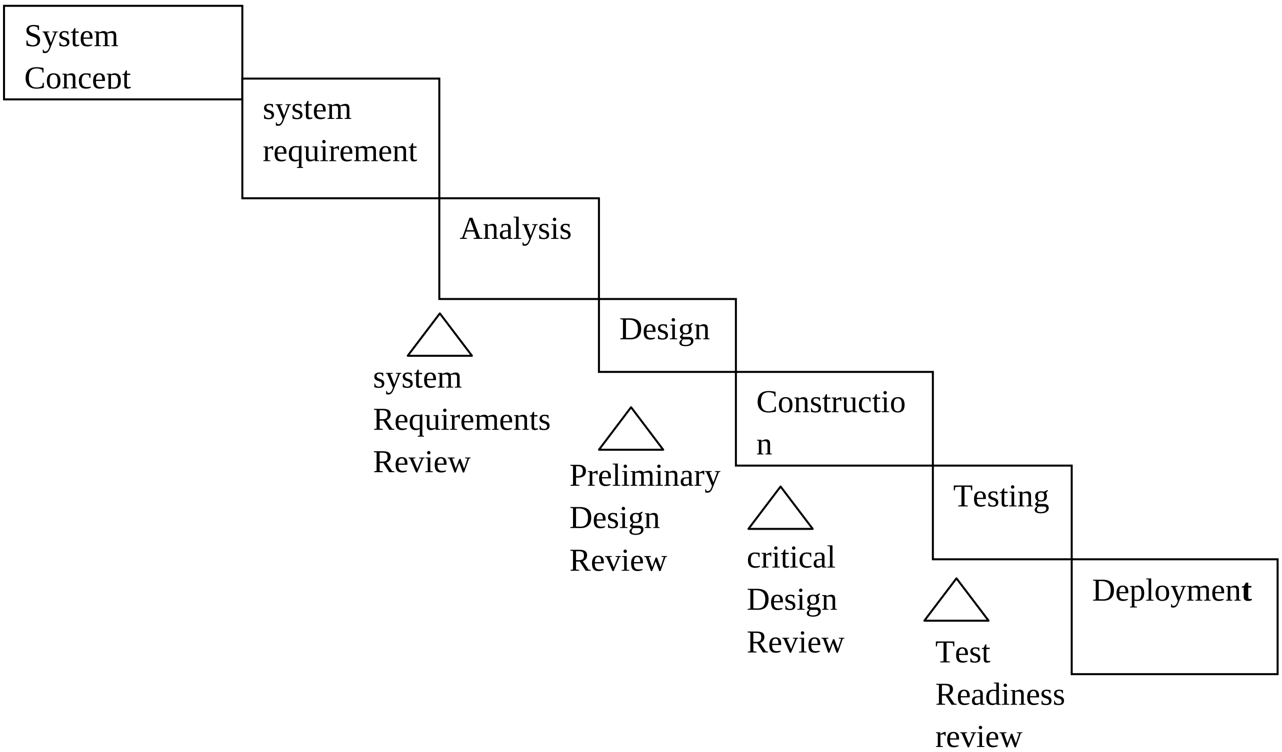


Figure 3.8 Process of system design

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1 Introduction

Table 4.1 shows results obtained using the present formulation to a curved membrane element and this are compared with those of Jerrel M. Thomas reported in 1970 and the solutions from the classical plate equations.Using the program developed, a parametric study is done to analyze the relationship of different plate parameters inherent in design of plates.

4.2 Sample problem

Variables:

1) Area (A) = 4.05 x $10^{-4}m^2$

2) Moment of area (I) = $1.31x10^{-6}$ M^4

3) Young's Modulus (E) =$6.9x10^{10}$ N/m^2

4) Radius (R) =2.54m

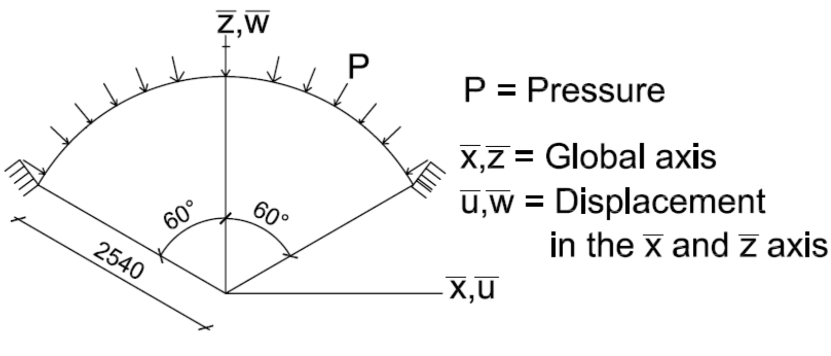


Figure 4.1 Circular arch under Uniform pressure p

4.3 Discussion of results

Table 4.1 Relationship number of elements and buckling pressure

Number of elements	Buckling pressure (N/mm^2)					
	Case I		Case II		Case III	
	This work	Thomas (1970)	This work	Thomas (1970)	This work	Thomas (1970)
2	16.580	16.007	16.580	16.007	16.580	16.007
3	9.952	10.000	11.300	11.331	11.859	11.331
6	10.012	10.052	10.806	10.963	11.964	11.278
9	9.962	9.912	10.760	10.840	11.300	11.086
12	9.962	9.877	10.801	10.788	11.089	11.086
Plate eq'n Solution	9.959		10.673		11.114	

From this table, it is seen that as the number of elements increases, the results have an oscillatory convergence to the exact solution.More divisions results into output of higher accuracy but requires more computational effort as there are more calculations. The model will increase the efficiency of the final result as the computations have been programmed.

ii) Effect of various plate properties

a) Cross sectional area

Table 4.2 Relation between number of elements and buckling pressure at cross sectional area of 3.05 x 10^{-4} m^2

Number of elements	Buckling pressure (N/m^2)		
	Case I	Case II	Case III
	This work	This work	This work
2	15035.443	15035.886	15035.886
3	8348.574	9695.646	10255.803
6	8408.561	9202.913	10360.651
9	8358.275	9157.455	9696.275
12	8362.302	9195.403	9487.40

Table 4.3 Relation between number of elements and buckling pressure at cross sectional area of 2.05 x $10^{-4}m^2$

Number of elements	Buckling pressure (N/m^2)		
	Case I	Case II	Case III
	This work	This work	This work
2	13738.393	13738.836	13738.836
3	7051.524	8398.596	8958.953
6	7111.511	7905.596	9063.511
9	7061.225	7860.405	8399.225
12	7065.252	7898.353	8190.353

Tables 4.2 and 4.3 show the relation between number of elements and buckling pressure at various cross sectional areas. It can be seen from these tables that a curved plate resists a higher load when it is directed towards the center of the arc. The loading cases vary as to the use of the plate element e.g. as a water structure or roof element. The program can be useful in a quick analysis considering the particular load case, given the load cases modeled are those frequently encountered.

Table 4.4 relationship of curvature and buckling pressure

	Buckling pressure (N/m^2)		
Curvature C in M^{-1}	Case I This work	Case II This work	Case III This work
0.4082	9962.706	10801.96	11089.7
0.4274	10108.31	10941.42	11233.42
0.5155	11284.06	12117.15	12409.16
0.6494	13070.34	13903.44	14195.44
0.7463	14362.83	15195.93	15487.93
1.0638	18600.95	19434.05	19726.05
1.3514	22436.5	23269.6	23561.7
1.8519	29114.52	29947.62	30239.62
2.9412	43649.38	44482.48	44774.48
7.1429	99716.68	100549.8	100841.8

Table 4.4 shows the relationship of curvature and buckling pressure for the three loading cases. It can be seen that curvature proportionately influences plate resistance to load. The higher the plate curvature the higher the load it resists. This

relationship can be used to optimize a curvature and load resistance in design of thin elements.

iii) Implication of the research to theory and practice

The formulation presented extends the finite element method to thin elements curved on plan. They include membranes and thin plates. As the world gears up into more production of smart structures some of which have curved elements, this model opens up room for further research to precision production of structural components.

iv) Contributionof this study to existing knowledge

The procedural analytical method arrived at is a new model as it is a combination of parameters of more than one referenced research work. The model has also been coded to run as a program in a visual basic platform, this will make research and relevant designs to be achieved at a sustainable cost.

CHAPTER FIVE

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusion

The overriding purpose of the study was to develop an Euler Bernoulli simulation model for analysis of thin curved plates with three specific objectives.This chapter presents the summary of the research work undertaken,the conclusions that were drawn,recommendations and areas of further research based on the analyzed data related to the general and specific objectives of the study.

A finite element method has been developed which is based on the Euler Bernoulli model. This model neglects shear deformations and hence the formulation is only applicable to thin elements including thin plates and membranes.

From the finite element analytical formulation a seven step chronological numerical procedure for analysis has been developed. From the finite element analysis the following conclusions can be drawn;

a) Curved plates resist a higher load when it is directed towards the center of the arc.

b) Curved plates with large curvatures resist higher loads than those with smaller curvatures.

A comparison is made between the result obtained in this research and those of other methods.There is a good agreement between the proposed and the existing methods,thus this confirms therefore that the proposed method is suitable for analysis of curved plates.

5.2 Recommendations

The program developed is useful for the study of thin curved plate elements because it manipulates the given plate and loading parameters to give output. This can be an essential tool in design and research work. It is recommended to use the program in design and study of thin curved elements.

5.3 Areas of further research

Plate structures exist under various supports and thicknesses depending on their usage. These systems affect the resistance of the plate to various load combinations and hence their design.

Further research to include the Timoshenko model into the theory will make the program developed more elaborate in its usage.

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APPENDICES

APPENDIX 1 Publication emanating from this work.

Analysis of Curved Plate Elements using Open Source Software

Mogire Philip, Dr.(Eng).G.N Mang'uriu, Prof.(Eng.) W.Oyawa

Department of Civil Engineering, School of Civil, Environment and Geospatial Engineering, Jomo Kenyatta University of Agriculture and Technology, Nairobi, Kenya

Abstract- Analysis of curved plate elements requires a high computational effort to obtain a reliable solution for a buckling load for design purposes. Available programs are expensive to acquire and they need thorough knowledge for effective use. There is therefore need to code cheaper and accessible programs in line with using sustainable methods to better the livelihood of mankind. To address this issue a theory is formulated based on the Euler-Bernoulli beam model. This model is applicable to thin elements which include plate and membrane elements.

This paper illustrates a finite element theory to calculate the master stiffness of a curved plate. The master stiffness takes into account the stiffness, the geometry and the loading of the element. The determinant of this matrix is established from which the buckling load which is unknown in the matrix is evaluated by the principal of bifurcation.

The curved element is divided into 2,3,6,9 and 12 elements; this demonstrates the computational effort to a reliable solution. As expected, that as you divide the curve into smaller constituent elements, the solution of the buckling load is tedious as more mathematical operations are involved hence the need to program the operations.

Numerical analysis is carried out by abstracting the procedural development of the theory and programming it to run in a visual basic platform. The results obtained are giving a good agreement with results obtained with classical plate equations. This program is proposed to increase computational efficiency in the analysis of curved plates at a sustainable cost. It can also be used to establish the relationship between buckling load and curvature of plates.

Index Terms- finite element method, analysis, curved plates, program

I. Introduction

A plate is a planar body whose thickness is small compared with its other dimensions. Curved plate structures are frequently used in; aerospace vehicles, domes, roof structures and pressure vessels. A plate structure may be as simple as the web of a stiffener or as complex as an integrally stiffened plate supported by heavy frames and rings.

Thin plates are characterized by a structure that is bounded by upper and lower surface planes separated by a distance h (figure 1). The x-y coordinate axes are located on the neutral plane of the plate and the z-axis is normal to the x-y plane.

For this paper it was assumed that h is a constant and those material properties are homogeneous through the thickness. Consequently, the location of the x-y axes(figure 2) lie at the mid-surface plane (z=0) with the upper and lower surfaces corresponding to z=h/2 and z=-h/2, respectively.

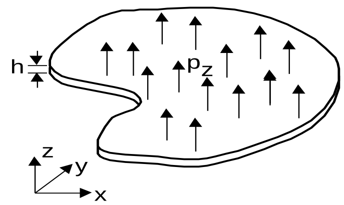

Figure 1 Structure of thin plate

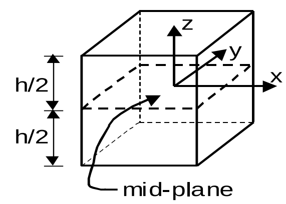

Figure 2 Location of thin plate axes

In the behavior of plate structures under in plane compression, a critical point exists where an infinitesimal increase in load can cause the plate surface to buckle; the load at this critical point defines the buckling strength of the plate.

Any further increase in load beyond the load at the initiation of buckling increases the buckling deformations until collapse occurs. Thus, the load at collapse defines the post buckling or crippling strength of the plate. The behavior of plate structures in this regard differs markedly from the behavior of columns and many other thin curved shell structures for which the buckling load corresponds closely to the collapse load.

Buckling of a plate structure can cause an unacceptable degradation. It can trigger general buckling of larger structures because of a redistribution of stresses; it can also affect the response by the structure to excessive displacement or fatigue which may be a cause of failure.

A lot of research has been done in this area of research which has been referenced.Available programs give criteria to do analysis of curved elements,but they are expensive to acquire and use. So, there is need to have an accessible criterion at an affordable cost.

1.1 Objectives

1) To develop an analytical program of curved plates.
2) To develop a numerical method for analysis of curved plates
3) To write a finite element analysis program to analyze curved plate elements using the method developed.

1.2 Scope for the work

An analytical formulation of the curved-plate non-linear equilibrium equations will be made. The analytical formulation will be implemented into a computer based program.

A convergence study using a segmented-plate approach will be performed for a simple example problem to obtain baseline results for use in future comparisons. Results will be compared with results from classical plate equations.

1.3 Methods of analysis

Finite element methods are now widely used to solve structural, fluid, and multi-physics problems numerically[1].The Euler –Bernoulli beam model applies since only thin elements are considered(shear deformations are neglected)[2].
Two methods of analysis of curved elements exist: the Eigenvalue Buckling Analysis and Nonlinear buckling Analysis.

1.3.1 The Eigenvalue Buckling Analysis

The Eigenvalue analysis predicts the theoretical buckling strength of an ideal linear elastic structure. This is analogous to the classical plate equation approach to elastic buckling analysis[3].However, imperfections and nonlinearities prevent most real-world structures from achieving their theoretical elastic buckling strength.

1.3.2 Nonlinear Buckling Analysis

This method takes account of imperfections and nonlinearities of real-world structures. In this method the load is increased until the solution fails to converge, indicating that the structure cannot support the applied load (or that numerical difficulties prevent solution)[4]. If the structure does not lose its ability to support additional load when it buckles, a nonlinear analysis can be used to track post-buckling behavior.

II. Basic Element Shapes

For the discretization of problems involving curved geometries, finite elements with curved sides are useful. The ability to model curved boundaries has been made possible by the addition of midsized nodes. Finite elements with straight sides are known as linear elements, whereas those with curved sides are called higher order elements[5].

2.1Size of Elements

The size of elements influences the convergence of the solution directly. If the size of the elements is small, the final solution is expected to be more accurate.

2.2 Number of Elements

The number of elements is related to the accuracy required and the number of degrees of freedom involved [5]. Although an increase in the number of elements generally means more accurate results, for any given problem, there will be a certain number of elements beyond which the accuracy cannot be improved by any significant amount shown graphically in figure 3[5].

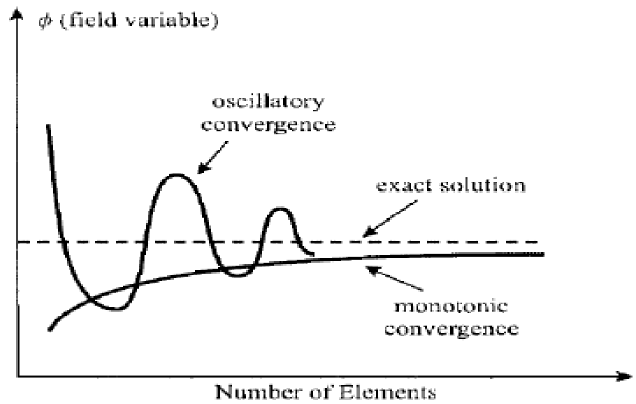

Fig 3 Relationship between the number of elements and accuracy[5]

III. Program Development

Computer programming languages are built around two approaches;
(1) Procedural programming and
(2) Object oriented programming.

In procedural programming, the program is prepared by a series of steps or routines that follow the data provided. The main drawback of the procedural programming languages is that they are not structured and the flow of the program largely depends on conditional statements that induce more chances of errors. These languages are good for small programs and are difficult to maintain when they become larger.

The object oriented programming languages are built on the concept of abstraction. Large complex procedures are subdivided into small procedures by abstraction, encapsulation and inheritance. Each of these sub procedures represents different objects with their own separate identity[6,7]. The program developed is object oriented and follows the following steps used in the formulation of the theory;

1. Identify the principal theory
$\{Q^1\} = [[K] + [G] + [L]] \{q^1\} = \{0\}$[8]

2. Divide the curved plate into appropriate elements and calculate the arc length (L) and the internal angle (ψ) and angle (α) of each element relative to the x axis.

It is noted that the unknown vector does not involve the rotation angle; the essential boundary condition can be imposed with the penalty function method [9,10].

3. Enter the following specific member properties (variables).
a) Area in m^2
b) Moment of inertia I in m^4
c) Length L in m
d) Young’s modulus E in N/m^2
e) (ψ) and (α) in degrees for each element

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4. For each element calculate:

a) The stiffness matrix K from the matrix below

$$[K] = \begin{array}{c} \\ \end{array}\begin{array}{cccccc} u_1 & u_2 & w_1 & w_2 & \theta_1 & \theta_2 \end{array}$$

$$[K] = \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} & 0 & 0 & 0 & 0 \\ -\frac{AE}{L} & \frac{AE}{L} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{12EI}{L3} & -\frac{12EI}{L3} & -\frac{6EI}{L2} & -\frac{6EI}{L2} \\ 0 & 0 & -\frac{12EI}{-L3} & \frac{12EI}{L3} & \frac{6E}{L2} & \frac{6EI}{L2} \\ 0 & 0 & -\frac{6EI}{L2} & \frac{6EI}{L2} & \frac{4EI}{L} & \frac{2EI}{L} \\ 0 & 0 & -\frac{6EI}{L2} & \frac{6EI}{L2} & \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \begin{array}{c} Fx^1 \\ Fx^2 \\ Fz^1 \\ Fz^2 \\ M_1 \\ M_2 \end{array}$$

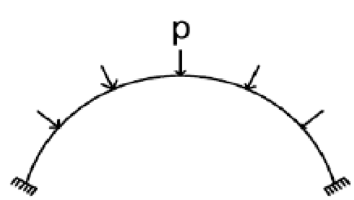
a) The geometric matrix from the matrix below

$$[G] = \left(\begin{array}{cc|cccc} u_1^1 & u_2^1 & w_1^1 & w_2^1 & \theta_1^1 & \theta_2^1 \\ \multicolumn{2}{c|}{3EA[u^0]\left[\gamma_k \gamma_j \gamma_1\right]} & \multicolumn{4}{c}{EA[w^0]\left[\phi_k \phi_j \gamma_1\right]} \\ \multicolumn{2}{c|}{3EA[u^0]\left[\gamma_k \gamma_j \gamma_2\right]} & \multicolumn{4}{c}{EA[w^0]\left[\phi_k \phi_j \gamma_2\right]} \\ \multicolumn{2}{c|}{EA[w^0]\left[\phi_1 \phi_k \gamma_j\right]} & \multicolumn{4}{c}{EA[u^0]\left[\phi_1 \phi_j \gamma_k\right]} \\ \multicolumn{2}{c|}{EA[w^0]\left[\phi_2 \phi_k \gamma_j\right]} & \multicolumn{4}{c}{EA[u^0]\left[\phi_2 \phi_j \gamma_k\right]} \\ \multicolumn{2}{c|}{EA[w^0]\left[\phi_3 \phi_k \gamma_j\right]} & \multicolumn{4}{c}{EA[u^0]\left[\phi_3 \phi_j \gamma_k\right]} \\ \multicolumn{2}{c|}{EA[w^0]\left[\phi_3 \phi_k \gamma_j\right]} & \multicolumn{4}{c}{EA[u^0]\left[\phi_2 \phi_j \gamma_k\right]} \end{array} \right)$$

b) Calculate the load matrix L selecting a specific load case

$[L] = [L_P] + [L_F]$

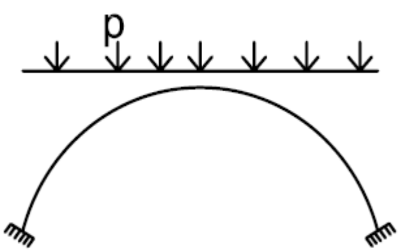
Load case i-Loads remain normal to the element



$$[L_P] = P \begin{pmatrix} 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{L}{12} & \frac{L}{12} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{L}{12} & -\frac{L}{12} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

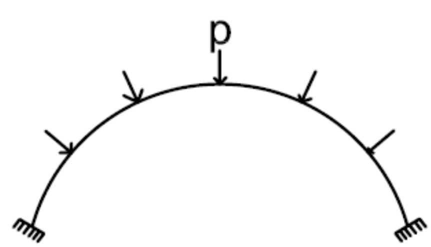
$$[L_F] = P \begin{pmatrix} 0 & 0 & 0 & 0 & -F_{Z1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -F_{Z2} \\ 0 & 0 & 0 & 0 & -F_{X1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -F_{X2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Load case ii-Loads remain parallel to the original direction



$$L_P = L_F = [0]$$

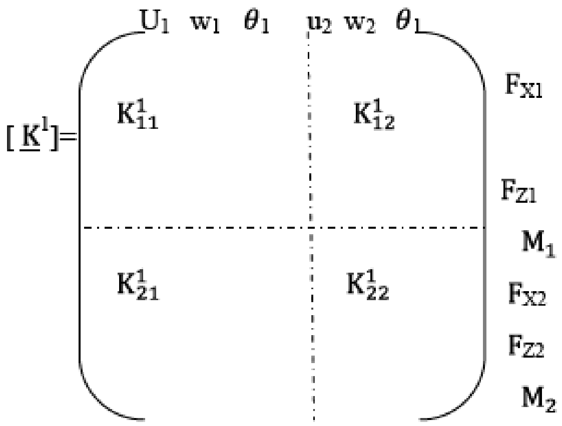
Load case iii-Loads remain directed towards a fixed point



$$[L_P] = P \begin{pmatrix} -\frac{L}{3R} & -\frac{L}{6R} & 0 & 0 & 0 & 0 \\ -\frac{L}{6R} & -\frac{L}{3R} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a) Rearrange all the matrices in the following order

$\underline{K}$, $\underline{G}$, and $\underline{L}_F$



5. Matrix transformation

Rotate all element matrices to the global coordinate system by the operation

$K^g = T^T K^{(e)} T$

Where

K^g = Global stiffness matrix

T^T = transpose of T matrix

$K^{(e)}$ = Local stiffness matrix

T= ɤφ where

$$\gamma = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\varphi = \begin{pmatrix} \cos\psi & \sin\psi & 0 & 0 & 0 & 0 \\ -\sin\psi & \cos\psi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\psi & \sin\psi & 0 \\ 0 & 0 & 0 & -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

4. Calculate the master stiffness from the summation of all the matrices of the system

Eg $K_1+K_2+K_3=k$, $G_1+G_2+G_3=G$, $L1+L_2+L_3 = L$ For three elements

6. Calculate the determinant of the resulting 6x6 matrix from the sum below

$|\bar{K} + p\,\bar{K}_0| = 0$ (Bifurcation theory)

Where

$\rho[\bar{K}_0] = [\bar{G}] + [\bar{L}_\rho] + [\bar{L}_F], [\bar{L}_F] = 0$

From the determinant the solution for p is the value for the buckling load.

IV. Sample Problem

The figure 4 is composed of a thin membrane forming a circular arch with uniform pressure. Determine the buckling pressure for three possible load cases.

Input data.

1) Area (A)= 4.05 x 10^{-4} m^2
2) Moment of inertia (I) = 1.31x10^{-6} m^4
3) Young's Modulus (E) =6.9x10^{10} N/m^2
4) Radius (R) =2.54m

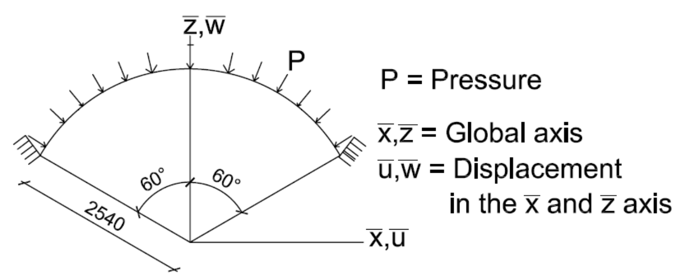


Figure 4 Uniform circular arch under uniform pressure p

4.1 DISCUSSION OF RESULTS

i) Summary of Results

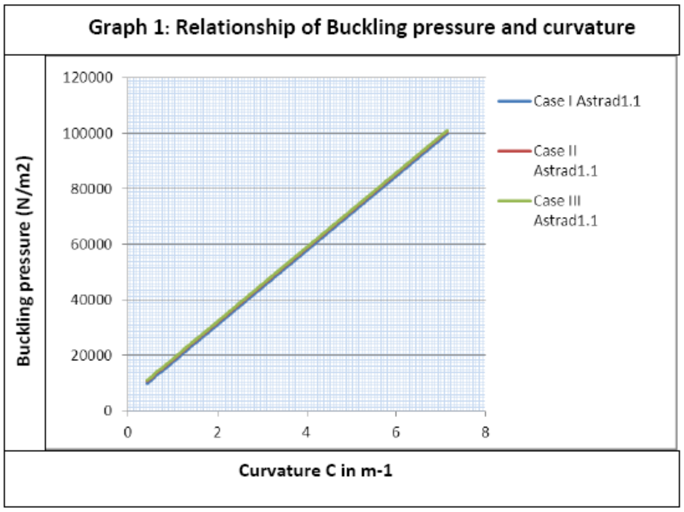
Table 1 Relationship of number of elements and resistance to buckling pressure

Number of elements	Buckling pressure (N/m^2)		
	Case I	Case II	Case III
	Astrad1.1	Astrad1.1	Astrad1.1
2	16580.425	16580.84	16580.85
3	9952.05	11299.77	11859.48
6	10012.20	10806.076	11964.198
9	9962.183	10760.107	11300.183
12	9962.706	10801.956	11089.702
Exact solution	9959.463	10673.981	11113.549

ii) Effect of varying plate properties

a) **CurvatureTable 2**
b) **:Relationship of Buckling pressure and Curvature**

Curvature C in m^{-1}	Buckling pressure (N/m^2)		
	Case I	Case II	Case III
	Astrad1.1	Astrad1.1	Astrad1.1
0.4082	9962.706	10801.96	11089.7
0.4274	10108.31	10941.42	11233.42
0.5155	11284.06	12117.15	12409.16
0.6494	13070.34	13903.44	14195.44
0.7463	14362.83	15195.93	15487.93
1.0638	18600.95	19434.05	19726.05
1.3514	22436.5	23269.6	23561.7
1.8519	29114.52	29947.62	30239.62
2.9412	43649.38	44482.48	44774.48
7.1429	99716.68	100549.8	100841.8



c) Cross sectional Area

i) Table 3 : Area = 3.05 x 10^{-4} m^2

Number of elements	Buckling pressure (N/m^2)		
	Case I	Case II	Case III
	Astrad1.1	Astrad1.1	Astrad1.1
2	15035.443	15035.886	15035.886
3	8348.574	9695.646	10255.803
6	8408.561	9202.913	10360.651
9	8358.275	9157.455	9696.275
12	8362.302	9195.403	9487.40

ii) Table 4:Area = 2.05 x 10^{-4} m^2

Number of elements	Buckling pressure (N/m^2)		
	Case I	Case II	Case III
	Astrad1.1	Astrad1.1	Astrad1.1
2	13738.393	13738.836	13738.836
3	7051.524	8398.596	8958.953
6	7111.511	7905.596	9063.511
9	7061.225	7860.405	8399.225
12	7065.252	7898.353	8190.353

4.2 DISCUSSION OF THE RESULTS

i) Nature of results

As the number of elements increases, the results have an oscillatory convergence to the exact solution.

More divisions results into output of higher accuracy but requires more computational effort as there are more calculations. The program will increase the efficiency of the final result as the computations have been programmed.

ii) Loading cases

The results show that a curved plate resists a higher load when it is directed towards the center of the arc. The loading cases vary as to the use of the plate element e.g. as a water structure or roof element. The program can be useful in a quick analysis considering the particular load case, given the load cases programmed are those frequently encountered.

ii) Relationship between load and curvature.

From the results, load resistance of a curved plate is directly proportional to curvature.

4.3 CONCLUSION AND RECOMMENDATIONS

The research had 3 specific objectives which have been achieved

a) Analytical program

An analysis criterion based on Euler Bernoulli theory was developed. This is applicable only to thin elements which includes thin plates and membranes.

b) Numerical method

The six step procedure of analysis to arrive at a buckling load forms a summary of the numerical method.

c) Finite element program

The six steps above were programmed to run on a visual basic platform. This program is referred as Astrad 1.1.This code was named with a future intention of redeveloping it to include analysis of thicker elements. The program is less costly and requires less effort to use.

In order to access the efficiency and accuracy of the program, an example is analyzed whose results is tabulated in tables 1,2,3 and 4.The analysis shows a good agreement with classical plate equations. The program

It can be seen that a curved element resists more loads directed towards its center than other loading cases. Curvature and plate thickness proportionately influence plate resistance to load. Use of the program is useful for the study of curved plate elements because it manipulates the given plate and loading parameters to give output.

The program should be developed further to carter for more attributes like modification of thin plates into a composite element and with stiffeners.

Further research into the inclusion of the Timoshenko theory into the program to carter for thicker elements should also be done.

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AUTHORS

First Author – Mogire Philip, Department of Civil Engineering School of Civil, Environment and Geospatial Engineering Jomo Kenyatta University of Agriculture and Technology, Nairobi, Kenya

Second Author – Dr.(Eng).G.N Mang'uriu, Department of Civil Engineering, School of Civil, Environment and Geospatial Engineering, Jomo Kenyatta University of Agriculture and Technology, Nairobi, Kenya

Third Author – Prof.(Eng.) W.Oyawa, Department of Civil Engineering, School of Civil, Environment and Geospatial Engineering, Jomo Kenyatta University of Agriculture and Technology, Nairobi, Kenya