Nonnegative P_0-matrix completion problem

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DECLARATION

This research is my original work and has not been presented elsewhere for a degree in any other University.

Signature..... **Date**.....
Juma Mwangi Munyiri

This thesis has been submitted for examination with our approval as University Supervisors.

Signature..... **Date**.....

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Signature..... **Date**.....

Dr. Lewis Nyaga,
JKUAT, Kenya

DEDICATION

I dedicate this work to my family, my wife and my sons. Your love and support was so wonderful such that this work was possible.

ACKNOWLEDGMENTS

To the Almighty God, am grateful for all the provisions and care as I did this study. May all Glory and Honour be unto Him the most High.

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I salute my colleagues in postgraduate studies for their company and encouragement.

Special thanks to my family for the love, care, support and encouragement.

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NOTATIONS

A ····· matrix

$[a_{ij}]$ ····· matrix with entry a in the i^{th} row and j^{th} column.

D ····· Diagonal matrix

P ····· Permutation matrix

$A[\alpha]$ ····· Principal submatrix obtained from a matrix A by deleting rows and columns not in α

Det A ····· Determinant of a matrix A

Q ····· Pattern for a matrix

$\prod$ ····· Class of matrices

a_{ij} ····· Specified entry in a matrix

x_{ij} ····· Unspecified entry in a matrix

d_{ii} ····· Diagonal entry in a matrix

LIST OF FIGURES

ABSTRACT

A partial matrix is a matrix with some entries specified while the other unspecified entries are free to be chosen. Completion of partial matrix is a specific choice of unspecified entries such that the resulting matrix satisfies a certain property. This study considers nonnegative P_0-matrix completion. The nonnegative P_0- matrix completion problem determines patterns of positions with the property that any partial nonnegative P_0-matrix that specifies the pattern can be completed to a nonnegative P_0-matrix. In particular the study focusses on real 5x5 partial nonnegative P_0-matrices specifying digraphs for p = 5 and q = 3, where p is the number of vertices and q is the number of arcs. The study establishes that all digraphs for p = 5, q = 3 which are either cycles or acyclic digraphs have zero completion to nonnegative P_0-completion.

CHAPTER ONE

INTRODUCTION

1.1 Introduction

1.1.1 Matrices

A **matrix** is a rectangular array of numbers or expressions arranged in rows and columns. When entries of a matrix are real such a matrix is referred to as a **real matrix**, and if some or all the entries are complex then such a matrix is referred to as **complex matrix**. The size of a matrix is defined by the number of rows and columns it contains. A matrix with the same number of rows and columns is called a **square matrix**, an n x n matrix is known as a square matrix of order n.

A **submatrix** of a matrix *A*, is a smaller matrix obtained by deleting a collection of some row(s) and/or column(s) from matrix *A*. Let *A* be an *n x n* matrix, for α a subset of $\{1, 2, \cdots\cdots\cdots, n\}$, the **principal sub-matrix** $A(\alpha)$ is obtained by deleting all rows and columns that are not in α.

Determinant is a scalar associated with square matrices. It is computed using any choosen row or column of the matrix using the formula det $A = \sum_{i=1}^{n} (-1)^{i+j}\ a_{ij}$ det A_{ij}, where A_{ij} denote the submatrix formed by deleting ith row and jth column of *A*. A **principal minor** is the determinant of a principal sub-matrix. The **leading principal minor** is the determinant of the leading principal sub-matrix obtained by deleting the last *n - k* rows and *n - k* columns of a nxn matrix A. For a *n x n* square matrix there are *n* leading principal minors, (Parthasaranthy,1994).

A **permutation matrix** is a matrix in which each entry is either 0 or 1 and there is exactly one 1 in each row and in each column of a matrix. A permutation matrix *P* is obtained by interchanging the rows on the identity matrix. The permutation of a matrix *A* is then PAP^T. This is represented on a digraph by renumbering the vertices.

Matrices are of various classes and each class specifies certain properties. Some of the classes of matrices are: positive definite matrices, P-matrices, P_0-matrices, M_0-matrices, nonnegative P-matrices, nonnegative P_0-matrices and others. A real n x n matrix is called a **P_0-matrix** if all its principal minors are nonnegative. On the other hand a real n x n matrix is called **nonnegative P_0-matrix** if all its entries are nonnegative and all its principal minors are nonnegative, i.e. it is a P_0-matrix whose entries are nonnegative.

1.1.2 Partial Matrix and Matrix Completion

A **partial matrix** is a matrix in which some entries are specified, while the remaining (unspecified) entries are free to be chosen. Example $A = \begin{bmatrix} 4 & 2 & x \\ 2 & 3 & -1 \\ 4 & y & 1 \end{bmatrix}$ is 3x3 partial matrix with elements in positions (1,1), (1,2), (2,1), (2,2), (2,3), (3,1), (3,3) specified while elements in positions (1,3), (3,2) are unspecified. A fully specified principal sub-matrix is a principal sub-matrix of a partial matrix which contains only the specified entries, example is principal submatrix $A(1,2)$ of matrix A above. A **pattern** for n x n matrix is a list of positions of the matrix, that is, a subset of $\{1,2,\ldots\ldots\ldots,n\}$ x $\{1,2,\ldots\ldots\ldots,n\}$. A partial matrix specifies a pattern if its specified entries are those exactly listed in the pattern, (Hogben, 2001).

A partial matrix is a **partial non-negative P_0-matrix** if determinants of all fully specified principal submatrices are nonnegative and all specified entries are nonnegative.

Completion of a partial matrix is a particular choice of values for the unspecified entries resulting in a conventional matrix. For a particular class $\prod$ of matrices, a pattern is said to have $\prod$-completion if every partial $\prod$-matrix specifying the pattern can be completed to $\prod$-matrix. Hence a pattern is said to have a nonnegative P_0-completion if every partial nonnegative P_0-matrix specifying the pattern can be completed to a nonnegative P_0-matrix, (Choi *et al*, 2002).

Graphs and digraphs have been used effectively to study matrix completion problems. For positionally symmetric pattern Q that includes all diagonal positions, the graph of Q (pattern graph) is used to carry out the study. For a pattern without positional symmetry,directed graph must be used. For patterns that do not include the diagonal entries mardigraphs are used, (Hogben, 2001).

A **Graph** is ordered pair *G=(V,E)* comprising of a set *V* of vertices or nodes together with a set *E* of edges or lines, while a **directed graph (Digraph)** is a graph in which each edge has been given a direction. The edges of directed graph are referred to as **arcs**. An arc that is directed from vertex u to v is denoted by an ordered pair (u,v).The **order** of a (di)graph is the number of vertices in the (di)graph while the **size** of the (di)graph is the number of arcs in the (di)graph. A (di)graph H is said to be a **sub-(di)graph** of D if every vertex of H is also a vertex of D and every edge/arc of H is also an edge/arc of D, (Roman, 1989).

Let D be a digraph, a path that begins and ends at the same vertex is called a **cycle**. A digraph that does not contain any cycles is called **acyclic digraph**. A chord is an arc joining two vertices of a **cycle**. A digraph is **chordal** if any cycle of length > 3 has a chord. A subset of a directed graph is called a **clique** if it contains at least three vertices and for each pair of vertices v_i and v_j in the subset, both $v_i \rightarrow v_j$ and $v_j \rightarrow v_i$ are true,

(Hogben, 2001).

1.2 Statement of the Problem

Hogben (2001) and Choi (2003) investigated on patterns with non-negative P_0-completion where 3x3 and 4x4 matrices were considered. This study extends their work by considering 5x5 matrices specifying digraphs for p = 5, q = 3.

1.3 Justification

Situations arise when a full set of data is not available or is not economical to collect. With the knowledge that the complete data set must have certain properties when arranged in a matrix the values of the unavailable data can be suggested.

According to Hogben, extensive research has been done on matrix completion for classes including positive definite matrices, M-matrices, M_0-matrices, completely positive matrices and doubly nonnegative matrices. Not much has been done on nonnegative P_0-matrices and hence the research.

1.4 Null Hypothesis

Digraphs for p=5, q=3 do not have nonnegative P_0-completion.

1.5 Objectives

1.5.1 General Objective

To determine which of the patterns specifying the digraph p=5, q=3 have nonnegative P_0-completion.

1.5.2 Specific Objectives

1. To determine all digraphs for p=5, q=3 that defines patterns of 5x5 partial nonnegative P_0-matrices.

2. To determine the 5x5 partial matrices specifying digraphs for p =5, q = 3 that have nonnegative P_0-matrix completion.

3. To determine the 5x5 partial matrices specifying digraphs for p = 5, q = 3 that do not have the nonnegative P_0-matrix completion.

CHAPTER TWO

LITERATURE REVIEW

In this section review of previous studies which are closely related to this work are discussed:

Johnson *et al* (1996) studied, The combinatorially symmetric P-matrix completion problem, The and established that every partial P -matrix with combinatorially symmetric specified entries has a P-matrix completion. Also shown in the study is that for every n $\geq$ 4, there is a partial P -matrix with exactly one unspecified entry for which there is no P -matrix completion. The study also indicate that the main result does not generalize to the P_0-case.

Dealba *et al* (2000) studied, Completions of P -matrix patterns. The study extended the work of Johnson (1996). According to this new study it is shown that a larger class of patterns has P-completion including any 4x4 patterns with eight or fewer off-diagonal positions. It is also shown that any pattern whose digraph contains minimally chordal symmetric-Hamiltonian induced subdigraph does not have P-completion.

Hogben (2001) studied, Graph theoretic methods for matrix completion problems. The study surveyed the current state of research on $\prod$ - matrix completion problems for many subclasses $\prod$ of P - and P_0- matrices, including positive definite matrices, M-matrices, inverse M-matrices, P-matrices and matrices defined by various sign symmetry and positivity conditions on P_0- and P-matrices. The study established that for nonnegative P_0-matrices, patterns of every nonseparable strongly connected induced subdigraph has nonnegative P_0-completion. In the same study it is shown that all 3x3 matrices have nonnegative P_0-completion.

Choi *et al* (2002) studied, The P_0- matrix completion problem. The study established that every asymmetric partial P_0- matrix has P_0- completion. It is shown that all 4x4 patterns that include all diagonal positions are classified as either having P_0- completion or not having P_0- completion. Also shown in the study is that any positionally symmetric pattern whose graph is an n-cycle with n$\geq$5 has a P_0- completion.

Hogben (2003) studied, Matrix completion problems for pairs of related classes of matrices. The study show that a pattern that has nonnegative P_0-completion also has nonnegative P -completion.

Choi *et al* (2003) studied, The non-negative P_0- matrix completion problem. The study gives the proof that any pattern for a 3x3 matrix has non-negative P_0- completion. Main results derived by the study is that a pattern for 4x4 matrices that includes all diagonal positions has non-negative P_0- matrix completion if and only if its digraph is complete when it has a 4-cycle. Also shown in the study is that any positionally symmetric pattern

that includes all diagonal positions and whose graph is an n-cycle has non-negative P_0-completion if and only if $n \neq 4$.

Kamaku P. (2008) studied, The M_0-matrix completion problem for p = 4, q = 4 digraphs. The study established that 4x4 partial matrices specified by digraphs p=4, q=4 which are either cycles or cliques do not have zero completion to M_0-matrix, while those that are neither cycles or cliques have zero completion to M_0-completion.

Njoroge J. M *et al* (2013) studied , M_0-matrix completion for digraphs of 5x5 matrices with less than four directed lines. The study showed that 5x5 M_0-partial matrices that specifies a digraph with less than four directed lines has M_0-completion when the digraph is not a cycle. The study further shows that for digraphs that are cycles the 5x5 partial matrix specified have have M_0-completion.

CHAPTER THREE

METHODOLOGY

Since the study consider situations where all diagonal entries are specified, digraphs are used. All possible digraphs that defines p=5, q=3 up to isomorphism are derived.

Then partial 5x5 matrices specified by the digraphs are constructed. The process of constructing the partial matrices involve using a specific entry a_{ij} where $i \neq j$ to signify presence of an arc and unspecified entry x_{ij} with $i \neq j$ to represent absence of an arc while d_{ii} represents diagonal entries.

Patterns specifying the partial matrices are analyzed to establish whether they have completion to non-negative P_0-matrix. The method used is:-

The zero completion method:- . The method involve setting all the unspecified entries to zero. Then computating determinants of principal sub-matrices to find out whether they are non-negative. If all the principal minors are non-negative the matrix is said to have completion. If otherwise it is said to be uncompletable.

By use of the lemma below:

Lemma 3.0.1. *The class of nonnegative P_0-matrices is closed under permutation.* (Choi, 2002)

It is therefore possible to permute a partial nonnegative P_0-matrix and hence renumber digraph vertices as convinient.

Example considering the following digraph for p =3 ,q =3

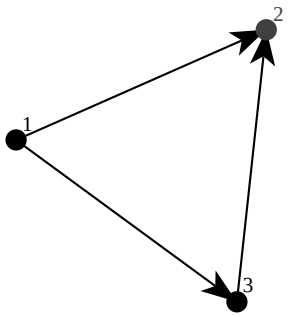

Figure 3.0.1: Digraph with vertices 1 to 2, 1 to 3 and 3 to 2

Permuting the vertices results to the following digraphs which are similar to the digraph in figure 2.1.1

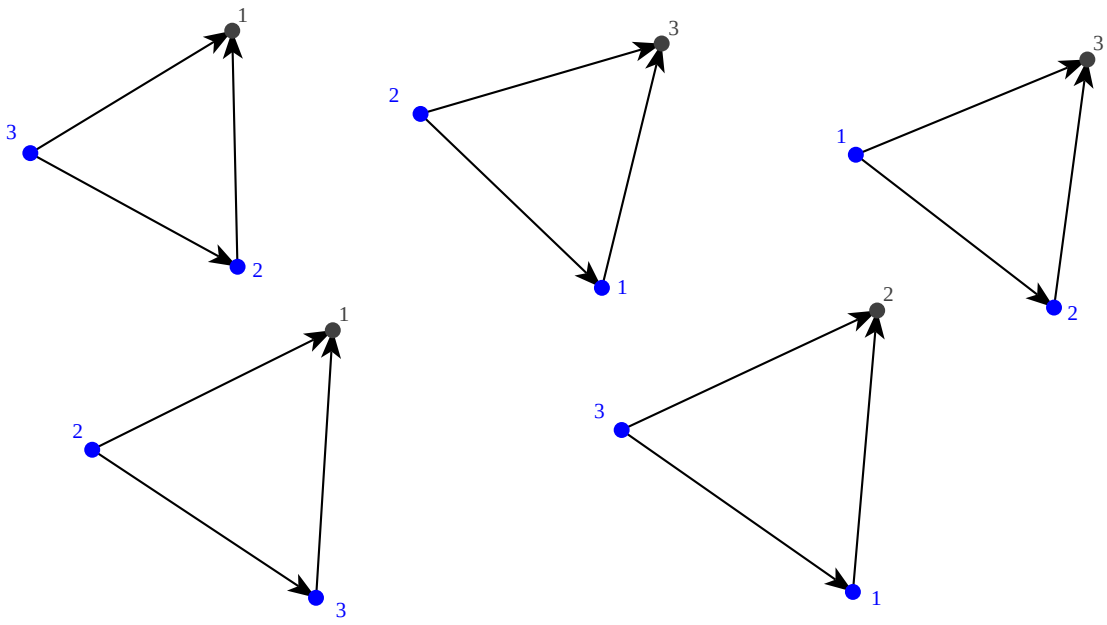

Figure 3.0.2: Permutations (123), (132), (23), (12), (13)

Hence each case considered in this study represents all the possible digraphs that result by permuting the vertices of the digraph.

CHAPTER FOUR

ACYCLIC DIGRAPHS

4.1 Introduction

In this section, all 5x5 matrices specifying digraphs with 5 vertices and 3 arcs that do not contain any cycles are discussed.The aim is to determine the acyclic digraphs with zero completion to non-negative P_0-matrix.

4.2 Acyclic digraphs with 5 vertices and 3 arcs specifying 5x5 matrices

1. Consider the digraph below:

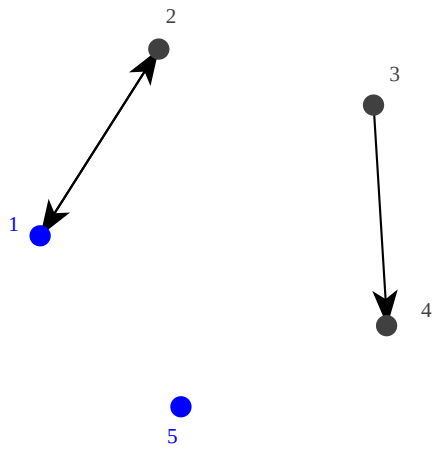

Figure 4.2.1: Digraph with directed lines 1 to 2, 2 to 1 and 3 to 4

Permuting the vertices gives other digraphs with similar results, such as the one that follows:

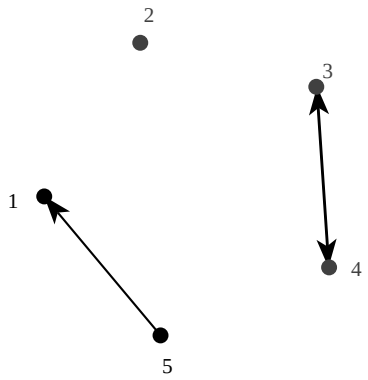

Figure 4.2.2: Digraph isomorphic to digraph in figure 2.2.1, obtained via permutation (13524)

Using the digraph in figure 2.2.1,

Let $A = \begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ a_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$ be a partial nonnegative P_0-matrix specifying the digraph.

Determinants of the principal submatrices

Det $A(1,2) = d_{11}d_{22} - a_{12}a_{21}$

Det $A(1,3) = d_{11}d_{33} - x_{13}x_{31}$

Det $A(1,4) = d_{11}d_{44} - x_{14}x_{41}$

Det $A(1,5) = d_{11}d_{55} - x_{15}x_{51}$

Det $A(2,3) = d_{22}d_{33} - x_{23}x_{32}$

Det $A(2,4) = d_{22}d_{44} - x_{24}x_{42}$

Det $A(2,5) = d_{22}d_{55} - x_{25}x_{52}$

Det $A(3,4) = d_{33}d_{44} - a_{34}x_{43}$

Det $A(3,5) = d_{33}d_{55} - x_{35}x_{53}$

Det $A(4,5) = d_{44}d_{55} - x_{45}x_{54}$

Det $A(1,2,3) = d_{11}(d_{22}d_{33} - x_{23}x_{32}) - a_{12}(a_{21}d_{33} - x_{23}x_{31}) + x_{13}(a_{21}x_{32} - d_{22}x_{31})$

Det $A(1,2,4) = d_{11}(d_{22}d_{44} - x_{24}x_{42}) - a_{12}(a_{21}d_{44} - x_{24}x_{41}) + x_{14}(a_{21}x_{42} - d_{22}x_{41})$

Det $A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - a_{12}(a_{21}d_{55} - x_{25}x_{51}) + x_{15}(a_{21}x_{52} - d_{22}x_{51})$

Det $A(1,3,4) = d_{11}(d_{33}d_{44} - a_{34}x_{43}) - x_{13}(x_{31}d_{44} - a_{34}x_{41}) + x_{14}(x_{31}x_{43} - d_{33}x_{41})$

Det $A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - x_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$

Det $A(1,4,5) = d_{11}(d_{44}d_{55} - x_{45}x_{54}) - x_{14}(x_{41}d_{55} - x_{45}x_{51}) + x_{15}(x_{41}x_{54} - d_{44}x_{51})$

Det $A(2,3,4) = d_{22}(d_{33}d_{44} - a_{34}x_{43}) - x_{23}(x_{32}d_{44} - a_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})$

Det $A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - x_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})$

Det $A(2,4,5) = d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})$

Det $A(3,4,5) = d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})$

Det $A(1,2,3,4) = d_{11}\{d_{22}(d_{33}d_{44} - a_{34}x_{43}) - x_{23}(x_{32}d_{44} - a_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})\} - a_{12}\{a_{21}(d_{33}d_{44} - a_{34}x_{43}) - x_{23}(x_{31}d_{44} - a_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41})\} + x_{13}\{a_{21}(x_{32}d_{44} - a_{34}x_{42}) - d_{22}(x_{31}d_{44} - a_{34}x_{41}) + x_{24}(x_{31}x_{42} - x_{32}x_{41})\} - x_{14}\{a_{21}(x_{32}x_{43} - d_{33}x_{42}) - d_{22}(x_{31}x_{43} - d_{33}x_{41}) + x_{23}(x_{31}x_{42} - x_{32}x_{41})\}$

Det $A(1,2,3,5) = d_{11}\{d_{22}(d_{33}d_{55} - x_{35}x_{53}) - x_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})\} - a_{12}\{a_{21}(d_{33}d_{55} - x_{35}x_{53}) - x_{23}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{53} - d_{33}x_{51})\} + x_{13}\{a_{21}(x_{32}d_{55} - x_{35}x_{52}) - d_{22}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{52} - x_{32}x_{51})\} - x_{15}\{a_{21}(x_{32}x_{53} - d_{33}x_{52}) - d_{22}(x_{31}x_{53} - d_{33}x_{51}) + x_{23}(x_{31}x_{52} - x_{32}x_{51})\}$

Det $A(1,2,4,5) = d_{11}\{d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})\} - a_{12}\{a_{21}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{a_{21}(x_{42}d_{55} - x_{45}x_{52}) - d_{22}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{52} - x_{42}x_{51})\} - x_{15}\{a_{21}(x_{42}x_{54} - d_{44}x_{52}) - d_{22}(x_{41}x_{54} - d_{44}x_{51}) + x_{24}(x_{41}x_{52} - x_{42}x_{51})\}$

Det $A(1,3,4,5) = d_{11}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{13}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{15}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{53} - x_{43}x_{51})\}$

Det $A(2,3,4,5) = d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{23}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + a_{34}(x_{42}x_{53} - x_{43}x_{52})\}$

Det $A = d_{11}[d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{23}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + a_{34}(x_{42}x_{53} - x_{43}x_{52})\}] -$

$a_{12}[a_{21}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{23}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{25}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{53} - x_{43}x_{51})\}] +$

$x_{13}[a_{21}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} - d_{22}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - x_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{52} - x_{42}x_{51})\}] -$

$x_{14}[\ a_{21}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} + x_{23}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - x_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}] +$

$x_{15}[a_{21}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + a_{34}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{53} - x_{43}x_{51})\} + x_{23}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{52} - x_{42}x_{51})\} - x_{24}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}]$

Performing zero completion by setting all the unspecified entries to zero i.e. $x_{13} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{23} = 0$, $x_{24} = 0$, $x_{25} = 0$, $x_{31} = 0$, $x_{32} = 0$, $x_{35} = 0$, $x_{41} = 0$, $x_{42} = 0$, $x_{43} = 0$, $x_{45} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$. The determinants of principal submatrices will be ;

Det $A(1,2) = d_{11}d_{22} - a_{12}a_{21} \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,3) = d_{11}d_{33} \geq 0$

Det $A(1,4) = d_{11}d_{44} \geq 0$

Det $A(1,5) = d_{11}d_{55} \geq 0$

Det $A(2,3) = d_{22}d_{33} \geq 0$

Det $A(2,4) = d_{22}d_{44} \geq 0$

Det $A(2,5) = d_{22}d_{55} \geq 0$

Det $A(3,4) = d_{33}d_{44} \geq 0$

Det $A(3,5) = d_{33}d_{55} \geq 0$

Det $A(4,5) = d_{44}d_{55} \geq 0$

Det $A(1,2,3) = d_{11}d_{22}\ d_{33} - a_{12}a_{21}d_{33} = d_{33}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,2,4) = d_{11}d_{22}\ d_{44} - a_{12}a_{21}d_{44} = d_{44}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,2,5) = d_{11}d_{22}\ d_{55} - a_{12}a_{21}d_{55} = d_{55}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,3,4) = d_{11}d_{33}d_{44} \geq 0$

Det $A(1,3,5) = d_{11}d_{33}d_{55} \geq 0$

Det $A(1,4,5) = d_{11}d_{44}d_{55} \geq 0$

Det $A(2,3,4) = d_{22}d_{33}d_{44} \geq 0$

Det $A(2,3,5) = d_{22}d_{33}d_{55} \geq 0$

Det $A(2,4,5) = d_{22}d_{44}d_{55} \geq 0$

Det $A(3,4,5) = d_{33}d_{44}d_{55} \geq 0$

Det $A(1,2,3,4) = d_{11}d_{22}\ d_{33}d_{44} - a_{12}a_{21}d_{33}d_{44} = d_{33}d_{44}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,2,3,5) = d_{11}d_{22}\ d_{33}d_{55} - a_{12}a_{21}d_{33}d_{55} = d_{33}d_{55}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,2,4,5) = d_{11}d_{22}\ d_{44}d_{55} - a_{12}a_{21}d_{44}d_{55} = d_{44}d_{55}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,3,4,5) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A = d_{11}d_{22}\ d_{33}d_{44}d_{55} - a_{12}a_{21}d_{33}d_{44}\ d_{55} = d_{33}d_{44}d_{55}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Hence all the determinants are nonnegative and since all entries are nonnegative the pattern therefore has zero completion into nonnegative P_0- matrix.

2. Consider the digraph below:

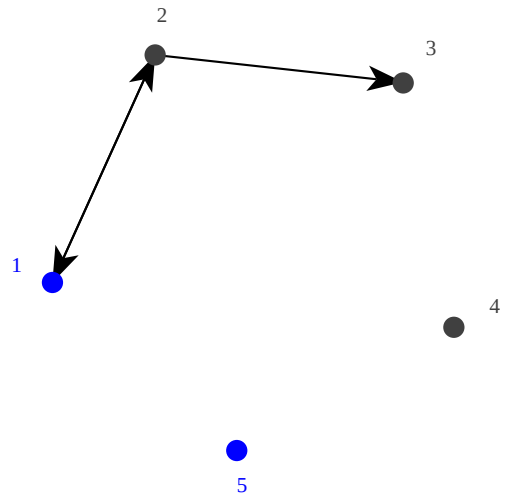

Figure 4.2.3: Digraph with directed lines 1 to 2, 2 to 1 and 2 to 3

The digraph has similar results with other digraphs obtained by permuting its vertices, such as the one that follows:

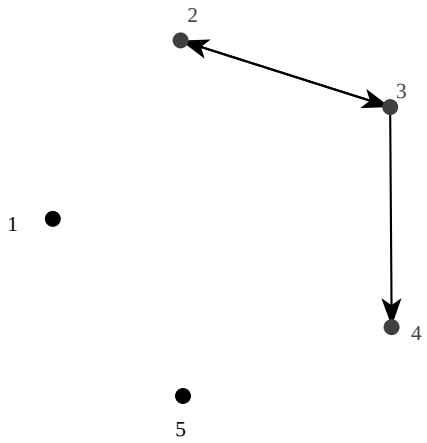

Figure 4.2.4: Digraph isomorphic to the digraph in figure 2.2.3, obtained via permutation (12345)

Using the digraph in figure 2.2.3,

Let $A = \begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ a_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$ be the partial nonnegative P_0-matrix specifying the digraph.

Determinants of the principal submatrices

Det $A(1,2) = d_{11}d_{22} - a_{12}a_{21}$

Det $A(1,3) = d_{11}d_{33} - x_{13}x_{31}$

Det $A(1,4) = d_{11}d_{44} - x_{14}x_{41}$

Det $A(1,5) = d_{11}d_{55} - x_{15}x_{51}$

Det $A(2,3) = d_{22}d_{33} - a_{23}x_{32}$

Det $A(2,4) = d_{22}d_{44} - x_{24}x_{42}$

Det $A(2,5) = d_{22}d_{55} - x_{25}x_{52}$

Det $A(3,4) = d_{33}d_{44} - x_{34}x_{43}$

Det $A(3,5) = d_{33}d_{55} - x_{35}x_{53}$

Det $A(4,5) = d_{44}d_{55} - x_{45}x_{54}$

Det $A(1,2,3) = d_{11}(d_{22}d_{33} - a_{23}x_{32}) - a_{12}(a_{21}d_{33} - a_{23}x_{31}) + x_{13}(a_{21}x_{32} - d_{22}x_{31})$

Det $A(1,2,4) = d_{11}(d_{22}d_{44} - x_{24}x_{42}) - a_{12}(a_{21}d_{44} - x_{24}x_{41}) + x_{14}(a_{21}x_{42} - d_{22}x_{41})$

Det $A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - a_{12}(a_{21}d_{55} - x_{25}x_{51}) + x_{15}(a_{21}x_{52} - d_{22}x_{51})$

Det $A(1,3,4) = d_{11}(d_{33}d_{44} - x_{34}x_{43}) - x_{13}(x_{31}d_{44} - x_{34}x_{41}) + x_{14}(x_{31}x_{43} - d_{33}x_{41})$

Det $A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - x_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$

Det $A(1,4,5) = d_{11}(d_{44}d_{55} - x_{45}x_{54}) - x_{14}(x_{41}d_{55} - x_{45}x_{51}) + x_{15}(x_{41}x_{54} - d_{44}x_{51})$

Det $A(2,3,4) = d_{22}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{32}d_{44} - x_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})$

Det $A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})$

Det $A(2,4,5) = d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})$

Det $A(3,4,5) = d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})$

Det $A(1,2,3,4) = d_{11}\{d_{22}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{32}d_{44} - x_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})\} - a_{12}\{a_{21}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41})\} + x_{13}\{a_{21}(x_{32}d_{44} - x_{34}x_{42}) - d_{22}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}x_{42} - x_{32}x_{41})\} - x_{14}\{a_{21}(x_{32}x_{43} - d_{33}x_{42}) - d_{22}(x_{31}x_{43} - d_{33}x_{41}) + a_{23}(x_{31}x_{42} - x_{32}x_{41})\}$

Det $A(1,2,3,5) = d_{11}\{d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})\} - a_{12}\{a_{21}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{53} - d_{33}x_{51})\} + x_{13}\{a_{21}(x_{32}d_{55} - x_{35}x_{52}) - d_{22}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{52} - x_{32}x_{51})\} - x_{15}\{a_{21}(x_{32}x_{53} - d_{33}x_{52}) - d_{22}(x_{31}x_{53} - d_{33}x_{51}) + a_{23}(x_{31}x_{52} - x_{32}x_{51})\}$

Det $A(1,2,4,5) = d_{11}\{d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})\} - a_{12}\{a_{21}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{a_{21}(x_{42}d_{55} - x_{45}x_{52}) - d_{22}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{52} - x_{42}x_{51})\} - x_{15}\{a_{21}(x_{42}x_{54} - d_{44}x_{52}) - d_{22}(x_{41}x_{54} - d_{44}x_{51}) + x_{24}(x_{41}x_{52} - x_{42}x_{51})\}$

Det $A(1,3,4,5) = d_{11}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{13}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{15}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}$

Det $A(2,3,4,5) = d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\}$

Det $A = d_{11}[d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\}] -$

$a_{12}[a_{21}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{25}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}] +$

$x_{13}[a_{21}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} - d_{22}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - x_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\}] -$

$x_{14}[a_{21}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} + a_{23}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - x_{32}(x_{41}d_{55} -$

$x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}] +$

$x_{15}[a_{21}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\} + a_{23}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\} - x_{24}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}]$

Performing zero completion by setting all the unspecified entries to zero i.e. $x_{13} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{24} = 0$, $x_{25} = 0$, $x_{31} = 0$, $x_{32} = 0$, $x_{34} = 0$, $x_{35} = 0$, $x_{41} = 0$, $x_{42} = 0$, $x_{43} = 0$, $x_{45} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$. The determinants of principal submatrices will be ;

Det $A(1,2) = d_{11}d_{22} - a_{12}a_{21} \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,3) = d_{11}d_{33} \geq 0$

Det $A(1,4) = d_{11}d_{44} \geq 0$

Det $A(1,5) = d_{11}d_{55} \geq 0$

Det $A(2,3) = d_{22}d_{33} \geq 0$

Det $A(2,4) = d_{22}d_{44} \geq 0$

Det $A(2,5) = d_{22}d_{55} \geq 0$

Det $A(3,4) = d_{33}d_{44} \geq 0$

Det $A(3,5) = d_{33}d_{55} \geq 0$

Det $A(4,5) = d_{44}d_{55} \geq 0$

Det $A(1,2,3) = d_{11}d_{22}\ d_{33} - a_{12}a_{21}d_{33} = d_{33}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,2,4) = d_{11}d_{22}\ d_{44} - a_{12}a_{21}d_{44} = d_{44}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,2,5) = d_{11}d_{22}\ d_{55} - a_{12}a_{21}d_{55} = d_{55}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,3,4) = d_{11}d_{33}d_{44} \geq 0$

Det $A(1,3,5) = d_{11}d_{33}d_{55} \geq 0$

Det $A(1,4,5) = d_{11}d_{44}d_{55} \geq 0$

Det $A(2,3,4) = d_{22}d_{33}d_{44} \geq 0$

Det $A(2,3,5) = d_{22}d_{33}d_{55} \geq 0$

Det $A(2,4,5) = d_{22}d_{44}d_{55} \geq 0$

Det $A(3,4,5) = d_{33}d_{44}d_{55} \geq 0$

Det $A(1,2,3,4) = d_{11}d_{22}\ d_{33}d_{44} - a_{12}a_{21}d_{33}d_{44} = d_{33}d_{44}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,2,3,5) = d_{11}d_{22}\ d_{33}d_{55} - a_{12}a_{21}d_{33}d_{55} = d_{33}d_{55}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,2,4,5) = d_{11}d_{22}\ d_{44}d_{55} - a_{12}a_{21}d_{44}d_{55} = d_{44}d_{55}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,3,4,5) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A = d_{11}d_{22}\ d_{33}d_{44}d_{55} - a_{12}a_{21}d_{33}d_{44}\ d_{55} = d_{33}d_{44}d_{55}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Hence the pattern has zero completion into nonnegative P_0- matrix since all the determinants are nonnegative.

3. Consider the digraph below:

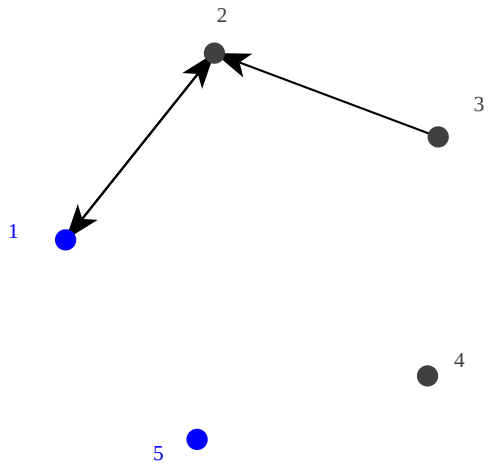

Figure 4.2.5: Diraph with arcs 1 to 2, 2 to 1 and 3 to 2

Digraphs achieved by permuting the vertices of the above digraph gives similar results, example of the digraph that follows:

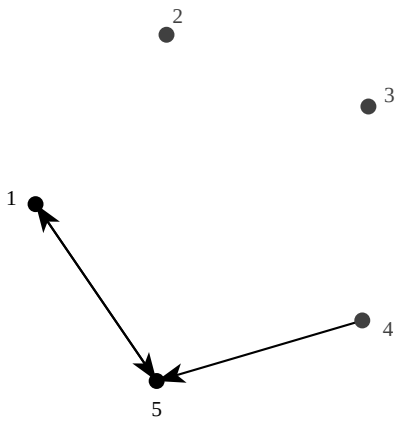

Figure 4.2.6: Digraph isomorphic to the digraph in figure 2.2.4, obtained via permutation (25)(34)

Using the digraph in the figure 2.2.5,

Let $A = \begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ a_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & a_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$ be the partial nonnegative P_0-matrix specifying the digraph.

Determinants of the principal submatrices

Det $A(1,2) = d_{11}d_{22} - a_{12}a_{21}$

Det $A(1,3) = d_{11}d_{33} - x_{13}x_{31}$

Det $A(1,4) = d_{11}d_{44} - x_{14}x_{41}$

Det $A(1,5) = d_{11}d_{55} - x_{15}x_{51}$

Det $A(2,3) = d_{22}d_{33} - x_{23}a_{32}$

Det $A(2,4) = d_{22}d_{44} - x_{24}x_{42}$

Det $A(2,5) = d_{22}d_{55} - x_{25}x_{52}$

Det $A(3,4) = d_{33}d_{44} - x_{34}x_{43}$

Det $A(3,5) = d_{33}d_{55} - x_{35}x_{53}$

Det $A(4,5) = d_{44}d_{55} - x_{45}x_{54}$

Det $A(1,2,3) = d_{11}(d_{22}d_{33} - x_{23}a_{32}) - a_{12}(a_{21}d_{33} - x_{23}x_{31}) + x_{13}(a_{21}a_{32} - d_{22}x_{31})$

Det $A(1,2,4) = d_{11}(d_{22}d_{44} - x_{24}x_{42}) - a_{12}(a_{21}d_{44} - x_{24}x_{41}) + x_{14}(a_{21}x_{42} - d_{22}x_{41})$

Det $A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - a_{12}(a_{21}d_{55} - x_{25}x_{51}) + x_{15}(a_{21}x_{52} - d_{22}x_{51})$

Det $A(1,3,4) = d_{11}(d_{33}d_{44} - x_{34}x_{43}) - x_{13}(x_{31}d_{44} - x_{34}x_{41}) + x_{14}(x_{31}x_{43} - d_{33}x_{41})$

Det $A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - x_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$

Det $A(1,4,5) = d_{11}(d_{44}d_{55} - x_{45}x_{54}) - x_{14}(x_{41}d_{55} - x_{45}x_{51}) + x_{15}(x_{41}x_{54} - d_{44}x_{51})$

Det $A(2,3,4) = d_{22}(d_{33}d_{44} - x_{34}x_{43}) - x_{23}(a_{32}d_{44} - x_{34}x_{42}) + x_{24}(a_{32}x_{43} - d_{33}x_{42})$

Det $A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - x_{23}(a_{32}d_{55} - x_{35}x_{52}) + x_{25}(a_{32}x_{53} - d_{33}x_{52})$

Det $A(2,4,5) = d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})$

Det $A(3,4,5) = d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})$

Det $A(1,2,3,4) = d_{11}\{d_{22}(d_{33}d_{44} - x_{34}x_{43}) - x_{23}(a_{32}d_{44} - x_{34}x_{42}) + x_{24}(a_{32}x_{43} - d_{33}x_{42})\} - a_{12}\{a_{21}(d_{33}d_{44} - x_{34}x_{43}) - x_{23}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41})\} + x_{13}\{a_{21}(a_{32}d_{44} - x_{34}x_{42}) - d_{22}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}x_{42} - a_{32}x_{41})\} - x_{14}\{a_{21}(a_{32}x_{43} - d_{33}x_{42}) - d_{22}(x_{31}x_{43} - d_{33}x_{41}) + x_{23}(x_{31}x_{42} - a_{32}x_{41})\}$

Det $A(1,2,3,5) = d_{11}\{d_{22}(d_{33}d_{55} - x_{35}x_{53}) - x_{23}(a_{32}d_{55} - x_{35}x_{52}) + x_{25}(a_{32}x_{53} - d_{33}x_{52})\} - a_{12}\{a_{21}(d_{33}d_{55} - x_{35}x_{53}) - x_{23}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{53} - d_{33}x_{51})\} + x_{13}\{a_{21}(a_{32}d_{55} - x_{35}x_{52}) - d_{22}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{52} - a_{32}x_{51})\} - x_{15}\{a_{21}(a_{32}x_{53} - d_{33}x_{52}) - d_{22}(x_{31}x_{53} - d_{33}x_{51}) + x_{23}(x_{31}x_{52} - a_{32}x_{51})\}$

Det $A(1,2,4,5) = d_{11}\{d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})\} - a_{12}\{a_{21}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{a_{21}(x_{42}d_{55} - x_{45}x_{52}) - d_{22}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{52} - x_{42}x_{51})\} - x_{15}\{a_{21}(x_{42}x_{54} - d_{44}x_{52}) - d_{22}(x_{41}x_{54} - d_{44}x_{51}) + x_{24}(x_{41}x_{52} - x_{42}x_{51})\}$

Det $A(1,3,4,5) = d_{11}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{13}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{15}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}$

Det $A(2,3,4,5) = d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{23}\{a_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{a_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{a_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\}$

Det $A = d_{11}[d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{23}\{a_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{a_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{a_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\}] -$

$a_{12}[a_{21}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{23}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{25}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}] +$

$x_{13}[a_{21}\{a_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} - d_{22}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - a_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - a_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\}] -$

$x_{14}[a_{21}\{a_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} + x_{23}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - a_{32}(x_{41}d_{55} -$

$x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - a_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}] +$

$x_{15}[a_{21}\{a_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\} + x_{23}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - a_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\} - x_{24}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - a_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}]$

Performing zero completion by setting all the unspecified entries to zero i.e. $x_{13} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{23} = 0$, $x_{24} = 0$, $x_{25} = 0$, $x_{31} = 0$, $x_{34} = 0$, $x_{35} = 0$, $x_{41} = 0$, $x_{42} = 0$, $x_{43} = 0$, $x_{45} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$. The determinants of principal submatrices will be ;

Det $A(1,2) = d_{11}d_{22} - a_{12}a_{21} \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,3) = d_{11}d_{33} \geq 0$

Det $A(1,4) = d_{11}d_{44} \geq 0$

Det $A(1,5) = d_{11}d_{55} \geq 0$

Det $A(2,3) = d_{22}d_{33} \geq 0$

Det $A(2,4) = d_{22}d_{44} \geq 0$

Det $A(2,5) = d_{22}d_{55} \geq 0$

Det $A(3,4) = d_{33}d_{44} \geq 0$

Det $A(3,5) = d_{33}d_{55} \geq 0$

Det $A(4,5) = d_{44}d_{55} \geq 0$

Det $A(1,2,3) = d_{11}d_{22}\ d_{33} - a_{12}a_{21}d_{33} = d_{33}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,2,4) = d_{11}d_{22}\ d_{44} - a_{12}a_{21}d_{44} = d_{44}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,2,5) = d_{11}d_{22}\ d_{55} - a_{12}a_{21}d_{55} = d_{55}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,3,4) = d_{11}d_{33}d_{44} \geq 0$

Det $A(1,3,5) = d_{11}d_{33}d_{55} \geq 0$

Det $A(1,4,5) = d_{11}d_{44}d_{55} \geq 0$

Det $A(2,3,4) = d_{22}d_{33}d_{44} \geq 0$

Det $A(2,3,5) = d_{22}d_{33}d_{55} \geq 0$

Det $A(2,4,5) = d_{22}d_{44}d_{55} \geq 0$

Det $A(3,4,5) = d_{33}d_{44}d_{55} \geq 0$

Det $A(1,2,3,4) = d_{11}d_{22}\ d_{33}d_{44} - a_{12}a_{21}d_{33}d_{44} = d_{33}d_{44}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,2,3,5) = d_{11}d_{22}\ d_{33}d_{55} - a_{12}a_{21}d_{33}d_{55} = d_{33}d_{55}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,2,4,5) = d_{11}d_{22}\ d_{44}d_{55} - a_{12}a_{21}d_{44}d_{55} = d_{44}d_{55}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

Det $A(1,3,4,5) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A = d_{11}d_{22}\ d_{33}d_{44}d_{55} - a_{12}a_{21}d_{33}d_{44}\ d_{55} = d_{33}d_{44}d_{55}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified..

Hence all the determinants are found to be nonnegative and therefore the pattern have zero completion into nonnegative P_0- matrix.

4. Consider the digraph below:

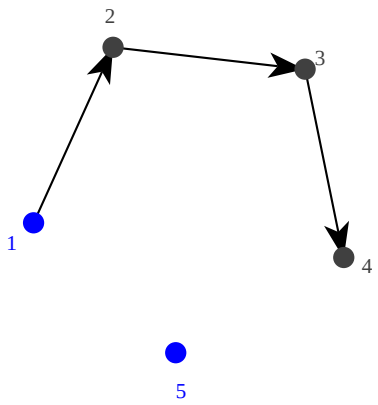

Figure 4.2.7: Digraph with arcs 1 to 2, 2 to 3 and 3 to 4

The digraph has similar results with the digraph that follows:

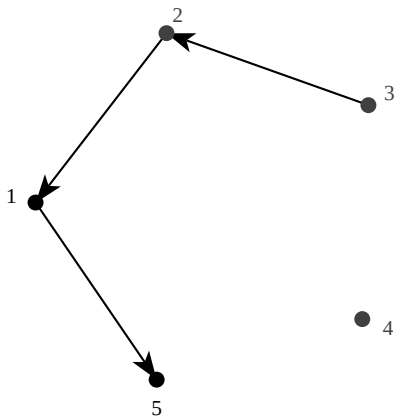

Figure 4.2.8: Digraph isomorphic to digraph in figure 2.2.7, obtained via permutation (13524)

Using the digraph in the figure 2.2.7,

Let $A = \begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$ be the partial nonnegative P_0-matrix specifying the digraph.

Determinants of the principal submatrices

Det $A(1,2) = d_{11}d_{22} - a_{12}x_{21}$

Det $A(1,3) = d_{11}d_{33} - x_{13}x_{31}$

Det $A(1,4) = d_{11}d_{44} - x_{14}x_{41}$

Det $A(1,5) = d_{11}d_{55} - x_{15}x_{51}$

Det $A(2,3) = d_{22}d_{33} - a_{23}x_{32}$

Det $A(2,4) = d_{22}d_{44} - x_{24}x_{42}$

Det $A(2,5) = d_{22}d_{55} - x_{25}x_{52}$

Det $A(3,4) = d_{33}d_{44} - a_{34}x_{43}$

Det $A(3,5) = d_{33}d_{55} - x_{35}x_{53}$

Det $A(4,5) = d_{44}d_{55} - x_{45}x_{54}$

Det $A(1,2,3) = d_{11}(d_{22}d_{33} - a_{23}x_{32}) - a_{12}(x_{21}d_{33} - a_{23}x_{31}) + x_{13}(x_{21}x_{32} - d_{22}x_{31})$

Det $A(1,2,4) = d_{11}(d_{22}d_{44} - x_{24}x_{42}) - a_{12}(x_{21}d_{44} - x_{24}x_{41}) + x_{14}(x_{21}x_{42} - d_{22}x_{41})$

Det $A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - a_{12}(x_{21}d_{55} - x_{25}x_{51}) + x_{15}(x_{21}x_{52} - d_{22}x_{51})$

Det $A(1,3,4) = d_{11}(d_{33}d_{44} - a_{34}x_{43}) - x_{13}(x_{31}d_{44} - a_{34}x_{41}) + x_{14}(x_{31}x_{43} - d_{33}x_{41})$

Det $A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - x_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$

Det $A(1,4,5) = d_{11}(d_{44}d_{55} - x_{45}x_{54}) - x_{14}(x_{41}d_{55} - x_{45}x_{51}) + x_{15}(x_{41}x_{54} - d_{44}x_{51})$

Det $A(2,3,4) = d_{22}(d_{33}d_{44} - a_{34}x_{43}) - a_{23}(x_{32}d_{44} - a_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})$

Det $A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})$

Det $A(2,4,5) = d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})$

Det $A(3,4,5) = d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})$

Det $A(1,2,3,4) = d_{11}\{d_{22}(d_{33}d_{44} - a_{34}x_{43}) - a_{23}(x_{32}d_{44} - a_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})\} - a_{12}\{x_{21}(d_{33}d_{44} - a_{34}x_{43}) - a_{23}(x_{31}d_{44} - a_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41})\} + x_{13}\{x_{21}(x_{32}d_{44} - a_{34}x_{42}) - d_{22}(x_{31}d_{44} - a_{34}x_{41}) + x_{24}(x_{31}x_{42} - x_{32}x_{41})\} - x_{14}\{x_{21}(x_{32}x_{43} - d_{33}x_{42}) - d_{22}(x_{31}x_{43} - d_{33}x_{41}) + a_{23}(x_{31}x_{42} - x_{32}x_{41})\}$

Det $A(1,2,3,5) = d_{11}\{d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})\} - a_{12}\{x_{21}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{53} - d_{33}x_{51})\} + x_{13}\{x_{21}(x_{32}d_{55} - x_{35}x_{52}) - d_{22}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{52} - x_{32}x_{51})\} - x_{15}\{x_{21}(x_{32}x_{53} - d_{33}x_{52}) - d_{22}(x_{31}x_{53} - d_{33}x_{51}) + a_{23}(x_{31}x_{52} - x_{32}x_{51})\}$

Det $A(1,2,4,5) = d_{11}\{d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})\} - a_{12}\{x_{21}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{21}(x_{42}d_{55} - x_{45}x_{52}) - d_{22}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{52} - x_{42}x_{51})\} - x_{15}\{x_{21}(x_{42}x_{54} - d_{44}x_{52}) - d_{22}(x_{41}x_{54} - d_{44}x_{51}) + x_{24}(x_{41}x_{52} - x_{42}x_{51})\}$

Det $A(1,3,4,5) = d_{11}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{13}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{15}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{53} - x_{43}x_{51})\}$

Det $A(2,3,4,5) = d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + a_{34}(x_{42}x_{53} - x_{43}x_{52})\}$

Det $A = d_{11}[d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + a_{34}(x_{42}x_{53} - x_{43}x_{52})\}] -$

$a_{12}[x_{21}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{25}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{53} - x_{43}x_{51})\}] +$

$x_{13}[x_{21}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} - d_{22}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - x_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{52} - x_{42}x_{51})\}] -$

$x_{14}[x_{21}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} + a_{23}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - x_{32}(x_{41}d_{55} -$

$x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}] +$

$x_{15}[x_{21}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + a_{34}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{53} - x_{43}x_{51})\} + a_{23}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{52} - x_{42}x_{51})\} - x_{24}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}]$

Performing zero completion by setting all the unspecified entries to zero i.e. $x_{13} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{21} = 0$, $x_{24} = 0$, $x_{25} = 0$, $x_{31} = 0$, $x_{32} = 0$, $x_{35} = 0$, $x_{41} = 0$ $x_{42} = 0$, $x_{43} = 0$, $x_{45} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$. The determinants of principal submatrices will be ;

Det $A(1,2) = d_{11}d_{22} \geq 0$

Det $A(1,3) = d_{11}d_{33} \geq 0$

Det $A(1,4) = d_{11}d_{44} \geq 0$

Det $A(1,5) = d_{11}d_{55} \geq 0$

Det $A(2,3) = d_{22}d_{33} \geq 0$

Det $A(2,4) = d_{22}d_{44} \geq 0$

Det $A(2,5) = d_{22}d_{55} \geq 0$

Det $A(3,4) = d_{33}d_{44} \geq 0$

Det $A(3,5) = d_{33}d_{55} \geq 0$

Det $A(4,5) = d_{44}d_{55} \geq 0$

Det $A(1,2,3) = d_{11}d_{22}\, d_{33} \geq 0$

Det $A(1,2,4) = d_{11}d_{22}\, d_{44} \geq 0$

Det $A(1,2,5) = d_{11}d_{22}\, d_{55} \geq 0$

Det $A(1,3,4) = d_{11}d_{33}d_{44} \geq 0$

Det $A(1,3,5) = d_{11}d_{33}d_{55} \geq 0$

Det $A(1,4,5) = d_{11}d_{44}d_{55} \geq 0$

Det $A(2,3,4) = d_{22}d_{33}d_{44} \geq 0$

Det $A(2,3,5) = d_{22}d_{33}d_{55} \geq 0$

Det $A(2,4,5) = d_{22}d_{44}d_{55} \geq 0$

Det $A(3,4,5) = d_{33}d_{44}d_{55} \geq 0$

Det $A(1,2,3,4) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(1,2,3,5) = d_{11}d_{22}\ d_{33}d_{55} \geq 0$

Det $A(1,2,4,5) = d_{11}d_{22}\ d_{44}d_{55} \geq 0$

Det $A(1,3,4,5) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(2,3,4,5) = d_{22}d_{33}d_{44}d_{55} \geq 0$

Det $A = d_{11}d_{22}\ d_{33}d_{44}d_{55} \geq 0$

With all the determinants are nonnegative, the pattern have zero completion into nonnegative P_0- matrix.

5. Consider the digraph below:

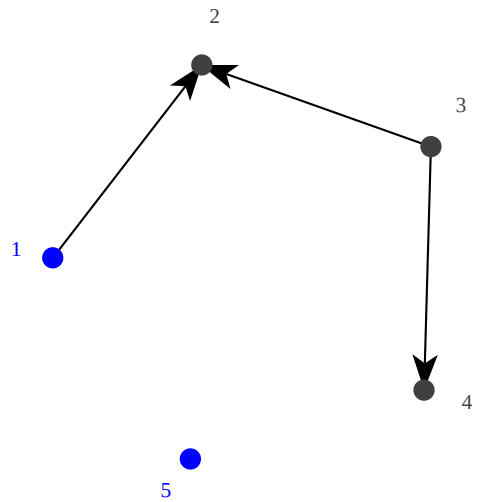

Figure 4.2.9: Digraph with arcs 1 to 2, 3 to 2 and 3 to 4

Permuting the vertices of the digraph gives other digraphs with similar results, such as the one that follows:

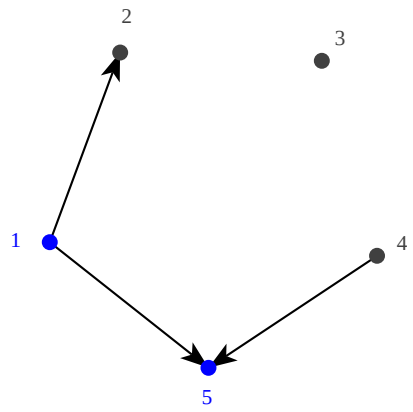

Figure 4.2.10: Digraph isomorphic to digraph in figure 2.2.9, obtained via permutation (14253)

Using the digraph in figure 2.2.9,

Let $A = \begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & a_{32} & d_{33} & a_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$ be the partial nonnegative P_0-matrix specifying the digraph.

Determinants of the principal submatrices

Det $A(1,2) = d_{11}d_{22} - a_{12}x_{21}$

Det $A(1,3) = d_{11}d_{33} - x_{13}x_{31}$

Det $A(1,4) = d_{11}d_{44} - x_{14}x_{41}$

Det $A(1,5) = d_{11}d_{55} - x_{15}x_{51}$

Det $A(2,3) = d_{22}d_{33} - x_{23}a_{32}$

Det $A(2,4) = d_{22}d_{44} - x_{24}x_{42}$

Det $A(2,5) = d_{22}d_{55} - x_{25}x_{52}$

Det $A(3,4) = d_{33}d_{44} - a_{34}x_{43}$

Det $A(3,5) = d_{33}d_{55} - x_{35}x_{53}$

Det $A(4,5) = d_{44}d_{55} - x_{45}x_{54}$

Det $A(1,2,3) = d_{11}(d_{22}d_{33} - x_{23}a_{32}) - a_{12}(x_{21}d_{33} - x_{23}x_{31}) + x_{13}(x_{21}a_{32} - d_{22}x_{31})$

Det $A(1,2,4) = d_{11}(d_{22}d_{44} - x_{24}x_{42}) - a_{12}(x_{21}d_{44} - x_{24}x_{41}) + x_{14}(x_{21}x_{42} - d_{22}x_{41})$

Det $A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - a_{12}(x_{21}d_{55} - x_{25}x_{51}) + x_{15}(x_{21}x_{52} - d_{22}x_{51})$

Det $A(1,3,4) = d_{11}(d_{33}d_{44} - a_{34}x_{43}) - x_{13}(x_{31}d_{44} - a_{34}x_{41}) + x_{14}(x_{31}x_{43} - d_{33}x_{41})$

Det $A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - x_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$

Det $A(1,4,5) = d_{11}(d_{44}d_{55} - x_{45}x_{54}) - x_{14}(x_{41}d_{55} - x_{45}x_{51}) + x_{15}(x_{41}x_{54} - d_{44}x_{51})$

Det $A(2,3,4) = d_{22}(d_{33}d_{44} - a_{34}x_{43}) - x_{23}(a_{32}d_{44} - a_{34}x_{42}) + x_{24}(a_{32}x_{43} - d_{33}x_{42})$

Det $A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - x_{23}(a_{32}d_{55} - x_{35}x_{52}) + x_{25}(a_{32}x_{53} - d_{33}x_{52})$

Det $A(2,4,5) = d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})$

Det $A(3,4,5) = d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})$

Det $A(1,2,3,4) = d_{11}\{d_{22}(d_{33}d_{44} - a_{34}x_{43}) - x_{23}(a_{32}d_{44} - a_{34}x_{42}) + x_{24}(a_{32}x_{43} - d_{33}x_{42})\} - a_{12}\{x_{21}(d_{33}d_{44} - a_{34}x_{43}) - x_{23}(x_{31}d_{44} - a_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41})\} + x_{13}\{x_{21}(a_{32}d_{44} - a_{34}x_{42}) - d_{22}(x_{31}d_{44} - a_{34}x_{41}) + x_{24}(x_{31}x_{42} - a_{32}x_{41})\} - x_{14}\{x_{21}(a_{32}x_{43} - d_{33}x_{42}) - d_{22}(x_{31}x_{43} - d_{33}x_{41}) + x_{23}(x_{31}x_{42} - a_{32}x_{41})\}$

Det $A(1,2,3,5) = d_{11}\{d_{22}(d_{33}d_{55} - x_{35}x_{53}) - x_{23}(a_{32}d_{55} - x_{35}x_{52}) + x_{25}(a_{32}x_{53} - d_{33}x_{52})\} - a_{12}\{x_{21}(d_{33}d_{55} - x_{35}x_{53}) - x_{23}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{53} - d_{33}x_{51})\} + x_{13}\{x_{21}(a_{32}d_{55} - x_{35}x_{52}) - d_{22}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{52} - a_{32}x_{51})\} - x_{15}\{x_{21}(a_{32}x_{53} - d_{33}x_{52}) - d_{22}(x_{31}x_{53} - d_{33}x_{51}) + x_{23}(x_{31}x_{52} - a_{32}x_{51})\}$

Det $A(1,2,4,5) = d_{11}\{d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})\} - a_{12}\{x_{21}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{21}(x_{42}d_{55} - x_{45}x_{52}) - d_{22}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{52} - x_{42}x_{51})\} - x_{15}\{x_{21}(x_{42}x_{54} - d_{44}x_{52}) - d_{22}(x_{41}x_{54} - d_{44}x_{51}) + x_{24}(x_{41}x_{52} - x_{42}x_{51})\}$

Det $A(1,3,4,5) = d_{11}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{13}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{15}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{53} - x_{43}x_{51})\}$

Det $A(2,3,4,5) = d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{23}\{a_{32}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{a_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{a_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + a_{34}(x_{42}x_{53} - x_{43}x_{52})\}$

Det $A = d_{11}[d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{23}\{a_{32}(d_{44}d_{55} -$

$x_{45}x_{54}) - a_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{a_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{a_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + a_{34}(x_{42}x_{53} - x_{43}x_{52})\}] -$

$a_{12}[x_{21}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{23}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{25}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{53} - x_{43}x_{51})\}] +$

$x_{13}[x_{21}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} - d_{22}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - a_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - a_{32}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{52} - x_{42}x_{51})\}] -$

$x_{14}[x_{21}\{a_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} + x_{23}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - a_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - a_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}] +$

$x_{15}[x_{21}\{a_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + a_{34}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{53} - x_{43}x_{51})\} + x_{23}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - a_{32}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{52} - x_{42}x_{51})\} - x_{24}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - a_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}]$

Performing zero completion by setting all the unspecified entries to zero i.e. $x_{13} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{21} = 0$, $x_{23} = 0$, $x_{24} = 0$, $x_{25} = 0$, $x_{31} = 0$, $x_{35} = 0$, $x_{41} = 0$, $x_{42} = 0$, $x_{43} = 0$, $x_{45} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$. The determinants of principal submatrices will be ;

Det $A(1,2) = d_{11}d_{22} \geq 0$

Det $A(1,3) = d_{11}d_{33} \geq 0$

Det $A(1,4) = d_{11}d_{44} \geq 0$

Det $A(1,5) = d_{11}d_{55} \geq 0$

Det $A(2,3) = d_{22}d_{33} \geq 0$

Det $A(2,4) = d_{22}d_{44} \geq 0$

Det $A(2,5) = d_{22}d_{55} \geq 0$

Det $A(3,4) = d_{33}d_{44} \geq 0$

Det $A(3,5) = d_{33}d_{55} \geq 0$

Det $A(4,5) = d_{44}d_{55} \geq 0$

Det $A(1,2,3) = d_{11}d_{22}\ d_{33} \geq 0$

Det $A(1,2,4) = d_{11}d_{22}\ d_{44} \geq 0$

Det $A(1,2,5) = d_{11}d_{22}\ d_{55} \geq 0$

Det $A(1,3,4) = d_{11}d_{33}d_{44} \geq 0$

Det $A(1,3,5) = d_{11}d_{33}d_{55} \geq 0$

Det $A(1,4,5) = d_{11}d_{44}d_{55} \geq 0$

Det $A(2,3,4) = d_{22}d_{33}d_{44} \geq 0$

Det $A(2,3,5) = d_{22}d_{33}d_{55} \geq 0$

Det $A(2,4,5) = d_{22}d_{44}d_{55} \geq 0$

Det $A(3,4,5) = d_{33}d_{44}d_{55} \geq 0$

Det $A(1,2,3,4) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(1,2,3,5) = d_{11}d_{22}\ d_{33}d_{55} \geq 0$

Det $A(1,2,4,5) = d_{11}d_{22}\ d_{44}d_{55} \geq 0$

Det $A(1,3,4,5) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(2,3,4,5) = d_{22}d_{33}d_{44}d_{55} \geq 0$

Det $A = d_{11}d_{22}\ d_{33}d_{44}d_{55} \geq 0$

Hence all the determinants are nonnegative and therefore the pattern have zero completion into nonnegative P_0- matrix.

6. Consider the digraph below:

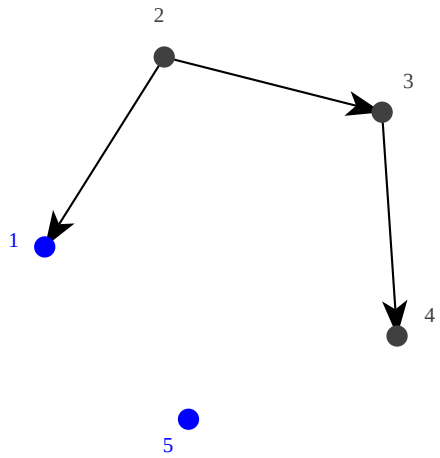

Figure 4.2.11: Dgraph with arcs 2 to 1, 2 to 3 and 3 to 4

Permuting the vertices of the digraph via permutation (14)(23) gives digraph that follows, which is amongst other digraphs with similar results.

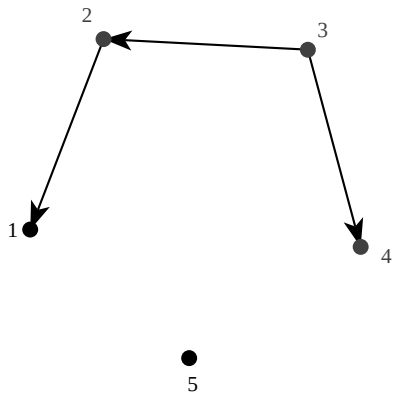

Figure 4.2.12: Digraph isomorphic to digraph in figure 2.2.11, obtained via permutation (14)(23)

Using the digraph in the figure 2.2.11,

Let $A = \begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ a_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$ be the partial nonnegative P_0-matrix specifying the digraph.

Determinants of the principal submatrices

Det $A(1,2) = d_{11}d_{22} - x_{12}a_{21}$

Det $A(1,3) = d_{11}d_{33} - x_{13}x_{31}$

Det $A(1,4) = d_{11}d_{44} - x_{14}x_{41}$

Det $A(1,5) = d_{11}d_{55} - x_{15}x_{51}$

Det $A(2,3) = d_{22}d_{33} - a_{23}x_{32}$

Det $A(2,4) = d_{22}d_{44} - x_{24}x_{42}$

Det $A(2,5) = d_{22}d_{55} - x_{25}x_{52}$

Det $A(3,4) = d_{33}d_{44} - a_{34}x_{43}$

Det $A(3,5) = d_{33}d_{55} - x_{35}x_{53}$

Det $A(4,5) = d_{44}d_{55} - x_{45}x_{54}$

Det $A(1,2,3) = d_{11}(d_{22}d_{33} - a_{23}x_{32}) - x_{12}(a_{21}d_{33} - a_{23}x_{31}) + x_{13}(a_{21}x_{32} - d_{22}x_{31})$

Det $A(1,2,4) = d_{11}(d_{22}d_{44} - x_{24}x_{42}) - x_{12}(a_{21}d_{44} - x_{24}x_{41}) + x_{14}(a_{21}x_{42} - d_{22}x_{41})$

Det $A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - x_{12}(a_{21}d_{55} - x_{25}x_{51}) + x_{15}(a_{21}x_{52} - d_{22}x_{51})$

Det $A(1,3,4) = d_{11}(d_{33}d_{44} - a_{34}x_{43}) - x_{13}(x_{31}d_{44} - a_{34}x_{41}) + x_{14}(x_{31}x_{43} - d_{33}x_{41})$

Det $A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - x_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$

Det $A(1,4,5) = d_{11}(d_{44}d_{55} - x_{45}x_{54}) - x_{14}(x_{41}d_{55} - x_{45}x_{51}) + x_{15}(x_{41}x_{54} - d_{44}x_{51})$

Det $A(2,3,4) = d_{22}(d_{33}d_{44} - a_{34}x_{43}) - a_{23}(x_{32}d_{44} - a_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})$

Det $A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})$

Det $A(2,4,5) = d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})$

Det $A(3,4,5) = d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})$

Det $A(1,2,3,4) = d_{11}\{d_{22}(d_{33}d_{44} - a_{34}x_{43}) - a_{23}(x_{32}d_{44} - a_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})\} - x_{12}\{a_{21}(d_{33}d_{44} - a_{34}x_{43}) - a_{23}(x_{31}d_{44} - a_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41})\} + x_{13}\{a_{21}(x_{32}d_{44} - a_{34}x_{42}) - d_{22}(x_{31}d_{44} - a_{34}x_{41}) + x_{24}(x_{31}x_{42} - x_{32}x_{41})\} - x_{14}\{a_{21}(x_{32}x_{43} - d_{33}x_{42}) - d_{22}(x_{31}x_{43} - d_{33}x_{41}) + a_{23}(x_{31}x_{42} - x_{32}x_{41})\}$

Det $A(1,2,3,5) = d_{11}\{d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})\} - x_{12}\{a_{21}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{53} - d_{33}x_{51})\} + x_{13}\{a_{21}(x_{32}d_{55} - x_{35}x_{52}) - d_{22}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{52} - x_{32}x_{51})\} - x_{15}\{a_{21}(x_{32}x_{53} - d_{33}x_{52}) - d_{22}(x_{31}x_{53} - d_{33}x_{51}) + a_{23}(x_{31}x_{52} - x_{32}x_{51})\}$

Det $A(1,2,4,5) = d_{11}\{d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})\} - x_{12}\{a_{21}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{a_{21}(x_{42}d_{55} - x_{45}x_{52}) - d_{22}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{52} - x_{42}x_{51})\} - x_{15}\{a_{21}(x_{42}x_{54} - d_{44}x_{52}) - d_{22}(x_{41}x_{54} - d_{44}x_{51}) + x_{24}(x_{41}x_{52} - x_{42}x_{51})\}$

Det $A(1,3,4,5) = d_{11}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{13}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{15}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{53} - x_{43}x_{51})\}$

Det $A(2,3,4,5) = d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + a_{34}(x_{42}x_{53} - x_{43}x_{52})\}$

Det $A = d_{11}[d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + a_{34}(x_{42}x_{53} - x_{43}x_{52})\}] -$

$x_{12}[a_{21}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{25}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{53} - x_{43}x_{51})\}] +$

$x_{13}[a_{21}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} - d_{22}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - x_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{52} - x_{42}x_{51})\}] -$

$x_{14}[a_{21}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} + a_{23}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - x_{32}(x_{41}d_{55} -$

$x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}] +$

$x_{15}[a_{21}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + a_{34}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{53} - x_{43}x_{51})\} + a_{23}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + a_{34}(x_{41}x_{52} - x_{42}x_{51})\} - x_{24}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}]$

Performing zero completion by setting all the unspecified entries to zero i.e. $x_{12} = 0$, $x_{13} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{24} = 0$, $x_{25} = 0$, $x_{31} = 0$, $x_{32} = 0$, $x_{35} = 0$, $x_{41} = 0$, $x_{42} = 0$, $x_{43} = 0$, $x_{45} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$. The determinants of principal submatrices will be ;

Det $A(1,2) = d_{11}d_{22} \geq 0$

Det $A(1,3) = d_{11}d_{33} \geq 0$

Det $A(1,4) = d_{11}d_{44} \geq 0$

Det $A(1,5) = d_{11}d_{55} \geq 0$

Det $A(2,3) = d_{22}d_{33} \geq 0$

Det $A(2,4) = d_{22}d_{44} \geq 0$

Det $A(2,5) = d_{22}d_{55} \geq 0$

Det $A(3,4) = d_{33}d_{44} \geq 0$

Det $A(3,5) = d_{33}d_{55} \geq 0$

Det $A(4,5) = d_{44}d_{55} \geq 0$

Det $A(1,2,3) = d_{11}d_{22}\, d_{33} \geq 0$

Det $A(1,2,4) = d_{11}d_{22}\, d_{44} \geq 0$

Det $A(1,2,5) = d_{11}d_{22}\, d_{55} \geq 0$

Det $A(1,3,4) = d_{11}d_{33}d_{44} \geq 0$

Det $A(1,3,5) = d_{11}d_{33}d_{55} \geq 0$

Det $A(1,4,5) = d_{11}d_{44}d_{55} \geq 0$

Det $A(2,3,4) = d_{22}d_{33}d_{44} \geq 0$

Det $A(2,3,5) = d_{22}d_{33}d_{55} \geq 0$

Det $A(2,4,5) = d_{22}d_{44}d_{55} \geq 0$

Det $A(3,4,5) = d_{33}d_{44}d_{55} \geq 0$

Det $A(1,2,3,4) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(1,2,3,5) = d_{11}d_{22}\ d_{33}d_{55} \geq 0$

Det $A(1,2,4,5) = d_{11}d_{22}\ d_{44}d_{55} \geq 0$

Det $A(1,3,4,5) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(2,3,4,5) = d_{22}d_{33}d_{44}d_{55} \geq 0$

Det $A = d_{11}d_{22}\ d_{33}d_{44}d_{55} \geq 0$

The pattern have all the determinants nonnegative and therefore have zero completion into nonnegative P_0- matrix.

7. Consider the digraph below:

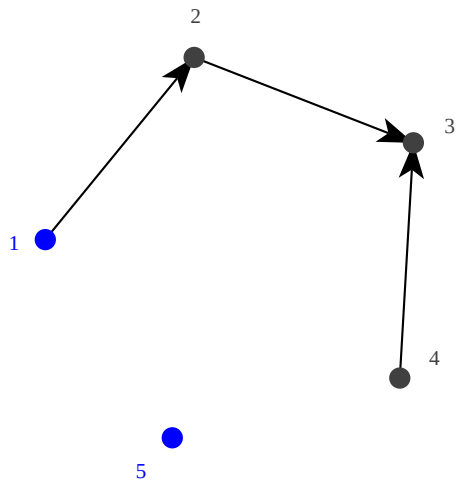

Figure 4.2.13: Digraph with arcs 1 to 2, 2 to 3 and 4 to 3

By using permutation (15)(24) the following digraph is obtained which is amongst others with similar results.

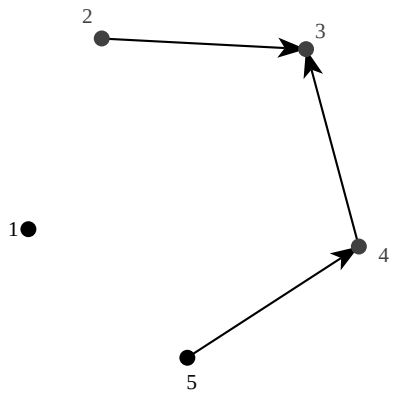

Figure 4.2.14: Digraph isomorphic to digraph in figure 2.2.13, obtained via permutation (15)(24)

Using the digraph in figure 2.2.13

Let $A = \begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & a_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$ be the partial nonnegative P_0-matrix specifying the digraph.

Determinants of the principal submatrices

Det $A(1,2) = d_{11}d_{22} - a_{12}x_{21}$

Det $A(1,3) = d_{11}d_{33} - x_{13}x_{31}$

Det $A(1,4) = d_{11}d_{44} - x_{14}x_{41}$

Det $A(1,5) = d_{11}d_{55} - x_{15}x_{51}$

Det $A(2,3) = d_{22}d_{33} - a_{23}x_{32}$

Det $A(2,4) = d_{22}d_{44} - x_{24}x_{42}$

Det $A(2,5) = d_{22}d_{55} - x_{25}x_{52}$

Det $A(3,4) = d_{33}d_{44} - x_{34}a_{43}$

Det $A(3,5) = d_{33}d_{55} - x_{35}x_{53}$

Det $A(4,5) = d_{44}d_{55} - x_{45}x_{54}$

Det $A(1,2,3) = d_{11}(d_{22}d_{33} - a_{23}x_{32}) - a_{12}(x_{21}d_{33} - a_{23}x_{31}) + x_{13}(x_{21}x_{32} - d_{22}x_{31})$

Det $A(1,2,4) = d_{11}(d_{22}d_{44} - x_{24}x_{42}) - a_{12}(x_{21}d_{44} - x_{24}x_{41}) + x_{14}(x_{21}x_{42} - d_{22}x_{41})$

Det $A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - a_{12}(x_{21}d_{55} - x_{25}x_{51}) + x_{15}(x_{21}x_{52} - d_{22}x_{51})$

Det $A(1,3,4) = d_{11}(d_{33}d_{44} - x_{34}a_{43}) - x_{13}(x_{31}d_{44} - x_{34}x_{41}) + x_{14}(x_{31}a_{43} - d_{33}x_{41})$

Det $A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - x_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$

Det $A(1,4,5) = d_{11}(d_{44}d_{55} - x_{45}x_{54}) - x_{14}(x_{41}d_{55} - x_{45}x_{51}) + x_{15}(x_{41}x_{54} - d_{44}x_{51})$

Det $A(2,3,4) = d_{22}(d_{33}d_{44} - x_{34}a_{43}) - a_{23}(x_{32}d_{44} - x_{34}x_{42}) + x_{24}(x_{32}a_{43} - d_{33}x_{42})$

Det $A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})$

Det $A(2,4,5) = d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})$

Det $A(3,4,5) = d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(a_{43}d_{55} - x_{45}x_{53}) + x_{35}(a_{43}x_{54} - d_{44}x_{53})$

Det $A(1,2,3,4) = d_{11}\{d_{22}(d_{33}d_{44} - x_{34}a_{43}) - a_{23}(x_{32}d_{44} - x_{34}x_{42}) + x_{24}(x_{32}a_{43} - d_{33}x_{42})\} - a_{12}\{x_{21}(d_{33}d_{44} - x_{34}a_{43}) - a_{23}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}a_{43} - d_{33}x_{41})\} + x_{13}\{x_{21}(x_{32}d_{44} - x_{34}x_{42}) - d_{22}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}x_{42} - x_{32}x_{41})\} - x_{14}\{x_{21}(x_{32}a_{43} - d_{33}x_{42}) - d_{22}(x_{31}a_{43} - d_{33}x_{41}) + a_{23}(x_{31}x_{42} - x_{32}x_{41})\}$

Det $A(1,2,3,5) = d_{11}\{d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})\} - a_{12}\{x_{21}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{53} - d_{33}x_{51})\} + x_{13}\{x_{21}(x_{32}d_{55} - x_{35}x_{52}) - d_{22}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{52} - x_{32}x_{51})\} - x_{15}\{x_{21}(x_{32}x_{53} - d_{33}x_{52}) - d_{22}(x_{31}x_{53} - d_{33}x_{51}) + a_{23}(x_{31}x_{52} - x_{32}x_{51})\}$

Det $A(1,2,4,5) = d_{11}\{d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})\} - a_{12}\{x_{21}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{21}(x_{42}d_{55} - x_{45}x_{52}) - d_{22}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{52} - x_{42}x_{51})\} - x_{15}\{x_{21}(x_{42}x_{54} - d_{44}x_{52}) - d_{22}(x_{41}x_{54} - d_{44}x_{51}) + x_{24}(x_{41}x_{52} - x_{42}x_{51})\}$

Det $A(1,3,4,5) = d_{11}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(a_{43}d_{55} - x_{45}x_{53}) + x_{35}(a_{43}x_{54} - d_{44}x_{53})\} - x_{13}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{31}(a_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{15}\{x_{31}(a_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - a_{43}x_{51})\}$

Det $A(2,3,4,5) = d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(a_{43}d_{55} - x_{45}x_{53}) + x_{35}(a_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{x_{32}(a_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - a_{43}x_{52})\} - x_{25}\{x_{32}(a_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - a_{43}x_{52})\}$

Det $A = d_{11}[d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(a_{43}d_{55} - x_{45}x_{53}) + x_{35}(a_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{x_{32}(a_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - a_{43}x_{52})\} - x_{25}\{x_{32}(a_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - a_{43}x_{52})\}] -$

$a_{12}[x_{21}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(a_{43}d_{55} - x_{45}x_{53}) + x_{35}(a_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(a_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - a_{43}x_{51})\} - x_{25}\{x_{31}(a_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - a_{43}x_{51})\}] +$

$x_{13}[x_{21}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} - d_{22}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - x_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\}] -$

$x_{14}[x_{21}\{x_{32}(a_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - a_{43}x_{52})\} - d_{22}\{x_{31}(a_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - a_{43}x_{51})\} + a_{23}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - x_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{53} - a_{43}x_{52}) - x_{32}(x_{41}x_{53} - a_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}] +$

$x_{15}[x_{21}\{x_{32}(a_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - a_{43}x_{52})\} - d_{22}\{x_{31}(a_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - a_{43}x_{51})\} + a_{23}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\} - x_{24}\{x_{31}(x_{42}x_{53} - a_{43}x_{52}) - x_{32}(x_{41}x_{53} - a_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}]$

Performing zero completion by setting all the unspecified entries to zero i.e. $x_{13} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{21} = 0$, $x_{24} = 0$, $x_{25} = 0$, $x_{31} = 0$, $x_{32} = 0$, $x_{34} = 0$, $x_{35} = 0$, $x_{41} = 0$, $x_{42} = 0$, $x_{45} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$. The determinants of principal submatrices will be ;

Det $A(1,2) = d_{11}d_{22} \geq 0$

Det $A(1,3) = d_{11}d_{33} \geq 0$

Det $A(1,4) = d_{11}d_{44} \geq 0$

Det $A(1,5) = d_{11}d_{55} \geq 0$

Det $A(2,3) = d_{22}d_{33} \geq 0$

Det $A(2,4) = d_{22}d_{44} \geq 0$

Det $A(2,5) = d_{22}d_{55} \geq 0$

Det $A(3,4) = d_{33}d_{44} \geq 0$

Det $A(3,5) = d_{33}d_{55} \geq 0$

Det $A(4,5) = d_{44}d_{55} \geq 0$

Det $A(1,2,3) = d_{11}d_{22}\ d_{33} \geq 0$

Det $A(1,2,4) = d_{11}d_{22}\ d_{44} \geq 0$

Det $A(1,2,5) = d_{11}d_{22}\ d_{55} \geq 0$

Det $A(1,3,4) = d_{11}d_{33}d_{44} \geq 0$

Det $A(1,3,5) = d_{11}d_{33}d_{55} \geq 0$

Det $A(1,4,5) = d_{11}d_{44}d_{55} \geq 0$

Det $A(2,3,4) = d_{22}d_{33}d_{44} \geq 0$

Det $A(2,3,5) = d_{22}d_{33}d_{55} \geq 0$

Det $A(2,4,5) = d_{22}d_{44}d_{55} \geq 0$

Det $A(3,4,5) = d_{33}d_{44}d_{55} \geq 0$

Det $A(1,2,3,4) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(1,2,3,5) = d_{11}d_{22}\ d_{33}d_{55} \geq 0$

Det $A(1,2,4,5) = d_{11}d_{22}\ d_{44}d_{55} \geq 0$

Det $A(1,3,4,5) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(2,3,4,5) = d_{22}d_{33}d_{44}d_{55} \geq 0$

Det $A = d_{11}d_{22}\ d_{33}d_{44}d_{55} \geq 0$

Determinants for the pattern are all nonnegative and therefore the pattern have zero completion into nonnegative P_0- matrix.

8. Consider the digraph below:

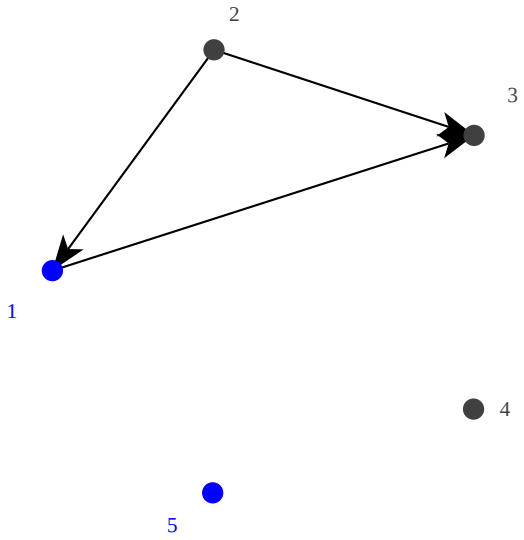

Figure 4.2.15: Digraph with arcs 1 to 3, 2 to 1 and 2 to 3

The digraph has similar results with the following digraph:

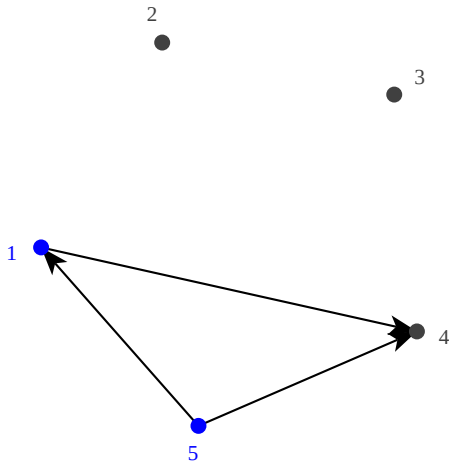

Figure 4.2.16: Digraph isomorphic to digraph in figure 2.2.15, obtained via permutation (2534)

Using the digraph in figure 2.2.15:

Let $A = \begin{bmatrix} d_{11} & x_{12} & a_{13} & x_{14} & x_{15} \\ a_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$ be the partial nonnegative P_0-matrix specifying the digraph.

Determinants of the principal submatrices

Det $A(1,2) = d_{11}d_{22} - x_{12}a_{21}$

Det $A(1,3) = d_{11}d_{33} - a_{13}x_{31}$

Det $A(1,4) = d_{11}d_{44} - x_{14}x_{41}$

Det $A(1,5) = d_{11}d_{55} - x_{15}x_{51}$

Det $A(2,3) = d_{22}d_{33} - a_{23}x_{32}$

Det $A(2,4) = d_{22}d_{44} - x_{24}x_{42}$

Det $A(2,5) = d_{22}d_{55} - x_{25}x_{52}$

Det $A(3,4) = d_{33}d_{44} - x_{34}x_{43}$

Det $A(3,5) = d_{33}d_{55} - x_{35}x_{53}$

Det $A(4,5) = d_{44}d_{55} - x_{45}x_{54}$

Det $A(1,2,3) = d_{11}(d_{22}d_{33} - a_{23}x_{32}) - x_{12}(a_{21}d_{33} - a_{23}x_{31}) + a_{13}(a_{21}x_{32} - d_{22}x_{31})$

Det $A(1,2,4) = d_{11}(d_{22}d_{44} - x_{24}x_{42}) - x_{12}(a_{21}d_{44} - x_{24}x_{41}) + x_{14}(a_{21}x_{42} - d_{22}x_{41})$

Det $A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - x_{12}(a_{21}d_{55} - x_{25}x_{51}) + x_{15}(a_{21}x_{52} - d_{22}x_{51})$

Det $A(1,3,4) = d_{11}(d_{33}d_{44} - x_{34}x_{43}) - a_{13}(x_{31}d_{44} - x_{34}x_{41}) + x_{14}(x_{31}x_{43} - d_{33}x_{41})$

Det $A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - a_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$

Det $A(1,4,5) = d_{11}(d_{44}d_{55} - x_{45}x_{54}) - x_{14}(x_{41}d_{55} - x_{45}x_{51}) + x_{15}(x_{41}x_{54} - d_{44}x_{51})$

Det $A(2,3,4) = d_{22}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{32}d_{44} - x_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})$

Det $A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})$

Det $A(2,4,5) = d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})$

Det $A(3,4,5) = d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})$

Det $A(1,2,3,4) = d_{11}\{d_{22}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{32}d_{44} - x_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})\} - x_{12}\{a_{21}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41})\} + a_{13}\{a_{21}(x_{32}d_{44} - x_{34}x_{42}) - d_{22}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}x_{42} - x_{32}x_{41})\} - x_{14}\{a_{21}(x_{32}x_{43} - d_{33}x_{42}) - d_{22}(x_{31}x_{43} - d_{33}x_{41}) + a_{23}(x_{31}x_{42} - x_{32}x_{41})\}$

Det $A(1,2,3,5) = d_{11}\{d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})\} - x_{12}\{a_{21}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{53} - d_{33}x_{51})\} + a_{13}\{a_{21}(x_{32}d_{55} - x_{35}x_{52}) - d_{22}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{52} - x_{32}x_{51})\} - x_{15}\{a_{21}(x_{32}x_{53} - d_{33}x_{52}) - d_{22}(x_{31}x_{53} - d_{33}x_{51}) + a_{23}(x_{31}x_{52} - x_{32}x_{51})\}$

Det $A(1,2,4,5) = d_{11}\{d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})\} - x_{12}\{a_{21}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{a_{21}(x_{42}d_{55} - x_{45}x_{52}) - d_{22}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{52} - x_{42}x_{51})\} - x_{15}\{a_{21}(x_{42}x_{54} - d_{44}x_{52}) - d_{22}(x_{41}x_{54} - d_{44}x_{51}) + x_{24}(x_{41}x_{52} - x_{42}x_{51})\}$

Det $A(1,3,4,5) = d_{11}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{13}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{15}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}$

Det $A(2,3,4,5) = d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\}$

Det $A = d_{11}[d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\}] -$

$x_{12}[a_{21}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{25}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}] +$

$a_{13}[a_{21}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} - d_{22}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - x_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\}] -$

$x_{14}[a_{21}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} + a_{23}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - x_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}] +$

$x_{15}[a_{21}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\} + a_{23}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\} - x_{24}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}]$

Performing zero completion by setting all the unspecified entries to zero i.e. $x_{12} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{24} = 0$, $x_{25} = 0$, $x_{31} = 0$, $x_{32} = 0$, $x_{34} = 0$, $x_{35} = 0$, $x_{41} = 0$, $x_{42} = 0$, $x_{43} = 0$, $x_{45} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$. The determinants of principal submatrices will be ;

Det $A(1,2) = d_{11}d_{22} \geq 0$

Det $A(1,3) = d_{11}d_{33} \geq 0$

Det $A(1,4) = d_{11}d_{44} \geq 0$

Det $A(1,5) = d_{11}d_{55} \geq 0$

Det $A(2,3) = d_{22}d_{33} \geq 0$

Det $A(2,4) = d_{22}d_{44} \geq 0$

Det $A(2,5) = d_{22}d_{55} \geq 0$

Det $A(3,4) = d_{33}d_{44} \geq 0$

Det $A(3,5) = d_{33}d_{55} \geq 0$

Det $A(4,5) = d_{44}d_{55} \geq 0$

Det $A(1,2,3) = d_{11}d_{22}\, d_{33} \geq 0$

Det $A(1,2,4) = d_{11}d_{22}\, d_{44} \geq 0$

Det $A(1,2,5) = d_{11}d_{22}\, d_{55} \geq 0$

Det $A(1,3,4) = d_{11}d_{33}d_{44} \geq 0$

Det $A(1,3,5) = d_{11}d_{33}d_{55} \geq 0$

Det $A(1,4,5) = d_{11}d_{44}d_{55} \geq 0$

Det $A(2,3,4) = d_{22}d_{33}d_{44} \geq 0$

Det $A(2,3,5) = d_{22}d_{33}d_{55} \geq 0$

Det $A(2,4,5) = d_{22}d_{44}d_{55} \geq 0$

Det $A(3,4,5) = d_{33}d_{44}d_{55} \geq 0$

Det $A(1,2,3,4) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(1,2,3,5) = d_{11}d_{22}\ d_{33}d_{55} \geq 0$

Det $A(1,2,4,5) = d_{11}d_{22}\ d_{44}d_{55} \geq 0$

Det $A(1,3,4,5) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(2,3,4,5) = d_{22}d_{33}d_{44}d_{55} \geq 0$

Det $A = d_{11}d_{22}\ d_{33}d_{44}d_{55} \geq 0$

All the determinants are nonnegative and therefore the pattern have zero completion into nonnegative P_0- matrix.

9. Consider the digraph below:

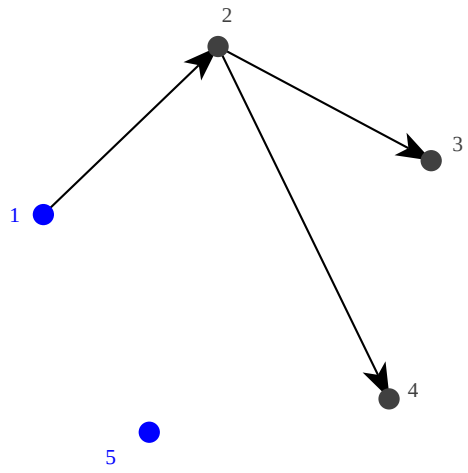

Figure 4.2.17: Digraph with arcs 1 to 2, 2 to 3 and 2 to 4

When the vertices of the digraph are permuted other digraphs with similar results are obtained, such as the one that follows:

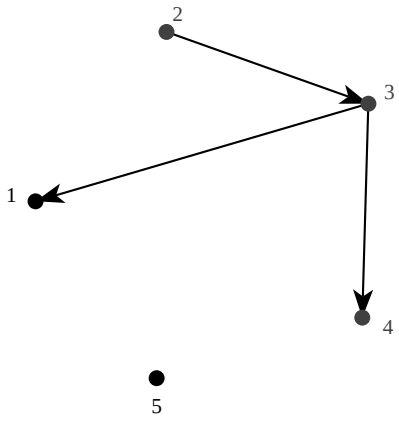

Figure 4.2.18: Digraph isomorphic to digraph in figure 2.2.17, obtained via permutation (1234)

Using the digraph in figure 2.2.17:

Let $A = \begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & a_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{44} \end{bmatrix}$ be a partial nonnegative P_0-matrix specifying the digraph.

Determinants of the principal submatrices

Det $A(1,2) = d_{11}d_{22} - a_{12}x_{21}$

Det $A(1,3) = d_{11}d_{33} - x_{13}x_{31}$

Det $A(1,4) = d_{11}d_{44} - x_{14}x_{41}$

Det $A(1,5) = d_{11}d_{55} - x_{15}x_{51}$

Det $A(2,3) = d_{22}d_{33} - a_{23}x_{32}$

Det $A(2,4) = d_{22}d_{44} - a_{24}x_{42}$

Det $A(2,5) = d_{22}d_{55} - x_{25}x_{52}$

Det $A(3,4) = d_{33}d_{44} - x_{34}x_{43}$

Det $A(3,5) = d_{33}d_{55} - x_{35}x_{53}$

Det $A(4,5) = d_{44}d_{55} - x_{45}x_{54}$

Det $A(1,2,3) = d_{11}(d_{22}d_{33} - a_{23}x_{32}) - a_{12}(x_{21}d_{33} - x_{23}x_{31}) + x_{13}(x_{21}x_{32} - d_{22}x_{31})$

Det $A(1,2,4) = d_{11}(d_{22}d_{44} - a_{24}x_{42}) - a_{12}(x_{21}d_{44} - a_{24}x_{41}) + x_{14}(x_{21}x_{42} - d_{22}x_{41})$

Det $A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - a_{12}(x_{21}d_{55} - x_{25}x_{51}) + x_{15}(x_{21}x_{52} - d_{22}x_{51})$

Det $A(1,3,4) = d_{11}(d_{33}d_{44} - x_{34}x_{43}) - x_{13}(x_{31}d_{44} - x_{34}x_{41}) + x_{14}(x_{31}x_{43} - d_{33}x_{41})$

Det $A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - x_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$

Det $A(1,4,5) = d_{11}(d_{44}d_{55} - x_{45}x_{54}) - x_{14}(x_{41}d_{55} - x_{45}x_{51}) + x_{15}(x_{41}x_{54} - d_{44}x_{51})$

Det $A(2,3,4) = d_{22}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{32}d_{44} - x_{34}x_{42}) + a_{24}(x_{32}x_{43} - d_{33}x_{42})$

Det $A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})$

Det $A(2,4,5) = d_{22}(d_{44}d_{55} - x_{45}x_{54}) - a_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})$

Det $A(3,4,5) = d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})$

Det $A(1,2,3,4) = d_{11}\{d_{22}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{32}d_{44} - x_{34}x_{42}) + a_{24}(x_{32}x_{43} - d_{33}x_{42})\} - a_{12}\{x_{21}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{31}d_{44} - x_{34}x_{41}) + a_{24}(x_{31}x_{43} - d_{33}x_{41})\} + x_{13}\{x_{21}(x_{32}d_{44} - x_{34}x_{42}) - d_{22}(x_{31}d_{44} - x_{34}x_{41}) + a_{24}(x_{31}x_{42} - x_{32}x_{41})\} - x_{14}\{x_{21}(x_{32}x_{43} - d_{33}x_{42}) - d_{22}(x_{31}x_{43} - d_{33}x_{41}) + a_{23}(x_{31}x_{42} - x_{32}x_{41})\}$

Det $A(1,2,3,5) = d_{11}\{d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})\} - a_{12}\{x_{21}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{53} - d_{33}x_{51})\} + x_{13}\{x_{21}(x_{32}d_{55} - x_{35}x_{52}) - d_{22}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{52} - x_{32}x_{51})\} - x_{15}\{x_{21}(x_{32}x_{53} - d_{33}x_{52}) - d_{22}(x_{31}x_{53} - d_{33}x_{51}) + a_{23}(x_{31}x_{52} - x_{32}x_{51})\}$

Det $A(1,2,4,5) = d_{11}\{d_{22}(d_{44}d_{55} - x_{45}x_{54}) - a_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})\} - a_{12}\{x_{21}(d_{44}d_{55} - x_{45}x_{54}) - a_{24}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{21}(x_{42}d_{55} - x_{45}x_{52}) - d_{22}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{52} - x_{42}x_{51})\} - x_{15}\{x_{21}(x_{42}x_{54} - d_{44}x_{52}) - d_{22}(x_{41}x_{54} - d_{44}x_{51}) + a_{24}(x_{41}x_{52} - x_{42}x_{51})\}$

Det $A(1,3,4,5) = d_{11}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{13}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{15}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}$

Det $A(2,3,4,5) = d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + a_{24}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} -$

$d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\}$

Det $A = d_{11}[d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + a_{24}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\}] -$

$a_{12}[x_{21}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + a_{24}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{25}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}] +$

$x_{13}[x_{21}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} - d_{22}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + a_{24}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - x_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\}] -$

$x_{14}[\ x_{21}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} + a_{23}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - x_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}] +$

$x_{15}[x_{21}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\} + a_{23}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\} - a_{24}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}]$

Performing zero completion by setting all the unspecified entries to zero i.e. $x_{13} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{21} = 0$, $x_{25} = 0$, $x_{31} = 0$, $x_{32} = 0$, $x_{34} = 0$, $x_{35} = 0$, $x_{41} = 0$, $x_{42} = 0$, $x_{43} = 0$, $x_{45} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$. The determinants of principal submatrices will be ;

Det $A(1,2) = d_{11}d_{22} \geq 0$

Det $A(1,3) = d_{11}d_{33} \geq 0$

Det $A(1,4) = d_{11}d_{44} \geq 0$

Det $A(1,5) = d_{11}d_{55} \geq 0$

Det $A(2,3) = d_{22}d_{33} \geq 0$

Det $A(2,4) = d_{22}d_{44} \geq 0$

Det $A(2,5) = d_{22}d_{55} \geq 0$

Det $A(3,4) = d_{33}d_{44} \geq 0$

Det $A(3,5) = d_{33}d_{55} \geq 0$

Det $A(4,5) = d_{44}d_{55} \geq 0$

Det $A(1,2,3) = d_{11}d_{22}\ d_{33} \geq 0$

Det $A(1,2,4) = d_{11}d_{22}\ d_{44} \geq 0$

Det $A(1,2,5) = d_{11}d_{22}\ d_{55} \geq 0$

Det $A(1,3,4) = d_{11}d_{33}d_{44} \geq 0$

Det $A(1,3,5) = d_{11}d_{33}d_{55} \geq 0$

Det $A(1,4,5) = d_{11}d_{44}d_{55} \geq 0$

Det $A(2,3,4) = d_{22}d_{33}d_{44} \geq 0$

Det $A(2,3,5) = d_{22}d_{33}d_{55} \geq 0$

Det $A(2,4,5) = d_{22}d_{44}d_{55} \geq 0$

Det $A(3,4,5) = d_{33}d_{44}d_{55} \geq 0$

Det $A(1,2,3,4) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(1,2,3,5) = d_{11}d_{22}\ d_{33}d_{55} \geq 0$

Det $A(1,2,4,5) = d_{11}d_{22}\ d_{44}d_{55} \geq 0$

Det $A(1,3,4,5) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(2,3,4,5) = d_{22}d_{33}d_{44}d_{55} \geq 0$

Det $A = d_{11}d_{22}\ d_{33}d_{44}d_{55} \geq 0$

Since all the determinants are nonnegative the pattern therefore have zero completion into nonnegative P_0- matrix.

10. Consider the digraph below:

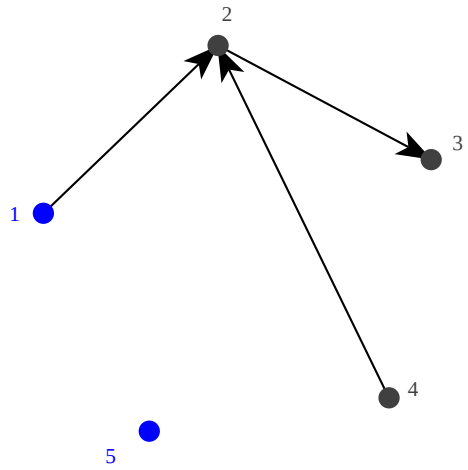

Figure 4.2.19: Digraph with arcs 1 to 2, 2 to 3 and 4 to 2

Similar results are obtained from digraphs got by permuting the vertices of the digraph, example digraph that follows:

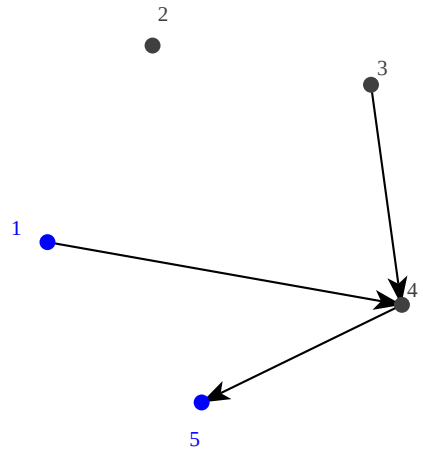

Figure 4.2.20: Digraph isomorphic to digraph in figure 2.2.19, obtained via permutation (13524)

Using the digraph in figure 2.2.19:

let $A = \begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & a_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$ be a partial nonnegative P_0-matrix specifying the digraph.

Determinants of the principal submatrices

Det $A(1,2) = d_{11}d_{22} - a_{12}x_{21}$

Det $A(1,3) = d_{11}d_{33} - x_{13}x_{31}$

Det $A(1,4) = d_{11}d_{44} - x_{14}x_{41}$

Det $A(1,5) = d_{11}d_{55} - x_{15}x_{51}$

Det $A(2,3) = d_{22}d_{33} - a_{23}x_{32}$

Det $A(2,4) = d_{22}d_{44} - x_{24}a_{42}$

Det $A(2,5) = d_{22}d_{55} - x_{25}x_{52}$

Det $A(3,4) = d_{33}d_{44} - x_{34}x_{43}$

Det $A(3,5) = d_{33}d_{55} - x_{35}x_{53}$

Det $A(4,5) = d_{44}d_{55} - x_{45}x_{54}$

Det $A(1,2,3) = d_{11}(d_{22}d_{33} - a_{23}x_{32}) - a_{12}(x_{21}d_{33} - a_{23}x_{31}) + x_{13}(x_{21}x_{32} - d_{22}x_{31})$

Det $A(1,2,4) = d_{11}(d_{22}d_{44} - x_{24}a_{42}) - a_{12}(x_{21}d_{44} - x_{24}x_{41}) + x_{14}(x_{21}a_{42} - d_{22}x_{41})$

Det $A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - a_{12}(x_{21}d_{55} - x_{25}x_{51}) + x_{15}(x_{21}x_{52} - d_{22}x_{51})$

Det $A(1,3,4) = d_{11}(d_{33}d_{44} - x_{34}x_{43}) - x_{13}(x_{31}d_{44} - x_{34}x_{41}) + x_{14}(x_{31}x_{43} - d_{33}x_{41})$

Det $A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - x_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$

Det $A(1,4,5) = d_{11}(d_{44}d_{55} - x_{45}x_{54}) - x_{14}(x_{41}d_{55} - x_{45}x_{51}) + x_{15}(x_{41}x_{54} - d_{44}x_{51})$

Det $A(2,3,4) = d_{22}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{32}d_{44} - x_{34}a_{42}) + x_{24}(x_{32}x_{43} - d_{33}a_{42})$

Det $A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})$

Det $A(2,4,5) = d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(a_{42}d_{55} - x_{45}x_{52}) + x_{25}(a_{42}x_{54} - d_{44}x_{52})$

Det $A(3,4,5) = d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})$

Det $A(1,2,3,4) = d_{11}\{d_{22}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{32}d_{44} - x_{34}a_{42}) + x_{24}(x_{32}x_{43} - d_{33}a_{42})\} - a_{12}\{x_{21}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41})\} + x_{13}\{x_{21}(x_{32}d_{44} - x_{34}a_{42}) - d_{22}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}a_{42} - x_{32}x_{41})\} - x_{14}\{x_{21}(x_{32}x_{43} - d_{33}a_{42}) - d_{22}(x_{31}x_{43} - d_{33}x_{41}) + a_{23}(x_{31}a_{42} - x_{32}x_{41})\}$

Det $A(1,2,3,5) = d_{11}\{d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})\} - a_{12}\{x_{21}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{53} - d_{33}x_{51})\} + x_{13}\{x_{21}(x_{32}d_{55} - x_{35}x_{52}) - d_{22}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{52} - x_{32}x_{51})\} - x_{15}\{x_{21}(x_{32}x_{53} - d_{33}x_{52}) - d_{22}(x_{31}x_{53} - d_{33}x_{51}) + a_{23}(x_{31}x_{52} - x_{32}x_{51})\}$

Det $A(1,2,4,5) = d_{11}\{d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(a_{42}d_{55} - x_{45}x_{52}) + x_{25}(a_{42}x_{54} - d_{44}x_{52})\} - a_{12}\{x_{21}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{21}(a_{42}d_{55} - x_{45}x_{52}) - d_{22}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{52} - a_{42}x_{51})\} - x_{15}\{x_{21}(a_{42}x_{54} - d_{44}x_{52}) - d_{22}(x_{41}x_{54} - d_{44}x_{51}) + x_{24}(x_{41}x_{52} - a_{42}x_{51})\}$

Det $A(1,3,4,5) = d_{11}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{13}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{15}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}$

Det $A(2,3,4,5) = d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(a_{42}d_{55} - x_{45}x_{52}) + x_{35}(a_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(a_{42}d_{55} - x_{45}x_{52}) + x_{35}(a_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(a_{42}x_{54} - d_{44}x_{52}) + x_{34}(a_{42}x_{53} - x_{43}x_{52})\}$

Det $A = d_{11}[d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(a_{42}d_{55} - x_{45}x_{52}) + x_{35}(_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(a_{42}d_{55} - x_{45}x_{52}) + x_{35}(a_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(a_{42}x_{54} - d_{44}x_{52}) + x_{34}(a_{42}x_{53} - x_{43}x_{52})\}] -$

$a_{12}[x_{21}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{25}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}] +$

$x_{13}[x_{21}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(a_{42}d_{55} - x_{45}x_{52}) + x_{35}(a_{42}x_{54} - d_{44}x_{52})\} - d_{22}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(a_{42}d_{55} - x_{45}x_{52}) - x_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - a_{42}x_{51})\} - x_{25}\{x_{31}(a_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - a_{42}x_{51})\}] -$

$x_{14}[x_{21}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(a_{42}d_{55} - x_{45}x_{52}) + x_{35}(a_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} + a_{23}\{x_{31}(a_{42}d_{55} - x_{45}x_{52}) - x_{32}(x_{41}d_{55} -$

$x_{45}x_{51}) + x_{35}(x_{41}x_{52} - a_{42}x_{51})\} - x_{25}\{x_{31}(a_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - a_{42}x_{51})\}] +$

$x_{15}[x_{21}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(a_{42}x_{54} - d_{44}x_{52}) + x_{34}(a_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\} + a_{23}\{x_{31}(a_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - a_{42}x_{51})\} - x_{24}\{x_{31}(a_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - a_{42}x_{51})\}]$

Performing zero completion by setting all the unspecified entries to zero i.e. $x_{13} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{21} = 0$, $x_{24} = 0$, $x_{25} = 0$, $x_{31} = 0$, $x_{32} = 0$, $x_{34} = 0$, $x_{35} = 0$, $x_{41} = 0$, $x_{43} = 0$, $x_{45} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$. The determinants of principal submatrices will be ;

Det $A(1,2) = d_{11}d_{22} \geq 0$

Det $A(1,3) = d_{11}d_{33} \geq 0$

Det $A(1,4) = d_{11}d_{44} \geq 0$

Det $A(1,5) = d_{11}d_{55} \geq 0$

Det $A(2,3) = d_{22}d_{33} \geq 0$

Det $A(2,4) = d_{22}d_{44} \geq 0$

Det $A(2,5) = d_{22}d_{55} \geq 0$

Det $A(3,4) = d_{33}d_{44} \geq 0$

Det $A(3,5) = d_{33}d_{55} \geq 0$

Det $A(4,5) = d_{44}d_{55} \geq 0$

Det $A(1,2,3) = d_{11}d_{22}\, d_{33} \geq 0$

Det $A(1,2,4) = d_{11}d_{22}\, d_{44} \geq 0$

Det $A(1,2,5) = d_{11}d_{22}\, d_{55} \geq 0$

Det $A(1,3,4) = d_{11}d_{33}d_{44} \geq 0$

Det $A(1,3,5) = d_{11}d_{33}d_{55} \geq 0$

Det $A(1,4,5) = d_{11}d_{44}d_{55} \geq 0$

Det $A(2,3,4) = d_{22}d_{33}d_{44} \geq 0$

Det $A(2,3,5) = d_{22}d_{33}d_{55} \geq 0$

Det $A(2,4,5) = d_{22}d_{44}d_{55} \geq 0$

Det $A(3,4,5) = d_{33}d_{44}d_{55} \geq 0$

Det $A(1,2,3,4) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(1,2,3,5) = d_{11}d_{22}\ d_{33}d_{55} \geq 0$

Det $A(1,2,4,5) = d_{11}d_{22}\ d_{44}d_{55} \geq 0$

Det $A(1,3,4,5) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(2,3,4,5) = d_{22}d_{33}d_{44}d_{55} \geq 0$

Det $A = d_{11}d_{22}\ d_{33}d_{44}d_{55} \geq 0$

The pattern have zero completion into nonnegative P_0- matrix since all the determinants are nonnegative.

11. Consider the digraph below:

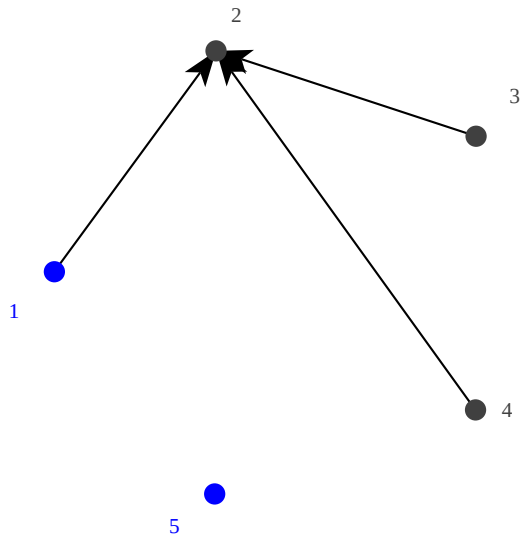

Figure 4.2.21: Digraph with arcs 1 to 2, 3 to 2 and 4 to 2

Permuting the vertices of the digraph gives other digraphs with similar results, example of the one that follows:

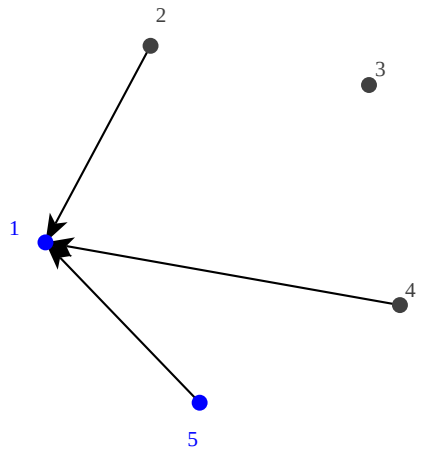

Figure 4.2.22: Digraph isomorpic to digraph in figure 2.2.21, obtained via permutation (12)(35)

Using the diagraph in figure 2.2.21

Let $A = \begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & a_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & a_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$ be the partial nonnegative P_0-matrix specifying the digraph.

Determinants of the principal submatrices

Det $A(1,2) = d_{11}d_{22} - a_{12}x_{21}$

Det $A(1,3) = d_{11}d_{33} - x_{13}x_{31}$

Det $A(1,4) = d_{11}d_{44} - x_{14}x_{41}$

Det $A(1,5) = d_{11}d_{55} - x_{15}x_{51}$

Det $A(2,3) = d_{22}d_{33} - x_{23}a_{32}$

Det $A(2,4) = d_{22}d_{44} - x_{24}a_{42}$

Det $A(2,5) = d_{22}d_{55} - x_{25}x_{52}$

Det $A(3,4) = d_{33}d_{44} - x_{34}x_{43}$

Det $A(3,5) = d_{33}d_{55} - x_{35}x_{53}$

Det $A(4,5) = d_{44}d_{55} - x_{45}x_{54}$

Det $A(1,2,3) = d_{11}(d_{22}d_{33} - x_{23}a_{32}) - a_{12}(x_{21}d_{33} - x_{23}x_{31}) + x_{13}(x_{21}a_{32} - d_{22}x_{31})$

Det $A(1,2,4) = d_{11}(d_{22}d_{44} - x_{24}a_{42}) - a_{12}(x_{21}d_{44} - x_{24}x_{41}) + x_{14}(x_{21}a_{42} - d_{22}x_{41})$

Det $A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - a_{12}(x_{21}d_{55} - x_{25}x_{51}) + x_{15}(x_{21}x_{52} - d_{22}x_{51})$

Det $A(1,3,4) = d_{11}(d_{33}d_{44} - x_{34}x_{43}) - x_{13}(x_{31}d_{44} - x_{34}x_{41}) + x_{14}(x_{31}x_{43} - d_{33}x_{41})$

Det $A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - x_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$

Det $A(1,4,5) = d_{11}(d_{44}d_{55} - x_{45}x_{54}) - x_{14}(x_{41}d_{55} - x_{45}x_{51}) + x_{15}(x_{41}x_{54} - d_{44}x_{51})$

Det $A(2,3,4) = d_{22}(d_{33}d_{44} - x_{34}x_{43}) - x_{23}(a_{32}d_{44} - x_{34}a_{42}) + x_{24}(a_{32}x_{43} - d_{33}a_{42})$

Det $A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - x_{23}(a_{32}d_{55} - x_{35}x_{52}) + x_{25}(a_{32}x_{53} - d_{33}x_{52})$

Det $A(2,4,5) = d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(a_{42}d_{55} - x_{45}x_{52}) + x_{25}(a_{42}x_{54} - d_{44}x_{52})$

Det $A(3,4,5) = d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})$

Det $A(1,2,3,4) = d_{11}\{d_{22}(d_{33}d_{44} - x_{34}x_{43}) - x_{23}(a_{32}d_{44} - x_{34}x_{42}) + x_{24}(a_{32}x_{43} - d_{33}a_{42})\} - a_{12}\{x_{21}(d_{33}d_{44} - x_{34}x_{43}) - x_{23}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41})\} + x_{13}\{x_{21}(a_{32}d_{44} - x_{34}a_{42}) - d_{22}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}a_{42} - a_{32}x_{41})\} - x_{14}\{x_{21}(a_{32}x_{43} - d_{33}a_{42}) - d_{22}(x_{31}x_{43} - d_{33}x_{41}) + x_{23}(x_{31}a_{42} - a_{32}x_{41})\}$

Det $A(1,2,3,5) = d_{11}\{d_{22}(d_{33}d_{55} - x_{35}x_{53}) - x_{23}(a_{32}d_{55} - x_{35}x_{52}) + x_{25}(a_{32}x_{53} - d_{33}x_{52})\} - a_{12}\{x_{21}(d_{33}d_{55} - x_{35}x_{53}) - x_{23}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{53} - d_{33}x_{51})\} + x_{13}\{x_{21}(a_{32}d_{55} - x_{35}x_{52}) - d_{22}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{52} - a_{32}x_{51})\} - x_{15}\{x_{21}(a_{32}x_{53} - d_{33}x_{52}) - d_{22}(x_{31}x_{53} - d_{33}x_{51}) + x_{23}(x_{31}x_{52} - a_{32}x_{51})\}$

Det $A(1,2,4,5) = d_{11}\{d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(a_{42}d_{55} - x_{45}x_{52}) + x_{25}(a_{42}x_{54} - d_{44}x_{52})\} - a_{12}\{x_{21}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{21}(a_{42}d_{55} - x_{45}x_{52}) - d_{22}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{52} - a_{42}x_{51})\} - x_{15}\{x_{21}(a_{42}x_{54} - d_{44}x_{52}) - d_{22}(x_{41}x_{54} - d_{44}x_{51}) + x_{24}(x_{41}x_{52} - a_{42}x_{51})\}$

Det $A(1,3,4,5) = d_{11}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{13}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{15}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}$

Det $A(2,3,4,5) = d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{23}\{a_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(a_{42}d_{55} - x_{45}x_{52}) + x_{35}(a_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{a_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(a_{42}d_{55} - x_{45}x_{52}) + x_{35}(a_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{a_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(a_{42}x_{54} - d_{44}x_{52}) + x_{34}(a_{42}x_{53} - x_{43}x_{52})\}$

Det $A = d_{11}[d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{23}\{a_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(a_{42}d_{55} - x_{45}x_{52}) + x_{35}(a_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{a_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(a_{42}d_{55} - x_{45}x_{52}) + x_{35}(a_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{a_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(a_{42}x_{54} - d_{44}x_{52}) + x_{34}(a_{42}x_{53} - x_{43}x_{52})\}] - a_{12}[x_{21}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{23}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{25}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}] + x_{13}[x_{21}\{a_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(a_{42}d_{55} - x_{45}x_{52}) + x_{35}(a_{42}x_{54} - d_{44}x_{52})\} - d_{22}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(a_{42}d_{55} - x_{45}x_{52}) - a_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - a_{42}x_{51})\} - x_{25}\{x_{31}(a_{42}x_{54} - d_{44}x_{52}) - a_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - a_{42}x_{51})\}] - x_{14}[x_{21}\{a_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(a_{42}d_{55} - x_{45}x_{52}) + x_{35}(a_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} + x_{23}\{x_{31}(a_{42}d_{55} - x_{45}x_{52}) - a_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - a_{42}x_{51})\} - x_{25}\{x_{31}(a_{42}x_{53} - x_{43}x_{52}) - a_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - a_{42}x_{51})\}] + x_{15}[x_{21}\{a_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(a_{42}x_{54} - d_{44}x_{52}) + x_{34}(a_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\} + x_{23}\{x_{31}(a_{42}x_{54} - d_{44}x_{52}) - a_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - a_{42}x_{51})\} - x_{24}\{x_{31}(a_{42}x_{53} - x_{43}x_{52}) - a_{32}(x_{41}x_{53} - x_{43}x_{52}) + d_{33}(x_{41}x_{52} - a_{42}x_{51})\}]$

Performing zero completion by setting all the unspecified entries to zero i.e. $x_{13} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{21} = 0$, $x_{23} = 0$, $x_{24} = 0$, $x_{25} = 0$, $x_{31} = 0$, $x_{34} = 0$, $x_{35} = 0$, $x_{41} = 0$, $x_{43} = 0$, $x_{45} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$. The determinants of principal submatrices will be ;

Det $A(1,2) = d_{11}d_{22} \geq 0$

Det $A(1,3) = d_{11}d_{33} \geq 0$

Det $A(1,4) = d_{11}d_{44} \geq 0$

Det $A(1,5) = d_{11}d_{55} \geq 0$

Det $A(2,3) = d_{22}d_{33} \geq 0$

Det $A(2,4) = d_{22}d_{44} \geq 0$

Det $A(2,5) = d_{22}d_{55} \geq 0$

Det $A(3,4) = d_{33}d_{44} \geq 0$

Det $A(3,5) = d_{33}d_{55} \geq 0$

Det $A(4,5) = d_{44}d_{55} \geq 0$

Det $A(1,2,3) = d_{11}d_{22}\, d_{33} \geq 0$

Det $A(1,2,4) = d_{11}d_{22}\, d_{44} \geq 0$

Det $A(1,2,5) = d_{11}d_{22}\, d_{55} \geq 0$

Det $A(1,3,4) = d_{11}d_{33}d_{44} \geq 0$

Det $A(1,3,5) = d_{11}d_{33}d_{55} \geq 0$

Det $A(1,4,5) = d_{11}d_{44}d_{55} \geq 0$

Det $A(2,3,4) = d_{22}d_{33}d_{44} \geq 0$

Det $A(2,3,5) = d_{22}d_{33}d_{55} \geq 0$

Det $A(2,4,5) = d_{22}d_{44}d_{55} \geq 0$

Det $A(3,4,5) = d_{33}d_{44}d_{55} \geq 0$

Det $A(1,2,3,4) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(1,2,3,5) = d_{11}d_{22}\ d_{33}d_{55} \geq 0$

Det $A(1,2,4,5) = d_{11}d_{22}\ d_{44}d_{55} \geq 0$

Det $A(1,3,4,5) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(2,3,4,5) = d_{22}d_{33}d_{44}d_{55} \geq 0$

Det $A = d_{11}d_{22}\ d_{33}d_{44}d_{55} \geq 0$

All the determinants are nonnegative and therefore the pattern have zero completion into nonnegative P_0- matrix.

12. Consider the digraph below:

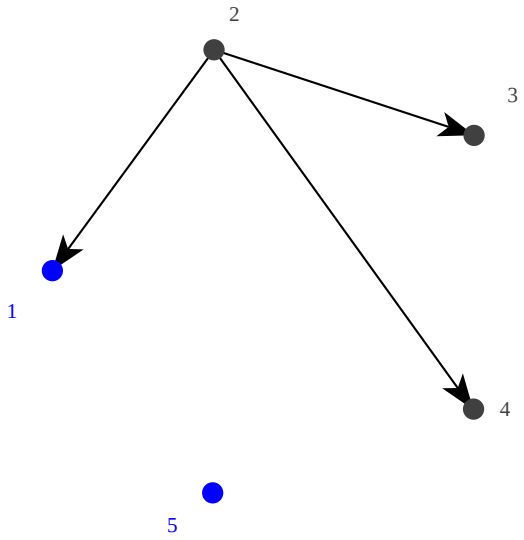

Figure 4.2.23: Digraph with arcs 2 to 1, 2 to 3 and 2 to 4

The digraph has similar results as the digraph that follows:

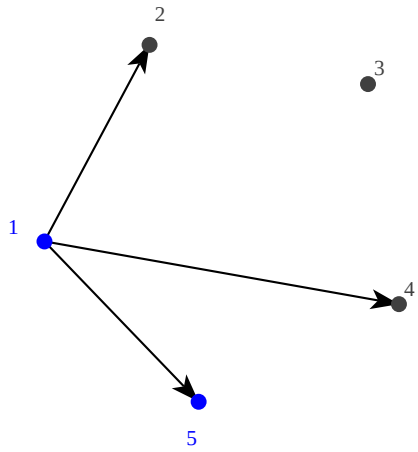

Figure 4.2.24: Digraph isomorphic to digraph in figure 2.2.23, obtained via permutation (12)(35)

Using the digraph in figure 2.2.23:

Let $A = \begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ a_{21} & d_{22} & a_{23} & a_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$ be the partial nonnegative P_0-matrix specifying the digraph.

Determinants of the principal submatrices

Det $A(1,2) = d_{11}d_{22} - x_{12}a_{21}$

Det $A(1,3) = d_{11}d_{33} - x_{13}x_{31}$

Det $A(1,4) = d_{11}d_{44} - x_{14}x_{41}$

Det $A(1,5) = d_{11}d_{55} - x_{15}x_{51}$

Det $A(2,3) = d_{22}d_{33} - a_{23}x_{32}$

Det $A(2,4) = d_{22}d_{44} - a_{24}x_{42}$

Det $A(2,5) = d_{22}d_{55} - x_{25}x_{52}$

Det $A(3,4) = d_{33}d_{44} - x_{34}x_{43}$

Det $A(3,5) = d_{33}d_{55} - x_{35}x_{53}$

Det $A(4,5) = d_{44}d_{55} - x_{45}x_{54}$

Det $A(1,2,3) = d_{11}(d_{22}d_{33} - a_{23}x_{32}) - x_{12}(a_{21}d_{33} - a_{23}x_{31}) + x_{13}(a_{21}x_{32} - d_{22}x_{31})$

Det $A(1,2,4) = d_{11}(d_{22}d_{44} - a_{24}x_{42}) - x_{12}(a_{21}d_{44} - a_{24}x_{41}) + x_{14}(a_{21}x_{42} - d_{22}x_{41})$

Det $A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - x_{12}(a_{21}d_{55} - x_{25}x_{51}) + x_{15}(a_{21}x_{52} - d_{22}x_{51})$

Det $A(1,3,4) = d_{11}(d_{33}d_{44} - x_{34}x_{43}) - x_{13}(x_{31}d_{44} - x_{34}x_{41}) + x_{14}(x_{31}x_{43} - d_{33}x_{41})$

Det $A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - x_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$

Det $A(1,4,5) = d_{11}(d_{44}d_{55} - x_{45}x_{54}) - x_{14}(x_{41}d_{55} - x_{45}x_{51}) + x_{15}(x_{41}x_{54} - d_{44}x_{51})$

Det $A(2,3,4) = d_{22}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{32}d_{44} - x_{34}x_{42}) + a_{24}(x_{32}x_{43} - d_{33}x_{42})$

Det $A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})$

Det $A(2,4,5) = d_{22}(d_{44}d_{55} - x_{45}x_{54}) - a_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})$

Det $A(3,4,5) = d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})$

Det $A(1,2,3,4) = d_{11}\{d_{22}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{32}d_{44} - x_{34}x_{42}) + a_{24}(x_{32}x_{43} - d_{33}x_{42})\} - x_{12}\{a_{21}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{31}d_{44} - x_{34}x_{41}) + a_{24}(x_{31}x_{43} - d_{33}x_{41})\} + x_{13}\{a_{21}(x_{32}d_{44} - x_{34}x_{42}) - d_{22}(x_{31}d_{44} - x_{34}x_{41}) + a_{24}(x_{31}x_{42} - x_{32}x_{41})\} - x_{14}\{a_{21}(x_{32}x_{43} - d_{33}x_{42}) - d_{22}(x_{31}x_{43} - d_{33}x_{41}) + a_{23}(x_{31}x_{42} - x_{32}x_{41})\}$

Det $A(1,2,3,5) = d_{11}\{d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})\} - x_{12}\{a_{21}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{53} - d_{33}x_{51})\} + x_{13}\{a_{21}(x_{32}d_{55} - x_{35}x_{52}) - d_{22}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{52} - x_{32}x_{51})\} - x_{15}\{a_{21}(x_{32}x_{53} - d_{33}x_{52}) - d_{22}(x_{31}x_{53} - d_{33}x_{51}) + a_{23}(x_{31}x_{52} - x_{32}x_{51})\}$

Det $A(1,2,4,5) = d_{11}\{d_{22}(d_{44}d_{55} - x_{45}x_{54}) - a_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})\} - x_{12}\{a_{21}(d_{44}d_{55} - x_{45}x_{54}) - a_{24}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{a_{21}(x_{42}d_{55} - x_{45}x_{52}) - d_{22}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{52} - x_{42}x_{51})\} - x_{15}\{a_{21}(x_{42}x_{54} - d_{44}x_{52}) - d_{22}(x_{41}x_{54} - d_{44}x_{51}) + a_{24}(x_{41}x_{52} - x_{42}x_{51})\}$

Det $A(1,3,4,5) = d_{11}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{13}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{15}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}$

Det $A(2,3,4,5) = d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + a_{24}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\}$

Det $A = d_{11}[d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + a_{24}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\}] -$

$x_{12}[a_{21}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + a_{24}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{25}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}] +$

$x_{13}[a_{21}\{x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} - d_{22}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + a_{24}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - x_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\}] -$

$x_{14}[a_{21}\{x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} + a_{23}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - x_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}] +$

$x_{15}[a_{21}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\} + a_{23}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\} - a_{24}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}]$

Performing zero completion by setting all the unspecified entries to zero i.e. $x_{12} = 0$, $x_{13} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{25} = 0$, $x_{31} = 0$, $x_{32} = 0$, $x_{34} = 0$, $x_{35} = 0$, $x_{41} = 0$, $x_{42} = 0$, $x_{43} = 0$, $x_{45} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$. The determinants of principal submatrices will be ;

Det $A(1,2) = d_{11}d_{22} \geq 0$

Det $A(1,3) = d_{11}d_{33} \geq 0$

Det $A(1,4) = d_{11}d_{44} \geq 0$

Det $A(1,5) = d_{11}d_{55} \geq 0$

Det $A(2,3) = d_{22}d_{33} \geq 0$

Det $A(2,4) = d_{22}d_{44} \geq 0$

Det $A(2,5) = d_{22}d_{55} \geq 0$

Det $A(3,4) = d_{33}d_{44} \geq 0$

Det $A(3,5) = d_{33}d_{55} \geq 0$

Det $A(4,5) = d_{44}d_{55} \geq 0$

Det $A(1,2,3) = d_{11}d_{22}\, d_{33} \geq 0$

Det $A(1,2,4) = d_{11}d_{22}\, d_{44} \geq 0$

Det $A(1,2,5) = d_{11}d_{22}\, d_{55} \geq 0$

Det $A(1,3,4) = d_{11}d_{33}d_{44} \geq 0$

Det $A(1,3,5) = d_{11}d_{33}d_{55} \geq 0$

Det $A(1,4,5) = d_{11}d_{44}d_{55} \geq 0$

Det $A(2,3,4) = d_{22}d_{33}d_{44} \geq 0$

Det $A(2,3,5) = d_{22}d_{33}d_{55} \geq 0$

Det $A(2,4,5) = d_{22}d_{44}d_{55} \geq 0$

Det $A(3,4,5) = d_{33}d_{44}d_{55} \geq 0$

Det $A(1,2,3,4) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(1,2,3,5) = d_{11}d_{22}\ d_{33}d_{55} \geq 0$

Det $A(1,2,4,5) = d_{11}d_{22}\ d_{44}d_{55} \geq 0$

Det $A(1,3,4,5) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(2,3,4,5) = d_{22}d_{33}d_{44}d_{55} \geq 0$

Det $A = d_{11}d_{22}\ d_{33}d_{44}d_{55} \geq 0$

With all the determinants nonnegative, the pattern has zero completion into nonnegative P_0- matrix.

13. Consider the digraph below:

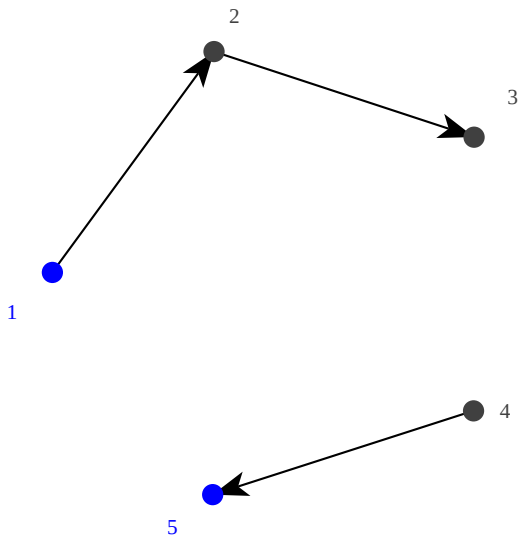

Figure 4.2.25: Digraph with arcs 1 to 2, 2 to 3 and 4 to5

Permuting the vertices of the digraph results to other digraphs with similar results, such as the following digraphs:

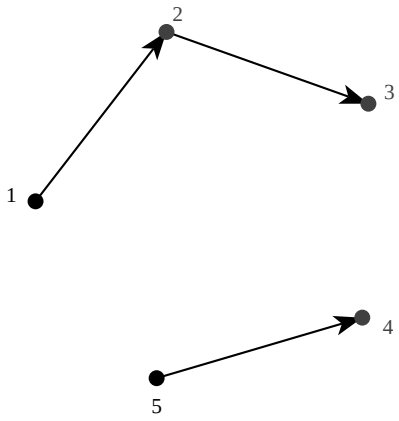

Figure 4.2.26: Digraph isomorphic to digraph in figure 2.2.25, obtained via peermutation (45)

Using the digraph in the figure 2.2.25:

Let $A = \begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$ be the partial nonnegative P_0matrix specifying the digraph.

Determinants of the principal submatrices

Det $A(1,2) = d_{11}d_{22} - a_{12}x_{21}$

Det $A(1,3) = d_{11}d_{33} - x_{13}x_{31}$

Det $A(1,4) = d_{11}d_{44} - x_{14}x_{41}$

Det $A(1,5) = d_{11}d_{55} - x_{15}x_{51}$

Det $A(2,3) = d_{22}d_{33} - a_{23}x_{32}$

Det $A(2,4) = d_{22}d_{44} - x_{24}x_{42}$

Det $A(2,5) = d_{22}d_{55} - x_{25}x_{52}$

Det $A(3,4) = d_{33}d_{44} - x_{34}x_{43}$

Det $A(3,5) = d_{33}d_{55} - x_{35}x_{53}$

Det $A(4,5) = d_{44}d_{55} - a_{45}x_{54}$

Det $A(1,2,3) = d_{11}(d_{22}d_{33} - a_{23}x_{32}) - a_{12}(x_{21}d_{33} - a_{23}x_{31}) + x_{13}(x_{21}x_{32} - d_{22}x_{31})$

Det $A(1,2,4) = d_{11}(d_{22}d_{44} - x_{24}x_{42}) - a_{12}(x_{21}d_{44} - x_{24}x_{41}) + x_{14}(x_{21}x_{42} - d_{22}x_{41})$

Det $A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - a_{12}(x_{21}d_{55} - x_{25}x_{51}) + x_{15}(a_{21}x_{52} - d_{22}x_{51})$

Det $A(1,3,4) = d_{11}(d_{33}d_{44} - x_{34}x_{43}) - x_{13}(x_{31}d_{44} - x_{34}x_{41}) + x_{14}(x_{31}x_{43} - d_{33}x_{41})$

Det $A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - x_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$

Det $A(1,4,5) = d_{11}(d_{44}d_{55} - a_{45}x_{54}) - x_{14}(x_{41}d_{55} - a_{45}x_{51}) + x_{15}(x_{41}x_{54} - d_{44}x_{51})$

Det $A(2,3,4) = d_{22}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{32}d_{44} - x_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})$

Det $A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})$

Det $A(2,4,5) = d_{22}(d_{44}d_{55} - a_{45}x_{54}) - x_{24}(x_{42}d_{55} - a_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})$

Det $A(3,4,5) = d_{33}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{43}d_{55} - a_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})$

Det $A(1,2,3,4) = d_{11}\{d_{22}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{32}d_{44} - x_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})\} - a_{12}\{x_{21}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41})\} + x_{13}\{x_{21}(x_{32}d_{44} - x_{34}x_{42}) - d_{22}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}x_{42} - x_{32}x_{41})\} - x_{14}\{x_{21}(x_{32}x_{43} - d_{33}x_{42}) - d_{22}(x_{31}x_{43} - d_{33}x_{41}) + a_{23}(x_{31}x_{42} - x_{32}x_{41})\}$

Det $A(1,2,3,5) = d_{11}\{d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})\} - a_{12}\{x_{21}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{53} - d_{33}x_{51})\} + x_{13}\{x_{21}(x_{32}d_{55} - x_{35}x_{52}) - d_{22}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{52} - x_{32}x_{51})\} - x_{15}\{x_{21}(x_{32}x_{53} - d_{33}x_{52}) - d_{22}(x_{31}x_{53} - d_{33}x_{51}) + a_{23}(x_{31}x_{52} - x_{32}x_{51})\}$

Det $A(1,2,4,5) = d_{11}\{d_{22}(d_{44}d_{55} - a_{45}x_{54}) - x_{24}(x_{42}d_{55} - a_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})\} - a_{12}\{x_{21}(d_{44}d_{55} - a_{45}x_{54}) - x_{24}(x_{41}d_{55} - a_{45}x_{51}) + x_{25}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{21}(x_{42}d_{55} - a_{45}x_{52}) - d_{22}(x_{41}d_{55} - a_{45}x_{51}) + x_{25}(x_{41}x_{52} - x_{42}x_{51})\} - x_{15}\{x_{21}(x_{42}x_{54} - d_{44}x_{52}) - d_{22}(x_{41}x_{54} - d_{44}x_{51}) + x_{24}(x_{41}x_{52} - x_{42}x_{51})\}$

Det $A(1,3,4,5) = d_{11}\{d_{33}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{43}d_{55} - a_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{13}\{x_{31}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{31}(x_{43}d_{55} - a_{45}x_{53}) - d_{33}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{15}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}$

Det $A(2,3,4,5) = d_{22}\{d_{33}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{43}d_{55} - a_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{42}d_{55} - a_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{x_{32}(x_{43}d_{55} - a_{45}x_{53}) - d_{33}(x_{42}d_{55} - a_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} -$

$d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\}$

Det $A = d_{11}[d_{22}\{d_{33}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{43}d_{55} - a_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{42}d_{55} - a_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{x_{32}(x_{43}d_{55} - a_{45}x_{53}) - d_{33}(x_{42}d_{55} - a_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\}] -$

$a_{12}[x_{21}\{d_{33}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{43}d_{55} - a_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{31}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{43}d_{55} - a_{45}x_{53}) - d_{33}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{25}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}] +$

$x_{13}[x_{21}\{x_{32}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{42}d_{55} - a_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} - d_{22}\{x_{31}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{42}d_{55} - a_{45}x_{52}) - x_{32}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\}] -$

$x_{14}[\ x_{21}\{x_{32}(x_{43}d_{55} - a_{45}x_{53}) - d_{33}(x_{42}d_{55} - a_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}d_{55} - a_{45}x_{53}) - d_{33}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} + a_{23}\{x_{31}(x_{42}d_{55} - a_{45}x_{52}) - x_{32}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}] +$

$x_{15}[x_{21}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\} + a_{23}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\} - x_{24}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}]$

Performing zero completion by setting all the unspecified entries to zero i.e. $x_{13} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{21} = 0$, $x_{24} = 0$, $x_{25} = 0$, $x_{31} = 0$, $x_{32} = 0$, $x_{34} = 0$, $x_{35} = 0$, $x_{41} = 0$, $x_{42} = 0$, $x_{43} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$. The determinants of principal submatrices will be ;

Det $A(1,2) = d_{11}d_{22} \geq 0$

Det $A(1,3) = d_{11}d_{33} \geq 0$

Det $A(1,4) = d_{11}d_{44} \geq 0$

Det $A(1,5) = d_{11}d_{55} \geq 0$

Det $A(2,3) = d_{22}d_{33} \geq 0$

Det $A(2,4) = d_{22}d_{44} \geq 0$

Det $A(2,5) = d_{22}d_{55} \geq 0$

Det $A(3,4) = d_{33}d_{44} \geq 0$

Det $A(3,5) = d_{33}d_{55} \geq 0$

Det $A(4,5) = d_{44}d_{55} \geq 0$

Det $A(1,2,3) = d_{11}d_{22}\ d_{33} \geq 0$

Det $A(1,2,4) = d_{11}d_{22}\ d_{44} \geq 0$

Det $A(1,2,5) = d_{11}d_{22}\ d_{55} \geq 0$

Det $A(1,3,4) = d_{11}d_{33}d_{44} \geq 0$

Det $A(1,3,5) = d_{11}d_{33}d_{55} \geq 0$

Det $A(1,4,5) = d_{11}d_{44}d_{55} \geq 0$

Det $A(2,3,4) = d_{22}d_{33}d_{44} \geq 0$

Det $A(2,3,5) = d_{22}d_{33}d_{55} \geq 0$

Det $A(2,4,5) = d_{22}d_{44}d_{55} \geq 0$

Det $A(3,4,5) = d_{33}d_{44}d_{55} \geq 0$

Det $A(1,2,3,4) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(1,2,3,5) = d_{11}d_{22}\ d_{33}d_{55} \geq 0$

Det $A(1,2,4,5) = d_{11}d_{22}\ d_{44}d_{55} \geq 0$

Det $A(1,3,4,5) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(2,3,4,5) = d_{22}d_{33}d_{44}d_{55} \geq 0$

Det $A = d_{11}d_{22}\ d_{33}d_{44}d_{55} \geq 0$

Hence the pattern have zero completion into nonnegative P_0- matrix since all the determinants are nonnegative

14. Consider the digraph below:

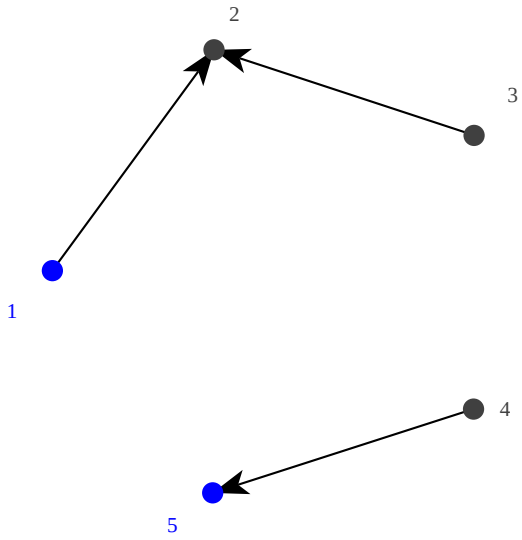

Figure 4.2.27: Digraph with arcs 1 to 2, 3 to 2 and 4 to 5

The following digraph obtained is among other digraphs with same results as one above.

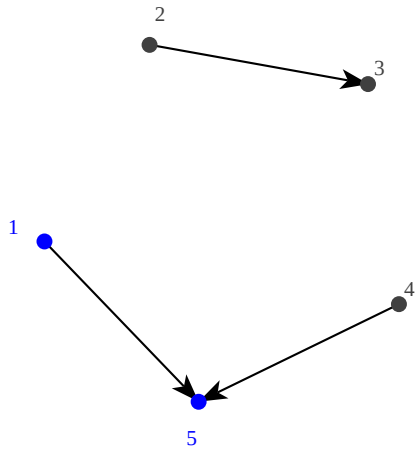

Figure 4.2.28: Digraph isomorphic to digraph in figure 2.2.17, obtained via permutation (2534)

Using the digraph in the figure 2.2.27

Let $A = \begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & a_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$ be the partial nonnegative P_0matrix specifying the digraph

Determinants of the principal submatrices

Det $A(1,2) = d_{11}d_{22} - a_{12}x_{21}$

Det $A(1,3) = d_{11}d_{33} - x_{13}x_{31}$

Det $A(1,4) = d_{11}d_{44} - x_{14}x_{41}$

Det $A(1,5) = d_{11}d_{55} - x_{15}x_{51}$

Det $A(2,3) = d_{22}d_{33} - x_{23}a_{32}$

Det $A(2,4) = d_{22}d_{44} - x_{24}x_{42}$

Det $A(2,5) = d_{22}d_{55} - x_{25}x_{52}$

Det $A(3,4) = d_{33}d_{44} - x_{34}x_{43}$

Det $A(3,5) = d_{33}d_{55} - x_{35}x_{53}$

Det $A(4,5) = d_{44}d_{55} - a_{45}x_{54}$

Det $A(1,2,3) = d_{11}(d_{22}d_{33} - x_{23}a_{32}) - a_{12}(x_{21}d_{33} - x_{23}x_{31}) + x_{13}(x_{21}a_{32} - d_{22}x_{31})$

Det $A(1,2,4) = d_{11}(d_{22}d_{44} - x_{24}x_{42}) - a_{12}(x_{21}d_{44} - x_{24}x_{41}) + x_{14}(x_{21}x_{42} - d_{22}x_{41})$

Det $A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - a_{12}(x_{21}d_{55} - x_{25}x_{51}) + x_{15}(x_{21}x_{52} - d_{22}x_{51})$

Det $A(1,3,4) = d_{11}(d_{33}d_{44} - x_{34}x_{43}) - x_{13}(x_{31}d_{44} - x_{34}x_{41}) + x_{14}(x_{31}x_{43} - d_{33}x_{41})$

Det $A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - x_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$

Det $A(1,4,5) = d_{11}(d_{44}d_{55} - a_{45}x_{54}) - x_{14}(x_{41}d_{55} - a_{45}x_{51}) + x_{15}(x_{41}x_{54} - d_{44}x_{51})$

Det $A(2,3,4) = d_{22}(d_{33}d_{44} - x_{34}x_{43}) - x_{23}(a_{32}d_{44} - x_{34}x_{42}) + x_{24}(a_{32}x_{43} - d_{33}x_{42})$

Det $A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - x_{23}(a_{32}d_{55} - x_{35}x_{52}) + x_{25}(a_{32}x_{53} - d_{33}x_{52})$

Det $A(2,4,5) = d_{22}(d_{44}d_{55} - a_{45}x_{54}) - x_{24}(x_{42}d_{55} - a_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})$

Det $A(3,4,5) = d_{33}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{43}d_{55} - a_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})$

Det $A(1,2,3,4) = d_{11}\{d_{22}(d_{33}d_{44} - x_{34}x_{43}) - x_{23}(a_{32}d_{44} - x_{34}x_{42}) + x_{24}(a_{32}x_{43} - d_{33}x_{42})\} - a_{12}\{x_{21}(d_{33}d_{44} - x_{34}x_{43}) - x_{23}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41})\} + x_{13}\{x_{21}(a_{32}d_{44} - x_{34}x_{42}) - d_{22}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}x_{42} - a_{32}x_{41})\} - x_{14}\{x_{21}(a_{32}x_{43} - d_{33}x_{42}) - d_{22}(x_{31}x_{43} - d_{33}x_{41}) + x_{23}(x_{31}x_{42} - a_{32}x_{41})\}$

Det $A(1,2,3,5) = d_{11}\{d_{22}(d_{33}d_{55} - x_{35}x_{53}) - x_{23}(a_{32}d_{55} - x_{35}x_{52}) + x_{25}(a_{32}x_{53} - d_{33}x_{52})\} - a_{12}\{x_{21}(d_{33}d_{55} - x_{35}x_{53}) - x_{23}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{53} - d_{33}x_{51})\} + x_{13}\{x_{21}(a_{32}d_{55} - x_{35}x_{52}) - d_{22}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{52} - a_{32}x_{51})\} - x_{15}\{x_{21}(a_{32}x_{53} - d_{33}x_{52}) - d_{22}(x_{31}x_{53} - d_{33}x_{51}) + x_{23}(x_{31}x_{52} - a_{32}x_{51})\}$

Det $A(1,2,4,5) = d_{11}\{d_{22}(d_{44}d_{55} - a_{45}x_{54}) - x_{24}(x_{42}d_{55} - a_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})\} - a_{12}\{x_{21}(d_{44}d_{55} - a_{45}x_{54}) - x_{24}(x_{41}d_{55} - a_{45}x_{51}) + x_{25}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{21}(x_{42}d_{55} - a_{45}x_{52}) - d_{22}(x_{41}d_{55} - a_{45}x_{51}) + x_{25}(x_{41}x_{52} - x_{42}x_{51})\} - x_{15}\{x_{21}(x_{42}x_{54} - d_{44}x_{52}) - d_{22}(x_{41}x_{54} - d_{44}x_{51}) + x_{24}(x_{41}x_{52} - x_{42}x_{51})\}$

Det $A(1,3,4,5) = d_{11}\{d_{33}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{43}d_{55} - a_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{13}\{x_{31}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{31}(x_{43}d_{55} - a_{45}x_{53}) - d_{33}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{15}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}$

Det $A(2,3,4,5) = d_{22}\{d_{33}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{43}d_{55} - a_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{23}\{a_{32}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{42}d_{55} - a_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{a_{32}(x_{43}d_{55} - a_{45}x_{53}) - d_{33}(x_{42}d_{55} - a_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{a_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\}$

Det $A = d_{11}[d_{22}\{d_{33}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{43}d_{55} - a_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{23}\{a_{32}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{42}d_{55} - a_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{a_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - a_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{a_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\}] -$

$a_{12}[x_{21}\{d_{33}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{43}d_{55} - a_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{23}\{x_{31}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{43}d_{55} - a_{45}x_{53}) - d_{33}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{25}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}] +$

$x_{13}[x_{21}\{a_{32}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{42}d_{55} - a_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} - d_{22}\{x_{31}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{42}d_{55} - a_{45}x_{52}) - a_{32}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - a_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\}] -$

$x_{14}[x_{21}\{a_{32}(x_{43}d_{55} - a_{45}x_{53}) - d_{33}(x_{42}d_{55} - a_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}d_{55} - a_{45}x_{53}) - d_{33}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} + x_{23}\{x_{31}(x_{42}d_{55} - a_{45}x_{52}) - a_{32}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - a_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}] +$

$x_{15}[x_{21}\{a_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\} + x_{23}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - a_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\} - x_{24}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - a_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}]$

Performing zero completion by setting all the unspecified entries to zero i.e. $x_{13} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{21} = 0$, $x_{23} = 0$, $x_{24} = 0$, $x_{25} = 0$, $x_{31} = 0$, $x_{34} = 0$, $x_{35} = 0$, $x_{41} = 0$, $x_{42} = 0$, $x_{43} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$. The determinants of principal submatrices will be ;

Det $A(1,2) = d_{11}d_{22} \geq 0$

Det $A(1,3) = d_{11}d_{33} \geq 0$

Det $A(1,4) = d_{11}d_{44} \geq 0$

Det $A(1,5) = d_{11}d_{55} \geq 0$

Det $A(2,3) = d_{22}d_{33} \geq 0$

Det $A(2,4) = d_{22}d_{44} \geq 0$

Det $A(2,5) = d_{22}d_{55} \geq 0$

Det $A(3,4) = d_{33}d_{44} \geq 0$

Det $A(3,5) = d_{33}d_{55} \geq 0$

Det $A(4,5) = d_{44}d_{55} \geq 0$

Det $A(1,2,3) = d_{11}d_{22}\, d_{33} \geq 0$

Det $A(1,2,4) = d_{11}d_{22}\, d_{44} \geq 0$

Det $A(1,2,5) = d_{11}d_{22}\, d_{55} \geq 0$

Det $A(1,3,4) = d_{11}d_{33}d_{44} \geq 0$

Det $A(1,3,5) = d_{11}d_{33}d_{55} \geq 0$

Det $A(1,4,5) = d_{11}d_{44}d_{55} \geq 0$

Det $A(2,3,4) = d_{22}d_{33}d_{44} \geq 0$

Det $A(2,3,5) = d_{22}d_{33}d_{55} \geq 0$

Det $A(2,4,5) = d_{22}d_{44}d_{55} \geq 0$

Det $A(3,4,5) = d_{33}d_{44}d_{55} \geq 0$

Det $A(1,2,3,4) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(1,2,3,5) = d_{11}d_{22}\ d_{33}d_{55} \geq 0$

Det $A(1,2,4,5) = d_{11}d_{22}\ d_{44}d_{55} \geq 0$

Det $A(1,3,4,5) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(2,3,4,5) = d_{22}d_{33}d_{44}d_{55} \geq 0$

Det $A = d_{11}d_{22}\ d_{33}d_{44}d_{55} \geq 0$

With all the determinants nonnegative, the pattern have zero completion into nonnegative P_0- matrix.

15. Consider the digraph below:

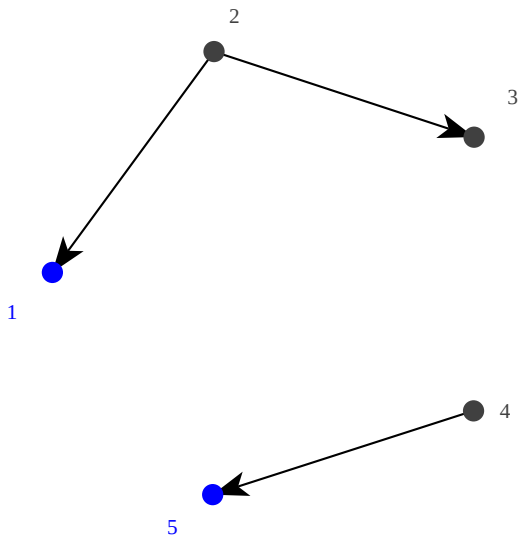

Figure 4.2.29: Digraph with arcs 2 to 1, 2 to 3 and 4 to 5

By permuting the vertices of the above digraph other digraphs are obtained with results similar to it, such as one that follows:

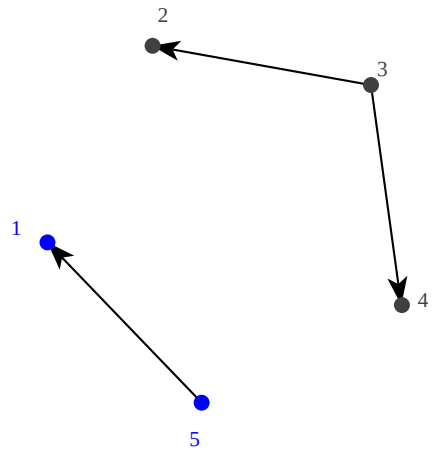

Figure 4.2.30: Digraph isomorphic to the digraph in figure 2.2.29, obtained via permutation (12345)

Using the digraph in figure 2.2.29:

Let $A = \begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ a_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$ be the partial nonnegative P_0-matrix specifying the digraph.

Determinants of the principal submatrices

Det $A(1,2) = d_{11}d_{22} - x_{12}a_{21}$

Det $A(1,3) = d_{11}d_{33} - x_{13}x_{31}$

Det $A(1,4) = d_{11}d_{44} - x_{14}x_{41}$

Det $A(1,5) = d_{11}d_{55} - x_{15}x_{51}$

Det $A(2,3) = d_{22}d_{33} - a_{23}x_{32}$

Det $A(2,4) = d_{22}d_{44} - x_{24}x_{42}$

Det $A(2,5) = d_{22}d_{55} - x_{25}x_{52}$

Det $A(3,4) = d_{33}d_{44} - x_{34}x_{43}$

Det $A(3,5) = d_{33}d_{55} - x_{35}x_{53}$

Det $A(4,5) = d_{44}d_{55} - a_{45}x_{54}$

Det $A(1,2,3) = d_{11}(d_{22}d_{33} - a_{23}x_{32}) - x_{12}(a_{21}d_{33} - a_{23}x_{31}) + x_{13}(a_{21}x_{32} - d_{22}x_{31})$

Det $A(1,2,4) = d_{11}(d_{22}d_{44} - x_{24}x_{42}) - x_{12}(a_{21}d_{44} - x_{24}x_{41}) + x_{14}(a_{21}x_{42} - d_{22}x_{41})$

Det $A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - x_{12}(a_{21}d_{55} - x_{25}x_{51}) + x_{15}(a_{21}x_{52} - d_{22}x_{51})$

Det $A(1,3,4) = d_{11}(d_{33}d_{44} - x_{34}x_{43}) - x_{13}(x_{31}d_{44} - x_{34}x_{41}) + x_{14}(x_{31}x_{43} - d_{33}x_{41})$

Det $A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - x_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$

Det $A(1,4,5) = d_{11}(d_{44}d_{55} - a_{45}x_{54}) - x_{14}(x_{41}d_{55} - a_{45}x_{51}) + x_{15}(x_{41}x_{54} - d_{44}x_{51})$

Det $A(2,3,4) = d_{22}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{32}d_{44} - x_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})$

Det $A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})$

Det $A(2,4,5) = d_{22}(d_{44}d_{55} - a_{45}x_{54}) - x_{24}(x_{42}d_{55} - a_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})$

Det $A(3,4,5) = d_{33}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{43}d_{55} - a_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})$

Det $A(1,2,3,4) = d_{11}\{d_{22}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{32}d_{44} - x_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})\} - x_{12}\{a_{21}(d_{33}d_{44} - x_{34}x_{43}) - a_{23}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41})\} + x_{13}\{a_{21}(x_{32}d_{44} - x_{34}x_{42}) - d_{22}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}x_{42} - x_{32}x_{41})\} - x_{14}\{a_{21}(x_{32}x_{43} - d_{33}x_{42}) - d_{22}(x_{31}x_{43} - d_{33}x_{41}) + a_{23}(x_{31}x_{42} - x_{32}x_{41})\}$

Det $A(1,2,3,5) = d_{11}\{d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})\} - x_{12}\{a_{21}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{53} - d_{33}x_{51})\} + x_{13}\{a_{21}(x_{32}d_{55} - x_{35}x_{52}) - d_{22}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{52} - x_{32}x_{51})\} - x_{15}\{a_{21}(x_{32}x_{53} - d_{33}x_{52}) - d_{22}(x_{31}x_{53} - d_{33}x_{51}) + a_{23}(x_{31}x_{52} - x_{32}x_{51})\}$

Det $A(1,2,4,5) = d_{11}\{d_{22}(d_{44}d_{55} - a_{45}x_{54}) - x_{24}(x_{42}d_{55} - a_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})\} - x_{12}\{a_{21}(d_{44}d_{55} - a_{45}x_{54}) - x_{24}(x_{41}d_{55} - a_{45}x_{51}) + x_{25}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{a_{21}(x_{42}d_{55} - a_{45}x_{52}) - d_{22}(x_{41}d_{55} - a_{45}x_{51}) + x_{25}(x_{41}x_{52} - x_{42}x_{51})\} - x_{15}\{a_{21}(x_{42}x_{54} - d_{44}x_{52}) - d_{22}(x_{41}x_{54} - d_{44}x_{51}) + x_{24}(x_{41}x_{52} - x_{42}x_{51})\}$

Det $A(1,3,4,5) = d_{11}\{d_{33}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{43}d_{55} - a_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{13}\{x_{31}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{31}(x_{43}d_{55} - a_{45}x_{53}) - d_{33}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{15}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}$

Det $A(2,3,4,5) = d_{22}\{d_{33}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{43}d_{55} - a_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{42}d_{55} - a_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{x_{32}(x_{43}d_{55} - a_{45}x_{53}) - d_{33}(x_{42}d_{55} - a_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\}$

Det $A = d_{11}[d_{22}\{d_{33}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{43}d_{55} - a_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{32}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{42}d_{55} - a_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{x_{32}(x_{43}d_{55} - a_{45}x_{53}) - d_{33}(x_{42}d_{55} - a_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\}] -$

$x_{12}[a_{21}\{d_{33}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{43}d_{55} - a_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{23}\{x_{31}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{43}d_{55} - a_{45}x_{53}) - d_{33}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{25}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}] +$

$x_{13}[x_{21}\{x_{32}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{42}d_{55} - a_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} - d_{22}\{x_{31}(d_{44}d_{55} - a_{45}x_{54}) - x_{34}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{42}d_{55} - a_{45}x_{52}) - x_{32}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\}] -$

$x_{14}[\ a_{21}\{x_{32}(x_{43}d_{55} - a_{45}x_{53}) - d_{33}(x_{42}d_{55} - a_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}d_{55} - a_{45}x_{53}) - d_{33}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} + a_{23}\{x_{31}(x_{42}d_{55} - a_{45}x_{52}) - x_{32}(x_{41}d_{55} - a_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}] +$

$x_{15}[a_{21}\{x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\} + a_{23}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\} - x_{24}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}]$

Performing zero competion by setting all the unspecified entries to zero i.e. $x_{12} = 0$, $x_{13} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{24} = 0$, $x_{25} = 0$, $x_{31} = 0$, $x_{32} = 0$, $x_{34} = 0$, $x_{35} = 0$, $x_{41} = 0$, $x_{42} = 0$, $x_{43} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$. The determinants of principal submatrices will be ;

Det $A(1,2) = d_{11}d_{22} \geq 0$

Det $A(1,3) = d_{11}d_{33} \geq 0$

Det $A(1,4) = d_{11}d_{44} \geq 0$

Det $A(1,5) = d_{11}d_{55} \geq 0$

Det $A(2,3) = d_{22}d_{33} \geq 0$

Det $A(2,4) = d_{22}d_{44} \geq 0$

Det $A(2,5) = d_{22}d_{55} \geq 0$

Det $A(3,4) = d_{33}d_{44} \geq 0$

Det $A(3,5) = d_{33}d_{55} \geq 0$

Det $A(4,5) = d_{44}d_{55} \geq 0$

Det $A(1,2,3) = d_{11}d_{22}\ d_{33} \geq 0$

Det $A(1,2,4) = d_{11}d_{22}\ d_{44} \geq 0$

Det $A(1,2,5) = d_{11}d_{22}\ d_{55} \geq 0$

Det $A(1,3,4) = d_{11}d_{33}d_{44} \geq 0$

Det $A(1,3,5) = d_{11}d_{33}d_{55} \geq 0$

Det $A(1,4,5) = d_{11}d_{44}d_{55} \geq 0$

Det $A(2,3,4) = d_{22}d_{33}d_{44} \geq 0$

Det $A(2,3,5) = d_{22}d_{33}d_{55} \geq 0$

Det $A(2,4,5) = d_{22}d_{44}d_{55} \geq 0$

Det $A(3,4,5) = d_{33}d_{44}d_{55} \geq 0$

Det $A(1,2,3,4) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(1,2,3,5) = d_{11}d_{22}\ d_{33}d_{55} \geq 0$

Det $A(1,2,4,5) = d_{11}d_{22}\ d_{44}d_{55} \geq 0$

Det $A(1,3,4,5) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(2,3,4,5) = d_{22}d_{33}d_{44}d_{55} \geq 0$

Det $A = d_{11}d_{22}\ d_{33}d_{44}d_{55} \geq 0$

Hence all the determinants are nonnegative and therefore the pattern have zero completion into nonnegative P_0- matrix.

Therefore it can be found that all patterns specifying acyclic digraphs for p = 5 and q = 3, have zero completion to non-negative P_0-matrix

CHAPTER FIVE

CYLES

5.1 Introduction

This section focuses on digraphs with 5vertices and 3 arcs whose paths start and end at the same point. The objective is to determine whether the cycles have nonnegative P_0-completion or not.

5.2 DIGRAPHS WITH 5 VERTICES AND THREE ARCS THAT ARE CYCLES

Consider the digraph below:

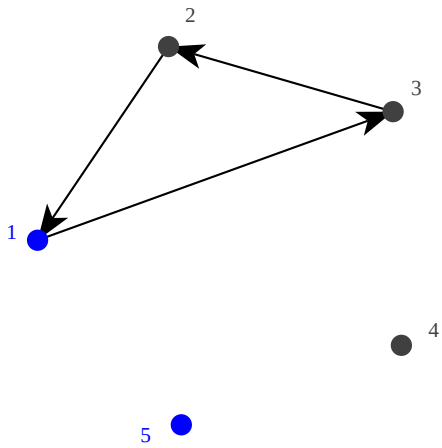

Figure 5.2.1: A cycle with arcs 1 to 3, 2 to 1 and 3 to 2

Permuting the vertices of the above digraph gives other digraphs with similar results, example of the one that follows:

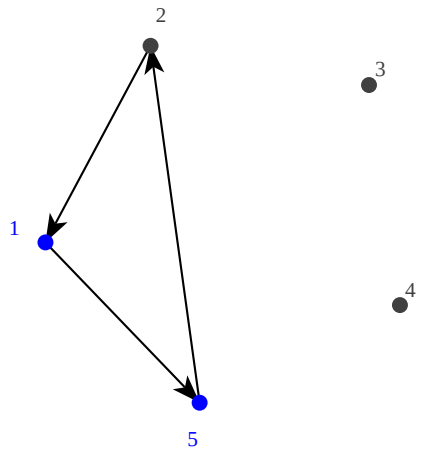

Figure 5.2.2: Cycle isomorphic to digraph in figure 3.2.1, obtained via permutation (1532)

Using the cycle in figure 3.2.1:

Let $A = \begin{bmatrix} d_{11} & x_{12} & a_{13} & x_{14} & x_{15} \\ a_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & a_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$ be the partial nonnegative P_0-matrix specifying the digraph.

Determinants of the principal submatrices

Det $A(1,2) = d_{11}d_{22} - x_{12}a_{21}$

Det $A(1,3) = d_{11}d_{33} - a_{13}x_{31}$

Det $A(1,4) = d_{11}d_{44} - x_{14}x_{41}$

Det $A(1,5) = d_{11}d_{55} - x_{15}x_{51}$

Det $A(2,3) = d_{22}d_{33} - x_{23}a_{32}$

Det $A(2,4) = d_{22}d_{44} - x_{24}x_{42}$

Det $A(2,5) = d_{22}d_{55} - x_{25}x_{52}$

Det $A(3,4) = d_{33}d_{44} - x_{34}x_{43}$

Det $A(3,5) = d_{33}d_{55} - x_{35}x_{53}$

Det $A(4,5) = d_{44}d_{55} - x_{45}x_{54}$

Det $A(1,2,3) = d_{11}(d_{22}d_{33} - x_{23}a_{32}) - x_{12}(a_{21}d_{33} - x_{23}x_{31}) + a_{13}(a_{21}a_{32} - d_{22}x_{31})$

Det $A(1,2,4) = d_{11}(d_{22}d_{44} - x_{24}x_{42}) - x_{12}(a_{21}d_{44} - x_{24}x_{41}) + x_{14}(a_{21}x_{42} - d_{22}x_{41})$

Det $A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - x_{12}(a_{21}d_{55} - x_{25}x_{51}) + x_{15}(a_{21}x_{52} - d_{22}x_{51})$

Det $A(1,3,4) = d_{11}(d_{33}d_{44} - x_{34}x_{43}) - a_{13}(x_{31}d_{44} - x_{34}x_{41}) + x_{14}(x_{31}x_{43} - d_{33}x_{41})$

Det $A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - a_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$

Det $A(1,4,5) = d_{11}(d_{44}d_{55} - x_{45}x_{54}) - x_{14}(x_{41}d_{55} - x_{45}x_{51}) + x_{15}(x_{41}x_{54} - d_{44}x_{51})$

Det $A(2,3,4) = d_{22}(d_{33}d_{44} - x_{34}x_{43}) - x_{23}(a_{32}d_{44} - x_{34}x_{42}) + x_{24}(a_{32}x_{43} - d_{33}x_{42})$

Det $A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - x_{23}(a_{32}d_{55} - x_{35}x_{52}) + x_{25}(a_{32}x_{53} - d_{33}x_{52})$

Det $A(2,4,5) = d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})$

Det $A(3,4,5) = d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})$

Det $A(1,2,3,4) = d_{11}\{d_{22}(d_{33}d_{44} - x_{34}x_{43}) - x_{23}(a_{32}d_{44} - x_{34}x_{42}) + x_{24}(a_{32}x_{43} - d_{33}x_{42})\} - x_{12}\{a_{21}(d_{33}d_{44} - x_{34}x_{43}) - x_{23}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41})\} + a_{13}\{a_{21}(a_{32}d_{44} - x_{34}x_{42}) - d_{22}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}x_{42} - a_{32}x_{41})\} - x_{14}\{a_{21}(a_{32}x_{43} - d_{33}x_{42}) - d_{22}(x_{31}x_{43} - d_{33}x_{41}) + x_{23}(x_{31}x_{42} - a_{32}x_{41})\}$

Det $A(1,2,3,5) = d_{11}\{d_{22}(d_{33}d_{55} - x_{35}x_{53}) - x_{23}(a_{32}d_{55} - x_{35}x_{52}) + x_{25}(a_{32}x_{53} - d_{33}x_{52})\} - x_{12}\{a_{21}(d_{33}d_{55} - x_{35}x_{53}) - x_{23}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{53} - d_{33}x_{51})\} + a_{13}\{a_{21}(a_{32}d_{55} - x_{35}x_{52}) - d_{22}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{52} - a_{32}x_{51})\} - x_{15}\{a_{21}(a_{32}x_{53} - d_{33}x_{52}) - d_{22}(x_{31}x_{53} - d_{33}x_{51}) + x_{23}(x_{31}x_{52} - a_{32}x_{51})\}$

Det $A(1,2,4,5) = d_{11}\{d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})\} - x_{12}\{a_{21}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{a_{21}(x_{42}d_{55} - x_{45}x_{52}) - d_{22}(x_{41}d_{55} - x_{45}x_{51}) + x_{25}(x_{41}x_{52} - x_{42}x_{51})\} - x_{15}\{a_{21}(x_{42}x_{54} - d_{44}x_{52}) - d_{22}(x_{41}x_{54} - d_{44}x_{51}) + x_{24}(x_{41}x_{52} - x_{42}x_{51})\}$

Det $A(1,3,4,5) = d_{11}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - a_{13}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{14}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{15}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}$

Det $A(2,3,4,5) = d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{23}\{a_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{a_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{a_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\}$

Det $A = d_{11}[d_{22}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{23}\{a_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} + x_{24}\{a_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - x_{25}\{a_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\}] -$

$x_{12}[a_{21}\{d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})\} - x_{23}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} - x_{25}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\}] +$

$a_{13}[a_{21}\{a_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})\} - d_{22}\{x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51})\} + x_{24}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - a_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - a_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\}] -$

$x_{14}[\ a_{21}\{a_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51})\} + x_{23}\{x_{31}(x_{42}d_{55} - x_{45}x_{52}) - a_{32}(x_{41}d_{55} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51})\} - x_{25}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - a_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}] +$

$x_{15}[a_{21}\{a_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52})\} - d_{22}\{x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51})\} + x_{23}\{x_{31}(x_{42}x_{54} - d_{44}x_{52}) - a_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51})\} - x_{24}\{x_{31}(x_{42}x_{53} - x_{43}x_{52}) - a_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51})\}]$

Performing zero completion setting all the unspecified entries to zero i.e. $x_{12} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{23} = 0$, $x_{24} = 0$, $x_{25} = 0$, $x_{31} = 0$, $x_{34} = 0$, $x_{35} = 0$, $x_{41} = 0$, $x_{42} = 0$, $x_{43} = 0$, $x_{45} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$. The determinants of principal submatrices will be ;

Det $A(1,2) = d_{11}d_{22} \geq 0$

Det $A(1,3) = d_{11}d_{33} \geq 0$

Det $A(1,4) = d_{11}d_{44} \geq 0$

Det $A(1,5) = d_{11}d_{55} \geq 0$

Det $A(2,3) = d_{22}d_{33} \geq 0$

Det $A(2,4) = d_{22}d_{44} \geq 0$

Det $A(2,5) = d_{22}d_{55} \geq 0$

Det $A(3,4) = d_{33}d_{44} \geq 0$

Det $A(3,5) = d_{33}d_{55} \geq 0$

Det $A(4,5) = d_{44}d_{55} \geq 0$

Det $A(1,2,3) = d_{11}d_{22}\ d_{33} + a_{13}a_{21}a_{32} \geq 0$, since all entries are nonnegative

Det $A(1,2,4) = d_{11}d_{22}\ d_{44} \geq 0$

Det $A(1,2,5) = d_{11}d_{22}\ d_{55} \geq 0$

Det $A(1,3,4) = d_{11}d_{33}d_{44} \geq 0$

Det $A(1,3,5) = d_{11}d_{33}d_{55} \geq 0$

Det $A(1,4,5) = d_{11}d_{44}d_{55} \geq 0$

Det $A(2,3,4) = d_{22}d_{33}d_{44} \geq 0$

Det $A(2,3,5) = d_{22}d_{33}d_{55} \geq 0$

Det $A(2,4,5) = d_{22}d_{44}d_{55} \geq 0$

Det $A(3,4,5) = d_{33}d_{44}d_{55} \geq 0$

Det $A(1,2,3,4) = d_{11}d_{22}\ d_{33}d_{44} + a_{13}a_{21}a_{32}d_{44} = d_{44}(d_{11}d_{22}\ d_{33} + a_{13}a_{21}a_{32}) \geq 0$, since Det $A(1,2,3) \geq 0$

Det $A(1,2,3,5) = d_{11}d_{22}\ d_{33}d_{55} + a_{13}a_{21}a_{32}d_{55} = d_{55}\ (d_{11}d_{22}\ d_{33} + a_{13}a_{21}a_{32}) \geq 0$, since Det $A(1,2,3) \geq 0$

Det $A(1,2,4,5) = d_{11}d_{22}\ d_{44}d_{55} \geq 0$

Det $A(1,3,4,5) = d_{11}d_{22}\ d_{33}d_{44} \geq 0$

Det $A(2,3,4,5) = d_{22}d_{33}d_{44}d_{55} \geq 0$

Det $A = d_{11}d_{22}\ d_{33}d_{44}d_{55} + a_{13}a_{21}a_{32}d_{44}d_{55} = d_{44}d_{55}(d_{11}d_{22}\ d_{33} + a_{13}a_{21}a_{32}) \geq 0$, since Det $A(1,2,3) \geq 0$

The pattern therefore have zero completion into nonnegative P_0- matrix since all the determinants are nonnegative.

Hence digraph which is a cycle has zero completion to nonnegative P_0-matrix.

CHAPTER SIX

CONCLUSION AND RECOMENDATIONS

6.1 Conclusion

The study established that all the partial matrices specifying digraphs p = 5, q = 3, which are either cycles or acyclic digraphs have zero completion to nonnegative P_0-matrix.

6.2 Recommendation

Further research can be done to extend the findings of this study to cover 5 x 5 partial matrices specifying digraphs with 5 vertices and more than 3 arcs and also 6 x 6, ..., n x n matrices partial matrices. For cases where $q \geq 4$, zero completion method would be not appropriate to carry out the completion since it would give results that are not conclusive, hence an approprite method should be derived.

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