SYNCHRONIZATION IN A NETWORK OF OSCILLATORS WITH DELAYED COUPLING

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Synchronization in a network of Oscillators with Delayed Coupling

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DECLARATION

This thesis is my original work and has not been presented for a degree in any other University;

Signature……………………………………… Date………………………….

James Mariita Okwoyo

This thesis has been submitted for examination with our approval as the University supervisors;

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DEDICATION

This dissertation is dedicated to my late father Joseph Okwoyo, my mother Rosa Okwoyo, my lovely wife Monicah Kemunto and my children Nancy, Heinrich, Raychelle and Ashley.

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To all I say may the Almighty God reward you abundantly.

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LIST OF SYMBOLS, NOTATIONS AND ABBREVIATIONS

$\mathbb{R}$	Set of real numbers , $(-\infty, \infty)$
$\mathbb{R}^n$	Real n-dimensional vector space
$T\mathbb{R}^n$	Tangent space over $\mathbb{R}^n$
$\mathfrak{M}$	Differentiable manifold embedded in $\mathbb{R}^n$
$T\mathfrak{M}$	Tangent space of the differentiable manifold $\mathfrak{M}$
$\boldsymbol{N}$	Bundle of vectors normal to $T\mathfrak{M}$
$T\mathbb{R}^n \mid \mathfrak{M}$	Restriction of the tangent space to $\mathfrak{M}$
π	Orthogonal projection
$\lVert . \rVert$	Euclidean norm or matrix norm
$\lvert . \rvert$	The norm in the c^r topology
$\mathfrak{C}^1$	Continuous function to the 1st derivative
c^r	Continuous function to the rth derivative
λ_M	Maximum Lyapunov exponent
$\oplus$	Kronecker sum
$\otimes$	Kronecker product
$\mathbb{R}^+{}_0$	Non-negative real numbers
$f \in c^k$	k-times differentiable function
$\downarrow$	Monotonically decreasing

$\uparrow$	Monotonically increasing
dde23	Delay Differential Equation Mat lab solver
max	Maximum
min	Minimum
DDEs	Delay Differential Equations
ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations

ABSTRACT

The study of coupled oscillators with time lag can get its applications in; Neurobiology, Laser arrays, Microwave devices, Communications satellites and electronic circuits, just to mention but few. That is why we studied a population of n oscillators each with an asymptotically stable limit cycle coupled all-to-all by a linear diffusive like path with a time lag, τ. The system of equations was inbuilt with symmetries which we exploited to get an analytical understanding of the dynamics of the system. The symmetries then helped us get two n-dimensional invariant manifolds: the diagonal manifold and the other orthogonal manifold. We exploited the symmetries in the coupling terms to establish the range of time delay τ for stability of synchronized state.

We did a rigorous study of the condition of stability and persistence of the synchronized manifold of diffusively coupled oscillators of linear and planar simple Bravais Lattices by considering $n \ (n \geq 2)$, d-dimensional oscillators each with an asymptotically stable limit cycle coupled all-to-all by a nearest neighbor linear diffusive like path. We used the invariant Manifold Theory and Lyapunov exponents to establish the range of coupling strength for stability and robustness of the synchronized manifold. The 4^{th} and 5^{th} order Runge-Kutta method, together with *ode*-45 and *dde*-23 Mat lab solvers were the numerical methods we used to get the numerical solution of our problem. We established the estimate for bound of τ for which the synchronized manifold remains stable when the oscillators are coupled in an all-to-all configuration. The synchronized state is seen to be stable when $\tau < 9$. Even for significant time delays, a stable synchronized state exists at a very low coupling strength.

From the study we realized that if synchronization exists for a certain coupling configuration, then there exist a $k_0 > 0$ such that for all $k_0 > k$, synchronization manifold is stable and persist under perturbation.

CHAPTER ONE

1.0 INTRODUCTION

Many non-linear dynamical systems in various scientific disciplines are influenced by the finite propagation time of signals in feedback loops. A typical physical system is provided by a laser system where the output light is reflected and fed back to the cavity. Time delays also occur in other situations; for example, in traffic flow including a driver's reaction time, in Biology due to physiological control mechanism, in economy where the finite velocity of information processing has to be taken into account. Moreover, realistic models in population dynamics or in Ecology include the duration for the replacement of resources. In some situations, systems with large-delay appear such as in lasers and electromechanical systems.

An oscillator is a system of model equations, normally Ordinary Differential Equations, with a non-constant solution that displays some repetitive phenomenon and its derivative with respect to time is non-constant. Observations of the phenomena of coupled oscillators date back to the early seventeenth century, when Christian Huygens noticed that the pendulums of two of his clocks, suspended side-by-side, always settled into swinging in opposite directions, even after he disturbed the position of the pendulums (Bennett *et al*., 2002; Strogatz, 2003). Another form of phased coupled oscillation, in the synchronous flashing of hundreds of fireflies on trees along the Chao Phraya River in Thailand was reported (Buck and Buck, 1976). Many similar instances of naturally occurring synchronization have since been discovered, such as in heart pacemaker cells and in neural networks (Camazine *et al*., 2001).

Fireflies generate light from the lantern in the abdomen; it usually takes about 800 milliseconds to recharge the lantern and 200 milliseconds to produce a spark; the process may then repeat. Formal models of this behavior describe a single firefly as an oscillator with a phase $0 \leq \theta \leq 2\pi$ and period $\acute{\omega}$. For a large proportion of each cycle, the oscillator is recharging and therefore discharging is impossible. For the remaining portion of the cycle, the firefly/oscillator is ready to discharge or "fire". If the firefly/oscillator is operating in isolation from other firefly/oscillators, then it fires at $\theta = 2\pi$. If a firefly/oscillator is not operating in isolation, has completed recharging, and sees sufficient light (stimulus) from neighboring fireflies, the firefly/oscillator can adjust its phase slightly so as to bring itself closer to synchronization with the other firefly/oscillators (Camazine *et al*., 2001).

Networks of oscillators have properties that make them an interesting approach to coordinating activity in large networks of simple computational elements. First, the synchronization mechanism of the oscillators is parallel and distributed – no global coordination is required. Second, the oscillators can be implemented in hardware with very simple circuitry, making it a promising approach for massive networks of tiny processing elements. In fact, the approach has already received some attention for synchronization in ad-hoc sensor networks (Hong *et al*.,2003; Lucarelli *et al.*, 2004; Werner-Allen *et al*., 2005) and the coordination of multi-agent systems (Spong, 2006).

It is important to understand the dynamics of a population of coupled oscillators, particularly how the collective behavior depends on the intrinsic properties of the individual oscillators and the mode of coupling between them. Sometimes individual oscillators display certain dynamical behavior, for instance when they synchronize and settle to a stationary point (Oscillator death) (Bar-Eli, 1990) or when chaotic dynamical behavior is possible (Mirollo *et al.*, 1990).

This problem is not easy to handle as Mathematical theory providing a detailed description of the behavior of the solutions does not exist for some model equations. Thus certain realistic simplifications are done on the type of oscillator, the nature of coupling or on both. Emphasis is usually on a special class of coupling functions and the exact nature of individual oscillators. The oscillators considered are those that have asymptotically stable periodic solutions, that is, solutions which tend to a particular periodic solution time t increases without bound. The coupling function can either be linear or non-linear and with or without delay.

An oscillator can be described by ordinary differential equations or partial differential equations. Herein, we shall consider oscillators described by ordinary differential equations. To obtain an asymptotically stable periodic solution for each oscillator, an appropriate system of ordinary differential equations, which must be non-linear and have at least two stable variables, has to be chosen. We will consider an oscillator described by differential equations of the form;

$$\dot{z}(t) = f(z(t), \mu) \tag{1.1}$$

where $z \in R^n$ is an n-dimensional $(n \geq 2)$ real state variable, $\mu \in R^p$ is a p-dimensional ($p \geq 1$) real parameter, n and p are positive natural numbers), f is a non-linear smooth function from $R^n \times R^p$ to R^n and the dot on z denotes differentiation with respect to time.

In many physical processes, oscillators are directly coupled. That is, an action which could be a variable in case of differential equations is transmitted from one oscillator to the other without going through an intermediate step. Frequently, systems of oscillators can be arranged in topological structures with nearest neighbor coupling, that is, the oscillators are linked to their immediate neighbours. In this case, a neighbor is a physical location of an oscillator relative to others in a topological structure and coupling is without going through an intermediate oscillator. There are different forms of nearest neighbor coupling topologies as in the figures below;

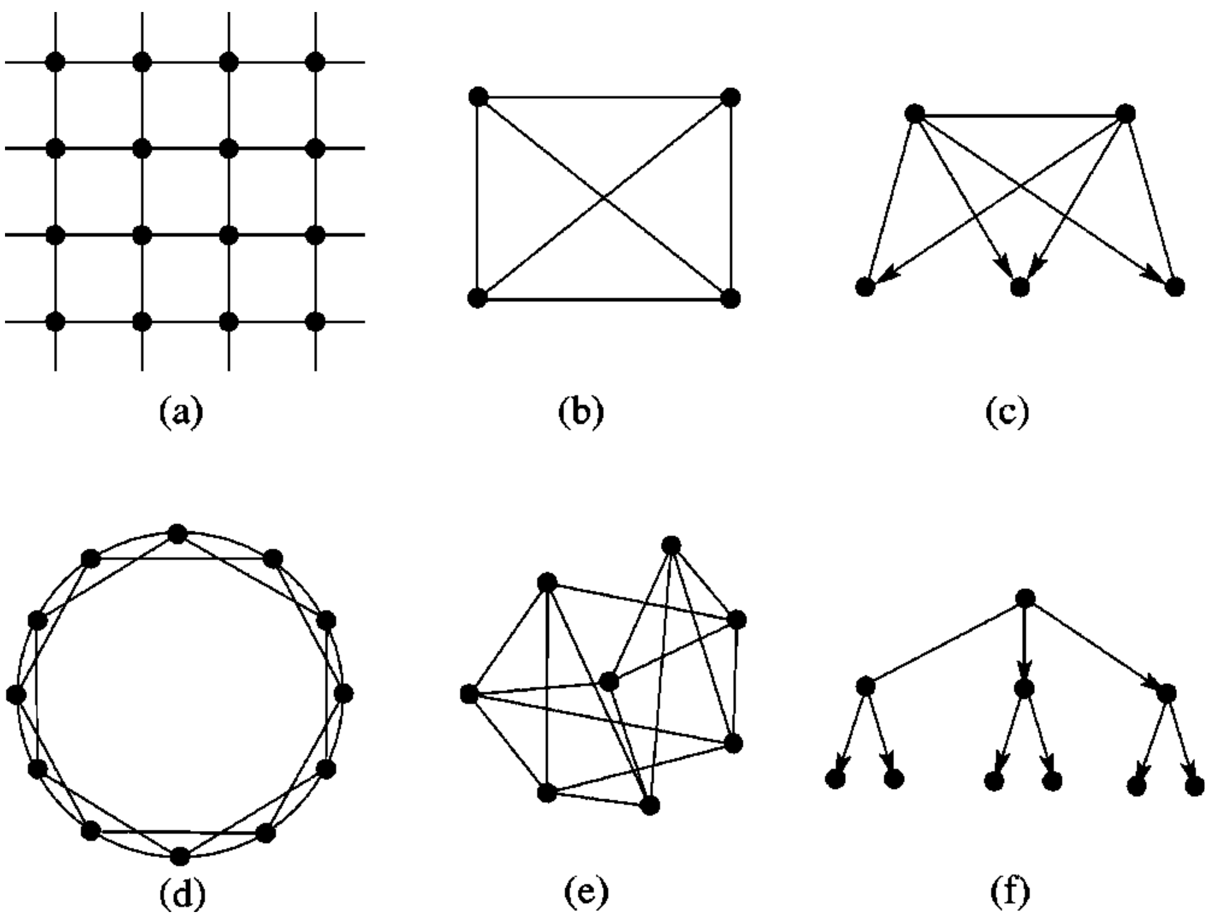

Figure. 1.1 Example of coupling topologies:

(a) Square grid with periodic boundary conditions

(b) Completely connected graph

(c) Directed graph where each oscillator receives signals from two others

(d) Ring with nearest neighbor and next to nearest neighbor coupling

(e) Randomly connected graph, and

(f) A tree in which the root node receives a signal from one of its siblings.

Arrows indicate direction of coupling along an edge; edges without arrows are coupled bi-directionally.

In this study, oscillators of the form given in equation (1.1), coupled all-to-all (oscillators are linked to other oscillators in all directions), were considered. This choice was taken because the resulting system of equations may be generalized to other types of topologies. The concept of time lag in all the variables involved in the coupling was considered. The n all-to-all coupled oscillators, with $(n \geq 2)$, can be described by the differential equation;

$$\dot{z}_j(t) = f(z_j(t), \mu) + g(z_i(t), z_j(t)) \tag{1.2}$$

where $z_i, z_j \in R^n$ are real state variables, $\mu \in R^p$ is a parameter, f is a non-linear smooth function from some open subset of $R^n \times R^p$ to R^n, and g is a non-linear or linear smooth function from $R^n \times R^n$ to R^n. Each system in (1.2) without g has an asymptotically stable periodic orbit that attracts the whole of R^2 except at the origin.

The system of Ordinary Differential Equations describing models of the example above require that cause and effect is simultaneous. That is, since only a single value of time occurs in the equation, the state of the system at time t, say, can only be influenced by its state at the same time t. Such systems are referred to as instantaneous models. However, in many physical situations the duration of transmission of an action cannot be ignored. In chemical reactors, which are diffusively coupled through mass transfer, a delay in mass transfer from one reactor to the other cannot be ignored. When the delay is considered, the coupling terms will depend not only on the present state of the system, but also on the previous one. In neural oscillators coupled at a synapse, it is possible that the coupling signal, which is either a chemical transfer or an electrical pulse or both between oscillators, may take some time before its effect is realized (Ermenrout *et al*, 1990).

Thus a more general formulation might allow events at one time to affect the state of the system at some other time, either at prior time($t-\tau$) or in future time($t+\tau$). ($\tau \in R_0^+$ is the delay and $R_0^+ = [0,\infty)$). Such generalization is provided for by use of Functional Differential Equations in place of familiar ordinary differential equations.

If we were to take into account the $\tau > 0$ in the transmission of variables between oscillators, equation (1.2) will take the form;

$$\dot{z}_j(t) = f(z_j(t),\mu) + g(z_i(t-\tau), z_j(t)), \; i, j = 1,2,....,n \tag{1.3}$$

This is an example of a functional differential equation called a delay differential equation (DDE).

1.1 BASIC THEORY OF DELAY DIFFERNTIAL EQUATIONS

After the First World War, the development and use of automatic control systems resulted in studies of an entirely different class of differential equations the so-called delay differential equations or difference differential equations (DDE). Any system involving a feedback control will almost certainly involve delays. A time delay arises because a finite time is required to sense information and the react to it. Stability problems, however, appear as soon as several mechanisms need to be controlled simultaneously.

Delay Differential Equations often arise in either natural or technological control problems. A controller monitors the state of the system, and makes adjustments to the system based on its observations. Since the adjustments can never be made instantaneously, a delay arises between the observation and the control action. For example a two- wheeled suitcase may begin to rock from side to side as it is pulled. When this happens, the person pulling it attempts to return it to the vertical position by applying a restoring moment to the handle. There is a delay in this response that can affect significantly the stability of the motion (Suherman *et al*., 1997).

There are different kinds of delay differential equations. We will focus on just one kind, that is;

$$\dot{z}(t) = f(z(t), z(t-\tau_1), z(t-\tau_2), \ldots\ldots, z(t-\tau_n)) \tag{1.4}$$

where the quantities $\tau_i (i = 1, 2, \ldots., n)$ are positive constants. In other words, we will focus on equations with fixed, discrete delays. There are other possibilities, notably equations with state dependent delays (the $\tau_i^{'}s$ depend on z) or with distributed delays (the right hand side of the differential equation is a weighted integral over past states).

When we give initial conditions for finite dimensional dynamical systems, we only need to specify a small set of numbers, namely the initial values of the state variables, and perhaps the initial time in non-autonomous systems. In order to solve delay equations we need more: at every time step, take into consideration the earlier value of z. We therefore need to specify an initial function which gives the behavior of the system prior to starting time (assuming that the starting time t = 0).

1.1.1 Properties of DDEs

A time independent solution of a DDE is not uniquely determined by its initial state at a given moment but, instead, the solution profile on an interval with length equal to the delay or time lag τ has to be given. That is, we need to define an infinite -dimensional set of initial conditions between $t = \tau$ and $t = 0$. Thus, DDEs are infinite-dimensional problems, even if we have only a single linear DDE.

Consider the initial value problem;

$$\frac{dz}{dt} = kz, z(0) = 1 \tag{1.5}$$

which can take the exponential solution;

$$z(t) = \exp(kt) \tag{1.6}$$

Physically the knowledge of the present (here: $z(0)=1$) allows us to predict the future at any time t. The past time is not involved in this solution. For a DDE, the past exerts its influence on the present and, hence, on the future. The following DDE;

$$\frac{dz}{dt} = kz(t-\tau), z(t)=1 \text{ when } -\tau \le t \le 0 \tag{1.7}$$

exhibits a right hand side that depends on z at time $(t-\tau)$ where τ is the delay or time lag. There are two important properties of this equation that need to be stressed;

1.1.2 Oscillatory Behavior

In contrast to the exponential solution (1.6), the solution of Eq. (1.7) can be oscillatory. This can be seen by seeking a particular solution of the form;

$$z = A\sin(\omega t) \tag{1.8}$$

Inserting (1.8) into Eq. (1.7), we find

$$\omega A\cos(\omega t) = kA\sin(\omega(t-\tau)) = kA[\sin(\omega t)\cos(\omega\tau) - \cos(\omega t)\sin(\omega\tau)] \tag{1.9}$$

Equating to zero the coefficients of $\cos(\omega t)$ and $\sin(\omega\tau)$, we get the following;

$$\cos(\omega\tau)=0 \text{ and } \omega = -k\sin(\omega\tau) \tag{1.10}$$

The first condition is satisfied if $\omega\tau = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ and with the second condition, we obtain the following possibilities;

(i) $\omega\tau = \frac{\pi}{2}$ and $k=-1$, (1.11)

(ii) $\omega\tau = \frac{3\pi}{2}$ and $k = -1$. (1.12)

For these particular values of k and τ the DDE (1.7) assumes the harmonic solution (1.8).

1.1.3 Short time solution

The second and most obvious difference between ODEs and DDEs is the initial data. The solution of an ODE is determined by its value at the initial point $t = a$. In evaluating the DDEs for $a \le t \le b$, a term like $y\,(t-\tau_j)$ may represent values of the solution at points prior to the initial point. For example, at $t = a$, we must have the solution at $(a - \tau_j)$. If T is the longest delay, the equations generally require the solution S (t) for $a - T \le t \le a$. For DDEs we must provide not just the value of the solution at the initial point, but also the "history", the solution at times prior to the initial point, that is, the solution $z_o(t)$ at times prior to the initial point. A long-time oscillatory solution is possible and that its initial history may have an effect on the short-time solution.

1.2 DIFFERENCES BETWEEN IVP FOR DDEs and ODEs.

In a system of ordinary differential equations

$$\dot{z}(t) = f(t, z(t)), \tag{1.13}$$

the derivative of the solution depend on the solution at the present time t. In a system of delay differential equation the derivative also depends on the solution at earlier times. Delay differential equations can take a general form;

$$\dot{z}(t) = f(t, z(t), z(t-\tau_1), z(t-\tau_2), \ldots\ldots\ldots\ldots\ldots\ldots z(t-\tau_k)) \qquad (1.14)$$

where the delay (lags) τ_j are positive constants

$$0 < \tau_1 < \tau_2 < \tau_3 < \tau_4 < \ldots\ldots\ldots\ldots\ldots < \tau_k$$

Delay differential equations arise in models throughout the sciences, but our solution will make it clear that they have been especially popular for biological models. Delay differential equations with constant delays are a large and important class. By restricting attention to problems with constant delays, it is possible to develop software that is more efficient and at the same time more provably reliable, than software available for more general problems. Methods used to solve ODE_s can generally to be extended to solve DDEs. In particular, the MATLAB DDE solver dde23 is based on the methods used in the MATLAB IVP solver ode 23. The user interface of dde23 is much like that of ode 23, yet owing the differences between DDE_s and ODE_s, it also resembles the MATLAB BVP solver bvp4c (type 4c of MATLAB BVP solver).

The numerical solution will be denoted by $S(t)$, so for $t \leq a$ was used to denote the given history. Since numerical methods for IVPs for both ODEs and DDEs are intended for problem with solutions that have several continuous derivations, discontinuities in low- order derivatives requires special attention. Such discontinuities are rare for ODEs, but they are almost always present for DDEs because the first derivative of the history function is almost always different from the first derivative of the solution at the initial point. That is, almost always;

$$\dot{z}(t) = \dot{s}(a-) \neq \dot{z}(a+) = f(a, s(a-\tau_1), s(a-\tau_2), \ldots\ldots\ldots, s(a-\tau_k))$$

There are other ways in which discontinuities in low – order derivatives commonly arise. Some problems have histories with discontinuities in low order derivatives. For instance in the solution of an immunology model due to Merchuk, one component of its history for $t \le 0$ is $\max(0, t+10^{-6})$, so there is discontinuity in the first derivative of this component at $t = 10^{-6}$. As with ODEs, a change in the model amounts to a restart and so introduces a discontinuity in the first derivative even when the solution is continuous through the change. This can happen at times known in advance or all times that must be determined by event location. Because they propagate, discontinuities are much more serious matter for DDEs than they are for ODEs. For a smooth function *f*, the equation (1.14) show that the smoothness of the derivative $\dot{z}$ at the current time *t* depends on the solution *z* at the past time $t - \tau_j$

Differentiating the equations show that the same is true for higher derivatives.

Consider the equation;

$$\dot{z}(t) = z(t-1) \tag{1.15}$$

Obviously we will have; $z^{(k+1)}(t) = z^{(k)}(t-1)$ for this equation. In general, if there is a discontinuity at time t^* of order *k*, meaning that $z^{(k)}$ has a jump at $t = t^*$ then as the variable *t* moves through $t^* + \tau_j$ there is a discontinuity in $z^{(k+1)}$ because of the term $z(t - \tau_j)$ in equation (1.14) with multiple delays, a discontinuity at the time t^* is propagated to the times;

$$t^* + \tau_1, t^* + \tau_2, t^* + \tau_3, \ldots\ldots\ldots\ldots\ldots t^* + \tau_k,$$

and each of these discontinuities is in turn propagated. If there is a discontinuity at t^* of order, *k*, then the discontinuity at each of the times $t^* + \tau_j$ is of at least order $(k+1)$, and so on. Because the effect of a delay appears in a derivative of higher order, the solution becomes smoother as the integration proceeds. This "smoothing" proves to be quite important to the numerical solution of DDEs. A method of steps is a technique for solving DDEs by reducing them to a sequence of ODEs. Using this technique, as an illustration, we can solve equation (1.15) with history $s(t)=1$ for $t \le 0$ on the interval $0 \le t \le 1$ the function $z(t-1)$ in (1.15) has known value $s(t-1)=1$ because $t-1 \le 0$. The DDE on this interval reduces to the ODE $\dot{z}(t)=1$ with initial value $z(0)=s(0)=1$. We solve this IVP to obtain $z(t)=(t+1)$ for $0 \le t \le 1$ .The solution of the DDE exhibits a typical discontinuity in its first derivative at $t=0$ because it is 0 to the left of the origin and 1 to the right. Now that we know the solution for $t \le 1$, we can reduce the DDE on the interval $1 \le t \le 2$ to an ODE $\dot{z}(t)=(t-1)+1=t$ with initial value $z(1)=2$ and solve this initial value problem to find that $z(t)=0.5t^2+1.5$ on this interval. The first derivative is continuous at $t=1$, but there is a discontinuity in the second derivative. The DDE's solution on the interval $[k,k+1]$ is a polynomial of degree $k+1$ and that the solution has a discontinuity at time $t=k$ of order $k+1$.

For the general equation (1.14), with the history function $S(t)$ defined for $t \le a$, the DDEs reduces to ODEs on the interval $[a, a+\tau]$ because, for each *j*, the argument $t-\tau_j \le t-\tau \le a$ and the $z(t-\tau_j)$ have the known values $s(t-\tau_j)$. Thus we have an

IVP for a system of ODEs with initial value $z(a) = s(a)$ .This problem is solved on $[a, a+\tau]$ and extend the definition of $s(t)$ to this interval by taking it to be the solution of this IVP. Now that the solution for $t \leq a+\tau$ is known, we can move on to the interval $[a+\tau, a+2\tau]$ and so forth. In this way we can solve the DDES on the whole interval of interest by solving a sequences of IVPs for ODEs. Although our concern is in problem with constant delays, the method of steps is clearly applicable to DDEs with delays that depend on both t and $z(t)$. The main requirement is simply that the delays all be bounded below by a constant $\tau > 0$.

1.3 NUMERICAL METHODS FOR DELAY DIFFERENTIAL EQUATIONS

The method of steps shows that we can solve DDEs with constant delays by solving a sequence of IVPs for ODEs. Because a lot is known about how to solve IVPs, this has been a popular approach to solving DDEs, both analytically and computationally. Solutions smooth out as the integration progresses, so if the shortest delay τ is small compared to the length of the interval of integration then there can be a good many IVPs, each of which may often be solved in just a few steps. In these circumstances, explicit Runge-Kutta methods are both effective and convenient. Because of this, most solvers are based on explicit Runge-Kutta methods; in particular, the MATLAB DDE solver dde23 is based on the BS (2,3) pair used by the ODE solver ode23. In what follow s consider how to make the approach practical and use dde23 to illustrate points.

In solving equation (1.15) for the interval $0 \le t \le 1$ the DDE reduces to an ODE with $z(t-1)$ equal to the given history $s(t-1)$ and $z(0)=1$. Solving this IVP with an explicit Runge-Kutta method is perfectly straightforward. A serious complication is revealed when we move to the next interval. The ODE on this interval depends on the solution in the previous interval. However, if we use a Runge-Kutta method in its classical form to compute this solution, we approximate the solution only on a mesh in the interval $[0,1]$. The first widely available DDE solver, DMRODE, (Naves.1975), used cubic Hermite interpolation to obtain the approximate solutions needed at other points in the interval. This approach is not entirely satisfactory because step sizes chosen for an accurate integration may be too large for accurate interpolation. What we need here is a continuous extension of the Runge—Kutta method.

The BS (2, 3) Runge—Kutta method used by the code ode23 was derived along with an accurate continuous extension that happens to be based on cubic Hermite interpolation. On reaching the current time *t*, we must be able to evaluate the approximate solution $S(t)$ as far back as the point $t-\tau^*$ (where τ^* is a large delay). This means that we must save the information necessary to evaluate the piecewise-polynomial function $S(t)$. The continuous extension of the BS (2, 3) pair is equivalent to cubic Hermite interpolation between mesh points, so it suffices to retain the mesh as well as the value and slope of the approximate solution at each mesh point. The code dde23 returns the solution as a structure that can have any name, but let us call it sol. The mesh is returned in the field sol.t. The solution and its slope at the mesh points are returned as sol.z and sol.p respectively. This form of output is an option for ode23, but it is the only form of output from dde23. Just as with the IVP solvers, the continuous extension is evaluated using the solution structure and the auxiliary function deval. It is often useful to be able to evaluate a solution anywhere in the interval of integration, but unlike the situation with the IVP solvers, here we need the capability in order to solve the problem. Representing the solution as a structure simplifies the user interface. We shall see that, in addition to the information needed for interpolation, we must have other information when solving DDEs — just what depends on the particular problem.

Holding this information as fields in a solution structure is both convenient and unobtrusive. All the early DDE solvers were written in versions of FORTRAN without dynamic storage allocation. This complicates the user interface greatly and there is a real possibility of allocating insufficient storage. The dynamic storage allocation of MATLAB and the use of structures allow a much simpler and more powerful user interface for dde23.

The DDE (1.15) leads to ODEs that are easy to integrate, so a code will try to use large step sizes for the sake of efficiency. Indeed, Runge—Kutta formulae are exact on the first interval, but a solver cannot be permitted to step past the point $t=1$ because the solution is not smooth there. If the discontinuity is ignored, the order of a Runge—Kutta method can be lowered. The numerical solution is then not as accurate as expected, but what is worse is that the error estimator is not valid. This is because the error is estimated by comparing the results of two formulae and neither has its usual order when the function f *is* not sufficiently smooth. We can deal with this difficulty by adjusting the step size so that all points where the solution $z(t)$ has a potential low-order discontinuity are mesh points. This implies that none of the functions $z(t), z(t-\tau_1), z(t-\tau_2), ..., z(t-\tau_k)$ can have a low-order discontinuity in the span of a step from t_n to $t_n + h$. Because we step to discontinuities of the solution $z(t)$, this is clear for $z(t)$ itself.

There cannot be a point ξ in $(t_n, t_n + h)$ where some function $z(\xi - \tau_j)$ is not smooth, because the discontinuity in $z(t)$ at the point $\xi - \tau_j$ would have propagated to the point ξ and we would have limited the step size h so that we did not step past this point. Runge-Kutta formulae are one-step Formulae and so, if we proceed in this way, they are applied to functions that are smooth in the span of a step and the formulae have the orders expected. As pointed out earlier, low-order discontinuities are a serious difficulty when solving DDEs because there is almost always one at the initial point and they propagate throughout the interval of integration. On the other hand, the order of a discontinuity increases each time it propagates forward, so we need to track discontinuities only as long as they affect the formula implemented. Before dde23 begins integrating, it locates all discontinuities of order low enough to affect the integration. It assumes that there will be a discontinuity in the first derivative at the initial point. Some problems have discontinuities at additional points known in advance. To inform dde23 of such derivative discontinuities, the points are provided as the value of the option Jumps. Options are set with the auxiliary function ddeset just as they are set with odeset for the

IVP solvers. For instance, the three discontinuities of the Hoppensteadt—Waltman model can be provided as;

```
C = 1/sqt(2);
Options = ddeset ('Jumps',[(1-c),(2-c)]);
```

No distinction is made between discontinuities in the history and in the rest of the integration, so the discontinuity of the Marchuk model is handled in the same way. Sometimes the initial value $z(a)$ has a value that is different from the value $s(a)$ of the history. This is handled by supplying $z(a)$ as the value of the option Initially. When there is a discontinuity in the solution itself at the initial point, we must track it to one level higher than usual.

Each of the initial discontinuities propagates to the points;

$$\xi + \tau_1, \xi + \tau_2, \xi + \tau_3, \ldots\ldots\ldots, \xi + \tau_k$$

where the order of the discontinuity is increased by 1. Each of the resulting discontinuities is, in turn, propagated in the same way. The locations of discontinuities form a tree that we can truncate when the order of the discontinuities is sufficiently high that they do not affect the performance of the formulae implemented. There is a practical difficulty in propagating discontinuities that is revealed by supposing that the DDE has the two lags $\frac{1}{3}$ and 1 and that the integration starts at $t = 0$. The first lag causes discontinuities to appear at the points $0, \frac{1}{3}, 2\times\frac{1}{3}, 3\times\frac{1}{3}, \ldots\ldots\ldots\ldots$ and the second causes discontinuities to appear at the points 0, 1, 2, 3….The difficulty is that the finite precision representation of $3 \times \frac{1}{3}$ is not quite equal to 1. It appears to the solver that there are two discontinuities that are extremely close together. This is catastrophic because the step size is limited by the distance between discontinuities. The solver `dde23` deals with this by regarding points that differ by no more than ten units of round off as being the same and purging one of them. This purging is done at each level of propagation in order to remove duplicates as early as possible.

The solution of a system of DDEs (1.14) becomes smoother as the integration progresses, which might lead us to expect a corresponding increase in step size. Certainly we must limit the step size so as not to step over a low-order discontinuity. The step size appears to be limited to the shortest delay, for if we were to step from t_n to $t_n + h$ with step size $h > \tau$ then at least one of the arguments $t - \tau_j$ would fall in the interval $(t_n, t_n + h)$. This means that we would need values of the solution at points in the span of the step, but we are trying to compute the solution there and don't yet know these values! Some solvers accept this restriction on the step size. Others, including `dde23` use whatever step size appears appropriate to the smoothness of the solution and iterate to evaluate the implicit formulae that arise in this way. On reaching t_n we have a piecewise-cubic polynomial approximation $s(t)$ to the solution for $t \le t_n$ When the BS (2, 3) formulae need values $z(t - \tau_j)$ for arguments $t - \tau_j > t_n$, these values are predicted by extrapolating the polynomial approximation of the preceding interval. After evaluating the formulae we have a new cubic polynomial approximation for the solution on the interval $(t_n, t_n + h)$ and we use it when correcting the solution by reevaluating the formulas. Evaluating the BS (2, 3) formula when the step size is larger than the shortest delay, is just like evaluating an implicit multistep method.

Earlier we mentioned the need for event location. This capability is available in dde23 exactly as in ode23 except that information about events is always returned as fields of the solution structure. When a terminal event is located, it is not unusual to continue the integration after modifying the equations and possibly modifying the final value of the solution for use as the initial value of the new integration. This is easy enough when solving ODEs with the IVP solvers of MATLAB because the solutions of the various IVPs can be aggregated easily to obtain a solution over the whole range of interest. The situation is quite different when solving DDEs. The most important difference is that a history must be supplied for the subsequent integration. This history is mainly the solution as computed up to the event, which is to be evaluated by interpolation, but it may also include the given history, which may be supplied in three different forms in dde23 and so be evaluated in different ways. The solver dde23 accepts a solution structure as history and uses information stored in this structure to evaluate properly the terms in the DDE that involve delays. Another issue is the propagation of discontinuities. It may be that the event occurs whilst some of the propagated discontinuities are still active. For the current integration we must reconstruct the tree of discontinuities and propagate them into the current interval of integration. This requires some information to be retained from the previous integration. There is, of course, a new discontinuity introduced at the new initial point. It is not unusual for the initial value of the solution to be different from the last solution value of the previous integration. This is handled with the Initialz option.

***Definition* 1.3.0**

Let $\tau > 0$ be a given real number, $R = (-\infty, \infty)$ R^n an n-dimensional linear vector space over the real field, and $C = \ell([-\tau, 0], R^n)$, where ℓ indicates continuous, is a Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^n. If Ω is an open subset of $R \times C$, $f \in \ell(\Omega, R^n)$ then the relation

$$\dot{z}(t) = f(t, z_t) \tag{1.16}$$

where the derivative at t denotes the right hand derivative, defines a Functional Differential Equation. This includes a special case namely;

$$\dot{z}(t) = f(t, z(t), z(t-\tau), \mu) \tag{1.17}$$

where $\mu \in R_0^+$ is some parameter. Equation (1.17) is an example of a parameterized autonomous delay differential equation.

The initial function would be a function $z(t)$ defined on the interval $[-\tau, 0]$. To understand the dynamics induced by this delay equation, one would think in the same terms as for ODEs, that is, the solution consists of a sequence of values of z at increasing values of t. However, from a purely theoretical perspective, this is not the best way to think of equations of this type. A much better way is to think of the solution of this DDE as a mapping from functions on the interval $[t-\tau, t]$ into the functions on the interval $[t, t+\tau]$. In other words, the solution of this dynamical system can be thought of as a sequence of functions $f_0(t), f_1(t), f_2(t),.......$ defined over a set of continuous time intervals of length τ.

***Definition* 1.3.1**

Let f in equation (1.17) above be such that $f \in \ell(0, R^n)$, 0 is an open subset of $C \times R$

(a) A continuous function $z:[-\tau,\infty]\times R \to R^n$ is called a solution of equation (1.17) if the following hold;

(i) $z(t,\mu)$ is continuous on $[-\tau,\infty)\times R$ and has a right hand derivative at $t=-\tau$

(ii) $z(t,\mu)$ is continuously differentiable for $t \geq 0$ and satisfies (1.17).

(b) Let $\varphi \in C$ be given, then a continuous function $z(t)=z(t,\mu,\varphi)$ is called a solution of the initial value problem;

$$\dot{z}(t) = f(z(t), z(t-\tau), \mu), \quad z(t)\big|_{[-\tau,0]} = \varphi(t) \tag{1.18}$$

if $z(t)$ satisfies equation (1.18) with

$$z(t)\big|_{[-\tau,0]} = \varphi(t). \tag{1.19}$$

So as to find the solution of (1.17), it is equivalent to solving the integral equation;

$$\left.\begin{aligned} z_0 &= \varphi(0) \\ z(t) &= \varphi(0) + \int_0^t f(z(s), z(s-\tau), \mu)ds, t \geq 0 \end{aligned}\right\} \tag{1.20}$$

Existence and uniqueness of the solution of equation (1.17) subject to condition (1.19) can easily be shown by the step-by-step method (Halaney, 1966).

Assume $\varphi_{k-1}(t)$ is defined for $t \in [(k-2)\tau,(k-1)\tau]$, $k=1,2,.......$ and form a system

$z(t) = f(z(t), \varphi_k(t-\tau), \mu)$ for $(k-1)\tau \leq t \leq k\tau$, $k=1,2,.......$

By virtue of continuity hypothesis, a solution of this system with initial condition;

$z_{k-1}(t) = \varphi_{k-1}(t) \qquad k = 1, 2, ...$, exists and is denoted by $z_k(t) = \varphi_k(t)$, where we have dropped the implicit indication of dependence of μ and φ for purposes of notation brevity.

A solution of (1.17) defined by the initial function $\varphi_0(t)$ will be given by the relation

$$z(t) = \begin{cases} \varphi_0(t), t \in [-\tau, 0] \\ z_{k-1}(t) + \int_{(k-1)\tau}^{k\tau} f(z(s), z(s-\tau), \mu) ds, t \in [(k-1)\tau, k\tau] \end{cases} \qquad k = 1, 2, ...$$

Clearly the function $z(t)$ constructed is continuous and differentiable in the interior points of the interval $[(k-1)\tau, k\tau]$, $k \geq 1$, At $t = 0$, $z(t)$ has only the right hand derivative, and in general if $f \in \ell^k$ (ℓ^k indicates k-times differentiable), $k = 1, 2, ...$, $z(t) \in \ell^j [j\tau, \infty]$ $\quad j = 1, 2, ..., k+1$.

1.4 OTHER DEFINITIONS AND TERMINOLOGIES

1.4.1 Invariant set

A set $\mathfrak{M}_1$ is said to be invariant under the flow defined by the equation;

$\dot{z}(t) = A_k z(t) + f(z(t))$

where $f(z(t)) = f_1(z_1), f_2(z_2), \ldots\ldots, f_n(z_n)$ and A_k is a real symmetric matrix depending upon a parameter $k = (k_1, k_2, k_3, \ldots\ldots, k_d) \in R^d$ and the state vector $z(t) \in R^{nN}$ with $k_j \geq 0$) if picking any initial point $z_0 \in \mathfrak{M}_1$, the solution $z(t, z_0) \in \mathfrak{M}_1$ for all $t \geq 0$.

1.4.2 Manifold

A manifold is a set which locally has a structure of the Euclidean space. They are *m*-dimensional surfaces embedded in $\mathbb{R}^n$, $m < n$. If the function $g(z)$ in the equation $\dot{z} = g(z_i)$, (where z_i is a state variable and g is some function), for instance, is describing a surface with maximal rank; that is the Jacobian of $g(z) \neq 0$, then by the implicit function theory, we can locally represent this surface as a graph. The surface is c^r manifold if the graph representing it is c^r

1.4.3 Flow

Let *E* be the open set of $\mathbb{R}^n$ and let $g \in c^1(E)$. For $z_0 \in E$, let $z(t; z_0)$ be the solution of the equation $\dot{z} = g(z_i)$ with initial condition $z(0) = z_0$ defined on its maximal interval of existence $I(z_0)$. Then for $t \in I(z_0)$, the set mappings $z(t)$ defined by $z(t) = z(t; z_0) = \Phi_t(z_0)$ is called the flow of the differential equation $\dot{z} = g(z_i)$. The evolution operator have the following properties that reflect the deterministic character behavior of the dynamical system $\dot{z} = g(z_i)$;

(i) $\Phi_0 = I_d$ where I_d is an identity map on E, with $I_d z = z$ for all $z \in E$; that is, the system does not change its state spontaneously.

(ii) $\Phi_{t+s}(z) = \Phi_t(z) + \Phi_s(z)$ for all $t, s \in I(z_0)$, $z \in E$; that is, the law governing the behavior of the system does not change with time.

(iii) $\Phi_{-t}(z_0)\Phi_t(z_0) = I_d$ for invertible maps.

(iv) $z(t)$ is also referred as the solution or flow of the vector field $g(z)$.

1.4.4 Inflowing Invariant Manifold

A manifold $\mathfrak{M}$ with boundary is inflowing invariant, under the flow of the vector field $g(z)$, if the vector field points strictly inwards on the boundary.

1.4.5 Overflowing Invariant Manifold

A manifold $\mathfrak{M}$ with boundary is overflowing invariant, under the flow of the vector field $g(z)$, if the vector field points strictly outwards on the boundary.

1.5 THE ROLE OF SYNCHRONIZATION

Synchronization occurs when oscillatory (or repetitive) systems via some kind of interaction adjust their behaviors relative to one another so as to attain a state where they work in unison. An essential aspect of many of the games played by children teaches them to coordinate their emotions. They skip and learn to jump in synchrony with the swinging rope.

One of the main problems in a swimming class is learning to breathe in synchrony with the strokes. Not necessarily one-to-one, as there are circumstances where it is advantageous to take two or more strokes per an inhalation. However, the phase relations must be correct if not to drown. In much the same way, a horse has different forms of motion (such as walk, trot, and gallop) and each of these gaits corresponds to a particular rhythm in the movement of the legs (Collins, J.J and Stewart, I., 1993; Strogatz, S.H. *et al*., 1993). At the trotting course, the jockey tries to keep the horse in trot to the highest possible speed. In its free motion, however, a horse is likely to choose the mode that is most comfortable to it (and, perhaps, least energy demanding). As the speed increases, the horse will make transitions from walk to trot and from trot to gallop.

Synchronization is a universal phenomenon in non-linear systems (Pikovsky, *et al*.,2001). Well-known examples are the synchronization of two (pendulum) clocks hanging on a wall, and the synchronization of the moon's rotation with the orbital motion so that the moon always turns the same side towards the earth. A radio receiver functions by synchronizing its internal oscillator with the period of the radio wave so that the difference, that is, the transmitted signal, can be detected and converted into sound. A microwave emitting diode is placed in cavity of a specific form and size so as to synchronize with a particular resonance frequency of the cavity.

Synchronization can also be observed between coupled laser systems and coupled biochemical reactors. At the assembly line one has to ensure an effective synchronization of the various processes for the production to proceed in an efficient manner, and engineers and scientists over and over again exploit the technique of modulating (or chopping) a test signal in order to benefit from the increased sensitivity of phase detection.

The history of synchronization dates back at least to Huygens' observations (Huygens, C. *et al* 1986). For regular (for example, limit cycle) oscillators, synchronization implies that the periodicities of the interacting systems precisely coincide and that differences in phase remain constant. In the presence of noise (or for chaotic systems) one can weaken the requirements such that the periodicities only have to coincide on average, and the phase differences are allowed to move within certain bounds. One may also accept occasional phase slips, provided that they do not occur too often (Stratonovich R.L.1983).

One-to-one synchronization is only a simple manifestation of a much more general phenomenon, also known as entrainment, mode locking or frequency locking. In non-linear systems, a periodic motion is usually accompanied by a series of harmonics at frequencies of p times the fundamental frequency, where p is an integer. When two non-linear oscillators interact, mode locking may occur whenever a harmonic frequency of one mode is close to a harmonic of the other. As a result, non-linear oscillators tend to lock to one another so that one subsystem completes precisely p cycles each time the other subsystem completes q cycles, with p and q as integers (Thompson, J. 1986; Glass, L. and Mackey, M.C. 1988). An early experience with this type of phenomenon is the way one excites a swing by forcing it at twice its characteristic frequency, that is, you move the body through two cycles of a bending and stretching mode for each swing. A similar phenomenon is utilized (in optics, electronics, etc.) in a wide range of so-called parametric devices.

Contrary to the conventional assumption of homeostasis, many physiological systems are unstable and operate in a pulsate or oscillatory mode (Glass, L. and Mackey, M.C. 1988 ; Leng, G.1988).

The beating of the heart, the respiratory cycle, the circadian rhythm, and the ovarian cycle are all examples of more or less regular self-sustained oscillations. The ventilatory signal is clearly visible in spectral analyses of the beat-to-beat variability of the heart signal, and in particular circumstances the two oscillators may lock together so that, for instance, the heart beats three or four times for each respiratory cycle (Schafer, C. *et al.*,1998). The jet lag experienced after a flight to a different time zone is related to the synchronization of the internal rhythm to the local day-and-night cycles, and it is often said that women can synchronize via specific scents (pheromones) if they live close together.

Rhythmic and pulsate signals are also encountered in intercellular communication (Goldbeter, A. (ed.). 1989). Besides neurons and muscle cells that communicate by trains of electric pulses, examples include the generation of cyclic pulses in slime mold cultures of *Dictostelium discoideum* (Goldbeter, A. and Wurster, B. 1989) and the newly discovered synchronization of the metabolic processes in suspensions of yeast cells (Dano, S. *et al*.,1999). Synchronization of the activity of the muscle cells in the heart is necessary for the cells to act in unison and produce a regular contraction. Similarly, groups of nerve cells must synchronize to produce the characteristic rhythms of the brain or to act as pacemakers for the glands of the hormonal systems (Kopell, N. *et al*.,2000). On the other hand, it is well known that synchronization of the electrical activity of large groups of cells in the brain plays an essential role in the development of epileptic seizures (Mormann, F. *et al.,* 2000). However, non-linear oscillators may also display more complicated forms of dynamics, and an interesting question that arises over and over again in the biological sciences concerns the collective behavior of a group of cells or functional units that each display strongly nonlinear phenomena (Kaneko, K.,1994).

In the economic realm, each individual production sector with its characteristic capital life time and inventory coverage parameters tends to exhibit an oscillatory response to changes in the external conditions. Overreaction, time delays, and reinforcing positive feedback mechanisms may cause the behavior to become destabilized and lead to complicated nonlinear dynamic phenomena. The sectors interact via the exchange of goods and services and via the competition for labor and other resources. A basic problem for the establishment of a dynamic macroeconomic theory is therefore to describe how the various interactions lead to a more or less complete entrainment of the sectors. Synchronization can be with or without delay.

1.5.1 Synchronization without Delay.

Consider n subsystems $z_j = z_j(t) \in R^n$, $(n \geq 2)$ $j = 1, 2, ..., n$ with the dynamics of z_j given by the solution of the Equation;

$$\dot{z}_j = f_j(z_j) \tag{1.21}$$

Let there be a compact global attractor for each j (Hale, J. 1997). That is, there is a compact set which is invariant under the flow defined by (1.21) and the ω-limit set of each orbit of (1.21) belong to this set.

Now, suppose that these sub-systems are coupled with linear terms to obtain the differential system for z as;

$$\dot{z}(t) = A_k z(t) + f(z(t)), \tag{1.22}$$

where $f(z(t)) = f_1(z_1), f_2(z_2), ..., f_n(z_n)$ and A_k is a real symmetric matrix depending upon a parameter $k = (k_1, k_2, k_3, ..., k_d) \in R^d$ and the state vector $z(t) \in R^n$ with $k \geq 0$.

Suppose also system (1.22) has a compact global attractor $\mathfrak{A}_k$ for each k, then we say that the system (1.22) is synchronized if the global attractor belongs to the diagonal in R^n defined as $z_1 = z_2 = ... = z_n$, (Fujisaka *et al.*, 1983). If z belongs to the attractor, it implies that the differences $z_j(t) - z_i(t) \to 0$ as $t \to \infty$ for all i, j. For the system to be synchronized , the diagonal must be an invariant set under the flow defined by Equation (1.22), which will be the case if all the f_j are the same (the dynamics of the sub-system are identical) and the matrix A_k has zero as an eigenvalue with the diagonal being the corresponding eigenspace. In this situation, all the other eigenvalues of A_k must be non-positive for the stability of the synchronous state.

For coupled identical systems, the diagonal of the system is invariant; synchronization is thus equivalent to the attracting property of the diagonal. The attractivity of the diagonal is determined by the Lyapunov exponents normal to the diagonal. If all the Lyapunov exponents normal to the diagonal are negative, then the coupled oscillators are synchronized (Enrico *et al.*, 2005).

1.5.2 Synchronization with Delay

Suppose the sub-systems in Equation (1.21) are coupled with linear terms to obtain the differential Equation for z_j in the form;

$$\dot{z}_j(t) = g(z_i(t-\tau), z_j(t)) + f_j(z_j(t)) \quad i \neq j \ , \ i, j = 1, 2, ..., n \tag{1.23}$$

where g is the coupling function, $\tau \geq 0$ is the time delay in the interaction. A synchronous state for the system is a solution of Equation (1.23) such that

$$z_j(t) = z_{j+1}(t) \, , \quad 1 \leq j \leq n-1 \tag{1.24}$$

for all t. This state lies on the synchronized manifold defined as;

$$\Theta_1 = \{z \in R^n : z_1 = z_2 = ... = z_n \neq 0\} \, .$$

Given asymmetric initial conditions, we say that the system synchronizes if Equation (1.24) holds asymptotically. It follows from the definition that, for a synchronous state, we have;

$$\lim_{t \to \infty}(z_j - z_{j+1}) = 0 \, , \qquad j = 1, 2, ..., (n-1)$$

More generally, let $C = \ell([-\tau, 0], R^n)$, be a set of continuous functions defined from $[-\tau, 0]$ to R^n. Let the norm of the element $\phi \in C$ be defined as $\|\phi\| = \sup\{|\phi(\theta)| : -\tau < \theta < 0\}$, where $|\phi(\theta)|$ denotes the Euclidean norm of $\phi(\theta)$. The set

C equipped with this norm is a Banach space. Let $z = (z_1, z_2, ...z_n)$ and take the space of initial function for Equation (1.23) to be in C. For any $\varphi \in C$, Eqn. (1.23) has a unique solution $z(t, \varphi)$. If $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)$ with $\varphi_1 = \varphi_2 = \varphi_3 = ... = \varphi_n$ an initial manifold (Hale, J. 1997), then the solutions of (1.23) will be synchronized if $(z_j(t) - z_{j+1}) \to 0$ as $t \to \infty$, $1 \le j \le n-1$

1.6 LITERATURE REVIEW

Coupled oscillators have been studied extensively. The emphasis in many of the studies has been on coupling of oscillators that have a stable limit cycle. The nature and type of coupling have played a significant role in determining the regime of the dynamical behavior of the population of oscillators. Most types of coupling have led to problems which are easy to treat both numerically and analytically. Coupling with time lag seems to have received little attention both numerically and analytically.

Perkel *et al., 1964,* measured the phase resetting function $F(\phi)$, in invertebrate neurons that act as pacemakers that they spontaneously and repetitively fire action potentials in a periodic manner. They induced precisely timed inhibitory or excitatory synaptic potentials in the pacemaker cell by evoking an action potential in an excitatory or inhibitory presynaptic neuron. The function $F(\phi)$ tabulates the change in cycle length as a function of the phase at which the perturbation was applied. This function is often normalized by the intrinsic unperturbed period. The opposite sign convention is used rather than the one used in that article in order to be consistent with the literature on this subject.

Peskin., *1975,* considered synchronization in a network of two mutually excitatory pulse-coupled leaky integrate and fire neural oscillators.

$$\frac{dv}{dt} = S_0 - \gamma V(t).$$

The pulsatile coupling resulting from the firing of one oscillator caused the voltage of the second oscillator to instantaneously become more depolarized by a fixed amount.

This implicitly defines a phase resetting curve because if a perturbation is received at time t, the next spike will be advanced by an amount that can be calculated using the explicit solution for the differential equation that describes the leaky integrate and fire oscillators,

$$F(\phi) = \frac{\ln(1-\gamma S_0 e^{A\phi})}{A}, \text{where } A = \ln\frac{S_0}{S_0-\gamma} \text{ with } S_0 \neq \gamma \text{ and } S_0 \neq 0.$$

He derived a return map for the phase of the non-firing oscillator immediately after its partner fired, with the oscillators reversing roles on each map iteration. This mapping was proved to have a unique and unstable fixed point. This fixed point repelled trajectories toward synchronization at a phase of zero or one. Once the oscillators synchronize, the coupling term drops out because each neuron is already at threshold when its partner fires, hence synchronization results for every set of initial conditions.

Mirollo *et al.*,*1990,* generalized these results to any two coupled oscillators in which the state variable (that is, the membrane potential V) is a smooth monotonically increasing and concave down function, as it is for the leaky integrator. The pulse coupling when one oscillator fired depolarized the other by an amount ε or pulled it up to the firing threshold, whichever was less. A return map was again formulated in terms of the phase of the non-firing oscillator immediately after its partner fired. This mapping was proven to have a unique fixed point which was unstable. Once again the firing is globally repelled toward synchrony where the coupling term disappears allowing the synchronized state to exist.

The work presented so far on two identical coupled oscillators focused exclusively on excitation that leads to synchrony, that is, a phase locking with no phase difference between the oscillators. For identical oscillators, in which the coupling term drops out at a phase of 0 and 1, synchrony is always a solution. However, other solutions are possible, and for systems in which the phase resetting does not disappear at 0 and 1 synchrony may not be a solution. Therefore this criterion is required for both existence and stability. The above derivation did not apply to exact synchrony because the assumed firing order could not be guaranteed for a small perturbation from synchrony.
Gransen *et al.*, 1979, studied the system of Equations of the form;

$$\left.\begin{aligned} \dot{U}_i &= \frac{(V_1 - f(U_1))}{\varepsilon} \\ \dot{V}_1 &= -(1-\delta q_1)U_1 + \delta U_2(t-\tau) \\ \dot{U}_2 &= \frac{V_2 - f(U_2)}{\varepsilon} \\ \dot{V}_2 &= -(1-\delta q_2)U_2 + \delta U_1(t-\tau) \end{aligned}\right\} \qquad (1.25)$$

where U_i, V_i, $i = 1,2$ are real state variables, $f(x) = \frac{x^3}{3} - x$ is a real valued smooth function of the real state variable, δ is a very small positive parameters , q_i, $i = 1,2$ are arbitrary constants representing the difference of the autonomous periods of the oscillators, $\tau \geq 0$ is a real constant representing a delay in the coupling.

Synchronization was investigated by means of a phase shift $z(\mu)$ that is T_0 periodic (T_0 being the period of a single oscillator) function of the phase difference μ of the oscillators. The following results were obtained;

(i) For $\tau = 0$, when $z^{'}(0^{+}) + z^{'}(0^{-}) < 0$. Two oscillators with equal free period have a stable synchronized solution with equal phases. The synchronized solution has a period greater than the free period when $z(0) < 0$. The synchronized solution has a period less than the free period when $z(0) > 0$.

(ii) With $\tau > 0$ but very small and identical oscillators, the stable synchronized stable $\mu = 0$ splits in two stable state $\mu = \pm\mu\tau$, with $\mu\tau \to 0$ as $\tau \to 0$.

(iii) With a small $\tau > 0$ and for oscillators with unequal free periods, synchronization only occurs if the difference of the free periods is within certain bounds.

Only a small delay in one variable was considered. The dynamics of the system in Equation (1.25) when coupling involves all the variables was not addressed.

Fujisaka *et al*., 1983, developed the general stability theory of synchronized motions of coupled oscillator systems with the use of extended Lyapunov matrix approach. They gave an explicit formula for a stability parameter of the synchronized state by considering the following system;

$$\dot{x}_j(t) = f(x_j(t)) + (\frac{\hat{D}}{2})\sum_i conf(x_i(t) - x_j(t)), \quad j = 1, 2, ..., n \qquad (1.26)$$

where $x_j(t)$ denotes the state vector of the j^{th} oscillator , $\hat{D}$ certain coupling constant, $\sum_i conf$ the summation over the given configuration of coupling. Three systems of configurations were considered;

(i) All oscillators coupled;

$$\sum_i conf\,(x_i - x_j) = \sum_{i=1}^{nN} (x_i - x_j)\,,$$

(ii) Only nearest neighbor oscillators coupled under periodic boundary conditions;

$$\sum_i conf\,(x_i - x_j) = x_{(j-1)} + x_{(j+1)} - 2x_j\,,\ x_n(t) = x_1(t) \qquad (1.27)$$

(iii) Only nearest neighbor oscillators coupled with Neumann boundary conditions;

$$\sum_i conf\,(x_i - x_j) = x_{(j-1)} + x_{(j+1)} - 2x_j\,,\ j = 2, ..., n-1 \quad (1.28)$$

The eigenvalues $\{c_j\}$ of the coupling matrix $\hat{A} = A_{ji}$ defined through $\sum_i conf\,(u_i - u_j) = \sum_i A_{ji} u_i$ where u_j is the deviation from the synchronized state, were determined as;

System I : $0, n, n, ..., n$ $\quad n \geq 2$

System II : $0, 4\sin^2(\frac{k\pi}{n})k = 1, 2, ..., n-1$ $\quad n \geq 3$

System III : $0, 4\sin^2(\frac{k\pi}{n})k = 1, 2, ..., n-1$ $\quad n \geq 2$ $\qquad (1.29)$

The stability of the synchronized state was studied by defining $\Lambda_j = \lim_{t\to\infty} t^{-1} \ln\{\hat{u}_j(\hat{D},t)\}$

where;

$$\hat{u}_j(\hat{D},t) = \exp\left\{\int_0^t \left\{\hat{G}(s) - \frac{c_j\hat{D}}{2}\right\} ds\right\},$$

$$G(t) = [G_{ij}(t)\Big|^n_{i,j=1}] = [\frac{\partial f_i}{\partial x_j(t)}],$$

where $u_j(t)$ is the deviation from the synchronized state and $\hat{u}(t)$ is a matrix satisfying;

$$\dot{\hat{u}}(t) = \hat{G}(t)\hat{u}(t), \qquad \hat{u}(0) = 1.$$

The largest eigenvalue (Lyapunov exponent) is given by;

$$\lambda_L = \max(\mathrm{Re}\,\lambda_j),$$

where λ_j are eigenvalues of Λ_j. The synchronized state is stable or unstable accordingly as;

$$\lambda_L \le 0 \ \text{(Or } \lambda_L > 0\text{).}$$

In this analysis finer points such as the effect of time lag in the stability of the synchronous state were left out.

Hale,1997, studied the dynamics of the system of the form;

$$\dot{z}_j = g(z_j) \tag{1.30}$$

where $z = (z_1, z_2, ..., z_n)^T$, $z_j \in R^d$, $1 \le j \le n$, $g \in \ell^r(R^d, R^d)$, $r \ge 1$.

For any initial condition in R^d, solution of Equation (1.30) has a compact global attractor. The system of Equation (1.30) was coupled to take the form;

$$\dot{z} = k\Delta z + f(z) \tag{1.31}$$

with $f(z) = (g(z_1), g(z_2), \ldots\ldots, g(z_n))$, k is a positive constant representing the coupling strength and the matrix Δ is given by;

$$\Delta = \begin{pmatrix} -I & I & 0 & 0 & . & . & . & 0 & 0 \\ 0 & I & -2I & I & . & . & . & 0 & 0 \\ . & . & . & . & . & & & . & . \\ . & . & . & . & & . & & . & . \\ . & . & . & .. & & & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & -2I & I \\ 0 & 0 & 0 & 0 & . & . & . & I & -I \end{pmatrix}$$

with $I = I_d$ The following results were obtained;

(i) For each $k > 0$ there is a global attractor $\mathfrak{A}_k$ which is uniformly bounded in k.

(ii) There is $k_0 \geq 0$ such that system (1.31) is synchronized for $k \geq k_0$.

The nearest neighbor scalar coupling was considered and no effort was made to address the dynamics of system (1.31) when time lag is involved in the coupling.

Pyragas, 1998 studied two directly coupled Mackay Glass equations of the form;

$$\left.\begin{aligned} \dot{x} &= f(x_\tau) - c \\ \dot{y} &= f(y_\tau) - c + k(x - y) \end{aligned}\right\} \tag{1.32}$$

where $f(x_\tau) = \dfrac{ax_\tau}{1 + x_\tau^{\ b}}$, $x_\tau = x(t - \tau)$, $y_\tau = y(t - \tau)$, a, b and c are parameters to be chosen. The term $k(x - y)$ represents a dissipative coupling, with k as the coupling strength. Analytic expression relating synchronization threshold to the maximum Lyapunov exponent of uncoupled driving and response subsystems was derived as

$$\lambda(k) = \lim_{t\to\infty} \frac{1}{t} \ln \left(\frac{\int_{-\tau}^{0} \Delta^2(t+\theta)d\theta}{\int_{-\tau}^{0} \Delta^2(\theta)d\theta} \right)^{\frac{1}{2}}$$

where $\Delta = y - x$ are small deviations from the synchronized manifold. $\Delta\theta$ is the initial solution on the interval $[-\tau, 0]$ and $\Delta(t+\theta)$ is the initial solution on the interval $[t-\tau, t]$. The analytical results were then compared with the numerical solutions for the system, by setting the parameters a, b and c fixed at $a = 0.2$, $b = 10$, $c = 0.1$. The following results were obtained;

(i) For $0.471 < \tau < 1.33$ there is a stable limit cycle.

(ii) A period double bifurcation sequence was observed for $1.33 < \tau < 1.68$.

(iii) For $\tau > 1.68$, numerical solutions show chaotic attractors at most parameter values.

Though the analysis was thorough numerically, only driving response subsystems were considered. The dynamics of two or more oscillators coupled all-to-all was not considered.

Wasike, A. 2003, studied oscillators whose dynamics were governed by the solution to the Equation;

$$\dot{z}_j = Az_j - z_j \left| z_j \right|^2 = g(z_j) \tag{1.33}$$

where $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $z_j = z_j(t) \in R^2$, $j = 1, 2, ..., n$.

The oscillations in Equation (1.33) were coupled in one, two, and three dimensional Bravis Lattice structure. On one dimensional lattice, the governing Equation was;

$$\dot{z} = B(k)z + f(z) \tag{1.34}$$

where $z = (z_1, z_2, ..., z_n)^T$ denotes the coordinates of a point on the lattice, $B(k)$ is a real symmetric matrix depending on the coupling strength $k \geq 0$ and is given by $B(k) = k\Delta_1 \otimes L$ with;

$$\Delta_1 = \begin{pmatrix} -1 & 1 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & & & . & . & . \\ . & . & . & . & & . & & . & . & . \\ . & . & . & . & & & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 1 & -1 \end{pmatrix}.$$

and $L = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

He analyzed Equation (1.33) considering the coordinate transformation; $w_j = z_j - z_{j+1}$, $1 \leq j \leq n$, $y = \frac{1}{n}\sum_{j=1}^{n} z_j$. The following results were found;

- There is $k_o > 0$ such that for n even, equation (1.33) has an anti-symmetric solution for all $0 < k < k_o$ and there is oscillator death for $k > k_o$, where the invariant sub-space $\Pi = \left\{ z \in R^{2n} : z_{j+1} = (-1)^j z_j \neq 0 \right\}$, $1 \leq j \leq n-1$ defines the anti-symmetric synchronization.
- Symmetric synchronization persists for all coupling strength. The invariant subspace $\Theta_1 = \left\{ z \in R^{2n} : z_1 = z_2 = ... = z_n \neq 0 \right\}$ defines symmetric synchronization.

In his study, nearest neighbor coupling was considered. However, in this case, the dynamics of the network of oscillators coupled together with the effect of time lag was not considered.

Enrico, R. *et al.*, 2005, studied the synchronization dynamics of a system of two identical Hodgkin-Huxley (HH) neurons coupled diffusively. The model equation they studied was of the form;

$$\dot{V}_i = I_{ion}(V_i, S_i) + I_{ext} + \varepsilon[V_j(t-\tau) - V_i], i \neq j,\ i, j = 1, 2 \qquad (1.35)$$

where ε is the coupling strength, $\tau \geq 0$ is the delay in the interaction, V is the membrane potential and $I_{ion} = -gNam^3h(V - V_{Na})dt - g_k n^4(V - V_k)dt - g_L(V - V_L)$ is the total ionic current, I_{ext} is an externally applied current which is assumed to be constant, and $S_i = (m_i, h_i, n_i)$ is the state variable. The stability of the synchronous state of equation (1.35) was studied by defining $z_i = (v_i, m_i, h_i, n_i)$ and $\varepsilon_1 = (\varepsilon_1, 0, 0, 0)$ so that the Equation (1.35) is written as;

$$\dot{z}_i(t) = F(z_i(t)) + \varepsilon_1[z_j(t-\tau) - z_i(t)] \quad i, j = 1,2 \ , \ i \neq j \tag{1.36}$$

where $\varepsilon_1 > 0$ is the coupling strength. The synchronous state z *(t)* is the solution of;

$$\dot{z}(t) = F(z(t)) + \varepsilon_1[z(t-\tau) - z(t)]$$

By defining $z_\perp = z_2 - z_1$ as the transverse vector to the synchronous state, and linearising equation (1.35) around the synchronous state, they obtained;

$$\dot{z}_\perp(t) = J(z(t)).z_\perp(t) - \varepsilon_1.(z_\perp(t-\tau) + z_\perp(t)) \tag{1.37}$$

where the matrix $J = DF(.)$ is the Jacobian of F. The stability of the synchronous state is thus related to the Lyapunov exponents associated with Eqn. (1.37). The synchronous state is stable if all the Lyapunov exponents are negative. They did not provide an analytical analysis for the stability of the synchronized state or range of values of τ for stability of synchronous state. The oscillators considered were only two.

I was motivated to carry out this research out of the Literature review cited. This is because much seems not to have been done on synchronization of n all-to-all coupled oscillators. That is why we studied a population of n oscillators each with an asymptotically stable limit cycle coupled all-to-all by a linear diffusive like path with a time lag τ. A limit cycle is asymptotically stable if any trajectory with initial value nearby of the limit cycle tends to the limit if the time goes to infinity. Each of the oscillators was considered to have its own intrinsic natural frequency and each is coupled equally to all other oscillators

The system of equations was inbuilt with symmetries which we were exploited to get an analytic understanding of the dynamics of the system. The symmetries in the coupling terms were studied to establish the range of time delay τ for stability of synchronized state, and then the trajectories of the two manifolds were computed.

Since models represent an approximation of the real phenomenon, perturbation is inevitable and thus we studied the ability of the synchronized manifold to be insensitive to some small perturbations.

1.7 STATEMENT OF THE PROBLEM

Oscillators described by ordinary differential equations were considered. To obtain an asymptotically stable periodic solution for each oscillator, an appropriate system of ODEs which must be necessarily non linear and have at least two state variables should be considered. These oscillators were described by a differential equation of the form:

$$\dot{z}\ (t)\ = f\ (z\ (t), \mu) \tag{1.38}$$

where $z \in \mathbf{R}^n$ is an n-dimensional real state variable, $\mu \in \mathbf{R}^p$ is a p-dimensional real parameter, n and p are natural numbers such that, $n \geq 2$ and $p \geq 1$, f in a non linear smooth function from

$\mathbf{R}^n \times \mathbf{R}^p \rightarrow \mathbf{R}^n$

The dynamical behavior of oscillators with a time lag in all variables involved in the coupling and oscillators coupled all-to-all i.e.,each oscillator coupled to all the others, was studied by considering n-oscillators each with an asymptotically stable limit cycle, coupled all-to-all by a linear diffusive path with time lag τ . Oscillators with d state variables, with $d \geq 2$ were considered so as to study dissipative oscillators coupled to their nearest neighbor by a diffusive like path via a linear and planar simple Bravais lattices. Each oscillator was described by a system of ODEs whose behavior is governed by the solution of the equation;

$$\dot{z}_j = g\left(z_j\right) \tag{1.39}$$

where g is a function taking $\mathbf{R}^2$ x $\mathbf{R} \rightarrow \mathbf{R}^2$ and is continuous with $z_j = z_j(t) \in \mathbf{R}^2$

Suppose that equation (1.36) is coupled, that is;

$$\left.\begin{aligned}
\dot{z}_1(t) &= kI\left[-(n-1)z_1 + \sum_{i=2}^{n} z_i(t-\tau)\right] + g\left(z_1(t)\right), \\
\dot{z}_2(t) &= kI\left[z_1(t-\tau) - (n-1)z_2(t) + \sum_{i=3}^{n} z_i(t-\tau)\right] + g\left(z_2(t)\right), \\
&\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
\dot{z}_n(t) &= kI\left[\sum_{i=1}^{n-1} z_i(t-\tau) - (n-1)z_n(t)\right] + g\left(z_n(t)\right)
\end{aligned}\right\} \tag{1.40}$$

where k is the coupling strength and $\tau > 0$ is delay.

Coupled oscillators dynamical behavior can generally be described by the following equation:

$$\dot{z}(t) = kB(z(t), z(t-\tau)) + f(z(t)), \tag{1.41}$$

where $k > 0$ is the coupling strength, $z(t) = (z_1(t),..,z_n(t))$, $kB(z(t), z(t-\tau))$ is a linear operator describing the coupling of different configurations, $f(z(t)) = (g(z_1(t),.................g(z_n(t)))$ and $\tau > 0$ is the time lag.

For situations where there is no time lag, Eq. (1.41), reduces to;

$$\dot{z}(t) = kB(z(t)) + f(z(t)), \tag{1.42}$$

where in this case $B(z(t))$ describes the diffusive like coupling configuration and $f(z(t))$ is the function describing the behavior of the site oscillator.

From the literature review, we notice that the researchers have considered the term $kB(z(t), z(t-\tau))$ for the following cases:

(i) For two oscillators with $\tau > 0$ and only one variable involved in the coupling,

(ii) For a large and finite number of oscillators without delay and all variables involved in the coupling.

Dynamical behavior of oscillators with $kB(z(t), z(t-\tau))$ involving a delay seems to have received little or no attention in research. We wish to study this particular scenario. Using Eq. (1.39) , we can choose $g\left(z_j(t)\right)$ of the form;

$$g\left(z_j(t)\right) = \begin{pmatrix} f_1(x, y) \\ \\ f_2(x, y) \end{pmatrix}$$

where;

$$f_1(x, y) = y - \alpha\left(\frac{x^3}{3} - x\right) \qquad \alpha > 0 \tag{1.43}$$

$$f_2(x, y) = -x$$

The choice of f_i was as equation (1.43) because in most diffusive path-like problems, for instance Chemistry where coupling is effected by the flow of reactants from one reactor to the other, the flow of various reactants through the connecting medium of the coupled oscillators is the same. Thus a time delay is required for x_i to be transferred from oscillator *i* to oscillator *j* and vice versa.

The coupled Eq. (1.42) can be written compactly as;

$$\dot{z}_j(t) = kI\left[-(n-1)z_j(t) + \sum_{i=1, i\neq j}^{n} z_i(t-\tau)\right] + g\left(z_j(t)\right) \tag{1.44}$$

Equation (1.44) describes oscillators coupled all to all by a linear diffusive terms. Equations (1.41), (1.42) and (1.44) will be studied with an aim of meeting the following objectives;

1.8 OBJECTIVES OF THE STUDY

We aimed at achieving the following objectives in our study;

- Investigating the dynamical behavior of n oscillators coupled to one another with a time lag τ in all variables involved in the coupling,
- Investigating, both analytically and numerically, the effects of the time lag in the stability of the synchronized manifold,
- Establishing the nature of synchronous frequency by considering very small coupling strength and when the coupling strength is increased and
- Studying the stability of synchronization manifold under perturbation and give the conditions under which the synchronization manifold is said to be robust.

1.9 SIGNIFICANCE OF THE STUDY

The nature and type of coupling have played a significant role in determining the regime of the dynamical behavior of a population of oscillators. Most types of coupling have led to problems which are easier to treat both numerically and analytically. Coupling with time lag appears to have received little attention .The above study can get its application in; Neurobiology, Laser arrays, Microwave devices, Communications satellites and electronic circuits. For instance in the case of microwave devices; When two microwave oscillators operate physically close to one another, the output signal of each may affect the behavior of the other. Since the frequencies are above the 10 GHz range, the time for light to travel from one oscillator to the other, a distance of the order of centimeters, represents a substantial portion of the period of the uncoupled microwave oscillator. This immediately leads to the inclusion of delay effects in the coupling terms.

Advances in understanding of non-linear dynamical systems have led to the interest in developing practical applications for chaotic dynamics in communications systems. Understanding semiconductor laser dynamics has been simulated by the possibilities of achieving fast secure communication systems which exploit the properties of chaos synchronization. Chaotic communications may, in particular, be effected by mixing the message signal with the output from a chaotic transmitter and then recovering the message from the received signal.

The possibility of encoding messages within a chaotic carrier was first proposed in electronic circuits (Cuomo, K.M. *et al*., 2008) and later in solid state lasers (Colet,P.*et al*., 1993), in semiconductor lasers (Mirasso,C. R. *et al*.,2005) The chaotic output of a transmitter system is used as a carrier in which a message is encoded. The amplitude of the message is much smaller than the typical fluctuations of the chaotic carrier, so that is very difficult to isolate the message from the chaotic carrier. Decoding is based on the fact that coupled chaotic systems are able to synchronize their output if the appropriate conditions are given. To decode the message, the transmitter signal is coupled to another chaotic system, the receiver, which is similar to the transmitter. The receiver synchronizes with the chaotic carrier itself, so that the message can be recovered by subtracting the input and output of the receiver.

Apart from electronic circuits, lasers remain the only physical systems for which laboratory experiments have demonstrated chaotic synchronization. Immediately after chaotic laser synchronization was demonstrated, efforts began to apply this new phenomenon to optical communications. A strong motivation was the possible “security” of such communication methods. A couple of years ago, the first experiments were performed to illustrate the concepts of communication using amplitude and frequency fluctuations of chaotic optical systems. The use of irregular chaotic waveforms and synchronization for communication is a generalization of century-old techniques for radio communication. Message recovery through synchronization may also be considered a variation of auto-correlation measurements in optics that recover a sequence of time-dependent perturbations made on a physical system, constituting the message. It offers the possibilities of increased privacy, recovery of distorted signals, multiplexing that will be realized in near future!

CHAPER TWO

2.0 LYAPUNOV EXPONENTS

In this chapter, a method due to Lyapunov, which is very useful in establishing the stability of a non-hyperbolic equilibrium point, is generally presented. It is the one used in the analysis of the analytical solution in chapter three.

2.1 MAXIMUM LYAPUNOV EXPONENT

Chaotic systems display a sensitive dependence on initial conditions. Such a property deeply affects the time evolution of trajectories starting from infinitesimally close initial conditions, and Lyapunov exponents are a measure of this dependence. These characteristic exponents give a coordinate independence measure of the local stability properties of a trajectory. If the trajectory evolves in an N-dimensional state space, there are N exponents arranged in decreasing order, referred to as the spectrum of Lyapunov exponents;

$$\lambda_1 \lambda_2 \lambda_3 ....\lambda_n \qquad (2.1)$$

Conceptually these exponents are a generalizations of eigenvalues used to characterize different types of equilibrium points.

A trajectory is chaotic if there is at least one positive exponent, the value of this exponent, said to be the maximum Lyapunov exponent, give a measure of the divergence rate of infinitesimally close trajectories and of the unpredictability of the system and gives a good characterization of the underlying dynamics (Walker, J. A. 1980).

For a dynamical system with evolution equation f^t (An equation that can be interpreted as the differential law of the development (evolution) in time of a system.) in n-dimensional phase space, the spectrum of Lyapunov exponents describe the behavior of vectors in the tangent space of the phase space and are defined from the Jacobian matrix;

$$J^t(x_o) = \left.\frac{df^t}{dx}\right|_{x_o} \tag{2.2}$$

The f^t matrix describes how small change at the point x_o propagates to the final point $f^t(x_o)$. Let the limit, $\lim_{t\to\infty}(J^t.(J^t)^T)^{\frac{1}{2}} = L(x_o)$ (2.3)

where $(J^t)^T$ is the transpose of the matrix (J^t), defines a matrix $L(x_o)$.

If $\Lambda_i(x_o)$ are the eigenvalues of $L(x_o)$, then the Lyapunov exponents λ_i are defined by;

$$\lambda_i(x_o) = \log \Lambda_i(x_o) \tag{2.4}$$

The set of Lyapunov exponents will be the same for almost all starting points of an ergodic component of the Dynamical system (Perko. L. 1991).

If a system is conservative (i.e., if there is no dissipation), a volume element of the phase space will stay the same along a trajectory. Thus the sum of all Lyapunov exponents must be zero. If the system is dissipative, the sum of the exponents is negative.

If the system is a flow, one exponent is always zero-the Lyapunov exponent corresponds to the eigenvalue of $L(x_o)$ with an eigenvector in the direction of the flow.

2.2 SIGNIFICANCE OF LYAPUNOV SPECTRUM

The Lyapunov spectrum can be used to give an estimate of the rate of entropy production and of the fractal dimension of the considered dynamical system. In particular, from the knowledge of the Lyapunov spectrum, it is possible to obtain the so called Kaplan-Yorke dimension D_{KY} that is defined by;

$$D_{KY} = k + \sum_{i=1}^{k} \frac{\lambda_i}{|\lambda_{k+1}|} \tag{2.5}$$

where k is the maximum integer such that the sum of the k-largest exponent is still non-negative (Grassberger, P. *et al.*,1983). The Kaplan-Yorke dimension D_{KY} represents the upper bound for the information dimension of the system. Moreover, the sum of all positive Lyapunov exponents gives an estimate of the Kolmogorov-Sinai entropy according to Pesin's theorem, (Pesin, Y.B. 1977).

The inverse of the largest Lyapunov exponent is sometimes referred to as Lyapunov time and it defines the characteristic e-folding time. For chaotic orbits, the Lyapunov time will be finite, whereas for regular orbits it will be infinite.

2.3 CALCULATION OF MAXIMUM LYAPUNOV EXPONENT

2.3.1 Introduction

Consider two orbits, a "reference" orbit and a "test" orbit, separated at time t_o by a small phase space d_o. We will use the test orbit as a means of calculating the value of the maximum Lyapunov exponent. Under the evolution of the equations of motion, the two orbits may (or may not) separate. If the motion is chaotic, the orbits will, by definition, separate at an exponential rate. The maximum Lyapunov exponent λ is a measure of this rate of separation and is given by;

$$\lambda = \lim_{t\to\infty} \frac{1}{t - t_o} \ln \frac{d(t)}{d_o} \tag{2.6}$$

Ideally, $\lambda(t)$ settles to approximately its asymptotic value, if indeed it is non-zero for the orbit of interest after some time (Marc, A.M. 1985). A simple method of calculating $\lambda(t)$ is as shown in section (2.3.2). Another practical problem is that, for chaotic orbits, the distance between reference and test particles, $d(t)$, quickly saturates. Hence we must periodically renormalize the orbit separation.

2.3.2 Exponent Calculation

Whenever the separation $d(t)$ has passed beyond a threshold value D, the test orbit is rescaled and not the reference orbit. It is important that D be set small enough that it is still in the linear regime (i.e., the regime in which the linearized equations of motion are an accurate description). Define a rescaling parameter:

$$\alpha_1 \equiv \frac{d(t_1)}{d(t_o)} \tag{2.7}$$

where t_1 is the time at which $d(t) \geq D$. Then we can write

$$\lambda_1 = \frac{1}{t_1 - t_o} \ln \frac{d_1}{d_o} = \frac{1}{t_1 - t_o} \ln \alpha_1 \tag{2.8}$$

where $\lambda_i \equiv \lambda(t_i)$ and $d_i \equiv d(t_i)$. At this point, the test orbit is then rescaled, as shown in section (2.3.3). Similarly, for successive threshold crossings and subsequent rescaling, we have;

$$\left.\begin{array}{l}
\lambda_2 = \dfrac{1}{t_2 - t_o} \ln \dfrac{d_2.\alpha_1}{d_o} = \dfrac{1}{t_2 - t_o} \ln(\alpha_1 \alpha_2) \\
\lambda_3 = \dfrac{1}{t_3 - t_o} \ln \dfrac{d_3.\alpha_2.\alpha_1}{d_o} = \dfrac{1}{t_3 - t_o} \ln(\alpha_1 \alpha_2 \alpha_3) \\
. \\
. \\
. \\
.
\end{array}\right\} \tag{2.9}$$

The multiplicative factors $\alpha_1, \alpha_1.\alpha_2, ...$ are derived in section (2.3.4), and in case it is not intuitively obvious. Therefore the instantaneous Lyapunov exponent is given by;

$$\lambda_n = \frac{1}{t_n - t_o} \sum_{i=1}^{n} \ln \alpha_i \tag{2.10}$$

where we have defined

$$\alpha_i \equiv \frac{d(t_i)}{d(t_o)} \tag{2.11}$$

This construction is valid as long as the rescaling takes place in the linear regime (Wolf, *et al.*, 1986). Notice that, in a computer, only the accumulative sum of the natural log of α_i, need to be stored. In addition, the time intervals need not be evenly spaced.

2.3.3 Renormalization of the Test Orbit

The rescaling of the test particle orbit is performed on the test-reference phase space distance vector. Whenever the distance $d(t)$ becomes greater than or equal to the threshold D, we scale the test particle distance from the reference particle by the factor $\frac{1}{\alpha_i}$, maintaining the current relative orientation between the two particles in phase space. Write the reference and test particle phase space vectors as;

$$\vec{R} = \begin{pmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{pmatrix}_{ref} \quad \text{and} \quad \vec{r} = \begin{pmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{pmatrix}_{test} \tag{2.12}$$

Define $\vec{\rho} \equiv \vec{r} - \vec{R}$. Then the adjustment to the test particle phase space coordinates at time t_i is ;

$$\vec{r} \leftarrow \vec{R}_i + \frac{\vec{\rho}_i}{\alpha_i} \tag{2.13}$$

Alternatively, one could write the equivalent expression as;

$$\vec{r}_i \leftarrow \vec{r}_i - \frac{\alpha_i - 1}{\alpha_i} . \rho_i \tag{2.14}$$

All we are doing is rescaling the distance $d(t)$,

$$d(t_i) \leftarrow \frac{d(t_i)}{\alpha_i} \tag{2.15}$$

in an appropriate direction in phase space. Any difference between using the full phase space distance;

$$\sqrt{x^2 + y^2 + z^2 + v_x{}^2 + v_y{}^2 + v_z{}^2} \tag{2.16}$$

and using only the configuration space distance

$$\sqrt{x^2 + y^2 + z^2} \tag{2.17}$$

is indiscernible.

2.3.4 Explanation of Summation

In this section for completeness, we derive the multiplicative factors in the distances in the logarithms in Eq. (2.10) and the equations leading up to it. Consider Eq. (2.6). At time t_1;

$$d_1 = d_o e^{\lambda_1 t_1} \tag{2.18}$$

Upon rescaling,

$$d_1 \leftarrow \frac{d_o e^{\lambda_1 t_1}}{\alpha_1} \tag{2.19}$$

At the next rescaling time t_2,

$$d_2 = d_1 e^{\lambda_1 (t_2 - t_1)} \tag{2.20}$$

where d_1 is the rescaled value (i.e., d_o) inserting equation (2.19) for d_1, we have

$$d_2 = \frac{d_o e^{\lambda_2 t_2}}{\alpha_1} \tag{2.21}$$

where we have assumed the increment in time is small so that $\lambda_1 \approx \lambda_2$. Upon rescaling, equation (2.21) becomes;

$$d_2 \leftarrow \frac{d_2}{\alpha_2} \tag{2.22}$$

where , using equation (2.20) ;

$$\alpha_2 = \frac{d_2}{d_o} = \frac{e^{\lambda_2 t_2}}{\alpha_1} \tag{2.23}$$

Hence, from equation (2.22), we have;

$$\lambda_2 = \frac{1}{t_2} \ln \alpha_1 \alpha_2 \tag{2.24}$$

which is equation (2.9). Extending this process further, we conclude equation (2.10).

2.4 STABILITY AND LYAPUNOV FUNCTIONS

The stability of any hyperbolic equilibrium point z_0 of;

$$\dot{z} = f(z) \tag{2.25}$$

is determined by the signs of the real part of the eigenvalues λ_j of the matrix $Df(z_0)$.

A hyperbolic equilibrium point z_0 is asymptotically stable if and only if $\mathrm{Re}(\lambda_j) < 0$

for $j = 1, 2, ..., n$; that is, if and only if z_0 is a sink. Similarly a hyperbolic equilibrium point z_0 is unstable if and only if it is either a source or a saddle. The stability of non-hyperbolic equilibrium point is typically more difficult to determine. A method, due to Lyapunov, that is very useful for deciding the stability of non-hyperbolic equilibrium points is presented in this section (Wolf, et al., 1985).

***Definition* 2.4.1**

Let ϕ_t denote the flow of the differential equation (2.25), defined for all $t \in \mathbb{R}$. An equilibrium point z_0 of (2.25) is *stable* if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $z \in N_\delta(z_o)$ and $t \geq 0$ we have;

$$\phi_t(z) \in N_\varepsilon(\phi_t(z_0))$$

The equilibrium point z_0 is *unstable* if it is not stable. Moreover, z_0 is *asymptotically stable* if it is stable and if there exists a $\delta > 0$ such that for all $z \in N_\delta(z_0)$ we have;

$$\lim_{t \to \infty} \phi_t(z) = z_0 .$$

Remark; The following two Theorems and a corollary provide a useful result in the subsection that is to follow;

Theorem 2.4.1 The stable Manifold Theorem (Perko, L.1991)

Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the non-linear system (2.25). Suppose that $f(0) = 0$ and that $Df(0)$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Then there exists a k-dimensional differentiable manifold S tangent to the stable subspace

E^s of the linear system $\dot{z} = Az$ where $A = Df(z_0)$, at 0 such that for all $t \geq 0$, $\phi_t(S) \subset S$ and for all $z_0 \in S$,

$$\lim_{t \to \infty} \phi_t(z_0) = 0;$$

and there exists an $n-k$ -dimensional differentiable manifold U tangent to the unstable subspace E^u of $\dot{z} = Az$ at 0 such that for all $t \leq 0$, $\phi_t(U) \subset U$ and for all $z_0 \in U$,

$$\lim_{t \to -\infty} \phi_t(z_0) = 0.$$

Corollary 2.4.1: Under the hypothesis of the Stable Manifold Theorem, if S and U are the stable and unstable manifolds of (2.25) at the origin and if $\mathrm{Re}(\lambda_j) < -\alpha < 0 < \beta < \mathrm{Re}(\lambda_m)$ for $j = 1, 2, ..., k$ and $m = k+1, ..., n,$ then given $\varepsilon > 0$ there exists a $\delta > 0$ such that if $z_0 \in N_\delta(0) \bigcap S$ then $\left|\phi_t(z_0)\right| \leq \varepsilon e^{-\alpha t}$ for all $t \geq 0$ and if $z_0 \in N_\delta(0) \bigcap U$ then $\left|\phi_t(z_0)\right| \leq \varepsilon e^{\beta t}$ for all $t \leq 0$.

Theorem 2.4.2 The Hartman-Grobman Theorem (Perko, L.1991)

Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1 E$ and let ϕ_t be the flow of the non-linear system (2.25). Suppose that $f(0) = 0$ and the matrix $A = Df(0)$ has no eigenvalue with zero real part. Then there exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that for each $z_0 \in U$, there is an open interval $I_0 \subset \mathbb{R}$ containing zero such that for all $z_0 \in U$ and $t \in I_0$;

$$H \circ \phi_t(z_0) = e^{At} H(z_0)$$

i.e., H maps trajectories of (2.25) near the origin onto trajectories of $\dot{z} = Az$ near the origin and preserves the parameterization by time.

A stable node or focus of a linear system in $\mathbb{R}^2$ is an asymptotically stable equilibrium point; an unstable node or focus or a saddle of a linear system in $\mathbb{R}^2$ is an unstable equilibrium point; and a center of a linear system in $\mathbb{R}^2$ is a stable equilibrium point which is not asymptotically stable.

It follows from the Stable Manifold Theorem and the Hartman-Grabman Theorem that any sink of (2.25) is asymptotically stable and any source or saddle of (2.25) is unstable. Hence, any hyperbolic equilibrium point of (2.25) is either asymptotically stable or unstable. The corollary above provides even more information concerning the local behavior of solutions near a sink.

Theorem 2.4.3. (Perko, L.1991)

If z_0 is a sink of the non-linear system (2.25) and $\mathrm{Re}(\lambda_j) < -\alpha < 0$ for all the eigenvalues λ_j of the matrix $Df(z_0)$, then given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $z \in N_\delta(z_0)$, the flow $\phi_t(z)$ of (2.25) satisfies $|\phi_t(z) - z_0| \le \varepsilon e^{-\alpha t}$ for all $t \ge 0$.

Since hyperbolic equilibrium points are either asymptotically stable or unstable, the only time that equilibrium point z_0 of (2.25) can be stable but not asymptotically stable is when $Df(z_0)$ has a zero eigenvalue or a pair of complex-conjugate, pure-imaginary eigenvalues $\lambda = \pm ib$ It follows from the next theorem, that all other eigenvalues λ_j of $Df(z_0)$ must satisfy $\mathrm{Re}(\lambda_j) \leq 0$ if z_0 is stable.

Theorem 2.4.4 (Perko, L.1991)

If z_0 is a stable equilibrium point of (2.25), no eigenvalue of $Df(z_o)$ has a positive real part.

Stable equilibrium points which are asymptotically stable can only occur at non-hyperbolic equilibrium points. But the question as to whether a non-hyperbolic equilibrium point is stable, asymptotically stable or unstable is a delicate question.

The following method due to Lyapunov, is very useful in answering the question.

Definition 2.4.2: If $f \in C^1(E)$, $V \in C^1(E)$ and ϕ_t is the flow of the differential equation (2.25), then for $z \in E$ the derivative of the function $V(z)$ along the solution $\phi_t(z)$;

$$\dot{V}(z) = \frac{d}{dt} V(\phi_t(z))\Big|_{t=0} = DV(z)f(z)$$

The last equality follows from the chain rule. If $\dot{V}(z)$ is negative in E then $V(z)$ decreases along the solution $\phi_t(z_0)$ through $z_0 \in E$ at $t = 0$. Furthermore , in $\mathbb{R}^2$, if $\dot{V}(z) \leq 0$ with equality only at $z = 0$, then for small positive C the family of curves $V(z) = C$ constitute a family of closed curves enclosing the origin and the trajectories of (2.25) cross these curves from their exterior to their interior with increasing; i.e., the origin of (2.25) is asymptotically stable.

2.5 NORMAL HYPERBOLICITY AND GENERALIZED LYAPUNOV NUMBERS

2.5.1 Normal Hyperbolicity

Synchronization and stability of an invariant manifold $\mathfrak{M}$ relates to the attraction and persistence of this manifold. A necessary and sufficient condition for such synchronization and persistence is Normal Hyperbolicity.

Let Z be a vector field on $\mathbb{R}^n$ with the flow F^t. Let $\mathfrak{M}$ be a compact connected manifold with a boundary which is overflowing invariant under the vector field *Z*. Let the tangent bundle $T\mathbb{R}^n \mid \mathfrak{M}$ restricted to split into three sub-bundles;

$T\mathbb{R}^n \mid \mathfrak{M} = N^u \oplus T\mathfrak{M} \oplus N^s$, invariant under the linearized flow $\Phi(t)$ for all t, where N^u is the unstable bundle and N^s is the stable normal bundle. When N^u is empty, the tangent space of $\mathfrak{M}$ decomposes as;

$$T\mathbb{R}^n \mid \mathfrak{M} = T\mathfrak{M} \oplus N^s$$

Let $\pi : T\mathbb{R}^n \mid \mathfrak{M} \to N^s$ be the orthogonal projection. Define the linearization $\Phi_c(t; z_0)$ of the flow parallel to the manifold, and $\Phi_s(t; z_0)$ normal to the manifold as;

$$\Phi_c(t; z_0) = D(\Phi_{-t} \mid \mathfrak{M}(z_0)),$$

$$\Phi_s(t; z_0) = \pi D\Phi_t(\Phi_{-t}(z_0))$$

for $z_0 \in \mathfrak{M}$

A manifold is normally hyperbolic if, under the dynamics linearized about the invariant manifold, the growth rate vectors transverse to the manifold dominates the growth rates of the vectors tangent to the manifold, that is, the contraction of the flows in the direction normal to the manifold is exponentially greater than the contraction of the flows tangent to the manifold.

2.5.2 Generalized Lyapunov Exponents

The generalized Lyapunov exponents are defined as;

$$\begin{aligned}\alpha(z_0) &= \lim_{t\to\infty}\sup\frac{1}{t}\ln\left\|\Phi_s(t;z_0)\right\| \\ \beta(z_0) &= \lim_{t\to\infty}\sup\frac{\ln\left\|\Phi_s(t;z_0)\right\|}{\ln m(\Phi_c(t;z_0))}\end{aligned} \tag{2.26}$$

where for a linear operator $L, m(L) = \min\{|Lx\| x|=1, x \in D(L)\}$.

The rate of contraction (growth) of the vectors in the direction normal to the manifold $\mathfrak{M}$ is measured by $\alpha(z_0)$ while $\beta(z_0)$ measures the ratio of the exponential rate of contraction (growth) of vectors in normal direction, and the exponential rate of contraction (growth) of vectors in $\mathfrak{M}$.

The growth rates of vectors can be characterized in terms of Generalized Lyapunov exponents determined from the linearised equations of motion around the synchronization manifold. This is because the asymptotic stability of the manifold $\mathfrak{M}$ can be inferred from the assumption that the asymptotic stability of the origin in the linearization of the equations describing the dynamics transverse to the manifold implies the transverse stability of $\mathfrak{M}$ under the full equations.

This splitting is called hyperbolic if $\alpha(z_0) < 1$ for all $z_0 \in \mathfrak{M}_1$, that is, $\Phi_s(t;z_0)$ contracts N^s more sharply than $\Phi_c(t;z_0)$ contracts $T\mathfrak{M}_1$.

The magnitude of these growth rates determines the strength of Robustness, i.e., condition under which a stable manifold remains stable even after a small perturbation.

CHAPTER THREE

3.0 COUPLED OSCILLATORS

3.1 INTRODUCTION

When different oscillators are mutually coupled, different types of synchronization phenomena may develop as the coupling strength is increased. This may be amplitude envelope synchronization, phase synchronization, and lag, identical or generalized synchronization

Similarly, when identical oscillators are coupled, the diagonal which is also referred to as synchronization manifold is usually invariant. The dynamical behaviour of oscillators described by the equation $\dot{z}(t) = f(z(t), \mu)$ in relation to synchronization, stability and persistence was studied. We considered one dimensional simple Bravais Lattice with Neumann boundary conditions and then two dimensional lattices.

3.2 DYNAMICS OF $n > 2$ ALL-TO-ALL COUPLED OSCILLATORS

Systems made of more than two coupled oscillators making a network are useful in studying the dynamics of physical, chemical and biological systems. These can be more or less complicated networks of interconnected elementary units, or continuous extended systems that are modeled by means of ordinary differential equations. In this last case, the numerical treatment of these equations may result in a reformulation of the mathematical problem in terms of networks of coupled oscillators. Collective behaviors such as turbulence, spatial variations of observables, pattern formation and clustering are the kind of phenomena that are of interest in this context. A desynchronized state is a model of turbulence in space and time, while synchronized state describes spatially ordered and turbulent systems. Intermediate states in which only groups of oscillators synchronize correspond to phenomena of pattern formation and clustering.

3.3 THEORY OF COUPLED OSCILLATORS

Consider the individual flow, $\dot{z} = F(z)$, the system used is described by;

$$\frac{dz_i}{dt} = F(z_i) + C\sum_{j=1}^{n} c_{ij}(z_j - z_i) \qquad i = 1, 2, ..., n \tag{3.1}$$

with C a scalar constant this gives an overall measure of the strength of interaction between the n oscillators, c_{ij} a matrix which describes the configuration of the interactions which is also known as the coupling matrix. This is frequently chosen, such that the following symmetry rule;

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}(z_j - z_i) = 0 \tag{3.2}$$

was verified (Fujsaka, *et al.*, 1983). There are many choices of c_{ij} that describe the configuration of the interactions. The two configurations frequently found are;

(i) $c_{ij} = 1$ for all $i, j \in n$, which describes an identical all-to-all interaction,

(ii) $c_{ij} = 1$ if $|i - j| = 1$ and $c_{ij} = 0$, otherwise, this describes a linear chain with interactions only to the nearest neighbors, i.e., diffusive coupling.

Approximations for the actions on the i^{th} oscillator that deviate from $C.\sum_{j=1}^{n} c_{ij}(z_j - z_i)$, can be used; one example is the mean field approximation, in which an all-to-all interaction can be averaged in the form;

$$C.\sum_{j=1}^{n} c_{ij}(z_j - z_i) \approx \frac{C}{n}\sum_{j=1}^{n} z_j \tag{3.3}$$

In all the above cases, it is assumed that the system of oscillators is isolated. In some applications, such as the study of continuous systems, or of specific experimental configurations found in practice, boundary conditions appropriate to the case studied can be used. A case frequently found is the use of periodic boundary conditions in which the finite system of *n* oscillators is assumed to be surrounded by a set of identical copies in a number and spatial arrangement given by the structure of the coupling matrix. For example, a linear chain of *n*-oscillators with only nearest neighbor interactions, where the coupling term is given by $C.\sum_{j=1}^{n} c_{ij}(z_j - z_i) = C.(z_{i-1} + z_{i+1} - 2z_i)$, the periodic boundary conditions are written as; $z_{n+1}(t) = z_1(t)$. This particular system is equivalent to a closed ring of oscillators, a geometrical configuration frequently found in experiments. The system of equations (3.1) allows the solution $z_1(t) = z_2(t) = z_3(t) = ... = z_n(t) = z(t)$. When $z(t)$ is a solution to the differential equation $\dot{z} = F(z)$, a state like this represents a very coherent form of motion in which all the oscillators are synchronized in the sense that there is identical synchronization between any pair of oscillators in the system.

3.4 SYNCHRONIZATION OF ALL-TO-ALL COUPLED OSCILLATORS

Coupled oscillators dynamical behavior can be described by the equation;

$$\dot{z}(t) = kB(z(t), z(t-\tau)) + f(z(t)) \tag{3.4}$$

where $z(t)=(z_1(t),z_2(t),............z_n(t))$, $kB(z(t),z(t-\tau))$ is a linear operator describing the coupling of different configurations, $f(z(t))=g(z_1(t)),g(z_2(t)),............g(z_n(t))$ and $\tau>0$ is a time lag. From the literature review, the types of $kB(z(t),z(t-\tau))$ considered are not as compressive as possible. They have only considered $kB(z(t),z(t-\tau))$ for the following cases;

(i) For two oscillators with $\tau>0$ and only one variable involved in the coupling,

(ii) For a large and finite number of oscillators with $\tau=0$ and the variables involved in the coupling

Dynamical behavior of oscillators with $kB(z(t),z(t-\tau))$ involving delay in all variables and for a large but finite number of oscillators was addressed.

Consider the ordinary differential equation;

$$\dot{z}_j=g(z_j) \tag{3.5}$$

where $z_j=z_j(t)\in\mathbb{R}^2$, g is an odd continuous function on $\mathbb{R}^2$. The solution of equation (3.5) is a limit cycle that attracts the whole of $\mathbb{R}^2$ except at the origin (0,0).

The equation (3.5) when coupled takes the form,

$$\dot{z}_j(t)=kI\left[-(n-1)z_j(t)+\sum_{i=1,i\neq j}^{n}z_i(t-\tau)\right]+g(z_j(t)) \tag{3.6}$$

This system describes oscillators coupled all-to-all by linear diffusive terms.

3.5 EFFECTS OF TIME LAG ON STABILITY

This sub-topic aimed at estimating the range of time lag for which the synchronized state is stable. We considered equation (3.6) and an orthogonal change of coordinates. Assuming that equation (3.6) has a compact global attractor $\mathfrak{A}_k$ for each *k*. Eq. (3.6) is synchronized if the global attractor $\mathfrak{A}_k$ belongs to the diagonal in R^n, where in this case the diagonal D is the set;

$$D = \left\{ z | z_1 = z_2 = z_3 = ... = z_n, z_i \in R^n, i = 1, 2, ..., n \right\}.$$

This implies that the differences $z_j(t) - z_i(t) \to 0$ as $t \to \infty$ for all i, j. Of course, for the system to be synchronized the diagonal has to be an invariant set, which is the case if we consider identical subsystems. In addition, the linear operator;

$kB(z(t), z(t-\tau)) = k\left[-(n-1)z_j(t) + \sum_{i=1, i\neq j}^{n} z_i(t-\tau) \right]$, must have a zero as an eigenvalue with the diagonal being the corresponding generalized eigenspace and all other eigenvalues have to lie to the left of the complex plane. We wish to make the following assumptions on $kB(z(t), z(t-\tau))$;

Assumption 1 ($\mathbf{H_1}$.)

For each *k,* zero is an eigenvalue of $kB(z(t), z(t-\tau))$ with the diagonal being a generalized eigenvector for zero. All other eigenvalues of $kB(z(t), z(t-\tau))$, lie to the left of the complex plane, for all $0 < \tau < \infty$. Equation (3.4) has to satisfy assumption $\mathbf{H_1}$.

The eigenvalues of $kB(z(t), z(t-\tau))$ are given by the zeroes of $p(\lambda) = (p_c(\lambda))^{n-1} p_s(\lambda)$, where

$$p_c(\lambda) = \lambda + k(n-1) + ke^{-\lambda\tau} \tag{3.7}$$

and

$$p_s(\lambda) = \lambda + k(n-1)(1-e^{-\lambda\tau}) \tag{3.8}$$

Clearly $\lambda = 0$ is an eigenvalue of $p_s(\lambda) = 0$ and the vector $col(1,1,1,......,1)$ with $n\times 2$ ones is the corresponding eigenvector. As long as $k > 0$ and $n \geq 2$, $p_s(\lambda)$ and $p_c(\lambda)$ have all their eigenvalues on the left half of the complex plane, regardless of the value of τ , $0 < \tau < \infty$.

Assumption 2 ($\mathbf{H_2}$.)

Let $T_{k(t)} : X \to X$, $t \geq 0$, $k > 0$ be a C^0 semi group generated by the solution of the equation (3.4) with initial condition $\varphi \in X = \mathrm{C}((-\tau,0),\mathbb{R}^n)$. X is a Banach space, $T_{k(t)}$ is dissipative and has a compact global attractor $\mathfrak{A}_k$:which is invariant under $T_{k(t)}$. Furthermore, $(T_{k(t)}, \mathfrak{A}_k) \to 0$ as $t \to \infty$ and $k > 0$, for all $0 < \tau < \infty$.

From assumption $\mathbf{H_1}$, a new coordinate system is introduced. Let e_j be the usual unit vector in R^n, and $\tilde{e}_j = \dfrac{1}{2}(e_j - e_{j+1})$, $1 \leq j \leq n-1$, The set e_j, $\tilde{e}_j$ is an orthogonal basis for R^n . If we let $\tilde{e} = (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, ..., \tilde{e}_{n-1})$ then in this basis, we can write z as;

$z = \tilde{e}\, w + ey$, $w = (w_1, w_2, ..., w_{n-1}) \in R^{2n-2}$, $y \in R^2$, $w \in M$, $y \in M$, where M is an inertial manifold with;

$w_j = z_j - z_{j+1}$, $1 \le j \le n-1$, $y = \frac{1}{n}\sum_{j=1}^{n} z_j$. Now, with this transformation, equation (3.6) becomes;

$$\left.\begin{aligned} \dot{y}(t) &= (n-1)kI[y(t-\tau) - y(t)] + \frac{1}{n}\sum_{j=1}^{n} g(z_j(t)) \\ \dot{w}(t) &= -kI[(n-1)w(t) + w(t-\tau)] + \frac{1}{2}[G(w, y)] \end{aligned}\right\} \tag{3.9}$$

where; $G(w, y) = g(z_j(t)) - g(z_{j+1}(t))$, $1 \le j \le n-1$.

For the synchronized state to be stable, all motion transverse to it must asymptotically dampen out. This is equivalent to saying that the zero solution of the second equation in Eq. (3.9) must be exponentially stable. Since the concern is the local dynamics, Eq. (3.9) can be linearized about $(y(t), w(t)) = y_o(t)$ with $y_o(t)$ solving the first equation of system (3.9) to get;

$$\left.\begin{aligned} \dot{y}(t) &= (n-1)kI[y(t-\tau) - y(t)] + D_z g(y_o(t))y(t) \\ \dot{w}(t) &= -kI[(n-1)w(t) + w(t-\tau)] + D_z g(y_o(t))w(t) \end{aligned}\right\} \tag{3.10}$$

The estimated value of $\tau = \tau_o$ such that for all $0 < \tau < \tau_o$, for the synchronization and the stability for system (3.4) can be inferred from $\dot{z}(t) = kB(z(t), z(t)) + f(z(t))$, that is the system (3.4) without τ.

Proposition 3.5.1

"There is a τ_o such for any given $n \ge 2$, and k, equation (3.4) has a stable synchronized solution for all $0 < \tau < \tau_o$".

Proof

The second equation in (3.10) can be written as

$$\dot{w}(t) = A(t)w(t) + B(t)w(t-\tau)$$

where

$A(t) = D_z g(y_o(t)) - k(n-1)I$, $B(t) = -kI$ .The system obtained for $\tau = o$ is;

$$\dot{w}(t) = (A(t) - kI)w(t) \tag{3.11}$$

Suppose that the trivial solution of equation (3.11) is uniformly asymptotically stable, which is the case for $\lambda_m - kI < 0$, λ_m being the maximum Lyapunov exponent for the matrix $A(t)$ (Chow *et al.,*1997).

If $W(t,s)$ is the fundamental matrix solution of equation (3.11), with $W(0) = I$, then;

$$|W(t,s)| \leq ke^{-\alpha(t-s)},\ \alpha > 0$$

Any solution $w(t)$ of the equation;

$\dot{w}(t) = A(t)w(t) + B(t)w(t-\tau) = (A(t) - B(t))w(t) + B(t)[w(t-\tau) - w(t)]$ is given by;

$$w(t) = W(t,0)w(t_o) + \int_0^t W(t,s)B[w(s-\tau) - w(s)]ds \tag{3.12}$$

For $t > \tau$ equation (3.12) can be written as;

$$w(t) = W(t,0)w(t_o) + \int_0^\tau W(t,s)B[w(s-\tau) - w(s)]ds + \int_\tau^t W(t,s)B[w(s-\tau) - w(s)]ds$$

For $0 \leq s \leq \tau$ we have that:

$w(s) = w(0) + \int_0^\tau [A(s)w(s) + Bw(s-\tau)]ds$ such that if norms are taken on both side;

$$\left|w(s)\right| \leq \left|w(0)\right| + \int_0^{\tau} [\left|A(s)w(s)\right| + \left|Bw(s-\tau)\right|]ds$$

Now, taking sup norm;

$$\left|w(s)\right| \leq \left|w(0)\right| + \int_0^{\tau} [\left|A(s)\right|\left|w(s)\right| + \left|B\right|\left|w(s-\tau)\right|]ds$$

or

$\left|w(s)\right| \leq \left|w(0)\right| + \int_0^{\tau} (L_1 + L_2)\left|w(s)\right| ds$, taking $\sup\limits_{-\tau \leq s \leq \tau}\left|w(s)\right| = \left|w(s)\right|$

On the basis of Gronwall's lemma, for $0 \leq s \leq \tau$;

$\left|w(s)\right| \leq \exp[(L_1 + L_2)\tau]\left\|\varphi\right\|$,

where φ is the initial function of the solution w given in $[-\tau, 0]$, that is, $\varphi \in$ C $[-\tau, 0, R^2]$. We have set;

$L_1 = \sup\limits_{t}\left|A(t)\right|$, $L_2 = \sup\limits_{t}\left|B\right|$.

This estimate is also valid for $s \leq 0$. It follows that for $0 \leq s \leq \tau$, we have in any case;

$\left|w(s-t) - w(s)\right| \leq 2\exp(L_1 + L_2)\tau\left\|\varphi\right\|$,

For $s \geq \tau$, it can be written as;

$w(s-\tau) - w(s) = \int_s^{s-\tau} \dot{w}(\sigma)d\sigma = \int_s^{s-\tau} [A(\sigma)w(\sigma) + Bw(\sigma - \tau)]d\sigma$, hence;

$\left\|w(s-\tau) - w(s)\right\| \leq \tau(L_1 + L_2) \sup\limits_{s-2\tau \leq \sigma \leq s} \left|w(\sigma)\right|$, thus we obtain;

$$\left|w(t)\right| \leq ke^{-\alpha t}\left\|\varphi\right\| + \int_0^{\tau} ke^{-\alpha(t-s)} L_2 2\exp[(L_1 + L_2)\tau]\left\|\varphi\right\| ds + \int_{\tau}^{t} ke^{-\alpha(t-s)} L_2 (L_1 + L_2)\tau \sup_{s-2\tau \leq \sigma \leq s} =$$

$$ke^{-\alpha t}\|\varphi\|+[2kL_2 2\exp[(L_1+L_2)\tau]e^{-\alpha t}\frac{1}{\alpha}(e^{-2\alpha t}-1)\|\varphi\|+k\tau L_2(L_1+L_2)\int_{\tau}^{t}e^{-\alpha(t-s)}\sup_{s-2\tau\le\sigma\le s}|w(\sigma)|ds$$

Setting;

$L_o = k\|\varphi\|[1+\frac{2}{\alpha}(e^{2\alpha t}-1)]L_2\exp(L_1+L_2)\tau$, and $m = k\tau L_2(L_1+L_2)$, then the above equation reduces to;

$$|w(t)| \le L_o e^{-\alpha t} + me^{-\alpha t}\int_{\tau}^{t} e^{\alpha s}\sup_{s-2\tau\le\sigma\le s}|w(\sigma)|ds \tag{3.13}$$

Let, $v(t) = e^{-\alpha t}[L_o + m\int_{\tau}^{t} e^{\alpha s}\sup_{s-\tau\le\sigma\le s}|w(\sigma)|ds]$, then;

$$\dot{v}(t) = -\alpha e^{-\alpha t}[L_o + m\int_{\tau}^{t} e^{\alpha s}\sup_{s-\tau\le\sigma\le s}|w(\sigma)|ds] + e^{-\alpha t}me^{\alpha t}\sup_{s-2\tau\le\sigma\le s}|w(\sigma)| \tag{3.14}$$

However $w(t) \le v(t)$, hence $\sup|w(\sigma)| \le v(\sigma)$. Thus $\dot{v}(t) \le -\alpha v(t) + m\sup_{s-\tau\le\sigma\le s} v(\sigma)$

The following Lemma is useful in establishing the bound for τ for stability of the synchronized manifold;

Lemma 3.5.1

If $\dot{f}(t) \le -\alpha f(t) + \beta\sup_{t-\tau\le\sigma\le t} F(\sigma)$ for $t \ge t_o$ and $\alpha > \beta > 0$, then there exist $\gamma > 0$ and $\kappa > 0$ such that $f(t) \le \kappa e^{-\gamma(t-t_o)}$ for all $t \ge t_o$.

Proof;

If $m < \alpha$, it follows that by virtue of lemma (3.5.1) that there exists constants N and γ such that;

$v(t) \le Ne^{-\gamma(t-t_o)}$

Hence a similar inequality holds for $w(t)$. Consequently, the trivial solution of system (3.6) is exponentially stable provided that $m < \alpha$, this leads to;

$$\tau < \frac{\alpha}{kL_2(L_1 + L_2)}$$

Taking;

$\tau_o < \dfrac{\alpha}{kL_2(L_1 + L_2)}$, completes the proof of the theorem.

3.6 ONE DIMENSIONAL LATTICE OSCILLATOR

In this case the oscillators are coupled in a line by diffusive like path where oscillators are allowed to interact with their immediate neighbor. The function $kB(z(t))$ in Eq. (1.38) will be a linear function of the form;

$$kB(z(t)) = A(k)z(t) = k\Delta_1 \otimes I_d z(t),$$

where $k = (k_1 = k_2 = k_3 = ... = k_n)$ and I_d is the identity matrix of order d and $\otimes$ is the Kronecker product. The choice of the constant coupling parameter is made in such a way that oscillators influence each other equally so that Eq. (1.38) can be written thus;

$$\dot{z} = k(\Delta_1 \otimes I_d)z + f(z) \tag{3.15}$$

where $z = (z_1, z_2, ..., z_n)^T$, with T denoting the transpose.

There is need to show that the matrix Δ_1 has $\lambda_0 = 0$ as an eigenvalue with the corresponding generalized eigenvectors whose span is the diagonal in $\mathbb{R}^{nd}$ and the other eigenvalues $\lambda_s, s = 1, 2, 3, ..., n-1.$ are bounded to the left hand side of the complex plane.

Consequently, the following assumptions about Δ_1, are made;

Assumption 3 (H_3)

For each k, Δ_1 is self adjoint, $\lambda_0 = 0$ is an eigenvalue of Δ_1 with the corresponding generalized eigenvector $e = (1,1,1,.........,1) \in \mathbb{R}^{nd}$. Its span is the diagonal in $\mathbb{R}^{nd}$

Assumption 4 (H_4)

For $n > 2$, there exist k_0 a and bounded set $U \in \mathbb{R}^{nd}$ such that for each k, Eq. (3.15) has a global attractor $\mathfrak{M}_1 \in U$.

We need to show that Eq. (3.15) satisfies the above two assumptions.

The eigenvalues of Δ_1 are $\lambda_0 = 0$ and $\lambda_s = (-2 - 2\cos\frac{s\pi}{2n})k, \quad s = 1, 2, ..., n-1$ which satisfies the first assumption.

For the stability of an invariant synchronized manifold, there is need to make a coordinate transformation that takes care of the transversal flow and the tangential flow to the manifold. Consider the transformation;

$z = ye + \tilde{e}w, \quad w = (w_1, w_2, w_3, ..., w_{n-1}), \quad w \in \mathbb{R}^{nd-d}, y \in \mathbb{R}^d$

$$w_j = z_j - z_{j+1}, \quad 1 \le j \le n-1,$$

$$w = \frac{1}{n}\sum_{j=1}^{n} z_j, \tag{3.16}$$

where e_j is the j^{th} column of an $n \times n$ identity matrix and $\tilde{e}_j = \sum_{j=1}^{n} e_i - \frac{j}{n} e$ with $\tilde{e} = (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, ..., \tilde{e}_{n-1})$. The set $e, \tilde{e}_j, 1 \le j \le n-1$ is an orthogonal basis for $\mathbb{R}^n$; that is, $e.\tilde{e}_j = 0$

Using transformation (3.16) in Eq. (3.15), we obtain;

$$\dot{w} = k(\Delta \otimes I_d)w + F(w, y),$$

$$\dot{y} = \frac{1}{n} \sum_{j=1}^{n} g(z_j), \tag{3.17}$$

where the function $F(w, y) = (F_1(w, y), F_2(w, y), F_3(w, y), ..., F_{n-1}(w, y))^T$ with $F_j(w, y) = (g(z_j) - g(z_{j+1})),\ 1 \le j \le n-1$ and the matrix Δ is given by;

$$\Delta = \begin{pmatrix} -2 & 1 & 0 & 0 & . & . & . & 0 & 0 \\ 1 & -2 & 1 & 0 & . & . & . & 0 & 0 \\ 0 & 1 & -2 & 1 & . & . & . & 0 & 0 \\ . & . & . & . & . & & & . & . \\ . & . & . & . & . & . & & . & . \\ . & . & . & . & & . & . & . & . \\ . & . & . & . & . & . & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & . & . & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & . & . & 0 & 1 & -2 \end{pmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}$$

The eigenvalues of the matrix Δ are $\lambda_s = -2 - 2\cos\frac{s\pi}{n}$ (Wasike, A. M. 2003).

The first equation in (3.17) describes the motion transverse to the synchronization manifold $\mathfrak{M}_1$; that is, it describes the deviations $z_j - z_{j+1}$. Synchronization means that the deviations dampen out as time evolves $(t \to \infty)$. This is equivalent as saying that the first equation in Eq. (3.17) is exponentially stable where stability refers to the attracting property of $\{w = 0\}$.

We are interested in local synchronization and thus local attractivity of the synchronization manifold. Synchronization and stability relates to the attractivity of the manifold, we thus say that a manifold is stable if it is locally attracting. This attraction property is achieved if the Lyapunov exponents of Eq. (3.17) linearized about the synchronization manifold $\mathfrak{M}_1$, are less than zero (Wasike, A. M. 2003).

Let $z_0 \in \mathfrak{M}_1$ and let $\Phi(t; z_0)$, be the fundamental matrix solution of;

$\dot{z} = A(z(t, z_0))z$,

where $A(z(t, z_0))$ is the linearized vector field of Eq. (3.15) about $\mathfrak{M}_1$ which has an invariant splitting with respect to $\Phi(t; z_0)$ such that;

$\Phi(t; z_0) = \Phi_c(t; z_0) \oplus \Phi_s(t; z_0)$;

where $\Phi_c(t; z_0)$ and $\Phi_s(t; z_0)$ are the restrictions of $\Phi(t; z_0)$ on $T_{z_0}\mathfrak{M}_1$ and N_{z_0} respectively, with $T_{z_0}\mathfrak{M}_1$ being the bundle of vectors tangent to the manifold at z_0 and is the bundle of vectors normal to the manifold at z_0. The criterion of stability and persistence is determined by looking at the growth rates of vectors transverse to the manifold and those tangential to the manifold. These growth rates are characterized by generalized Lyapunov exponents defined by Eq. (2.26). The following theorem relates to the stability and Lyapunov exponents;

Theorem 3.6.1

If system (3.17) satisfies $\mathbf{H_3}$ and $\mathbf{H_4}$, then there exist a k_0 such that for all $k > k_0$, there is a positively invariant synchronized manifold $\mathfrak{M}_1$ that is attracting and $\mathfrak{C}^1$ stable or persistent.

To prove theorem (3.6.1), we shall need some results due to Chow and Liu. These results are obtained in the following Lemmas:

Lemma 3.6.2. (Chow, *et al.*, 1997).

Consider the system (3.15), suppose that the synchronized manifold $\mathfrak{M}_1$ is invariant, then if $\alpha(z_0) < 0$, for all $z_0 \in \mathfrak{M}_1$, then the manifold $\mathfrak{M}_1$ is attracting and hence system (3.15) is synchronized.

Lemma 3.6.3. (Chow, *et al.*, 1997).

Suppose that $\mathfrak{M}_1$ is locally synchronized, then the synchronization is $\mathfrak{C}^1$ stable if and only if $\alpha(z_0) < 0$ and $\beta(z_0) < 1$ for all $z_0 \in \mathfrak{M}_1$.

Proof

Linearization of Eq. (3.17) along the solution $(0, y_0(t))$ which corresponds to $(w(t), y(t)) \in \mathfrak{M}_1$ yields;

$$\begin{pmatrix} \dot{w} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} I_{n-1} D_z g(y_0(t)) + k\Delta \otimes I_d & 0 \\ 0 & D_z g(y_0(t)) \end{pmatrix} \begin{pmatrix} w \\ y \end{pmatrix} \tag{3.18}$$

where $[D_z g(y_0(t)] = Jf(y_0(t))$ is the Jacobian matrix of f at $y_0(t)$, the solution of Eq. (1.38).

We realize that there is an invariant splitting of $T\mathbb{R}^{nd}$ into $T\mathfrak{M}_1 \oplus N$ (here $\oplus$ refers to the direct sum of subspaces). System (3.18) is uncoupled, and thus we can solve each equation individually. Now, $\{w=0\}$ is locally attracting if the maximum Lyapunov exponents of

$$\dot{w} = [I_{n-1} \otimes D_z g(y_0(t)) + k\Delta \otimes I_d]w \tag{3.19}$$

is less than zero.

Since $I_{n-1} \otimes D_z g(y_0(t))$ and $k\Delta \otimes I_d$ commute, the fundamental matrix solution of Eq. (3.18) is of the form;

$$\begin{pmatrix} w(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{(k\lambda_s+\lambda_l)t} & 0 \\ 0 & e^{\lambda_l t} \end{pmatrix} = \Phi(t;t_0), \tag{3.20}$$

where $\lambda_s, s \in \{1, 2, ..., nd-d\}$ are the eigenvalues of the coupling configuration $\Delta \otimes I_d$ and λ_l, $1 \le l \le d$ is the Lyapunov exponents of the trajectory defined by $\dot{y} = D_z g(y_0(t))y$.

In our case, the invariant manifold $w=0$ is the one generated by the second equation in (3.18). Therefore $\Phi_c(t;t_0)$ corresponds to $e^{\lambda_l t}$ while $\Phi_s(t;t_0)$ corresponds to $e^{(k\lambda_s+\lambda_l)t}$.

The maximum eigenvalue of Δ is $\lambda_1 = -4\sin^2\dfrac{\pi}{2n}$ and from the definition of Lyapunov exponents,

$\alpha(z_0) = (-k\xi_1 + \lambda_M)$, where $\xi_1 = -\lambda_1$ is the maximum eigenvalue of the coupling configuration and λ_M is the maximum Lyapunov exponent of the trajectory defined by $D_z g(y_0(t))$ in $\mathfrak{M}_1$

Thus by Lemma 3.6.2, $\{w=0\}$ is attracting and the coupled system is locally synchronized if $\alpha(z_0)<0$; that is, k satisfies the expression;

$$k > \frac{1}{\xi_1}\lambda_M$$

To ensure robustness of the synchronized manifold, the condition for normal hyperbolicity must be satisfied. $\mathfrak{M}_1$ is normally hyperbolic if and only if for all $z_0 \in \mathfrak{M}_1$.

$$\sup_{z_0}\left\|\Phi_s(t;t_0)\right\| < \inf_{z_0} m(\Phi_c(t;t_0)).$$

This is equivalent to saying that;

$$\beta(z_0) = \lim_{t\to\infty}\sup\frac{\ln\left\|\Phi_s(t;z_0)\right\|}{\ln m\left\|\Phi_c(t;z_0)\right\|} < 1.$$

By Lemma 3.6.3, we require that $\alpha(z_0)<0$ and $\beta(z_0)<1$ for persistence of the synchronized manifold $\mathfrak{M}_1$

This is equivalent to;

$$\frac{\lambda_M - k\xi_1}{\lambda_m} < 1$$

This is satisfied if $k > \frac{1}{\xi_1}(\lambda_M - \lambda_m)$. Thus we can conclude that, for synchronization, stability and robustness of the synchronization manifold $\mathfrak{M}_1$, k must satisfy the inequality;

$$k_0 > \max\{\frac{1}{\xi_1}(\lambda_M, \lambda_M - \lambda_m)\}$$

Taking $k_0 = \max\{\frac{1}{\xi_1}(\lambda_M, \lambda_M - \lambda_m)\}$ completes our proof.

3.7 TWO DIMENSION AL LATTICE OSCILLATOR

In this section, we explore the existence and stability of synchronized manifold and its persistence after perturbation for the case of a two dimensional lattice oscillators. The synchronization manifold exists by making appropriate coordinate transformation which decomposes the flow linearized along the invariant manifold into transversal and tangential flows. The condition of normal hyperbolicity using the Lyapunov exponents is studied.

Consider the coupling of $n \times n$ oscillators on a simple square lattice. Let $z_{i,j}, 1 \le i, j \le n$ be the coordinate of the $(i,j)^{th}$ site oscillators and let $Z = [z_{i,j}]_{i,j=1}^{n} \in \mathbb{R}^{nd \times nd}$ be the matrix formed when the oscillators are coupled on a square Lattice. Also, let $Z = (Z_1, Z_2, Z_3, ..., Z_n)$ with $Z_r \in \mathbb{R}^{nd}$, $1 \le r \le n$ be its r^{th} column. We define the vector valued function of Z as;

$$vecZ = \begin{pmatrix} Z_1 \\ Z_2 \\ . \\ . \\ . \\ Z_n \end{pmatrix} \in \mathbb{R}^{n^2 d} \; , \; Z_r = (z_{1r}, z_{2r}, ..., z_{nr})^T \tag{3.21}$$

Since on a square Lattice, oscillators interact with each other on each column, and at the same time interact with each other on each row, the system of differential equations describing the dynamics on the $n \times n$ lattice is given by;

$$\dot{Z} = k[(\Delta_1 \otimes I_d)Z + Z\Delta_1] + G(Z) \tag{3.22}$$

where, $G(Z) = [g(z_{i,j})]_{i,j=1}^{n} = (f(Z_1), f(Z_2), ..., f(Z_n))$,

with; $f(Z_r) = (g(z_{1,r}), g(z_{2,r}), ..., g(z_{n,r}))^T$, where T denotes the transpose.

The interpretation of Eq. (3.22) is important. The first term $k[(\Delta_1 \otimes I_d)Z]$ in the coupling matrix, indicates the near neighbor coupling of elements within a column of *Z,* and the second term $kZ\Delta_1$, corresponds to the nearest neighbor diffusive coupling of elements within a row of Z

Eq. (3.22) can be written in vector form as;

$$vec(\dot{Z}) = k\{vec[(\Delta_1 \otimes I_d)Z + Z\Delta_1] + vecG(Z) \tag{3.23}$$

Note that $(\Delta_1 \otimes I_d) \in \mathbb{R}^{nd \times nd}$ and $\Delta_1 \in \mathbb{R}^{n \times n}$. Eq. (3.20) is therefore equivalent to;

$$vec(\dot{Z}) = B(k)vecZ + vecG(Z), \tag{3.24}$$

where $B(k) = k(\Delta_1 \oplus \Delta_1) \otimes I_d$ and $\oplus$ is the Kronecker sum.

For system (3.24) to be synchronized, the diagonal must be invariant set which will be the case if all the $g(z_{i,j})$ are the same, that is, the dynamics of the subsystems are identical, and the matrix $B(k)$ has zero as an eigenvalue with the diagonal being the corresponding eigenspace. In this situation, all the other eigenvalues of $B(k)$ must be less than zero. We can make the following assumptions about $B(k)$;

Assumption 5. (H_5)

For each k ,$B(k)$ is self adjoint, $\lambda_0 = 0$ is an eigenvalue of $B(k)$ with the corresponding generalized eigenvector $e = (1,1,.......,1) \in \mathbb{R}^{n^2 d}$ which spans the diagonal in $\mathbb{R}^{n^2 d}$.

Assumption 6. (H_6)

For $n > 2$, there exist a k_0 and bounded set $U \in \mathbb{R}^{n^2 d}$ such that for each k , Eq. (3.24) has a global attractor $\mathfrak{M}_2 \in U$.

It is trivial to show that Eq. (3.24) satisfies assumption $\mathbf{H_5}$. Symmetric synchronization occurs when $z_{p,q} = z_{r,s} \neq 0$ for all $p,q,r,s \in \mathbb{Z}$. This is only possible when;

$B(k) vecZ = \lambda vecZ = 0, \qquad Z \neq 0$

Thus we require that $\lambda = 0$ be an eigenvalue of $B(k)$. The eigenvalue of $B(k)$ are;

$\sigma(B(k)) = \{\mu : \mu = (\lambda_p + \lambda_q)k, 0 \leq p,q \leq n-1\}$,

with $\lambda_0 = 0$, $\lambda_\zeta = -2 - 2\cos\dfrac{\zeta\pi}{2n}$, $\zeta = p,q$, $1 \leq \zeta \leq n-1$, each occurring d times and the corresponding eigenvectors is given by;

$V_{pq} = (V_p \otimes V_q) \otimes I_d$

where V_p and V_q are eigenvectors corresponding to the eigenvalues λ_p and λ_q respectively. Because of the existence of zero eigenvalue, there exist an invariant manifold spanned by diagonal, hence assumption H_5 is satisfied.

Synchronization is said to occur if the solutions of Eq. (3.24) belongs to the set;

$$\mathfrak{M}_2 = \{Z \in \mathbb{R}^{n\times n} \mid Z_j = Z_{j+1}, Z_j \in \mathfrak{M}_\infty, 1 \le j \le n-1\}$$

This set is referred to as synchronization manifold

In order to understand the dynamics of the set $\mathfrak{M}_2$, we define a transformation that decomposes Eq. (3.24) into equations describing the transverse and tangential flows. The transformation used in the one dimensional case can be rewritten for the two dimensional case as;

$$W_j = Z_j + Z_{j+1},\ 1 \le j \le n-1,\ Z_j \in \mathbb{R}^{nd}$$

$$Y = \frac{1}{n}\sum_{j=1}^{n} Z_j,\ \ Y \in \mathbb{R}^{nd} \tag{3.25}$$

where $W = [w_{ij}]_{i,j=1}^{n,n-1} = (W_1, W_2, W_3, ..., W_{n-1})$ with $W_r \in \mathbb{R}^{nd}$, $1 \le r \le n-1$ as its r^{th} column, is the matrix formed by finding the differences between the interacting oscillators coupled on a square lattice and $Y = (y_1, y_2, ..., y_n)^T$, with $y_i = \sum_{j=1}^{n} z_{ij}$

The first equation in (3.25) describes the differences between the interacting oscillators, thus defines the deviations from the invariant manifold.

Let e_j be the usual unit vector in $\mathbb{R}^n$. The set $e, \tilde{e}_j,\ 1 \le j \le n-1$ is an orthogonal basis for $\mathbb{R}^n$, and in this basis we can write Z as;

$$vecZ = e \otimes Y + (\tilde{e} \otimes I_n) vecW, \quad W \in \mathbb{R}^{n^2 d - nd} \tag{3.26}$$

Using equations (3.25) and (3.26) in equation (3.24) we get;

$$vec\dot{W} = k[(\Delta_1 \oplus \Delta) \otimes I_d] vecW + vec\tilde{F}(W, Y)$$

$$\dot{Y} = k\Delta_1 \otimes I_d Y + \frac{1}{n}\sum_{j=1}^{n} f(Z_j), \tag{3.27}$$

where $\tilde{F}(W,Y) = (\tilde{F}_1(W,Y), \tilde{F}_2(W,Y), ..., \tilde{F}_{n-1}(W,Y))$ with

$\tilde{F}_j(W,Y) = f(Z_j) - f(Z_{j+1})$

The first equation in (3.27) describes the transverse motion to the synchronization manifold $\mathfrak{M}_2$ and the second equation describes the motion tangent to the manifold $\mathfrak{M}_2$.

Symmetric synchronization is equivalent to $z_{p,q} - z_{r,s} \to 0$ as $t \to \infty$ for all $p,q,r,s \in \mathbb{Z}_+$. This means that the deviations w_{ij} dampen out as $t \to 0$, and we have $W = 0$. Thus synchronization implies that the zero solution of the first equation in (3.27) is asymptotically stable and therefore the synchronization manifold is attracting.

Since our interest is local synchronization of the invariant manifold, we shall seek to show that the manifold is locally attracting. Attractivity of the manifold is measured using Lyapunov exponents.

The manifold $\mathfrak{M}_2$ is stable if the Lyapunov exponents of the flow defined by the first equation in Eq. (3.25) are less than zero.

Consider the linearization of Eq. (3.24) along $\mathfrak{M}_2$ thus;

$$vec\dot{Z} = [k(\Delta_1 \oplus \Delta_1) \otimes I_d + \hat{A}(Z(t,z_0))]vecZ, \tag{3.28}$$

where $Z(t,z_0)$ is the solution of Eq. (3.24) with $Z(0,z_0) = z_0 \in \mathfrak{M}_2$ and $\hat{A}(Z)$ is the Jacobian matrix of *G* at *Z*.

Let $\hat{\Phi}(t;z_0)$, be the fundamental matrix solution of Eq. (3.28) which has an invariant splitting with respect to $\hat{\Phi}(t;z_0)$, such that;

$\hat{\Phi}(t;z_0) = \hat{\Phi}_c(t;z_0) \oplus \hat{\Phi}_s(t;z_0)$;

Where the indices c denotes centre manifold and s denotes stable manifold, while $\hat{\Phi}_c(t;z_0)$ and $\hat{\Phi}_s(t;z_0)$ are restrictions of $\hat{\Phi}(t;z_0)$ on $T_{z_0}\mathfrak{M}_2$ and N_{z_0} respectively, with $T_{z_0}\mathfrak{M}_2$ being the bundle of vectors tangent to the manifold at z_0 and N_{z_0} is the bundle of vectors normal to the manifold at z_0. Stability and persistence is determined by the growth rates of vectors transverse to the manifold and those tangential to the manifold. These growth rates are characterized by generalized Lyapunov type numbers defined in Eq. (2.26)

The following theorem relates stability and Lyapunov type numbers;

Theorem 3.7.1 (Chow, *et al*., 1997).

If system (3.22) satisfies assumptions $\mathbf{H_5}$ and $\mathbf{H_6}$, then there exist a k_o such that for all $k > k_o$, there is a positively invariant synchronized manifold $\mathfrak{M}_2$ that is attracting and c^1 stable or persistent.

Proof

The proof of this theorem is done using the results due to Chow and Liu which are contained in Lemmas (3.6.2) and (3.6.3) which can be rephrased to refer to $\mathfrak{M}_2$.

The same results as in one dimensional coupling holds; that is, the corresponding invariant manifold $\mathfrak{M}_2$ is locally attracting hence synchronized if $\alpha(z_0) < 0$. Also, if

$\mathfrak{M}_2$ is locally synchronized, the synchronization is persistent if and only if $\alpha(z_0) < 0$ and $\beta(z_0) < 1$ for all $z_0 \in \mathfrak{M}_2$.

To characterize growth rates in terms of Lyapunov type numbers, we need to linearize system (3.27) along the solution $(W(t), Y(t)) = (0, Y_0(t))$;

where $Y_0(t) = (y_{01}(t), y_{02}(t), y_{03}(t), ..., y_{0n}(t))$ a solution to Eq. (3.22); gives;

$$\begin{pmatrix} vec\dot{W} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} k(\Delta_1 \oplus \Delta) \otimes I_d + I_{n-1} \otimes DG(Y_0(t)) & 0 \\ 0 & DG(Y_0(t)) \end{pmatrix} \begin{pmatrix} vecW \\ Y \end{pmatrix} \tag{3.29}$$

where $DG(Y_0(t)) = I_n diag D_z g(y_0(t))$ is the Jacobian matrix of f at $Y_0(t)$.

Notice that there is an invariant splitting of $T\mathbb{R}^{n^2 d}$ into $T\mathfrak{M}_2 \oplus N$, where $T\mathfrak{M}_2 \in \mathbb{R}^{n^2 d - nd}$ and $N \in \mathbb{R}^{nd}$, where $\oplus$ refers to the direct sum of subspaces.

From Lemma (3.6.2) $W = 0$ is locally synchronized if the Lyapunov exponents of the trajectory defined by the first equation in Eq. (3.29) is less than zero.

Since Eq. (3.29) is uncoupled, and the matrices $k(\Delta_1 \oplus \Delta) \otimes I_d$ and $I_{n-1} \otimes DG(Y_0(t))$ commute, the fundamental matrix solution of Eq. (3.29) is of the form;

$$\begin{pmatrix} vecW(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} e^{(k\hat{\lambda}_\zeta + \lambda_i)t} & 0 \\ 0 & e^{\lambda_j t} \end{pmatrix} = \hat{\Phi}(t; t_0)$$

where $k\hat{\lambda}_\zeta$, $1 \le \zeta \le n^2 d - nd$ are the eigenvalues of $k(\Delta_1 \oplus \Delta)$, λ_i and λ_j, $1 \le i \le n^2 d - nd$, $1 \le j \le nd$ are the Lyapunov exponents over $\mathfrak{M}_2$. The maximum eigenvalues $\hat{\lambda}_\zeta$ of $(\Delta_1 \oplus \Delta)$ will be the maximum eigenvalue of $\Delta_1 = 0$ plus the maximum eigenvalue of $\Delta = \xi_1$; that is, $-\xi_2 = -4\sin^2\frac{s\pi}{2n} = \xi_1$. Let λ_M and λ_m be the maximal and minimal Lyapunov exponents over $\mathfrak{M}_2$. Then from the definition of Lyapunov type numbers in (2.26), we realize that the Lyapunov type number $\alpha(z_0)$ corresponding to $-k\xi_1 + \lambda_M$ and it is less than zero if $k\xi_1 > \lambda_M$, thus the synchronization manifold is locally attracting if;

$$k > \frac{\lambda_M}{\xi_1},$$

According to Lemma (3.6.3), synchronization manifold $\mathfrak{M}_2$ is stable if $\alpha(z_0) < 0$ and $\beta(z_0) < 1$.

Since $\beta(z_0)$ corresponds to $\frac{-k\xi_1 + \lambda_M}{\lambda_m}$, it follows that;

$\beta(z_0) < 1$ if $k > \frac{1}{\xi_1}(\lambda_M - \lambda_m)$, thus the manifold is stable and robust if satisfies the inequality;

$$k > \max\{\frac{1}{\xi_1}(\lambda_M, \lambda_M - \lambda_m)\}$$

Taking $k_0 > \max\{\frac{1}{\xi_1}(\lambda_M, \lambda_M - \lambda_m)\}$ completes the proof.

CHAPTER FOUR

4.0 INTRODUCTION

Explicit Runge-Kutta method can be used to solve delay differential equations. The dde23 mat lab solver is closely related to ode23 solver (Shampine, A. *et al*., 1997), which implements the BS (2, 3) triple (Shampine, A. *et al*., 1989). A triple of s stages involves three formulas. Let g_n be the approximation to $g(z)$ at z_n and an approximation at $z_{n+1} = z_n + h_n$ is computed . For $i = 1, 2, ..., s$ the stages $f_{ni} = f(x_{ni}, y_{ni})$ were defined in terms of $z_{ni} = z_n + c_i h_n$ and $z_{ni} = z_n + h_n \sum_{j=1}^{i-1} a_{ij} f_{nj}$ where $c_i = \sum_{j=1}^{i-1} a_{ij} f_{nj}$ and h_n is the step size.

The approximation used to advance the integration is given by;

$$g_{n+1} = g_n + h_n \sum_{i=1}^{s} b_i f_{ni} \tag{4.1}$$

The solution satisfies the following formula $g(z_{n+1}) = g(z_n) + h_n \phi(z_n, g(z_n)) + t_e$ with a truncation error t_e. For sufficiently smooth f and $g(z)$ this error is $\bigcirc(h_n^{p+1})$. The triple includes another formula;

$$g^*{}_{n+1} = g_n + h_n \sum_{i=1}^{s} b^*{}_i f_{ni} = g_n + h_n \phi^*(z_n, g_n) \tag{4.2}$$

This second formula was used only for selecting the step size. The third formula has the form;

$$g_{n+\sigma} = g_n + h_n \sum_{i=1}^{s} b_i(\sigma) f_{ni} \tag{4.3}$$

The coefficients $b_i(\sigma)$ are polynomials in σ, so this represents a polynomial approximation to $g(z_n + \sigma h_n)$ for $0 \le \sigma \le 1$. We assumed that this formula yielded the value g_n when $\sigma = o$ and g_{n+1} when $\sigma = 1$. For this reason the third formula is described as a continuous extension of the first. A much more serious assumption is that the order of the continuous extension is the same as that of the first formula. These assumptions hold for the BS (2,3) triple. For such triples we regard the formula used to advance the integration as just the special case $\sigma = 1$ of the continuous extension we can write;

$$g_{n+\sigma} = g_n + h_n \phi(z_n, g_n, \sigma) \tag{4.4}$$

When using the explicit Runge-Kutta triple in solving the DDE given, a strategy for handling the history terms $g(z_n - \tau_j)$ that appear in $f_{ni} = f(z_{ni}, g_{ni}, g(z_{ni} - \tau_1), \ldots\ldots, g(z_{ni} - \tau_k))$ is needed. Two situations must be distinguished; $h_n < \tau_j$ and $h_n > \tau_j$ for some j. Suppose that we have the approximation $S(z)$ to $g(z)$ for all $z \le z_n$, if $h_n \le \tau$, then all the $z_{ni} - \tau_j \le z_n$ and $f_{ni} = f(z_{ni}, g_{ni}, S(z_{ni} - \tau_1), \ldots\ldots, S(z_{ni} - \tau_k))$ is an explicit recipe for the stage and the formulae are explicit. The function $S(z)$ is the initial history for $z \le a$. After taking the step to z_{n+1}, we use the continuous extension to define $S(z)$ on $[z_n, z_{n+1}]$ as $S(z_n + \sigma h_n) = g_{n+\sigma}$. This suffices for proving convergence as the maximum step size tends to zero, but we must take up the other situation because in practice, we may very well want to use a step size larger than the smallest delay.

When $h_n < \tau_j$ for some *j*, the "history" term *S* (*z*) is evaluated in the span of the current step and the formulas are defined implicitly. In this situation the formulas were evaluated with simple iteration. On reaching z_n, we will have defined *S* (*z*) for $z \le z_n$. The definition is extended somehow to $(z_n, z_n + h_n)$ and the resulting function called $S^{(0)}(z)$. A typical stage of simple iteration begins with the approximate solution $S^{(m)}(z)$. The next iterate is then computed with the explicit formula;

$$S^{(m+1)}(z_n + \sigma h_n) = g_n + h_n \phi(z_n, g_n, \sigma; S^{(m)}(z)). \tag{4.5}$$

4.1 NUMERICAL RESULT

The numerical solution of system (1.44) for various values of τ was computed by choosing $g(z_j(t))$ of the form:

$$g(z_j(t)) = \begin{pmatrix} f_1(x_i, y_i) \\ f_2(x_i, y_i) \end{pmatrix}$$

where

$$f_1(x, y) = y - \alpha(\frac{x^3}{3} - x) \qquad \alpha > 0$$

$$f_2(x, y) = -x \tag{4.6}$$

The choice of f_i is plausible because in most diffusive path-like problems, for instance Chemistry where coupling is effected by the flow of reactants from one reactor to the other, the flow of various reactants through the connecting medium of the coupled oscillators is the same. Thus a time delay is required for x_i to be transferred from oscillator *i* to oscillator *j* and vice versa.

We compare the solution of system (1.44) by plotting the trajectories of its solution for the various values of τ and that of $\tau_o = 0$.

4.1.1 The One Dimensional Case

We wish to illustrate the importance of the results with a specific example, we consider the coupled Van der pol oscillator for the choice of the parameter $\alpha = 1$. Where we chose $n = 4$, and $d = 2$.

We compute and compare numerical solution of system (1.44) by using Eq. (4.4);

The arrangement of oscillators as described by Eq. (3.15) with $n = 4$, is given by;

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \\ \dot{z}_3(t) \\ \dot{z}_4(t) \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{pmatrix} + \begin{pmatrix} g(z_1(t)) \\ g(z_2(t)) \\ g(z_3(t)) \\ g(z_4(t)) \end{pmatrix} \tag{4.7}$$

The eigenvalues of the coupling matrix (4.7) are;

$$\lambda_s = (-2 - 2\cos\frac{s\pi}{8}), \qquad s = 0,1,2,3.$$

where $\lambda_0 = 0,\ \ \lambda_1 = -3.8476k,\ \ \lambda_2 = -34142k,\ \ \lambda_3 = -2.7654k$. Thus, the existence of zero eigenvalues shows $\mathbf{H_3}$ that is satisfied.

To investigate whether Eq. (4.7) satisfies $\mathbf{H_4}$, we will have to use transformation (3.14) with;

$z = ye + \tilde{e}w$, $w = (w_1, w_2, w_3, w_4)^T$, $w \in \mathbb{R}^6$, $y \in \mathbb{R}^2$,

where in this case,

$$e=\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \text{ and } \tilde{e}=\frac{1}{4}\begin{pmatrix}3 & 2 & 1\\-1 & 2 & 1\\-1 & -2 & 1\\-1 & -2 & -3\end{pmatrix}$$

Using the transformation in Eq. (4.7), we get;

$$\begin{pmatrix}\dot{w}_1(t)\\\dot{w}_2(t)\\\dot{w}_3(t)\end{pmatrix}=k\begin{pmatrix}-2 & 1 & 0\\1 & -2 & 1\\0 & 1 & -2\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}w_1(t)\\w_2(t)\\w_3(t)\end{pmatrix}+\begin{pmatrix}g(z_1(t))-g(z_2(t))\\g(z_2(t))-g(z_3(t))\\g(z_3(t))-g(z_4(t))\end{pmatrix},$$

$$\dot{y}(t)=\frac{1}{4}\{g(z_1(t))+g(z_2(t))+g(z_3(t))+g(z_4(t))\}$$

which is equivalent to;

$$\begin{aligned}\dot{w}(t)&=(k\Delta\otimes I_2)w+F(w,y)\\\dot{y}(t)&=\frac{1}{4}\{g(z_1(t))+g(z_2(t))+g(z_3(t))+g(z_4(t))\}\end{aligned}\tag{4.8}$$

With $F(w,y)=g(z_1(t))-g(z_2(t)), g(z_2(t))-g(z_3(t))$, and $g(z_3(t))-g(z_4(t))$.

The first equation in Eq. (4.8) describes motion transverse to the manifold $\mathfrak{M}_1=\{z: z_1=z_2=z_3=z_4\}$ and the second equation describes the motion tangential to the manifold.

The eigenvalues of the coupling matrix in (4.8) are;

$$\lambda_\zeta=(-2-2\cos\frac{\zeta\pi}{4})k \qquad \zeta\in\{1,2,3\},$$

each occurring two times, and all these are less than zero. Synchronization occurs when the deviations have dampen out; that is, $\{w=0\}$.

To understand the dynamics on the manifold $\mathfrak{M}_1$, we will have to linearize Eq. (4.8) along the solution $(0, y_0(t))$ on the manifold $\mathfrak{M}_1$. This linearization yields;

$$\begin{pmatrix} \dot{w}_1(t) \\ \dot{w}_2(t) \\ \dot{w}_3(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} k\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + I_3 \otimes \tilde{w} & 0 \\ 0 & \tilde{w} \end{pmatrix} \begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \\ y(t) \end{pmatrix}$$

This is equivalent to;

$$\begin{pmatrix} \dot{w} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} (k\Delta \otimes I_2 + \varphi) & 0 \\ 0 & \tilde{w} \end{pmatrix} \begin{pmatrix} w \\ y \end{pmatrix} \tag{4.9}$$

where $\tilde{w} = D_z g(y_0(t))$ and $\varphi = I_3 \otimes \tilde{w}$.

System (4.9) is uncoupled, and thus each equation can be solved independently. Since $k\Delta \otimes I_2$ and φ commute, the solution of the first equation in (4.9) is of the form;

$$w(t) = \Phi_s(t, t_0) \simeq e^{(k\lambda_\zeta + \lambda_i)t}, \quad \zeta = 1,2,3, \quad \text{and} \quad i = 1,2,3$$

where $k\lambda_\zeta$ is the eigenvalue of $k\Delta \otimes I_2$ and λ_i is the Lyapunov exponents over φ.

The solution of the second equation similarly will be of the form;

$$y(t) = \Phi_c(t; t_0) \simeq e^{\lambda t}$$

Note that $\{w = 0\}$ is locally attracting and thus the invariant manifold $\mathfrak{M}_1$ is stable if the maximum of $k\lambda_\zeta + \lambda_i$ is less than zero. But we know that the $\max \lambda_\zeta = -4k \sin^2 \frac{\pi}{8}$, let λ_M be the maximum Lyapunov exponent of the trajectory defined by $\dot{y} = \tilde{w} y$. Using the 4th and 5th order Runge-Kutta method, with a step size of 0.01 and up to 5000 iterations, we obtain;

$\lambda_M \simeq 1.6361$,

and using the same algorithm, we obtain;

$\lambda_m \simeq 0.1993$.

If z_0 is a point on $\mathfrak{M}_1$, then from the definition of generalized Lyapunov exponents, we have;

$$\alpha(z_0) = -4k\sin^2\frac{\pi}{8} + \lambda_M \text{ or } -0.5858 + 1.6361$$

Stability criterion is therefore satisfied if $\alpha(z_0) < 0$; that is,

$$k > \frac{1}{0.5858}(1.6361) \simeq 2.8,$$

For persistence, we require that;

$$\sup_{z_0}\|\Phi_s(t;t_0)\| < \inf_{z_0} m(\Phi_c(t;t_0)),$$

This is the same as saying that;

$$\lim_{t\to\infty}\sup\frac{\ln\|\Phi_s(t;z_0)\|}{\ln m(\Phi_c(t;z_0))} < 1 = \beta(z_0)$$

If $\alpha(z_0) < 0$ and $\beta(z_0) < 1$, the manifold $\mathfrak{M}_1$ persists under small perturbations. This implies that;

$k > \frac{1}{0.5858}(\lambda_M - \lambda_m)$,which is equivalent to; $k > \frac{1}{0.5858}(1.6361 - 0.1993) \simeq 2.45$.

Thus we conclude that for synchronization, stability and robustness of the synchronization manifold $\mathfrak{M}_1$, k must satisfy the inequality;

$$k > \max\{\frac{1}{0.5858}(\lambda_M, \lambda_M - \lambda_m)\},$$

$$k > \max(2.8, 2.45)$$

$$k_0 = 2.8.$$

4.1.2 The Two Dimensional Case

Consider coupling of 4×4 oscillators on a simple square lattice. Let ;

$Z = \begin{pmatrix} z_{1,1} & z_{1,2} & z_{1,3} & z_{1,4} \\ z_{2,1} & z_{2,2} & z_{2,3} & z_{2,4} \\ z_{3,1} & z_{3,2} & z_{3,3} & z_{3,4} \\ z_{4,1} & z_{4,2} & z_{3,4} & z_{4,4} \end{pmatrix} = (Z_1, Z_2, Z_3, Z_4)$ be the matrix formed when oscillators are coupled on a square Lattice. Since a square lattice oscillators interact amongst themselves on each column and at the same time interact amongst themselves on each row, the system of differential equations describing the dynamics on the 4×4 lattice is given by;

$$\dot{Z} = k\left[\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Z + Z \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}\right] + G(Z) \tag{4.10}$$

where;

$$G(Z) = \begin{pmatrix} g(z_{1,1}) & g(z_{1,2}) & g(z_{1,3}) & g(z_{1,4}) \\ g(z_{2,1}) & g(z_{2,2}) & g(z_{2,3}) & g(z_{2,4}) \\ g(z_{3,1}) & g(z_{3,2}) & g(z_{3,3}) & g(z_{3,4}) \\ g(z_{4,1}) & g(z_{4,2}) & g(z_{3,4}) & g(z_{4,4}) \end{pmatrix} = (f(Z_1), f(Z_2), f(z_3), f(Z_4)).$$

Let $vecZ = \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{pmatrix}$ with Z_r as its r^{th} column, thus Eq. (4.8) can be written in vector notation as;

$$vec\dot{Z} = k(\Delta_1 \oplus \Delta_1) \otimes I_2 vecZ + vecG(Z) \tag{4.11}$$

The eigenvalues of matrix $k(\Delta_1 \oplus \Delta_1) \otimes I_2$ are $k(\lambda_p + \lambda_q)$, $p,q = 0,1,2,3$ each appearing two times, where;

$\lambda_q = (-2 - 2\cos\frac{q\pi}{8})k$, $q = 0,1,2,3$ and $\lambda_p = (-2 - 2\cos\frac{p\pi}{8})k$, $p = 0,1,2,3$, where

$\lambda_0 = 0$, $\lambda_1 = -3.8476k$, $\lambda_2 = -3.4142k$, $\lambda_3 = -2.7654k$

Clearly, there exist a zero eigenvalue and thus H_5 is satisfied.

To investigate if H_6 holds, we use the transformation (3.23) and (3.24) with;

$vecZ = e \otimes Y + (\tilde{e} \otimes I_n) vecW, \quad W \in \mathbb{R}^{24}$

to split Eq. (4.11) into tangential and transversal components to $\mathfrak{M}_2$, where in this case;

$$e = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \tilde{e} = \frac{1}{4}\begin{pmatrix} 3 & 2 & 1 \\ -1 & 2 & 1 \\ -1 & -2 & 1 \\ -1 & -2 & -3 \end{pmatrix}$$

Using this transformation, Eq. (4.11) satisfies Eq. (3.27). This can be written in compact form as;

$$\begin{pmatrix} vec\dot{W} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} k(\Delta_1 \oplus \Delta) \otimes I_2 + \hat{w} \otimes I_3 & 0 \\ 0 & k\Delta_1 \otimes I_2 + \hat{w} \end{pmatrix} \begin{pmatrix} vecW \\ Y \end{pmatrix} + \begin{pmatrix} vec\hat{F}(W,Y) \\ F(\tilde{Y}) \end{pmatrix} \tag{4.12}$$

where for this specific example;

$$\Delta_1 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \Delta = \begin{pmatrix} -2 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}, \quad vec\hat{F}(W,Y) = \begin{pmatrix} f(Z_1(t)) - f(Z_2(t)) \\ f(Z_2(t)) - f(Z_3(t)) \\ f(Z_3(t)) - f(Z_4(t)) \end{pmatrix}$$

and $\hat{w} = \begin{pmatrix} \hat{w}_1 & 0 & 0 & 0 \\ 0 & \hat{w}_2 & 0 & 0 \\ 0 & 0 & \hat{w}_3 & 0 \\ 0 & 0 & 0 & \hat{w}_4 \end{pmatrix}$

with $\hat{w}_j = D_z g(y_{0,j}(t))$, $1 \le j \le 4$ while;

$$F(\tilde{Y}) = \frac{1}{4}\{f(Z_1(t)) + f(Z_2(t)) + f(Z_3(t)) + f(Z_4(t))\}$$

Linearization of Eq. (4.12) about $(W(t), Y(t)) = (0, Y_0(t)) \in \mathfrak{M}_2$ yields;

$$\begin{pmatrix} vec\dot{W} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} k(\Delta_1 \oplus \Delta) \otimes I_2 + \hat{w} \otimes I_3 & 0 \\ 0 & k\Delta_1 \otimes I_2 + \hat{w} \end{pmatrix} \begin{pmatrix} vecW \\ Y \end{pmatrix}$$

which on $\mathfrak{M}_2$ Eq. (4.12) reduces to;

$$\begin{pmatrix} vec\dot{W} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} k(\Delta_1 \oplus \Delta) \otimes I_2 + \hat{w} \otimes I_3 & 0 \\ 0 & \hat{w} \end{pmatrix} \begin{pmatrix} vecW \\ Y \end{pmatrix} \tag{4.13}$$

Since Eq. (4.13) is decoupled, the solution is of the form;

$$\begin{pmatrix} vec\dot{W} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} e^{(k\lambda_\varsigma + \lambda_i)t} & 0 \\ 0 & e^{\lambda_j t} \end{pmatrix} = \Phi(t, t_0) \tag{4.14}$$

where; $1 \le i \le n^2 d - nd$, $1 \le j \le nd$

The synchronization manifold $\mathfrak{M}_2$ is attracting if;

$\max(k\lambda_{\zeta} + \lambda_i) < 0$

This is satisfied if;

$k > \frac{1}{\xi}\lambda_M$,

where λ_M being the largest Lyapunov exponent of the trajectory defined by $\dot{Y} = \hat{w}Y$ and $\xi = -4\sin^2\frac{\pi}{8}$. We know that $\xi = -0.5858$. Computation using 4th and 5th order Runge-Kutta method with step size of 0.01 up to 5000 iterations, we obtain the maximum and minimum Lyapunov exponents as $\lambda_M = 1.6361$ and $\lambda_m = 0.1993$ respectively. The condition that guarantees synchronization is thus given as;

$k > \frac{1}{0.5858}(1.6361)$, or $k > 2.8$.

For persistence, we require that;

$\frac{\lambda_M - k\xi}{\lambda_m} < 1$; that is, $k > \frac{1}{\xi}(\lambda_M - \lambda_m)$

where λ_m is the minimum Lyapunov exponent of the Jacobian matrix $\hat{w}$. Thus stability and persistence is satisfied if;

$k > \max\frac{1}{\xi}\{\lambda_M, (\lambda_M - \lambda_m)\}$, that is, $k > \max\frac{1}{0.5858}\{1.6361, (1.6361 - 0.1993)\}$ or $k > \max(2.8, 2.45)$,which implies that; $k_0 = 2.8$.

4.2 DISCUSSION OF RESULTS

The following figures (1.2) to (1.6) depict the effect of varying the values of the coupling strength and perturbation on stability and persistence of the synchronized manifold.

Firstly we considered the weakly coupled Van der pol system which satisfies assumptions H_1 and not H_2 that is, the coupling strength k is zero;

$$\dot{z}_1 = \begin{bmatrix} \dot{x}_1 = y_1 - (\frac{1}{3}x_1^3 + x_1) + k(x_2 - x_1), \\ \dot{y}_1 = -x_1 + k(y_2 - y_1), \end{bmatrix}$$
$$\dot{z}_2 = \begin{bmatrix} \dot{x}_2 = y_2 - (\frac{1}{3}x_2^3 + x_2) + k(x_1 - x_2), \\ \dot{y}_2 = -x_1 + k(y_1 - y_2) \end{bmatrix}. \tag{4.15}$$

The 4th order Runge-Kutta numerical integration of the trajectory with initial conditions on the orbit $(1, 2, 1, 2)$ up to 18,000 iterations plotted in figures (1.2) – (1.6). In the figures, figure (1.2) – (1.6) (a) shows the orbit, where we pick the initial conditions, figure (1.2) – (1.6) (b) shows the invariant manifold, figure (1.2) – (1.6) (c) shows two graphs, x_1 versus t and x_2 versus t, and figure (1.2) – (1.6) (d) shows the graph of the differences (x_1-x_2) versus time.

Figure (1.2) shows the aspects (a)-to-(d) for uncoupled system (4.15) without any perturbation. Adding a small perturbation which is $\ll 1$ to the uncoupled Eq. (4.15), we obtain the system;

$$\begin{aligned}
\dot{x}_1 &= y_1 - (\frac{1}{3}x_1^3 + x_1) + k(x_2 - x_1) + 0.005x_1 + 0.002x_2, \\
\dot{y}_1 &= -x_1 + k(y_2 - y_1) + 0.01x_1 - 0.015x_2, \\
\dot{x}_2 &= y_2 - (\frac{1}{3}x_2^3 + x_2) + k(x_1 - x_2) + 0.003y_1 - 0.015x_2, \\
\dot{y}_2 &= -x_1 + k(y_1 - y_2) + 0.01y_1.
\end{aligned} \tag{4.16}$$

Applying the same numerical integration with the same initial conditions, figure (1.3) exhibits a chaotic oscillatory behavior and the magnitude of the deviations are rather large. Figure (1.4) shows weak coupling which satisfies assumption H_1 and not assumption H_2, with some perturbation. Although the differences are rather chaotic, their magnitude is small, which reflects a slight perturbation of the diagonal. The deviations' magnitude reduces further with a strong coupling as in figure (1.5). If the coupling strength is increased to k = 2.8 which satisfies assumption H_1 and assumption H_2, as in figure (1.6) the deviations will dampen out almost completely and we conclude that the manifold, for all coupling strength k >2.8 persist under small perturbations. This is because for synchronization to be of physical interest, the synchronized manifold needs to persist under small perturbations both the coupling and component systems.

From the numerical calculation of the maximum Lyapunov exponent, the delay times can result in unsynchronized systems. Such configurations can both enhance the absolute stability of the synchronized manifolds and minimize effects of convective instabilities. As the delay is increased, there is an effect on the stability of the synchronized manifold as can be seen from figures (1.7) to (2.7). Stability of the synchronization manifold normally relates with the time delay. If the coupling delays are less than a positive threshold, then the network of oscillators will be synchronized.

On the other hand, with the increase in coupling delays, the synchronizability of the network of oscillators will be restrained and even eventually desynchronized. As can be seen from figures (1.7) through (2.5), whenever the delay is less than nine, the synchronized manifold is stable. Once the time lag is greater than nine, the synchronized manifold is unstable as can be inferred from figures (2.6) and (2.7)

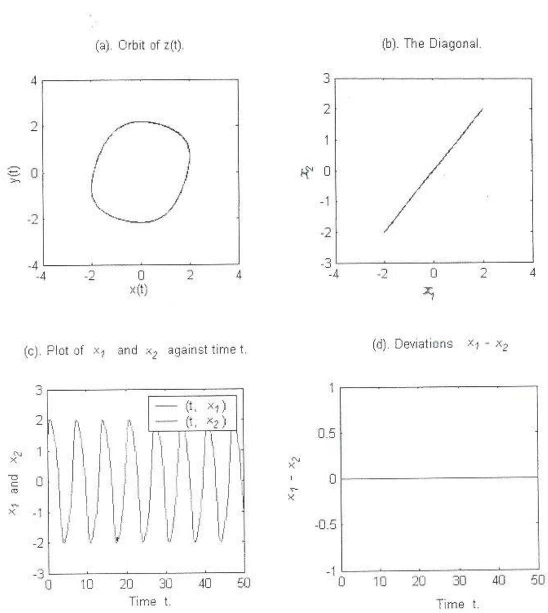

Figure 1.2 Uncoupled , unperturbed system

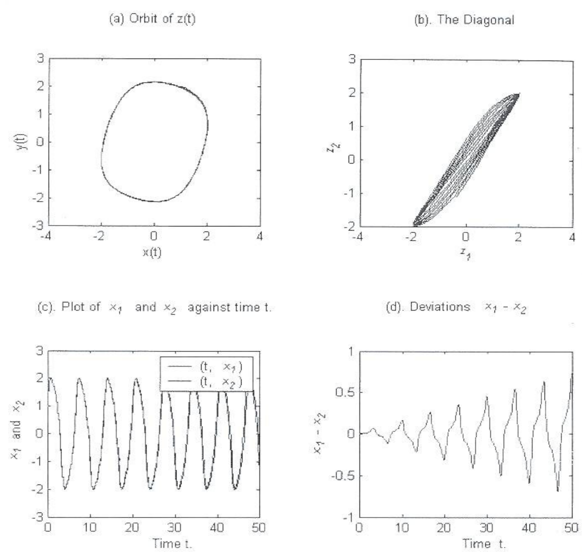

Figure 1.3 Uncoupled systems with small perturbation

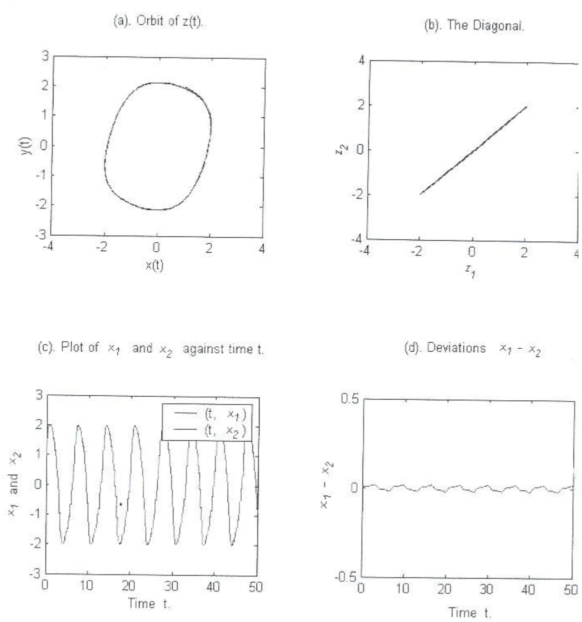

Figure 1.4 Weakly Coupled Systems with small perturbation k = 1

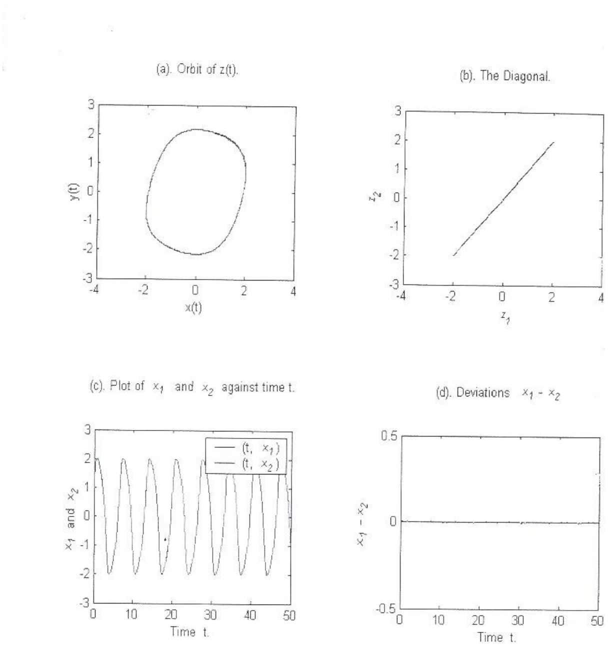

Figure 1.5 Perturbed systems with Coupling Strength k = 2

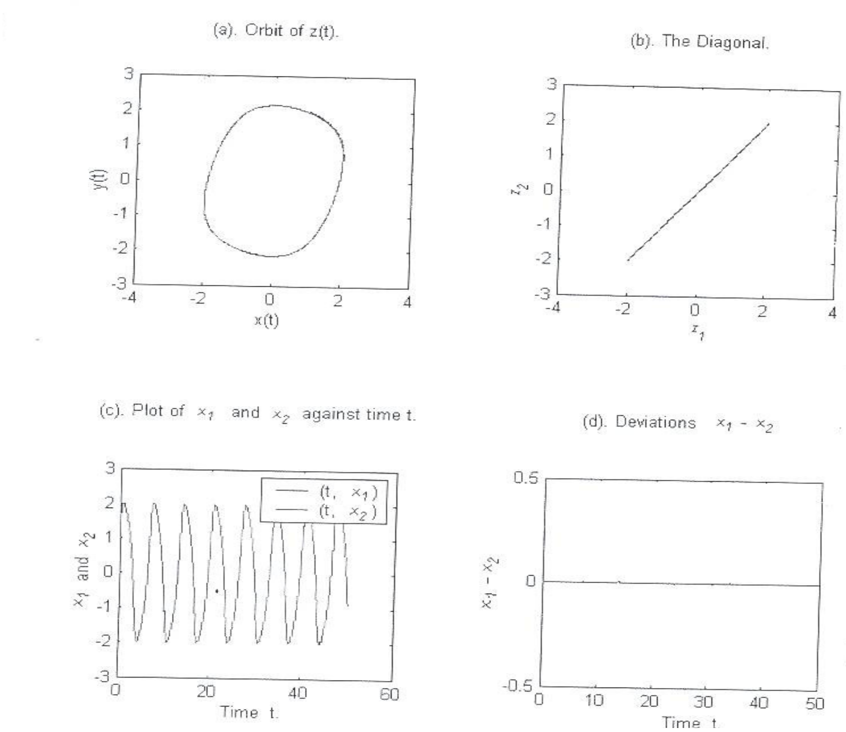

Figure 1.6 Perturbed Systems with Coupling Strength $k = 2.8$

From the numerical calculation of the maximum Lyapunov exponent, the delay times can result in unsynchronized systems. Such configurations can both enhance the absolute stability of the synchronized manifolds and minimize effects of convective instabilities. As the delay is increased, there is an effect on the stability of the synchronized manifold as can be seen from figures (1.7) to (2.7). Stability of the synchronization manifold normally relates with the time delay. If the coupling delays are less than a positive threshold, then the network of oscillators will be synchronized. On the other hand, with the increase in coupling delays, the synchronizability of the network of oscillators will be restrained and even eventually desynchronized. As can be seen from figures (1.7) through (2.5), whenever the delay is less than nine, the synchronized manifold is stable. Once the time lag is greater than nine, the synchronized manifold is unstable as can be inferred from figures (2.6) and (2.7).

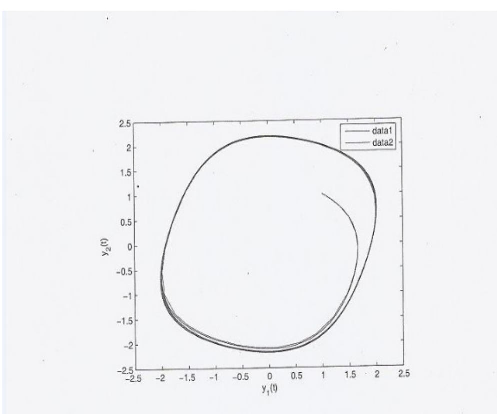

Figure 1.7. Trajectory System for System (1.44) for τ = 0.05 and $\tau_o = 0$

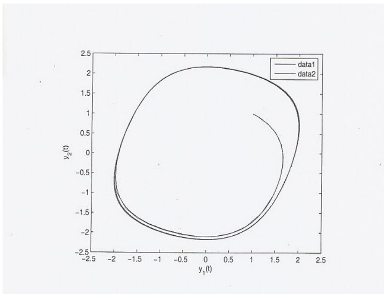

Figure 1.8. Trajectory System for System (1.44) for $\tau = 0.09$ and $\tau_o = 0$

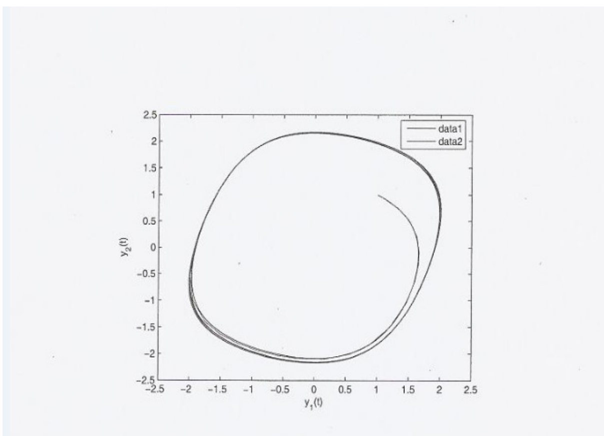

Figure 1.9. Trajectory System for System (1.44) for $\tau = 0.1$ and $\tau_o = 0$

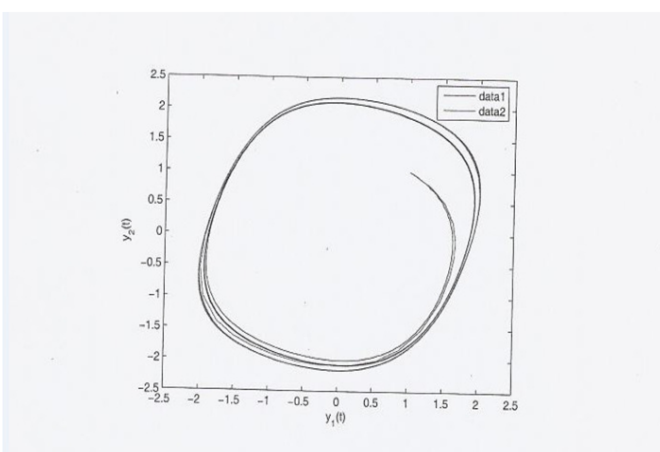

Figure 2.0. Trajectory System for System (1.44) for $\tau = 0.5$ and $\tau_o = 0$

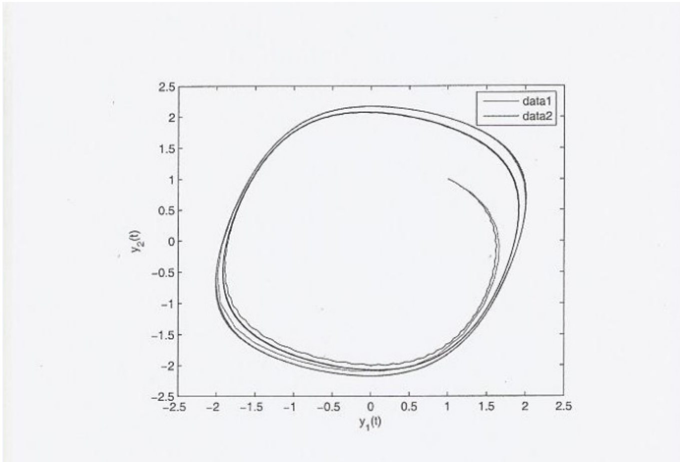

Figure 2.1. Trajectory System for System (1.44) for τ = 1 and $\tau_o = 0$

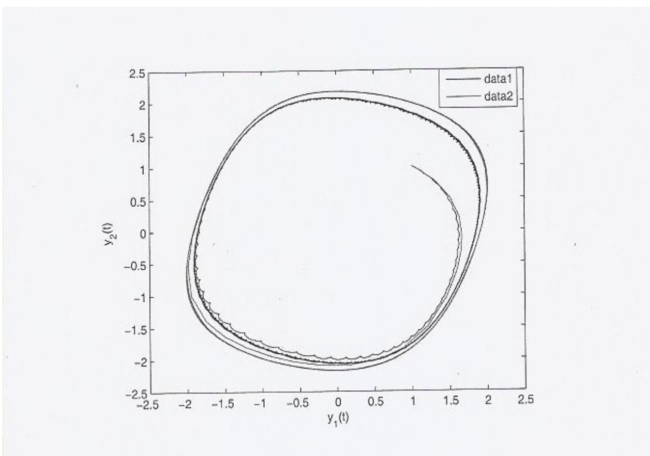

Figure 2.2. Trajectory System for System (1.44) for τ = 3 and $\tau_o = 0$

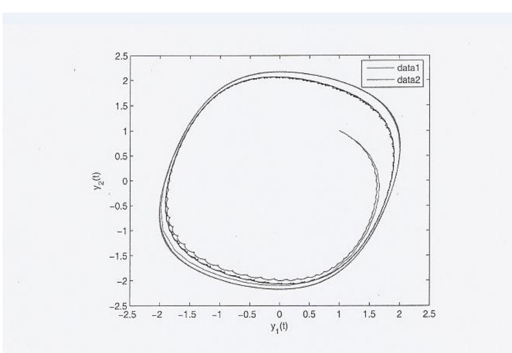

Figure 2.3. Trajectory System for System (1.44) for τ = 5 and $\tau_o = 0$

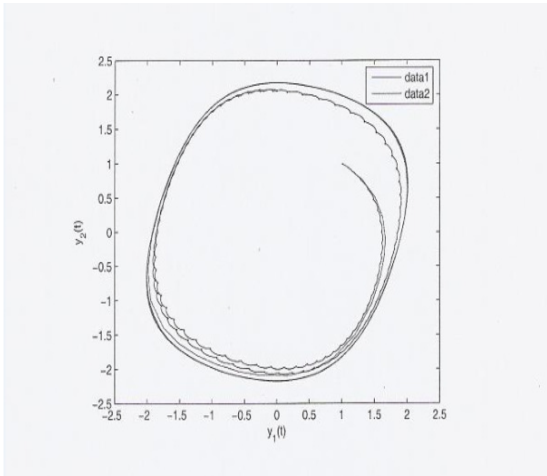

Figure 2.4. Trajectory System for System (1.44) for τ = 8 and $\tau_o = 0$

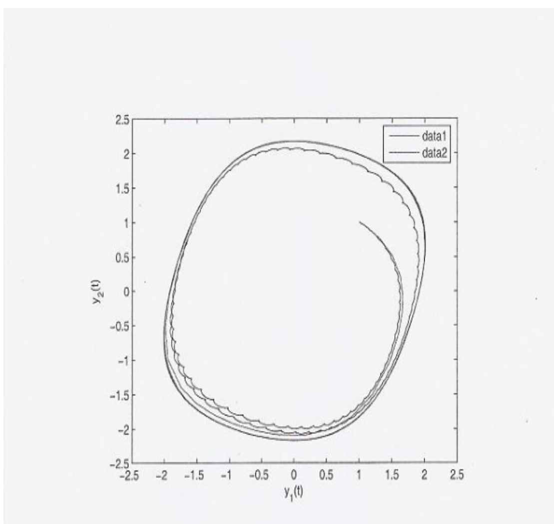

Figure 2.5. Trajectory System for System (1.44) for τ = 9 and $\tau_o = 0$

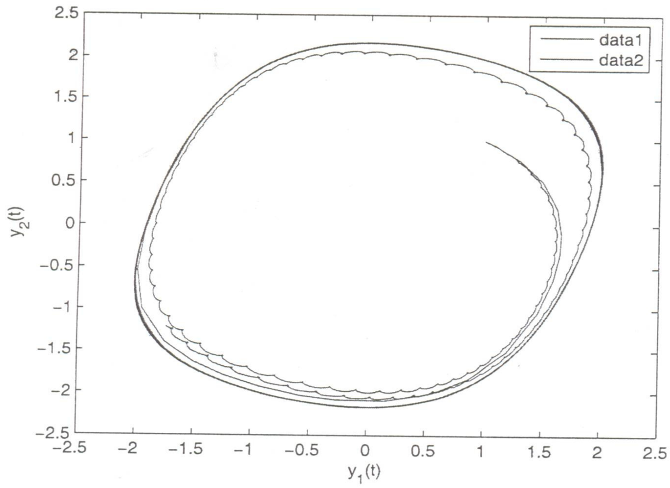

Figure 2.6 Trajectory System for system (1.44) τ = 10 and $\tau_o = 0$

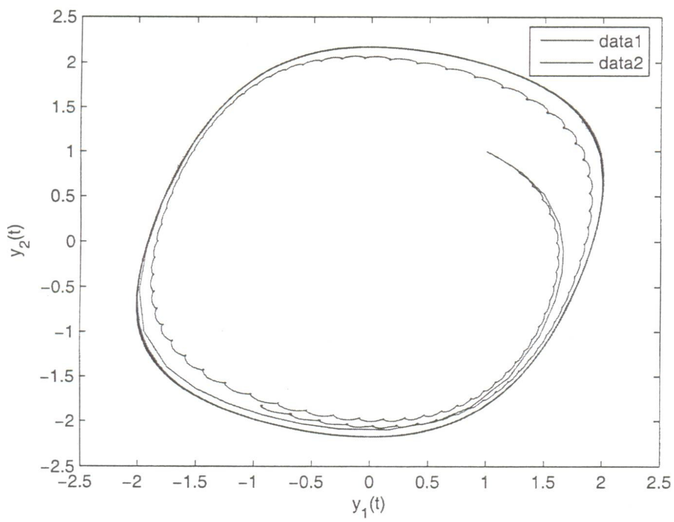

Figure 2.7 Trajectory for system (1.44) for τ = 10.5 and $\tau_o = 0$

CHAPTER FIVE

5.0 CONCLUSION

In this research it has been shown if synchronization exists for a certain coupling configuration, then there exist a $k_0 > 0$ such that for all $k > k_0$, synchronization manifold is stable and persist under perturbation. It is seen that this value depends on the maximum and minimum Lyapunov exponents over the invariant manifolds $\mathfrak{M}_1$ and $\mathfrak{M}_2$ and it also depends on the maximum and minimum eigenvalues of the coupling configuration. Thus the number of oscillators coupled, contribute to the determination of this value.

We established the estimate for bound of τ for which the synchronized manifold remain stable when the oscillators are coupled in an all-to-all configuration. The time lag τ from this study can be seen to be a function of the number oscillators (*n*), the coupling strength (*k*), and the maximum Lyapunov exponent of the system (3.11). For small *k*, we see that the stability range is big and for large *k* stability range is small and may not exist for very large *k*. Several ranges of coupling strengths and time delays that caused the oscillators to poop out (Oscillator death).

The effect of the time delay on the stability of the synchronized manifold can clearly be seen from the trajectories of system (1.43). Figures (1.7) to (2.5) show that the synchronized state is stable when $\tau < 9$, thus confirming our proposition. Synchrony of networks with time-delayed connections can be achieved at lower coupling strengths than within the same network with instantaneous couplings. Even for significant time delays, a stable synchronized state exists at a very low coupling strength, which may account for long-range neural synchrony observed in experiments.

Remarkably, an infinite number of eigenvalues—corresponding to the infinite dimensionality of the delay-differential equation linearized about the synchronized state—is kept in the left half plane. Stability condition holds for any network in which each oscillator receives k signals, independent of all other details of its topology.

5.1 FURTHER RESEARCH

Further research needs to be done on stability and persistence of systems coupled via periodic boundary conditions where time lag is incorporated. As far as this study is concerned, results were obtained from linearization and by considering the local dynamics. The effects of non-linear terms have not been addressed. This needs further investigation. The study has concentrated on synchronization, stability and persistence of oscillators coupled via one and two dimensional Bravis Lattice. There is need to consider three or more dimensional lattice coupling.

PAPERS PUBLISHED AND ACCEPTED FOR PUBLICATION

The following three papers from this research have been published and the forth one has been accepted for publication;

- Okwoyo, M.J., Abonyo, J.O. "*Synchronization of all-to-all Coupled Oscillators*" Journal of Mathematical Sciences 19, **2** (2008) 239-252
- Okwoyo, M.J. and Kinyanjui M. "*Mode locking (synchronization) in a coupled two Sector model*" Journal of Mathematical Sciences 21, **4** (2010) 483-491
- Okwoyo, M.J. Kinyanjui, M. Okelo, A.J. "*Stability and Persistence of Synchronized Manifold of Diffusively Coupled Oscillators*" Far East Journal of Dynamical Systems **15** (2) 2011, 113-128.
- Okwoyo, M.J. "*Modification of Parameters for weakly non-Linear Time Delay Systems*" has been accepted for Publication in the Journal of Mathematical Sciences.

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