ON THE SPECTRA OF NÖRLUND Q AND ALMOST NÖRLUND Q OPERATORS

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Declaration

This thesis is my original work and has not been presented for a degree in any other University.

Signature: ____________________ Date: ________________

Jotham Raymond Akanga

This thesis has been submitted for examination with my approval as University Supervisor.

Signature: ____________________ Date: ________________

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Dedication

This work is dedicated to my parents, Robert Nanjelo and Pauline Akhaboosa; My wife Helen Achungo; my sons and daughters: Jacqueline, Anne, Cleophas, Rispah, Davies, Antony and Hughes. My parents for having for ever toiled and incessantly prayed for the continued survival and success of their only son! My wife for having been encouraging and supportive. My kids for having always given me a reason to strive to achieve the best. By so doing, I have always hoped to provide a challenge and a strong role model.

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Notations

1. $\mathbb{N}; \mathbb{R}; \mathbb{R}^+; \mathbb{C}; \exists!; O(1); o(1); Re(z); \begin{matrix}\cup\\ \\ \cap\end{matrix}; \|.\|; \to; \not\to; \sharp$; will denote the set of all natural numbers; the set of real numbers; the set of all positive real numbers; the set of all complex numbers; there exists a unique; capital order, i.e., $x_n = O(1)$ if there exists $M \in \mathbb{R}^+$ such that $|x_n| \leq M, \forall n$; small order, i.e.,$x_n = o(1)$ as $n \to \infty$ if $x_n \to 0$ as $n \to \infty$; real part of the complex number z; lies between two positive constant multiplies, e.g $x_N \begin{matrix}\cup\\ \\ \cap\end{matrix} \frac{1}{N^\alpha}$ means that there exists $m, M \in \mathbb{R}^+$ such that $\frac{m}{N^\alpha} \leq x_N \leq \frac{M}{N^\alpha}$; norm of; tends to; does not tend to and end of proof respectively. In general,$\{\dots\}$ will denote "the set of" $(\dots)$ "the set sequence of" and $(\dots)^t$ "the transpose of the sequence of" unless otherwise specified.

2. $s; c_0; \ell_p (0 < p < \infty); c; \ell_\infty; bv; bv_0; bs; cs; E^\infty; w_p(0 < p < \infty); w_p(0); C_1$; or $(\mathrm{C}_1, 1); X^*, T^*$; Will denote the set of all sequences; the set of all sequences which converge to zero - null sequences; sequences such that

$$\sum_{k=0}^{\infty} |x_k|^p < \infty;$$

convergent sequences; bounded sequences,i.e., sequences x such that $= sup_k\, |x_k| < \infty$; sequences of bounded variation, i.e., sequence x such that $\sum_{k=0}^{\infty} |x_{k+1} - x_k| < \infty$; sequences of bounded variation with $x_k \to 0$ as $k \to \infty$; bounded series, i.e., sequences x such that $\sup_{n\geq 0} \{\sum_{k=0}^{n} |x_k|\} < \infty$; convergent series, i.e., sequences x such that $\sum_{k=0}^{\infty} x_k$ is convergent; finite sequences; the space of strongly Cesaro summable complex sequences of order 1 index p, i.e., the set of all sequences $x = (x_k)_{k=1}^{\infty}$ such that there exists a number ℓ depending on x for which $\sum_{k=0}^{\infty} |x_k - \ell|^p = o(n)$; the space of strongly Cesaro summable complex sequences of order 1 index p such that $\ell = 0$; Cesaro matrix of order 1, i.e.,

$$C_1 = (a_{nk}) = \begin{cases} \frac{1}{1+n} & 0 \leq k \leq n \\ 0 & k > n \end{cases};$$

the space of all continuous linear functionals on X i.e., the continuous dual of X and the adjoint operator of T respectively.

3. We shall frequently make use of the following special sequences

$$\delta = (1, 1, 1, \ldots); \delta^k = (\delta_n^k)_{n=0}^{\infty} = (0, 0, \ldots, 0, 1, 0, \ldots);$$

$$\triangle = (\delta^0, \delta^1 \delta^2, \ldots); \triangle^+ = (\delta, \delta^0, \delta^1, \ldots);$$

$\theta = (0, 0, 0, \ldots)$ - the zero sequence or operator;

4. $\sigma(T, X)$ will denote the spectrum of T on X, were $T \in B(X)$ and $\sigma(T)$ will simply denote the spectrum of T.

Abstract

The main objectives in this study are to investigate and determine the spectra of Nörlund means acting as operators on Banach sequence spaces. We also aim to determine the spectra of almost triangular matrices. Specifically we determine the spectrum of the Nörlund Q operator on the Banach spaces c_0, c, bv_0, bv. We also determine the spectrum of an almost Nörlund Q matrix operator on c_0 and c.

In all the cases mentioned we show that the spectrum comprises of the disc centered at the point $(\frac{1}{2}, 0)$ of radius $\frac{1}{2}$. We also construct the fine spectrum of the Q operator on c.

Apart from the more obvious benefit i.e., the solution of systems of linear equation of which the spectrum of operators is all about; there are other more subtle, but equally important applications of the research. A central problem in the whole of mathematics and even science and engineering; is the determination of the convergence or non- convergence of sequences and series. Mathematics, especially Mathematical analysis, develops and is maintained via the concept of convergence of sequences and series. Even in applied science and Engineering, one is interested in the convergence of a sequence or a series of results generated during experimentation. Established theorems such as the ratio theorems and integral theorem, are not applicable in a variety of sequences and series. Even where they apply, they just determine convergence and not the limit or sum of the convergent sequence or series. Tauberian theorems in Summability Theory handle this problem well.The convergence and even limit of a convergent sequence or series is determined from the convergence of some transform of it together with a side condition, (Maddox, 1970); (Boos, 2000) pp. 167 - 204; (Hardy, 1948) pp. 148 - 177; (Powell and Shah, 1972) pp. 75 - 92; or (Maddox, 1980) pp. 65 - 80, e.t.c. The spectrum of an operator plays a crucial role in the development of a Tauberian theory for the operator, (Dunford and Schwartz, 1957) pp. 593 - 597. It is evident from the mentioned books that a Tauberian theory for Nörlund operators is almost non-existent. Therefore we are confident the results

developed in the thesis will open a floodgate for such theorems for Nörlund means. In turn this will find application in diverse fields such as, integral transforms and Fourier analysis; and in probability and statistics through such areas involving central limit theorem, almost sure convergence, summation of random series, Markov chains e.t.c; (Boos, 2000) pp. 256 - 257.

Chapter 1 deals with literature review, a summary of Functional Analysis material; as well as classical summability methods; especially those that are pertinent to our study.

Chapter II deals with the spectrum of the Q matrix on c_0 and c. In chapter III we investigate the spectrum of the Nörlund Q operator on the spaces bv_0 and bv. Chapter IV is concerned with the fine spectrum of the Q matrix operator on c. In Chapter V we investigate the spectrum of an almost Nörlund Q matrix operator on c_0 and c. Chapter VI gives an overview of the results obtained and points the way forward for future research interests.

In achieving the results, we used a combination of classical and modern functional analytic methods as well as Summability methods. Functional analytic methods usually appeal to the powerful Banach space theorems, such as Hahn - Banach; Banach-Steinhaus; exetra. Classical Summability methods employ sequence space mapping theorems such as Silverman - Toeplitz; Kojima - Shur; exetra.

Chapter 1

Literature review and preliminaries

1.1 Introduction

This chapter is divided into five parts. In part two, we give an overview of the results achieved on the spectra of infinite matrices. The third part deals with classical Summability, particularly the concepts that are needed later in the thesis. In part four and five, we summarize functional analysis material that are needed in the thesis.

1.2 Literature Review

In 1960, E.K. Dorff and A. Wilansky showed that the spectrum of a certain mercerian Nörlund matrix with $a_{nn} = 1$, contains negative numbers, (Dorff and Wilansky, 1960) and (Wilansky, 1984), theorem 3. In 1965, A. Brown, P.R. Halmos and H.L. Shields, determined the spectrum and eigenvalues of the Cesaro operator (C_1 operator) on space ℓ^2 of square summable sequences , (Brown et al., 1965). Boyd (1968) extended the work by determining the spectrum of the same C_1 operator on $L^p(R^+)$ for $p \neq 2$, (Boyd, 1968). Sharma (1972) determined the spectra of conservative matrices and in particular showed that the spectrum of any Hausdorff method is either uncountable or finite. Sharma (1975) determined the isolated points of the spectra of conservative matrices. Wenger (1975), computed the fine spectra of Holder summability operators on c - the space of convergent sequences . Deddens (1978) computed the spectra of all Hausdorff operators onℓ_1^2. Rhoades (1983) extended Wenger's work by determining the fine spectra of weighted mean operators on c. Reade (1985) determined the spectrum of the Cesaro operator on c_0 - the space of null sequences. Okutoyi (1985) determined the spectrum of C_1 on $w_p(0)(1 \leq p < \infty)$. Gonzalez (1985) computed the fine spectrum of the C_1 operator on $\ell_p(1 < p < \infty)$. In 1989,Okutoyi, J.I. and Thorpe, B. computed the spectrum of the Cesaro operator of order two (C_{11} operator) on $c_0(c_0)$- the space of double null sequences, (Okutoyi and Thorpe, 1989). Okutoyi (1990) determined the spectrum of the C_1 operator on bv_0 space. In 1992, Okutoyi extended his work by determining the spectrum of C_1 operator on bv space, (Okutoyi, 1992). In 1996, Shafiquel Islam obtained the spectrum of the C_1 operator on ℓ_∞ - the space of bounded sequences, (Shafiquel, 1996). In his PhD thesis Mutekhele, J.S.K. extended Okutoyi's work by determining the spectrum of C_{11} operator on $c(c)$ - the space of double sequences which converge. He went further and determined the fine spectrum of C_{11} operator on $c(c)$, (Mutekhele, 1999). In 2003, Coskun. C., determined the set of eigenvalues of a special Nörlund matrix as a bounded operator over some sequence spaces, (Coskun, 2003). In the abstract of the paper, Coskun remarked that as far as he was concerned there was no investigation on the spectrum of Nörlund means! In 2005, Okutoyi, J. I and Akanga, J. R., computed the spectrum of the C_1 operator on $w_p(1 \leq p < \infty)$ - the space of strongly Cesaro summable complex sequences of order 1, index p, (Okutoyi and Akanga, 2005).

1.3 Classical Summability

The central problem of summability theory is to find means of assigning a limit to a divergent sequence or sum to a divergent series. In such a way that the sequence or series can be manipulated as though it converges, (Ruckle, 1981), pp. 159 - 161. The commonest means of summing divergent series or sequences, is that of using an infinite matrix of complex numbers.

Definition 1.3.1 *(Sequence to sequence transformation)*

Let $A = (a_{nk}), n, k = 0, 1, 2, \ldots$ be an infinite matrix of complex numbers. Given a sequence $x = (x_k)_{k=0}^{\infty}$ define

$$y_n = \sum_{k=0}^{\infty} a_{nk} x_k, n = 0, 1, 2, \ldots \tag{1.1}$$

If the series (1.1), converges for all n, then we call the sequence $(y_n)_{n=0}^{\infty}$, the A - transform of the sequence $(x_k)_{k=0}^{\infty}$. If further, $y_n \to a$ as $n \to \infty$, we say that $(x_k)_{k=0}^{\infty}$ is summable A to a.

There are numerous examples of sequence to sequence transformations. We give a few well known examples.

Example 1.3.1 *(Cesaro matrix - means)*

Consider the matrix $A = (a_{nk})$, where

$$a_{nk} = \begin{cases} \frac{1}{1+n}, & 0 \leq k \leq n \\ 0, & k > n \end{cases} \tag{1.2}$$

and a sequence $(x_k)_{k=0}^{\infty} = (1, 0, 1, 0, \ldots)$, then the sequence $(x_k)_0^{\infty}$ is summable by A to $\frac{1}{2}$. The matrix A is called the Cesaro matrix of order 1 and is usually denoted by $(C, 1)$ or C_1. Cesaro means of other orders are also well known. The most general of them is the (C, α) means which are given by;

$$a_{nk} = \begin{cases} \frac{A_{n-k}^{\alpha-1}}{A_n^{\alpha}}, & 0 \leq k \leq n \\ 0, & k > n \end{cases} \tag{1.3}$$

Where $A_n^{\alpha} = \begin{pmatrix} \alpha + n \\ n \end{pmatrix} = \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)\Gamma(n+1)}, \alpha > -1.$

Example 1.3.2 *(Hölder means)*

Closely related to the Cesaro means $(C, k), k \in \mathbb{N}$, is the Hölder means (H, k). This is simply the product of $(C, 1)$ means k - times. Its matrix is given by:

$$h_{nk}^{(m)} = (h_{nk}^{(1)})(h_{nk}^{(m-1)}) \tag{1.4}$$

where $h_{nk}^{(1)} = (C, 1)$, (Powell and Shah, 1972) pp.46-49

Example 1.3.3 *(Nörlund means)*

The transformation given by

$$y_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}x_k, n = 0, 1, 2, \dots \tag{1.5}$$

,where $P_n = p_0 + p_1 + \cdots + p_n \neq 0$*, is called a Nörlund means and is denoted by* (N, p)*. Its matrix is given by*

$$a_{nk} = \begin{cases} \frac{p_{n-k}}{P_n}, & 0 \leq k \leq n \\ 0, & k > n \end{cases} \tag{1.6}$$

In matrix (1.6) if $p_0 = 1, p_1 = -2, p_2 = p_3 = \cdots = 0$, then $A = (a_{nk})$ transforms the unbounded sequence $(x_k)_{k=0}^{\infty} = (1, 2, 4, 8, 16, \dots)$ to zero. If $p_n = 1$ for each $n = 0, 1, 2, \dots$,then $(a_{nk}) = (C, 1)$, (Powell and Shah, 1972) pp. 45 - 46. Similarly in matrix (1.6) if $p_0 = 1, p_1 = 1, p_2 = p_3 = p_4 = \cdots = 0$, then $(a_{nk}) = (q_{nk})$ - the Q matrix given by

$$q_{nk} = \begin{cases} 1, & n = k = 0 \\ \frac{1}{2}, & n - 1 \leq k \leq n \\ 0, & \text{Otherwise} \end{cases} \tag{1.7}$$

That is

$$Q = (q_{nk}) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \dots \\ 0 & 0 & 0 & \frac{1}{2} & \dots \\ & & \dots & & \end{pmatrix} \tag{1.8}$$

Which is the matrix of our interest in this thesis.

If $p_n = \begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix} = \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}, k > 0$ then (N, p_n) is the (C, k) means, (Hardy, 1948) pp. 64 - 65.

Definition 1.3.2 *(Series to series transformation).*

The transformation of the series $\sum_{k=0}^{\infty} x_k$ into a convergent series $\sum_{n=0}^{\infty} y_n$ by an infinite matrix $A = (a_{nk})$ so that

$$y_n = \sum_{k=0}^{\infty} a_{nk} x_k \tag{1.9}$$

is called series to series transformation. For further details on series transformations, see (Vermes, 1949).

Definition 1.3.3 *(Triangle and triangular matrices).*

A matrix A is called triangular if $a_{nk} = 0$ for $k > n$, it is called a triangle if it is triangular and $a_{nn} \neq 0$ for all n (Wilansky, 1984) pp.7.

Definition 1.3.4 *(Almost triangular matrix).*

A matrix A is called almost triangular if $a_{nk} = 0$, $\forall$ $k > n + m$ for some $m\in \mathbb{N}$,m$\geq$ 1. A shall be called almost triangle if A is almost triangular and $a_{nk} \neq 0$, $\forall k = n$.

1.4 General Results in Classical Summability

Definition 1.4.1 *(Regular Method, Conservative method)*

Let $A = (a_{nk}), n = 0, 1, 2, \ldots$ be an infinite matrix of complex numbers.

i. If the A - transform of any convergent sequence of complex numbers exists and converges then A is called a conservative method. We them write $A \in (c, c)$

ii. If A is conservative and preserves limits, i.e.

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n = a, a \in \mathbb{C}; \text{where}(y_n)_{n=0}^{\infty} \tag{1.10}$$

is the A transform of the convergent sequence $(x_n)_{n=0}^{\infty}$, then A is called regular. We then write $A \in (c, c; P)$

Theorem 1.1 *(Silverman - Toeplitz)$A \in (c, c; P)$ if and only if*

i. $\lim_{n\to\infty} a_{nk} = 0$ *for each fixed* $k, k = 0, 1, 2, \ldots;$

ii. $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1;$

iii. $\sup_{n\geq 0}\{\sum_{k=0}^{\infty} |a_{nk}|\} \leq M < \infty, M \in \mathbb{R}^{+}.$

Proof: (Hardy, 1948), pp.44 - 46, (Petersen, 1966) and Maddox (1970), pp 165 - 166.

Remark 1.4.1 *The Silverman - Toeplitz theorem gives the complete class of matrices* (a_{nk}) *which transform all convergent sequences* $(x_n)_0^{\infty}$ *such that* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ *by means of the linear equations 1.1*

Theorem 1.2 *(Kojima - Shur)$A \in (c, c)$ if and only if*

i. $a_{nk} \to a_k$ *as* $n \to \infty$ *for each fixed* $k \geq 0;$

ii. $\sum_{k=0}^{\infty} a_{nk} \to a$ *as* $n \to \infty;$

iii. $\sup_{n\geq 0}\{\sum_{k=0}^{\infty} |a_{nk}|\} < \infty.$

Proof: (Maddox, 1970), pp. 166 - 167; (Ruckle, 1981), pp. 104 - 105; (Powell and Shah, 1972) and (Wilansky, 1984), pp. 5 - 6.

Theorem 1.3 $A \in (c_0, c_0)$ *if and only if*

i. $\lim_{n\to\infty} a_{nk} = 0$ *for each fixed* $k;$

ii. $\sup_{n\geq 0}\{\sum_{k=0}^{\infty} |a_{nk}|\} < \infty.$

Proof: (Hardy, 1948), pp. 42 - 60; (Maddox, 1970), pp. 165 - 167.

Theorem 1.4 *$A \in (\ell_1, \ell_1)$ if and only if*

i. $\sum_{n=0}^{\infty} |a_{nk}| < \infty$*; for each* k*;*

ii. $\sup_k \sum_{n=0}^{\infty} |a_{nk}| < \infty$.

Proof: (Limaye, 1996), pp. 88 - 90 as well as pp. 154 - 156

Definition 1.4.2 *(Spaces bv and bv_0)*

The sequence space bv is such that x is $\in bv$ if

$$\sum_{k=0}^{\infty} |x_{k+1} - x_k| < \infty \tag{1.11}$$

And $x \in bv_0$ if

$$\sum_{k=0}^{\infty} |x_{k+1} - x_k| < \infty,$$

with $x_k \to 0$ as $k \to \infty$. That is, bv_0 is the space of sequence of bounded variation with limit zero.

Theorem 1.5 *A matrix $A = (a_{nk}) \in (bv_0, bv_0)$ if and only if*

i. $\lim_{n\to\infty} a_{nk} = 0$ *for each* $k \geq 0$*;*

ii. $\sup_{m\geq 0} \sum_{n=0}^{\infty} |\sum_{k=0}^{m} (a_{nk} - a_{n-1,k})| < \infty$.

Theorem 1.6 *A matrix $A = (a_{nk}); n, k \geq 0 \in (bv, bv)$ if and only if*

i. $\sup_{m\geq 0} \sum_{n=0}^{\infty} |\sum_{k=0}^{m} (a_{nk} - a_{n-1,k})| < \infty$*;*

ii. $\sum_{k=0}^{\infty} a_{nk}$ *Converges, for all* $n \geq 0$.

Moreover, $\|A\|_{(bv,bv)} = \|A\|_{(bv_0,bv_0)} = \sup_{m\geq 0} \sum_{n=0}^{\infty} |\sum_{k=0}^{m} (a_{nk} - a_{n-1,k})|$

Proof:For the proof of theorems (1.5) and (1.6), see (Stieglitz and Tietz, 1977) , pp. 1 - 16 and (Jakimovski and Russel, 1972) , pp. 345 - 353.

Remark 1.4.2 *For a more comprehensive characterization of matrix classes , one may also consult (Maddox, 1980), pp. 9 - 18.*

1.5 Banach Spaces

Definition 1.5.1 *(Paranorm)*

A paranorm p, on a linear space X, is a function $p : X \to \mathbb{R}$ such that

i. $p(\theta) = 0$

ii. $p(x) \geq 0$

iii. $p(x) = p(-x)$

iv. $p(x + y) \leq p(x) + p(y)$

v. If $(\lambda_n)_0^\infty$ is a sequence of scalars with $\lambda_n \to \lambda$ and $(x_n)_0^\infty$ is a sequence of points in X with $x_n \to x$, then $p(\lambda_n x_n - \lambda x) \to 0$ (continuity of multiplication)

Definition 1.5.2 *(Seminorm/norm)*

A seminorm p, on a linear space X, is a function $p : X \to \mathbb{R}$ such that

i. $p(x) \geq 0$

ii. $p(x + y) \leq p(x) + p(y)$

iii. $p(\lambda x) = |\lambda|\, p(x), \lambda \in \mathbb{K}(\mathbb{R} \text{or} \mathbb{C})$

If in addition to these conditions if a seminorm satisfies the condition that $p(x) = \theta$ iff $x = \theta$, then we call it a norm; θ denotes the zero vector.

Definition 1.5.3 *(Linear topological space)*

A linear topological space is a linear space X which has a topology T, such that addition and scalar multiplication in X are continuous. If T is given a metric, we speak of a linear metric space.

Definition 1.5.4 *(Schauder basis)*

Let X be a paranormed or normed space with a paranorm p or norm $\|.\|$. A sequence $(b_k)_0^\infty$ of elements of X is called a Schauder basis if and only if for every $x \in X, \exists!$ Sequence $(\lambda_k)_0^\infty$ of scalars such that

$$x = \sum_{k=0}^{\infty} \lambda_k b_k \tag{1.12}$$

That is, $p\left(x - \sum_{k=0}^{n} \lambda_k b_k\right) \to 0$ as $n \to \infty$; or in norm notation $\|x - \sum_{k=0}^{n} \lambda_k b_k\| \to 0$ as $n \to \infty$.

Example 1.5.1 $\triangle = \left(\delta^k\right)_0^\infty = (\delta^0, \delta^1, \delta^2, \dots)$ *is a Schauder basis for the spaces* $c_0; bv_o; \ell_p(0 < p < 1); cs; s$.
$\triangle^+ = (\delta, \delta^0, \delta^1, \delta^2, \dots)$ *is a Schauder basis for the spaces* $c; bv$. *The spaces* ℓ_∞ *and* bs *have no Schauder basis.*

The requirement that $X \supset \mathrm{E}^\infty$ should have a basis $\triangle = (\delta^0, \delta^1 \delta^2, \dots)$, is equivalent to $\|x^{[n]} - x\|_x \to 0$ as $n \to \infty, \forall x \in X$, where $x^{[n]} = (x_0, x_1, x_2, \dots, x_n, 0, 0, \dots)$ is the n^{th}section of x. In this case we say that X has AK (abschnittskonvergent).

Example 1.5.2 $c_0; c; \ell_p(p \geq 1); \ell_\infty; bv; bv_0; cs; bs; w_p(p \geq 1)$ *are all normed linear spaces.*
Their norms are as follows: $c_0; c; \ell_\infty$ *have the same natural norm, namely* $\|x\| = \sup_{n\geq 0}\{|x_n|\}$; $\ell_p(1 \leq p < \infty)$ *has a natural norm*

$$\|x\| = \left(\sum_{n=0}^{\infty} |x_n|^p\right)^{\frac{1}{p}}; bv \tag{1.13}$$

has a natural norm

$$\|x\| = \lim_{n\to\infty} |x_n| + \sum_{k=0}^{\infty} |x_{k+1} - x_k|; \tag{1.14}$$

bv_0 *has a norm* $\|x\| = \sum_{k=0}^{\infty} |x_{k+1} - x_k|$; cs *and* bs *have the same natural norm given by*

$$\|x\| = \sup_{n\geq 0}\left\{\left|\sum_{k=0}^{n} x_k\right|\right\}. \tag{1.15}$$

Definition 1.5.5 *(Banach Space)*

A Banach space is a complete normed linear space. Completeness means that if $\|x_m - x_n\| \to 0$ as $m, n \to \infty$, where $x_n \in X$, then there exists $x \in X$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.

Example 1.5.3 *(Banach spaces). All the spaces in example 1.5.2 are Banach spaces under their natural norms.*

Definition 1.5.6 *(Frechet space), - FK space*

A Frechet space is a complete linear metric space. An FK - space is a Frechet space with continuous coordinates. A normed FK-space is called a BK- space.

NOTE: Every Frechet space with a Schauder basis is an FK - space. Obvious examples of FK - spaces are $c_0; c; \ell_p(p \geq 1); cs; bv; bv_0; w_p(p \geq 1)$. See (Bennett, 1971), (Bennett, 1972b), (Bennett, 1972a), (Bennett and Kalton, 1972), (Brown et al., 1969), and (Maddox, 1970).

1.6 Linear Operators and Functionals

Definition 1.6.1 *(Linear operator)*

Let X and Y be linear spaces. Then a function $f : X \to Y$ is called a linear operator or map or transformation if and only if for all $x, y \in X$ and all $\lambda, \mu \in \mathbb{K}$

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y). \tag{1.16}$$

Definition 1.6.2 *(Linear functional)*

f is a linear functional on X if $f : X \to \mathbb{K}$ is a linear operator, i.e., a linear functional is a real or complex valued linear operator.

Definition 1.6.3 *(Bounded linear operator)*

A linear operator $A : X \to Y$ is called bounded if there exists a constant M such that

$$\|A(x)\| \leq M\|x\|, \forall x \in X \tag{1.17}$$

NOTE: A bounded functional on X satisfies

$$|f(x)| \leq M\|x\|, \forall x \in X. \tag{1.18}$$

NOTATION: Let X and Y be linear spaces. Then $L(X,Y)$ denotes the set of all linear operators on X into Y. $L(X,\mathbb{K})$, the set of all linear functionals on X. It is usual to denote this by X^+ and call it the algebraic dual of X.

Definition 1.6.4 *(Continuous dual of X)*

Let X and Y be normed spaces. Then $B(X,Y)$ denotes the set of all bounded (or continuous) linear operators on X into Y. $B(X,\mathbb{K})$, the set of all bounded (continuous) functionals on X. It is usual to denote this by X^* and call it the continuous dual of X.

Remark 1.6.1 *Let X be a Banach space, then it is well known, (Maddox, 1970), page 107 theorem 7, that $B(X,X) = B(X)$, the linear space of all bounded linear operators T on X into itself is a Banach space with norm.*

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\| \tag{1.19}$$

This norm induces a metric topology, the uniform operator topology on $B(X)$. See (Dunford and Schwartz, 1957), page 475.

Definition 1.6.5 *(Adjoint operator T^*)*

The adjoint T^* of linear operator $T \in B(X,Y)$ is the mapping from Y^* to X^* defined by

$$T^* o f = f o T, f \in Y^* \tag{1.20}$$

Theorem 1.7 *T^* is linear and bounded. Moreover, $\|T^*\| = \|T\|$.*

Proof: (Dunford and Schwartz, 1957), p.478 ; (Goldberg, 1966), page 54; and (Kreyszig, 1980), page 232.

Theorem 1.8 *A Linear operator $T \in B(X, Y)$ has a bounded inverse T^{-1} defined on all of Y if and only if its adjoint T^* has a bounded inverse $(T^*)^{-1}$ defined on all of X^*. When these inverses exist $(T^{-1})^* = (T^*)^{-1}$.*

Proof: (Dunford and Schwartz, 1957), page 479 and (Goldberg, 1966), page 60.

Definition 1.6.6 *(Resolvent operator $R_\lambda = (T - \lambda I)^{-1}$)*

Let X be a non-empty Banach space and suppose that $T : X \to X$. With T, we associate the operator $T_\lambda = T - \lambda I, \lambda \in \mathbb{C}, I$ the identity operator on X. If $T_\lambda = T - \lambda I$ has an inverse, we denote it by $R_\lambda(T)$ or simply R_λ and call it the resolvent operator of T.

Definition 1.6.7 *(Resolvent set $\rho(T)$, spectrum $\sigma(T)$)*

Let X be a non - empty Banach space and suppose that $T : X \to X$. The resolvent set $\rho(T)$ of T is the set of complex numbers λ for which $(T - \lambda I)^{-1}$ exists as a bounded operator with domain X. The spectrum $\sigma(T)$ of T is the complement of $\rho(T)$ in $\mathbb{C}$.

Theorem 1.9 *The resolvent set $\rho(T)$ of a bounded linear operator T on a Banach space X is open; hence the spectrum $\sigma(T)$ of T is closed.*

Proof:(Taylor and Lay, 1980) , page 273 and (Kreyszig, 1980) page 376.

Theorem 1.10 *If X is any Banach space and $T \in B(X)$, then $\sigma(T) \neq \phi$.*

Proof: Kreyszig (1980), page 390; (Taylor and Lay, 1980) , page 278.

Theorem 1.11 *The Spectrum $\sigma(T)$ of a bounded linear operator $T : X \to X$ on a Banach space X is compact and lies in the disc given by:*

$$|\lambda| \leq \|T\| \tag{1.21}$$

Proof: See (Kreyszig, 1980), page 377.

Theorem 1.12 *Let $T \in B(X)$, where X is any Banach space, then the spectrum of T^* is identical with the spectrum of T. Furthermore, $R_\lambda(T^*) = (R_\lambda(T))^*$ for $\lambda \in \rho(T) = \rho(T^*)$.*

Proof: See (Dunford and Schwartz, 1957) , page 568 and (Goldberg, 1966), page 71.

Remark 1.6.2 *The key theorems of functional analysis; Baire category theorem; Banach - steinhaus theorem (Uniform boundedness theorem); open mapping theorem; closed graph theorem and Hahn-Banach extension theorem are assumed as necessary for this work although not stated here. They are proved in most standard text books of functional analysis.See (Maddox, 1970); (Wilasky, 1964); (Wilansky, 1978);(Wilansky, 1984); (Lusternik and Sobolev, 1961) and (Jain et al., 1995).*

In the next chapter we determine the spectrum of the Q operator on c and c_0 spaces.

Chapter 2

The Spectrum of Q Operator on c_0 and c spaces

The chapter is divided into two sections. Section one deals with the spectrum of Q as an operator on c_0. In section two we determine the spectrum of Q matrix acting as an operator on c.

2.1 The spectrum of Q on c_0

In this section we show that $Q \in B(c_0)$ and determine its spectrum. The following corollary arises from theorem (1.3), chapter I.

Corollary 2.1.1 *It is clear that* $Q \in B(c_0)$ *since* $\lim_n q_{nk} = 0$ *for each* k, *(see matrix* (1.7)*)*

$$||Q|| = \sup_n \sum_{k=0}^{\infty} |q_{nk}| = \sup(1, 1, 1 \ldots) = 1 \tag{2.1}$$

By theorem (1.7) , $||Q|| = ||Q^*|| = 1$

Lemma 2.1.1 *Each bounded linear operator* $T : X \longrightarrow Y$, *where* $X = c_0, \ell_1, c$ *and* $Y = c_0, \ell_p (1 \leq p < \infty), \ell_\infty$ *determines and is determined by an infinite matrix of* *complex numbers.*

Proof: See (Taylor, 1958) pages 217 - 219

Lemma 2.1.2 *Let $T : c_0 \longrightarrow c_0$ be a linear map and define $T^* : \ell_1 \longrightarrow \ell_1$ by $T^*\ o\ g = g\ o\ T$, $g \in c_0^* = \ell_1$. Then T must be given by a matrix by lemma*(2.1.1) *and moreover $T^* : \ell_1 \longrightarrow \ell_1$ is the transposed matrix of T.*

Proof: See (Wilansky, 1984) page 266.

Corollary 2.1.2 *Let $Q : c_0 \longrightarrow c_0$ where Q is the Nörlund matrix* (1.8). *Then $Q^* \in B(\ell_1)$, moreover*

$$Q^* = \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & \dots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \dots \\ & & \dots & & \end{pmatrix} \tag{2.2}$$

Theorem 2.1 *$Q \in B(c_0)$ has no Eigenvalues.*

Proof: Suppose $Qx = \lambda x$ for $x \neq \theta$ in c_0 and $\lambda \in \mathbb{C}$ then

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \dots \\ & & \dots & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} \tag{2.3}$$

Implies

$$\begin{aligned} x_0 &= \lambda x_0 \\ \frac{1}{2}(x_0 + x_1) &= \lambda x_1 \\ \frac{1}{2}(x_1 + x_2) &= \lambda x_2 \\ &\dots \end{aligned}$$

$$\frac{1}{2}(x_{n-1}+x_n)=\lambda x_n,\ n\geq\ 1 \tag{2.4}$$

Solving system (2.4) , we have that if x_0 is the first non-zero entry of x, then $\lambda=1$. But $\lambda=1$ implies that $x_0=x_1=x_2=\cdots=x_n=\ldots$

Which shows that x is in the span of $\delta=(1,1,1\ldots)$ and hence does not tend to zero as n tends to infinity. Hence $\lambda=1$ is not an eigenvalue of $Q\in B(c_0)$. When $x_{n+1},\ n=0,1,2,3,\ldots,$ is the first non-zero entry of x, then $\lambda=\frac{1}{2}$. Solving system (2.4) with $\lambda=\frac{1}{2}$ results in $x_n=0,\ n=0,\ 1,2,\ldots$ which is a contradiction. Hence $\lambda=\frac{1}{2}$ cannot be an eigenvalue of $Q\in B(c_0)$.

Theorem 2.2 *The eigenvalues of $Q^*\in B(\ell_1)$ is the set*

$$\left\{\lambda\in\mathbb{C}:|\lambda-\frac{1}{2}|<\frac{1}{2}\right\}\cup\{1\}$$

Proof: Suppose $Q^*x=\lambda x$, for $x\neq\theta$ and $\lambda\in\mathbb{C}$

Then

$$\begin{pmatrix}1&\frac{1}{2}&0&0&\ldots\\0&\frac{1}{2}&\frac{1}{2}&0&\ldots\\0&0&\frac{1}{2}&\frac{1}{2}&\ldots\\0&0&0&\frac{1}{2}&\ldots\\&&\ldots&&\end{pmatrix}\begin{pmatrix}x_0\\x_1\\x_2\\x_3\\\vdots\end{pmatrix}=\lambda\begin{pmatrix}x_0\\x_1\\x_2\\x_3\\\vdots\end{pmatrix} \tag{2.5}$$

That is,

$$\begin{aligned}x_0+\tfrac{1}{2}x_1&=\lambda x_0\\\tfrac{1}{2}x_1+\tfrac{1}{2}x_2&=\lambda x_1\\\tfrac{1}{2}x_2+\tfrac{1}{2}x_3&=\lambda x_2\\\vdots&\qquad\vdots\\\tfrac{1}{2}x_n+\tfrac{1}{2}x_{n-1}&=\lambda x_n,\ n\geq 1\end{aligned} \tag{2.6}$$

Solving system(2.6) for $x_1,\ x_2,\ x_3, \ldots, x_n, \ldots$ in terms of x_0 gives

$$\begin{aligned} x_1 &= 2\lambda(1-\tfrac{1}{\lambda})x_0 \\ x_2 &= 2^2\lambda^2(1-\tfrac{1}{2\lambda})(1-\tfrac{1}{\lambda})x_0 \\ x_3 &= 2^3\lambda^3(1-\tfrac{1}{2\lambda})^2(1-\tfrac{1}{\lambda})x_0 \\ &\cdots \end{aligned} \tag{2.7}$$

In general,

$$x_n = (2\lambda)^n(1-\frac{1}{2\lambda})^{n-1}(1-\frac{1}{\lambda})x_0,\ n \geq 1 \tag{2.8}$$

Now,

$$\lim_{n\longrightarrow\infty}|\frac{x_{n+1}}{x_n}| = \lim_{n\longrightarrow\infty}\left|\frac{(2\lambda)^{n+1}(1-\frac{1}{2\lambda})^n(1-\frac{1}{\lambda})}{(2\lambda)^n(1-\frac{1}{2\lambda})^{n-1}(1-\frac{1}{\lambda})}\right| \tag{2.9}$$

$$= \lim_{n\longrightarrow\infty}\left|2\lambda(1-\frac{1}{2\lambda})\right| \tag{2.10}$$

$$= \left|2\lambda(1-\frac{1}{2\lambda}\right| = m \tag{2.11}$$

say for some $m \in \mathbb{R}$ such that $m \geq 0$.
By the ratio test $(x_n) \in \ell_1$ iff $m < 1$
That is, iff $\left|2\lambda(1-\frac{1}{2\lambda})\right| < 1$. Or iff $|\lambda - \frac{1}{2}| < \frac{1}{2}$.

That is, the series$\sum_{n=0}^{\infty}|x_n|$ converges for all λ in the circular disc centered at the point $(\frac{1}{2}, 0)$ of radius $\frac{1}{2}$.

It is clear that $\lambda = 1$ is an eigenvalue corresponding to the eigenvector $(x_0, 0, 0, \ldots)^t$. Where x_0 is any real or complex number. This is the case since $(x_0, 0, 0, \ldots)^t \subset \ell_1$ for any $x_0 \in \mathbb{C}$

Theorem 2.3 *The spectrum $\sigma(Q)$ of $Q\in B(c_0)$ is the set*

$$\left\{\lambda \in \mathbb{C} \;\middle|\; \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$$

Proof: By virtue of theorem (2.2) and the fact that $\sigma(Q^*) = \sigma(Q)$ by theorem (1.12), we show that $(Q - \lambda I)^{-1} \in B(c_0)$ for all $\lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$. To this end we solve the system $(Q - \lambda I)x = y$ for x in terms of y to obtain:

$$\begin{aligned}
x_0 &= \frac{-1}{\lambda(1-\frac{1}{\lambda})}y_0 \\
x_1 &= \frac{-1}{2\lambda^2(1-\frac{1}{2\lambda})(1-\frac{1}{\lambda})}y_0 - \frac{1}{\lambda(1-\frac{1}{2\lambda})}y_1 \\
x_2 &= \frac{-1}{4\lambda^3(1-\frac{1}{2\lambda})^2(1-\frac{1}{\lambda})}y_0 - \frac{1}{2\lambda^2(1-\frac{1}{2\lambda})^2}y_1 - \frac{1}{\lambda(1-\frac{1}{2\lambda})}y_2 \\
x_3 &= \frac{-1}{8\lambda^4(1-\frac{1}{2\lambda})^3(1-\frac{1}{\lambda})}y_0 - \frac{1}{4\lambda^3(1-\frac{1}{2\lambda})^3}y_1 - \frac{1}{2\lambda^2(1-\frac{1}{2\lambda})^2}y_2 - \frac{1}{\lambda(1-\frac{1}{2\lambda})}y_3 \\
x_4 &= \frac{-1}{2^4\lambda^5(1-\frac{1}{2\lambda})^4(1-\frac{1}{\lambda})}y_0 - \frac{1}{2^3\lambda^4(1-\frac{1}{2\lambda})^4}y_1 - \frac{1}{2^2\lambda^3(1-\frac{1}{2\lambda})^3}y_2 \\
&- \frac{1}{2\lambda^2(1-\frac{1}{2\lambda})^2}y_3 - \frac{1}{\lambda(1-\frac{1}{2\lambda})}y_4 \\
&\dots
\end{aligned} \tag{2.12}$$

So that in general,

$$x_n = \frac{-1}{2^n\lambda^{n+1}(1-\frac{1}{2\lambda})^n(1-\frac{1}{\lambda})}y_0 - \frac{1}{2^{n-1}\lambda^n(1-\frac{1}{2\lambda})^n}y_1 -$$
$$\frac{1}{2^{n-2}\lambda^{n-1}(1-\frac{1}{2\lambda})^{n-1}}y_2 - \frac{1}{2^{n-3}\lambda^{n-2}(1-\frac{1}{2\lambda})^{n-2}}y_3 -$$
$$\dots - \frac{1}{\lambda(1-\frac{1}{2\lambda})}y_n, n \geq 0 \tag{2.13}$$

That is,

$$x_n = -\frac{1}{2^n\lambda^{n+1}(1-\frac{1}{2\lambda})^n(1-\frac{1}{\lambda})}y_0 - \frac{1}{2^{n-1}\lambda^n(1-\frac{1}{2\lambda})^n}y_1 -$$
$$\frac{1}{2^{n-k}\lambda^{n-k+1}(1-\frac{1}{2\lambda})^{n-k+1}}y_k,\ 0 \leq k \leq n. \tag{2.14}$$

The system(2.14) yields the matrix of $(Q-\lambda I)^{-1}$ which we denote by M. That is

$$M = \begin{pmatrix} -\frac{1}{\lambda(1-\frac{1}{\lambda})} & 0 & 0 & 0 & \dots \\ -\frac{1}{2\lambda^2(1-\frac{1}{2\lambda})(1-\frac{1}{\lambda})} & -\frac{1}{\lambda(1-\frac{1}{2\lambda})} & 0 & 0 & \dots \\ -\frac{1}{2^2\lambda^3(1-\frac{1}{2\lambda})^2(1-\frac{1}{\lambda})} & -\frac{1}{2\lambda^2(1-\frac{1}{2\lambda})^2} & -\frac{1}{\lambda(1-\frac{1}{2\lambda})} & 0 & \dots \\ -\frac{1}{2^3\lambda^4(1-\frac{1}{2\lambda})^3(1-\frac{1}{\lambda})} & -\frac{1}{2^2\lambda^3(1-\frac{1}{2\lambda})^3} & -\frac{1}{2\lambda^2(1-\frac{1}{2\lambda})^2} & -\frac{1}{\lambda(1-\frac{1}{2\lambda})} & \dots \\ & & \dots & & \end{pmatrix}$$

Note that,

$$(Q-\lambda I) = M^{-1} = \begin{pmatrix} (1-\lambda) & 0 & 0 & 0 & \dots \\ \frac{1}{2} & (\frac{1}{2}-\lambda) & 0 & 0 & \dots \\ 0 & \frac{1}{2} & (\frac{1}{2}-\lambda) & 0 & \dots \\ 0 & 0 & \frac{1}{2} & (\frac{1}{2}-\lambda) & \dots \\ & & \dots & & \end{pmatrix} \tag{2.15}$$

It can be shown that $MM^{-1} = M^{-1}M = I$

We now check that $M \in B(c_0)$. Note that matrix M may be given the formula

$$M = (m_{nk}) = \begin{cases} -\frac{1}{2^n\lambda^{n+1}\left(1-\frac{1}{2\lambda}\right)^n\left(1-\frac{1}{\lambda}\right)}, & k=0 \\ -\frac{1}{2^{n-k}\lambda^{n-k+1}\left(1-\frac{1}{2\lambda}\right)^{n-k+1}}, & 1 \le k \le n \\ 0, & k > n \end{cases} \tag{2.16}$$

The columns converge to zero if

$$\left|\frac{(n+1)^{th}\text{term}}{n^{th}\text{term}}\right| < 1$$

Using formula (2.16), when $k=0$, we have

$$\left|\frac{-2^n\lambda^{n+1}(1-\frac{1}{2\lambda})^n(1-\frac{1}{\lambda})}{-2^{n+1}\lambda^{n+2}(1-\frac{1}{2\lambda})^{n+1}(1-\frac{1}{\lambda})}\right| < 1 \tag{2.17}$$

Which implies that, $\left|\frac{1}{2\lambda(1-\frac{1}{2\lambda})}\right| < 1$

or $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$

Similarly, for columns $k \geq 1$, we have

$$\left|\frac{-2^{n-k}\lambda^{n-k+1}(1-\frac{1}{2\lambda})^{n-k+1}}{-2^{n-k+1}\lambda^{n-k+2}(1-\frac{1}{2\lambda})^{n-k+2}}\right| < 1 \tag{2.18}$$

$$\Rightarrow \left|\frac{1}{2\lambda(1-\frac{1}{2\lambda})}\right| < 1$$

or

$$\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$$

So that all columns converge to zero for all $\lambda \in \mathbb{C}$ such that

$$\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$$

Hence condition (i) of theorem(1.3) is satisfied when $\lambda \in \mathbb{C}$ is such that $|\lambda - \frac{1}{2}| > \frac{1}{2}$. We now check condition (ii)of the same theorem. But before that, we have a remark.

Remark 2.1.1 *For any matrix $A = (a_{nk})_{n,k\geq 0}$, if $\lim_n a_{nk} = 0$, $\forall k \geq 0$; then $\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$.*

See (Maddox, 1970) page 164 or (Reade, 1985) page 266.

Summing the entries of the matrix(2.16) along the n^{th} row, we have

$$\sum_{k=0}^{\infty} |m_{nk}| = \left|\frac{-1}{2^n\lambda^{n+1}(1-\frac{1}{2\lambda})^n(1-\frac{1}{\lambda})}\right| + \sum_{k=1}^{n}\left|\frac{-1}{2^{n-k}\lambda^{n-k+1}(1-\frac{1}{2\lambda})^{n-k+1}}\right| \tag{2.19}$$

$$= \varepsilon_n, \text{ say, for } n \geq 0$$

By Remark (2.1.1)

$$\sup_n \{\varepsilon_n\} \leq K < \infty,$$

provided $\lambda \in \mathbb{C}$ is such that

$$\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$$

So that $M = (Q - \lambda I)^{-1} \in B(c_0)$ if $\lambda \in \mathbb{C}$ is such that $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$.
Which implies that $(Q - \lambda I)^{-1} \notin B(c_0)$ for all $\lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}$

2.2 The Spectrum of Q as an Operator On c

In this section the spectrum of $Q \in B(c)$ is determined.

Corollary 2.2.1 *$Q \in B(c)$, moreover*

$$\|Q\| = \|Q^*\| = 1$$

Proof: The validity of parts (i) and (iii) of theorem(1.2) for Q matrix easily follows from corollary (2.1.1). From matrix (1.8), it is evident that

$$\sum_{k=0}^{\infty} q_{nk} = \sum_{k=0}^{n} q_{nk} = 1, \text{for each } n \tag{2.20}$$

So that $\lim_n \sum_{k=0}^{n} q_{nk} = 1$

Therefore all the conditions of theorem (1.2) are satisfied . Hence $Q \in B(c)$.

Theorem 2.4 *Let $T : c \longrightarrow c$ be a linear map and define $T^* : c^* \longrightarrow c^*$ i.e., $T^* : \ell_1 \longrightarrow \ell_1$ by $T^*(g) = goT,\ g \in c^* \equiv \ell_1$. Then both T and T^* must be given by a matrix. See Lemma (2.1.2). More over $T^* : \ell_1 \longrightarrow \ell_1$ is given by the matrix*

$$A^* = T^* = \begin{pmatrix} \chi(\lim_A) & (v_n)_0^\infty \\ (a_k)_0^\infty & A^t \end{pmatrix} \tag{2.21}$$

$$
= \begin{pmatrix} \chi(\lim_A) & v_0 & v_1 & v_2 & \dots \\ a_0 & a_{00} & a_{10} & a_{20} & \dots \\ a_1 & a_{01} & a_{11} & a_{21} & \dots \\ a_2 & a_{02} & a_{12} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix} \tag{2.22}
$$

Where $\chi(\lim_A)=\lim_A(\delta) - \sum_{k=0}^{\infty} \lim_A \delta^k$;

$$
v_n = \chi(P_n o T);
$$

$$
a_{nk} = P_n(T(\delta^k)) = (T(\delta^k))_n
$$

$$
\textit{and } a_k = \lim_{n \longrightarrow \infty} a_{nk} \tag{2.23}
$$

Proof : See (Wilansky, 1984) page 267.

Corollary 2.2.2 *Let* $Q : c \longrightarrow c$. *Then* $Q^* \in \mathbf{B}(\ell_1)$ *and*

$$
Q^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & \frac{1}{2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \dots \\ & & & \dots & & \end{pmatrix} \tag{2.24}
$$

Proof: By theorem (2.4)

$$
Q^* = \begin{pmatrix} \chi(\lim_A) & (v_n)_0^{\infty} \\ (a_k)_0^{\infty} & Q^t \end{pmatrix} \tag{2.25}
$$

But for Q matrix, $v_n = \theta$ and $(a_k)_0^{\infty} = \theta$, since $\lim_{n \longrightarrow \infty} q_{nk} = 0 \; \forall \; k \geq 0$.
$(P_n o T)\delta = 1, \; \forall \; n \geq 0$;
and $\sum_{k=0}^{\infty} (P_n o T)\delta^k = 1$, so that

$$
v_n = \chi(P_n o T)
$$

$$= (P_n o T)\delta - \sum_{k=0}^{\infty}(P_n o T)\delta^k$$

which implies that

$$\begin{aligned}
v_0 &= 1 - (1 + 0 + 0 + \dots) = 1 - 1 = 0 \\
v_1 &= 1 - (\frac{1}{2} + \frac{1}{2} + 0 + 0 + \dots) = 1 - 1 = 0 \\
v_2 &= 1 - (0 + 0 + \frac{1}{2} + \frac{1}{2} + 0 + 0 + \dots) = 1 - 1 = 0 \\
&\vdots \\
v_n &= 0,\ n \geq 0
\end{aligned} \tag{2.26}$$

Hence matrix (2.25) becomes

$$Q^* = \begin{pmatrix} \chi & \theta \\ \theta & Q^t \end{pmatrix} \tag{2.27}$$

Where

$$\chi = (\lim o\, Q)\delta - \sum_{k=0}^{\infty}(\lim o\, Q)\delta^k, \tag{2.28}$$

$\lim \in c^*$. That is

$$\chi = \lim \delta - \sum_{k=0}^{\infty} a_k = 1 - 0 = 1 \tag{2.29}$$

So that matrix (2.27) becomes matrix (2.24)

Theorem 2.5 *$Q \in B(c)$ has one Eigenvalue, namely $\lambda = 1$. Which corresponds to the Eigenvector $x = \delta = (1, 1, 1 \dots)$*

Proof: Suppose $Qx = \lambda x, x, \neq \theta$ in c and $\lambda \in \mathbb{C}$. Then

solving the system as in the proof of theorem (2.1) we have that if x_0 is the first non-zero entry of the vector x, then $\lambda = 1$. But $\lambda = 1$ implies that

$$x_0 = x_1 = x_2 = \cdots = x_n = \dots$$

Which shows that x is in the span of δ. But $\delta = (1, 1, 1, \dots) \in c$. Hence $\lambda = 1$ is an eigenvalue of Q corresponding to the eigenvector $\delta = (1, 1, 1, \dots)$.

When $x_{n+1},\ n = 0, 1, 2, \dots$ is the first non-zero entry of x, then $\lambda = \frac{1}{2}$.
Solving the system with $\lambda = \frac{1}{2}$ results in $x_n = 0,\ n = 0, 1, 2, \dots$. Which is a contradiction. Hence $\lambda = \frac{1}{2}$ cannot be an eigenvalue of $Q \in B(c)$.

Therefore, $\lambda = 1$ is the only eigenvalue of $Q \in B(c)$.

Theorem 2.6 *The Eigenvalues of $Q^* \in B(\ell_1)$ form the set*

$$\left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| < \frac{1}{2}\right\} \cup \{1\}$$

Proof: Suppose $Q^*x = \lambda x, x \neq \theta$ and $\lambda \in \mathbb{C}$. Then

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & \frac{1}{2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \dots \\ & & & \dots & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} \tag{2.30}$$

i.e.

$$\begin{aligned} x_0 &= \lambda x_0 \\ x_1 + \tfrac{1}{2}x_2 &= \lambda x_1 \\ \tfrac{1}{2}(x_2 + x_3) &= \lambda x_2 \\ \tfrac{1}{2}(x_3 + x_4) &= \lambda x_3 \\ &\dots \end{aligned} \tag{2.31}$$

In general $\frac{1}{2}(x_n + x_{n+1}) = \lambda x_n,\ n \geq 2$
Solving system (2.31) with $\lambda = 1$ and $x_0 \neq 0$ gives the vectors

$$x^{(1)} = (x_0, 0, 0 \dots) \tag{2.32}$$

$$x^{(2)} = (x_0, x_1, 0, 0, \dots), \tag{2.33}$$

Where $x_0, x_1 \in \mathbb{C}$

Clearly, both $x^{(1)}$ and $x^{(2)} \in \ell_1$ for any $x_1, x_2 \in \mathbb{C}$. So that $\lambda = 1$ is an Eigenvalue of $Q^* \in B(\ell_1)$.

Similarly, solving the system for $x_n,\ n \geq 2$ in terms of x_1, yields

$$\begin{aligned} x_2 &= 2(\lambda - 1)x_1 \\ x_3 &= 2^2(\lambda - 1)(\lambda - \tfrac{1}{2})x_1 \\ x_4 &= 2^3(\lambda - 1)(\lambda - \tfrac{1}{2})^2 x_1 \\ x_5 &= 2^4(\lambda - 1)(\lambda - \tfrac{1}{2})^3 x_1 \\ &\dots \end{aligned} \tag{2.34}$$

In general,

$$x_n = 2^{n-1}\lambda^{n-1}(1 - \frac{1}{\lambda})(1 - \frac{1}{2\lambda})^{n-2} x_1,\ n \geq 2. \tag{2.35}$$

Now,

$$\lim_{n \longrightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| = \lim_{n \longrightarrow \infty} \left| \frac{2^n \lambda^n (1 - \frac{1}{\lambda})(1 - \frac{1}{2\lambda})^{n-1}}{2^{n-1}\lambda^{n-1}(1 - \frac{1}{\lambda})(1 - \frac{1}{2\lambda})^{n-2}} \right| \tag{2.36}$$

$$= \left| 2\lambda(1 - \frac{1}{2\lambda}) \right| = m, \tag{2.37}$$

Say for some $m \in \mathbb{R}$ such that $m \geq 0$. By the ratio test, $x \in \ell_1$ if $m < 1$;
i.e., iff $\left|2\lambda(1 - \frac{1}{2\lambda})\right| < 1$. or iff $\left|\lambda - \frac{1}{2}\right| < \frac{1}{2}$. So that $\left\{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| < \frac{1}{2}\right\} \cup \{1\}$ form the set of Eigenvalues of $Q^* \in B(\ell_1)$

Theorem 2.7 *The spectrum $\sigma(Q)$ of $Q \in B(c)$ is the set*

$$\left\{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2}\right\}$$

Proof: By virtue of theorem (2.6) and the fact that $\sigma(Q^*) = \sigma(Q)$, it is enough to show that

$$M = (Q - \lambda I)^{-1} \in B(c)$$

for all $\lambda \in \mathbb{C}$ such that $|\lambda - \frac{1}{2}| > \frac{1}{2}$. But from the proof of theorem (2.3), $\lim_n m_{nk} = 0$ for each k and $\sup_n |m_{nk}| < \infty$, provided $|\lambda - \frac{1}{2}| > \frac{1}{2}$. Which deals with parts (i) and (iii) of theorem (1.2). Summing the rows of matrix M in equation (2.16) and comparing with the sequence $(\epsilon_n)_{n\geq 1}$ of equation (2.19), we get

$$\left|\sum_{k=0}^{n} m_{nk}\right| = \left|\frac{-1}{2^n\lambda^{n+1}(1-\frac{1}{2\lambda})^n(1-\frac{1}{\lambda})}\right| +$$

$$\left|\sum_{k=1}^{n} \frac{-1}{2^{n-k}\lambda^{n-k+1}(1-\frac{1}{2\lambda})^{n-k+1}}\right|$$

$$\leq \left|\frac{-1}{2^n\lambda^{n+1}(1-\frac{1}{2\lambda})^n(1-\frac{1}{\lambda})}\right| +$$

$$\sum_{k=1}^{n} \left|\frac{-1}{2^{n-k}\lambda^{n-k+1}(1-\frac{1}{2\lambda})^{n-k+1}}\right| = \epsilon_n, \; n \geq 0 \qquad (2.38)$$

So that $(\sum_{k=0}^{\infty} m_{nk})_{n=1}^{\infty}$ is a decreasing sequence of numbers which is bounded provided $\lambda \in \mathbb{C}$ is such that $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$. All these implies that

$$\lim_{n\longrightarrow\infty} \sum_{k=0}^{\infty} m_{nk}$$

exists, provided $\lambda \in \mathbb{C}$ is such that $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$. Which deals with part (ii) of theorem (1.2). That is $M = (Q - I\lambda)^{-1} \in B(c)$, $\forall\lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$.
Hence $M = (Q - \lambda I)^{-1} \notin B(c)$ for all $\lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}$. Hence the theorem.

Remark 2.2.1 $Q : c_0 \longrightarrow c_0$; $Q : c \longrightarrow c$ *have the same spectrum despite the fact that* $Q \in B(c_0)$ *has no Eigenvalues and* $Q \in B(c)$ *has* $\lambda = 1$ *as its only Eigenvalue.*

In the next chapter we determine the spectrum of Q on bv_0 and bv.

Chapter 3

The Spectrum of Q as an Operator On bv_0 And bv spaces

This chapter is divided into two sections. Section One deals with bv_0 space while section two deals with bv space.

3.1 The Spectrum of $Q \in B(bv_0)$

The section is concerned with the spectrum of $Q \in B(bv_0)$. The following corollary arises from Theorem (1.5) of Chapter 1.

Corollary 3.1.1 *$Q : bv_0 \longrightarrow bv_0$ and $Q \in B(bv_0)$ with $\|Q\|_{bv_0} = 1$*

Proof: Using equation (1.7) for matrix Q and Letting $y_n = \sum_{k=0}^{\infty} q_{nk} x_k$, where $x_n \in bv_0$;

we have:

$$\begin{aligned} y_0 &= x_0 \\ y_1 &= \tfrac{1}{2}(x_0+x_1) \\ y_2 &= \tfrac{1}{2}(x_1+x_2) \\ y_3 &= \tfrac{1}{2}(x_2+x_3) \\ &\vdots \\ y_n &= \tfrac{1}{2}(x_{n-1}+x_n), n \geq 1 \end{aligned} \tag{3.1}$$

In general,

$$|y_n - y_{n+1}| = \frac{1}{2}|x_{n-1} - x_n + x_n - x_{n+1}|,\ n \geq 1 \tag{3.2}$$

So we have,

$$\begin{aligned} \sum_{n=0}^{\infty} |y_n - y_{n+1}| &= \frac{1}{2}\,|x_0 - x_1| + \frac{1}{2}\,|x_0 - x_1 + x_1 - x_2| + \\ &\quad \frac{1}{2}|x_1 - x_2 + x_2 - x_3| + \cdots + \frac{1}{2}|x_{n-1} - x_n + x_n - x_{n+1}| + \ldots \\ &\leq \frac{1}{2}|x_0 - x_1| + \frac{1}{2}|x_0 - x_1| + \frac{1}{2}|x_1 - x_2| + \frac{1}{2}|x_1 - x_2| + \cdots + \\ &\quad \frac{1}{2}|x_{n-1} - x_n| + \frac{1}{2}|x_{n-1} - x_n| + \ldots \\ &= |x_0 - x_1| + |x_1 - x_2| + |x_2 - x_3| + \cdots + |x_{n-1} - x_n| + \ldots \\ &= \sum_{n=0}^{\infty} |x_n - x_{n+1}| \end{aligned} \tag{3.3}$$

i.e.,

$$\sum_{n=0}^{\infty} |y_n - y_{n+1}| \leq \sum_{n=0}^{\infty} |x_n - x_{n+1}| < \infty \tag{3.4}$$

Moreover,

$$\left|\frac{y_{n+1}}{y_n}\right| = \left|\frac{\frac{1}{2}(x_n + x_{n+1})}{\frac{1}{2}(x_{n-1} + x_n)}\right| < \left|\frac{2x_n}{2x_n}\right| = 1,\ n \geq 1$$

i.e.,

$$\left|\frac{y_{n+1}}{y_n}\right| < 1,\ n \geq 1. \tag{3.5}$$

This is the case since $x_n \longrightarrow 0$ as $n \longrightarrow \infty$ so that x$_n > x_{n+1}$ and $x_n < x_{n-1}$. Hence $y_n \longrightarrow 0$ as $n \longrightarrow \infty$

Therefore $y = Qx \in \ bv_0$

Direct manipulation shows that

$$\sup_m \sum_{n=0}^{\infty} \left| \sum_{k=0}^{m} (q_{nk} - q_{n-1,k}) \right| = \sup(1, 1, 1, \dots) = 1 \tag{3.6}$$

So that $\|Q\|_{(bv_0, bv_0)} = 1$

Clearly,

$$\lim_{n \longrightarrow \infty} q_{nk} = 0, \ \forall k \geq 0$$

(See matrix (1.8))

Which checks out all conditions in theorem(1.5). Therefore $Q \in B(bv_0, bv_0)$.

Lemma 3.1.1 *The most general continuous linear functional on bv_0 is given by*

$$f(x) = \sum_{n=0}^{\infty} x_n t_n \tag{3.7}$$

Where $t_n = f(\delta^n), t \in bs$ and $(\delta^n)_{n=0}^{\infty}$ is a Schauder basis for bv_0 with norm

$$\|x\|_{bv_0} = \sum_{k=0}^{\infty} |x_{k+1} - x_k| \tag{3.8}$$

Moreover bv_0^* is isomorphic to bs via the map

$$h : bv_0^* \longrightarrow bs, h(f) = (t_0, t_1, t_2, \dots) \tag{3.9}$$

Proof: See (Okutoyi, 1984) pp 31 - 33.

Theorem 3.1 *Let $T : bv_0 \longrightarrow bv_0$ be given by a matrix $A = (a_{nk})$. Then $T^* : bv_0^* \longrightarrow bv_0^*$ is also given by a matrix. Moreover the matrix determined by T^* is the transposed* *matrix A^t of A. That is,*

$$T^* = A^t = \begin{pmatrix} a_{00} & a_{10} & a_{20} & \dots \\ a_{01} & a_{11} & a_{21} & \dots \\ a_{02} & a_{12} & a_{22} & \dots \\ & \dots & & \end{pmatrix} \tag{3.10}$$

Proof: Since bv_0 has AK and bv_0^* is isomorphic to bs via the map $h : bv_0^* \longrightarrow bs,\ h(f) = (t_0, t_1, t_2, \dots)$ where $t_n = f(\delta^n),\ n \geq 0,\ f \in bv_0^*$ (see lemma (3.1.1)); we define $S = hoT^*oh^{-1} : bs \longrightarrow bs$; that is

$$S : bs \longrightarrow bs, where\ \|x\|_{bs} = \sup_{n\geq 0}\left|\sum_{k=0}^{n} x_k\right|. \tag{3.11}$$

In particular,

$$h(\mathbb{P}_k) = \delta^k\ for\ k \geq 0 \tag{3.12}$$

This is so since,

$$h(P_k) = (P_k(\delta^0), P_k(\delta^1), P_k(\delta^2), \dots, P_k(\delta^n), \dots) = \delta^k. \tag{3.13}$$

Thus the k^{th} column (S_{nk}) of $S, k \geq 0$ is

$$S(\delta^k) = h\left(T^*(h^{-1}\delta^k)\right) = h\left(T^*(h^{-1}hP_k)\right) = h\left(T^*(P_k)\right) \text{ (by equation(3.12))}$$

$$= h(P_k oT) \text{ (by definition 1.8.5 of } T^*)$$

$$= \left(P_kT(\delta^0), P_kT(\delta^1), P_kT(\delta^2), \dots, P_kT(\delta^n), \dots\right) \tag{3.14}$$

i.e., $S(\delta^k) = (a_{k0}, a_{k1}, a_{k2}, a_{k3}, \dots, a_{kn}, \dots)$, where $a_{kn} = P_kT(\delta^n)$.

Thus S which is identified with T^* is given by

$$S = T^* = A^t = \begin{pmatrix} a_{00} & a_{10} & a_{20} & \dots \\ a_{01} & a_{11} & a_{21} & \dots \\ a_{02} & a_{12} & a_{22} & \dots \\ a_{03} & a_{13} & a_{23} & \dots \\ & \dots & & \end{pmatrix} : bs \longrightarrow bs. \tag{3.15}$$

It is also clear that $\|S\| = \|T^*\|$, since $S = hoT^*oh^{-1}, h : bv_0^* \longrightarrow bs$ is an isometry and

$$\|S\| \leq \|h\| \; \|T^*\| \; \|h^{-1}\| \;=\; \|T^*\| \tag{3.16}$$

i.e., $\|S\| \leq \|T^*\|$

But also $S = hoT^*oh^{-1}$ implies that $T^* = h^{-1}oSoh$ and thus

$$\|T^*\| \leq \|h^{-1}\| \; \|S\| \; \|h\| \;=\; \|S\| \tag{3.17}$$

i.e., $\|T^*\| \leq \|S\|$

Hence from (3.16) and (3.17), $\|S\| = \|T^*\|$

Corollary 3.1.2 *In theorem* (3.1) *if* $T : bv_0 \longrightarrow bv_0$ *is given by the* Q *matrix, then* $T^* : bv_0^* \longrightarrow bv_0^*$ *is given by* $Q^* = Q^t$*, the transposed matrix of* Q *acting on* bs*.*

Proof: Replace A by Q in theorem (3.1) and the result follows immediately.

Corollary 3.1.3 $Q \in B(bv_0)$ *has no Eigenvalues.*

Proof: Since $bv_0 \subset c_0$, $Q \in B(c_0)$ has no Eigenvalues, see theorem (2.1).

Lemma 3.1.2 *A series* $\sum U_n(x)$ *is uniformly convergent if there exists a positive number* r *less than one such that* $\left|\frac{U_{n+1}(x)}{U_n(x)}\right|$*, for all values of* n*, provided that* $U_1(x)$ *is bounded.*

Proof: See (Titchmarsh, 1939) pp. 1 - 5

Lemma 3.1.3 *Let* $Z_n = \Pi_{\nu=0}^{n}(1 - \frac{1}{\lambda(\nu+1)}),\ \lambda \neq 0,\ \lambda \in \mathbb{C}$*. Then the partial sums of* $\sum_{n=1}^{\infty} Z_n$ *are bounded iff* $Re(\frac{1}{\lambda}) \geq 1,\ \lambda \neq 1$*.*

Proof: See (Okutoyi, 1986) pp. 54 - 57

Theorem 3.2 *The Eigenvalues of* $Q^* \in B(bv_0^*)$*, i.e.,* $Q^t \in B(bs)$ *are all* $\lambda \in \mathbb{C}$ *satisfying the inequality* $|\lambda - \frac{1}{2}| < \frac{1}{2}$*.*

Proof: Suppose $Q^t x = \lambda x,\ x \in bs,\ x \neq \theta$ then solving the system for x_N in terms of x_0 as in proof of theorem (2.2) obtains $x_N = 2\lambda(2\lambda-1)^{N-1}(1-\frac{1}{\lambda})x_0, N \geq 1$. Let $Z_N = 2\lambda(2\lambda-1)^{N-1}(1-\frac{1}{\lambda})x_0,\ \lambda \neq 0$, then the partial sums of $\sum_{N=1}^{\infty} Z_N$ are certainly unbounded, when $Re(\frac{1}{\lambda}) = 1$.

Using Lemma (3.1.2), we have that the series $\sum_{N-1}^{\infty} Z_N$ is uniformly convergent when

$$\left|\frac{Z_{N+1}}{Z_N}\right| < 1, N \geq 1 \tag{3.18}$$

that is when,

$$\left|\frac{(2\lambda)^{N+1}(1-\frac{1}{2\lambda})^N(1-\frac{1}{\lambda})}{(2\lambda)^N(1-\frac{1}{2\lambda})^{N-1}(1-\frac{1}{\lambda})}\right| < 1 \tag{3.19}$$

or

$$\left|2\lambda(1-\frac{1}{2\lambda})\right| < 1 \tag{3.20}$$

or when,

$$\left|\lambda-\frac{1}{2}\right| < \frac{1}{2}. \tag{3.21}$$

Therefore, the series $\sum_{N=1}^{\infty} Z_N$ is bounded for all $\lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{1}{2}\right| < \frac{1}{2}$
Alternatively, the partial sums of the series $\sum_{N=1}^{\infty} Z_N$ are bounded whenever

$$|2\lambda - 1| < 1; or, 2\left|\lambda-\frac{1}{2}\right| < 1 \tag{3.22}$$

i.e., whenever $\left|\lambda - \frac{1}{2}\right| < \frac{1}{2}$
$\sum_{N=1}^{\infty} Z_N$ are certainly unbounded when $Re(\frac{1}{\lambda}) = 1$. So that the partial sums of $\sum_{N=1}^{\infty} Z_N$ are bounded whenever $\lambda \in \mathbb{C}$ is such that $Re(\frac{1}{\lambda}) > 1$ or $\left|\lambda - \frac{1}{2}\right| < \frac{1}{2}$.

Theorem 3.3 *Let $Q : bv_0 \longrightarrow bv_0$, then the spectrum $\sigma(Q)$ of $Q \in B(bv_0)$ is the set*

$$\left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$$

Proof: By virtue of theorem (3.2) and the fact that $\sigma(Q) = \sigma(Q^*)$ (See theorem (1.12)), it is enough to show that $(Q-\lambda I)^{-1} \in B(bv_0),\ \forall\lambda \in \mathbb{C}$ such that $|\lambda - \frac{1}{2}| > \frac{1}{2}$.

Now, solving the system $(Q-\lambda I)(x) = y$ for x in terms of y as in the proof of theorem (2.3) obtains the matrix of $(Q-\lambda I)^{-1}$ which we denote by M. Which is the matrix given by the formula (2.16). So that columns of M are null when,

$$\left|\lambda - \frac{1}{2}\right| > \frac{1}{2} \tag{3.23}$$

So that condition (i) of theorem (1.5) is satisfied by $M = (m_{nk})$, $\forall \lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$. Condition (ii) of the same theorem (1.5) follows from the calculations done below: Using the formula (2.16) of matrix M in the summations,

$$\sum_{n=0}^{\infty} \left| \sum_{k=0}^{N} (m_{nk} - m_{n-1,k}) \right| \tag{3.24}$$

We have,

$$\sum_{n=0}^{\infty} \left| \sum_{k=0}^{N} (m_{nk} - m_{n-1,k}) \right| = \sum_{1} + \sum_{2} + \sum_{3} \tag{3.25}$$

Where

$$\sum_{1} = \sum_{n=0}^{N} \left| \sum_{k=0}^{n} m_{nk} - \sum_{k=0}^{n-1} m_{n-1,k} \right|, \ 0 \le n \le N \tag{3.26}$$

$$\sum_{2} = \left| \sum_{k=0}^{N+1} m_{N+1,k} - m_{N+1,N+1} - \sum_{k=0}^{N} m_{Nk} \right|, \ n = N+1 \tag{3.27}$$

and

$$\sum_{3} = \sum_{n=N+2}^{\infty} \left| \sum_{k=0}^{N} (m_{nk} - m_{n-1,k}) \right|, \ N+2 \le n < \infty \tag{3.28}$$

Now,

$$\sum_{1} = \sum_{n=0}^{N} \left| \sum_{k=0}^{n} m_{nk} - \sum_{k=0}^{n-1} m_{n-1,k} \right| \tag{3.29}$$

So we have, $M = (Q-\lambda I)^{-1}$, $(Q-\lambda I)(Q-\lambda I)^{-1} = I$. But $M\delta = \sum_{k=0}^{n} m_{nk}$, where $\delta = (1, 1, 1, \dots)^t$

Also $(Q-\lambda I)^{-1}(Q-\lambda I)=I$, i.e.$M(Q-\lambda I)(\delta)=\delta$. Now, since $Q(\delta)=\delta$, we have $M(Q-\lambda I)\delta=\delta$, implies that $M(\delta-\lambda\delta)=\delta$ or $M(1-\lambda)\delta=\delta$.

Therefore,

$$M\delta=\frac{1}{1-\lambda}\delta \tag{3.30}$$

That is,

$$\sum_{k=0}^{n} m_{nk}=\frac{1}{1-\lambda} \tag{3.31}$$

Therefore,

$$\sum_{1}=|m_{00}|+\sum_{n=1}^{N}\left|\frac{1}{1-\lambda}-\frac{1}{1-\lambda}\right|=|m_{00}|=\left|\frac{1}{1-\lambda}\right| \tag{3.32}$$

$$\begin{aligned}\sum_{2} &= \left|\sum_{k=0}^{\infty} m_{N+1,k}-m_{N+1,N+1}-\sum_{k=0}^{N} m_{N,k}\right| \\ &= \left|\frac{1}{1-\lambda}-m_{N+1,N+1}-\frac{1}{1-\lambda}\right|=|-m_{N+1,N+1}| \\ &= \left|\frac{1}{\lambda(1-\frac{1}{2\lambda})}\right| \end{aligned} \tag{3.33}$$

$$\sum_{3}=\sum_{n=N+2}^{\infty}\left|\sum_{k=0}^{N}(m_{nk}-m_{n-1,k})\right| \tag{3.34}$$

$$=\sum_{n=N+2}^{\infty}\left|\sum_{k=0}^{N}\left(\frac{2\lambda}{2^{n}\lambda^{n+1}(1-\frac{1}{2\lambda})^{n}}+\frac{-1}{2^{n-k}\lambda^{n-k+1}(1-\frac{1}{2\lambda})^{n-k+1}}+\frac{1}{2^{n-1-k}\lambda^{n-k}(1-\frac{1}{2\lambda})^{n-k}}\right)\right|$$

But,

$$\left|\frac{2\lambda}{2^{n}\lambda^{n+1}\left(1-\frac{1}{2\lambda}\right)^{n}}\right|\leq\left|\frac{2\lambda}{n^{2}\prod_{i=1}^{n}\left(1-\frac{1}{i\lambda}\right)}\right|\leq\frac{2\left|\lambda\right|n^{\alpha}}{n^{2}O\left(1\right)} \tag{3.35}$$

(by (Reade, 1985), page 267; where $\alpha=\mathrm{Re}\left(\frac{1}{\lambda}\right)$)

Also,

$$\begin{aligned}\left|\frac{-1}{2^{n-k}\lambda^{n-k+1}\left(1-\frac{1}{2\lambda}\right)^{n-k+1}}\right| &\leq \left|\frac{-1}{n^{2}\prod_{i=1}^{n}\left(1-\frac{1}{i\lambda}\right)}\right|, n\geq k+1 \\ &\leq \frac{n^{\alpha}}{n^{2}O\left(1\right)}\end{aligned} \tag{3.36}$$

Similarly,

$$\left|\frac{-1}{2^{n-k-1}\lambda^{n-k}\left(1-\frac{1}{2\lambda}\right)^{n-k}}\right| \leq \left|\frac{-1}{n^2 \underset{i=1}{\overset{n}{\pi}}\left(1-\frac{1}{i\lambda}\right)}\right| \leq \frac{1}{n^2\frac{O(1)}{n^\alpha}} = \frac{n^\alpha}{n^2 O\left(1\right)} \tag{3.37}$$

Adding the inequalities(3.35),(3.36),and(3.37) gives

$$\frac{2\left|\lambda\right| n^\alpha}{n^2 O\left(1\right)} + \frac{n^\alpha}{n^2 O\left(1\right)} + \frac{n^\alpha}{n^2 O\left(1\right)} = 2\left(\left|\lambda\right|+1\right)\frac{n^\alpha}{n^2 O\left(1\right)} \tag{3.38}$$

so that

$$\left|\sum_{k=0}^{N}\left(\frac{2\lambda}{2^n\lambda^{n+1}\left(1-\frac{1}{2\lambda}\right)^n} - \frac{1}{2^{n-k}\lambda^{n-k+1}\left(1-\frac{1}{2\lambda}\right)^{n-k+1}} + \frac{1}{2^{n-1-k}\lambda^{n-k}\left(1-\frac{1}{2\lambda}\right)^{n-k}}\right)\right|$$
$$\leq \sum_{k=0}^{N}\frac{2\left(\left|\lambda\right|+1\right)n^\alpha}{n^2 O\left(1\right)} = N\frac{2\left(\left|\lambda\right|+1\right)n^\alpha}{n^2 O\left(1\right)} \tag{3.39}$$

Hence,

$$\begin{aligned}\sum\nolimits_3 \leq \sum_{n=N+2}^{\infty}\frac{N2\left(\left|\lambda\right|+1\right)n^\alpha}{n^2 O\left(1\right)} &= \frac{KN2\left(\left|\lambda\right|+1\right)N^\alpha}{N^2 O\left(1\right)}, \text{if } \alpha < 2 \\ &= \frac{2K\left(\left|\lambda\right|+1\right)N^{\alpha-1}}{O\left(1\right)}, K \in \mathbb{R}, K > 0. \end{aligned} \tag{3.40}$$

So that,

$$\sup_{N\geq 0}\left\{\left|\frac{1}{1-\lambda}\right| + \left|\frac{1}{\lambda\left(1-\frac{1}{2\lambda}\right)}\right| + \frac{2K\left(\left|\lambda\right|+1\right)N^{\alpha-1}}{O\left(1\right)}\right\} \tag{3.41}$$

exists for all $\alpha \in \mathbb{C}$, such that $\alpha - 1 < 0$ or $\alpha < 1$. Or equivalently $\lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$. Therefore, $M = (Q - \lambda I)^{-1} \in B\left(bv_0\right)$, for all $\lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$. And so $M = (Q - \lambda I)^{-1} \notin B\left(bv_0\right)$ for $\lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}$.

3.2 The Spectrum of $Q \in B(bv)$

In this section the spectrum of $Q \in B(bv)$ is obtained.

Theorem 3.4 *Let $T : bv \longrightarrow bv$ be given by a matrix $A = (a_{nk})$, then $T^* : bv^* \longrightarrow bv^*$ is also given by a matrix. Moreover the matrix of T^* is*

$$T^* = \begin{pmatrix} \bar{\chi} & \nu_0 - \bar{\chi} & \nu_1 - \bar{\chi} & \nu_2 - \bar{\chi} & \dots \\ a_0 & a_{00} - a_0 & a_{10} - a_0 & a_{20} - a_0 & \dots \\ a_1 & a_{01} - a_1 & a_{11} - a_1 & a_{21} - a_1 & \dots \\ a_2 & a_{02} - a_2 & a_{12} - a_2 & a_{22} - a_2 & \dots \\ & & \dots & & \end{pmatrix} \tag{3.42}$$

Where

$$\bar{\chi} = \lim_{n \longrightarrow \infty} \sum_{\nu=0}^{\infty} a_{n\nu}, a_k = \lim_{n \longrightarrow \infty} a_{nk} \tag{3.43}$$

for $k \geq 0$, $\nu_n = \sum_{\nu=0}^{\infty} a_{n\nu}, n \geq 0$ and $a_{kn} = P_k T(\delta^n)$.
Furthermore, bv^* is isomorphic to $\mathbb{C} \oplus bs$.

Proof: See (Okutoyi, 1986) pp. 62 - 63 as well as (Okutoyi, 1984).

Corollary 3.2.1 *Let $Q : bv \longrightarrow bv$, then $Q^* : bv^* \longrightarrow bv^*$ and $\|Q\|_{(bv,bv)} = \|Q\|_{(bv^*,bv^*)} = 1$. So that Q is bounded over bv. Moreover,*

$$Q^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & \frac{1}{2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \dots \\ & & \dots & & & \end{pmatrix}$$

Proof: We merely choose

$$A = Q = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \dots \\ & & \dots & & \end{pmatrix} \tag{3.44}$$

in theorem (3.4). Using the Q matrix , we have

$$\bar{\chi} = \lim_{n \longrightarrow \infty} (1, 1, 1, \dots)^t = 1, \tag{3.45}$$

$$a_k = \lim_{n \longrightarrow \infty} = 0, \ \forall k \geq 0. \tag{3.46}$$

And

$$\nu_n = \sum_{\nu=0}^{\infty} a_{n\nu} = 1, \ \forall n \geq 0 \tag{3.47}$$

Also,

$$(a_{kn}) = \begin{pmatrix} a_{00} & a_{10} & a_{20} & \dots \\ a_{01} & a_{11} & a_{21} & \dots \\ a_{02} & a_{12} & a_{22} & \dots \\ & \dots & & \end{pmatrix} \tag{3.48}$$

$$= \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & \dots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \dots \\ & & \dots & & \end{pmatrix} \tag{3.49}$$

Substituting these values in the matrix (3.42) of T^* gives the matrix of Q^* which acts on $\mathbb{C} \oplus bs$. That is $Q^* : \mathbb{C} \oplus bs \longrightarrow \mathbb{C} \oplus bs$. Moreover

$$\|Q\|_{(bv,bv)} = \|Q\|_{(bv_0,bv_0)} = \|Q^*\|_{(bv^*,bv^*)} = 1. \tag{3.50}$$

Corollary 3.2.2 *The only Eigenvalue of* $Q \in B(bv)$ *is* $\lambda = 1$

Proof: The proof follows from theorem (2.5) since $bv \subset c$ and both bv and c are BK-spaces with $\triangle^+$ as their Schauder basis. See definitions (1.5.4) and (1.5.6).

Theorem 3.5 *The Eigenvalues of* $Q^* \in B(bv^*) = B(\mathbb{C} \oplus bs)$ *are all* $\lambda \in \mathbb{C}$ *satisfying the inequality*

$$\left|\lambda - \frac{1}{2}\right| < \frac{1}{2}$$

Proof: Suppose $Q^*x = \lambda x,\ x \in \mathbb{C} \oplus bs,\ x \neq \theta.$ Then

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & \frac{1}{2} & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \dots \\ 0 & 0 & 0 & \frac{1}{2} & \dots \\ & & \dots & & \dots \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} \tag{3.51}$$

gives the system

$$\begin{array}{ccccc} & & x_0 & = & \lambda x_0 \\ x_1 & + & \frac{1}{2}x_2 & = & \lambda x_1 \\ \frac{1}{2}x_2 & + & \frac{1}{2}x_3 & = & \lambda x_2 \\ \frac{1}{2}x_3 & + & \frac{1}{2}x_4 & = & \lambda x_3 \\ \frac{1}{2}x_4 & + & \frac{1}{2}x_5 & = & \lambda x_4 \\ & \dots & & & \end{array} \tag{3.52}$$

Solving the above system, we obtain

$$\begin{array}{ccc} x_0 & = & 0 \text{ or } \lambda = 1 \\ x_2 & = & 2\lambda(1-\frac{1}{\lambda})x_1 \\ x_3 & = & (2\lambda)^2(1-\frac{1}{\lambda})(1-\frac{1}{2\lambda})x_1 \\ x_4 & = & (2\lambda)^3(1-\frac{1}{\lambda})(1-\frac{1}{2\lambda})^2x_1 \\ x_5 & = & (2\lambda)^4(1-\frac{1}{\lambda})(1-\frac{1}{2\lambda})^3x_1 \\ & \dots & \end{array} \tag{3.53}$$

In general, we have

$$x_N = (2\lambda)^{N-1}(1-\frac{1}{2\lambda})^{N-2}(1-\frac{1}{\lambda})x_1,\ N \geq 2 \tag{3.54}$$

Hence the series $\sum_{N=2}^{\infty} x_N$ is bounded, i.e., its partial sums are bounded iff $Re(\frac{1}{\lambda}) > 1$ or $|\lambda - \frac{1}{2}| < \frac{1}{2}$ (see the proof of theorem (3.2)).

Theorem 3.6 *Let $Q : bv \longrightarrow bv$, then the spectrum $\sigma(Q)$ of Q comprises the set*

$$\left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$$

Proof: By virtue of theorem (3.5) and the fact that $\sigma(Q) = \sigma(Q^*)$, it is enough to show that $(Q - \lambda I)^{-1} \in B(bv)$ for all $\lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$.
Solving system $(Q - \lambda I)(x) = y$ for x in terms of y as in the proof of theorem (2.3), we obtain the matrix of $(Q - \lambda I)^{-1}$. See formula (2.16). Condition (i) of theorem (1.6) is satisfied for all $\lambda \in \mathbb{C}$ such that $Re(\frac{1}{\lambda}) < 1$ or $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$ (see the proof of theorem (3.3). Condition (ii) of the same theorem (1.6) is automatically satisfied since

$$\sum_{k=0}^{\infty} m_{nk} = \sum_{k=0}^{n} m_{nk} = \frac{1}{1-\lambda} \tag{3.55}$$

for all $n \geq 0$. See the proof of theorem (3.3). So that

$$\sum_{k=0}^{n} m_{nk} < \infty,\ n \geq 0 \text{ provided } \lambda \neq 1. \tag{3.56}$$

Therefore $M \in B(bv)$ for all $\lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$. Which implies that the spectrum $\sigma(Q)$ of Q on bv is given by

$$\sigma(Q) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$$

Hence $Q \in B(bv_0)$ and $B(bv)$ has the same spectrum despite having different sets of eigenvalues. And this ends the chapter.In the following chapter the fine spectrum of Q on c is constructed.

Chapter 4

The Fine Spectrum of The N$\ddot{o}$rlund Q Matrix as an Operator On c

4.1 Introduction

In this chapter we derive the fine spectrum of the Q matrix acting as an operator on c - the space of convergence sequences. In so doing a state diagram is employed. The state diagram is found in (Goldberg, 1966) page 61 or(Taylor, 1958) page 237.

For this purpose, we use the following notation which is also found in (Goldberg, 1966) page 58. The diagram is a book-keeping device for keeping track of some theorems concerning the range and inverse of T as well as T^*.
If $T \in B(X)$, where X is a Banach space, we have three possibilities for $R(T)$, the range of T:

I. $R(T) = X$

II. $R(T) \neq X,$ but $\overline{R(T)} = X$

III. $\overline{R(T)} \neq X$

and three possibilities for T^{-1}:

1. T^{-1} exists and is continuous
2. T^{-1} exists and is discontinuous
3. T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3. If an operator is in state III_2, for example, then $\overline{R(T)} \neq X$ and T^{-1} exists but is discontinuous. We then write $T \in III_2$.

Similarly, we write $T \in 2$ to indicate that T^{-1} exists and is discontinuous or $T \in III$ when $\overline{R(T)} \neq X$.

If $\lambda \in \mathbb{C}$ is such that $\lambda I - T \in I_1$ or $\lambda I - T \in II_1$, then λ is in the resolvent set of T. This set of values is denoted by $\rho(T)$. All scalar values of λ not in $\rho(T)$ comprise the spectrum of T. If $\lambda I - T$ is in state III_1 (say), then we shall write $\lambda \in III_1\sigma(T)$.

4.2 The Fine Spectrum of $Q \in B(c)$

Since $||Q||_c = 1$(see corollary (2.2.1)), $\lambda \in \rho(Q)$ if $|\lambda| > 1$. We now obtain an enlargement of $\rho(Q)$ in the following theorem.

Theorem 4.1 *If $Re(\frac{1}{\lambda}) < 1$, then $\lambda \in \rho(Q)$.*

Proof: Let $\lambda \in \mathbb{C}$ be such that $Re(\frac{1}{\lambda}) < 1$. Since $\lambda \neq 1, \frac{1}{2}; \lambda I - Q$ is a triangle and therefore as an operator is one - one. We now need to establish that $\lambda I - Q$ is onto. We take $y = (y_1, y_2, y_3, \dots)$, an arbitrary element in c and let

$$(\lambda I - Q)(x) = y \tag{4.1}$$

Solving equation (4.1) for x in terms of y as in the proof of theorem (2.3), we obtain the matrix of $(\lambda I - Q)^{-1} = B$. So that

$$B = \begin{pmatrix} \frac{1}{\lambda(1-\frac{1}{\lambda})} & 0 & 0 & 0 & \cdots \\ \frac{1}{2\lambda^2(1-\frac{1}{2\lambda})(1-\frac{1}{\lambda})} & \frac{1}{\lambda(1-\frac{1}{2\lambda})} & 0 & 0 & \cdots \\ \frac{1}{2^2\lambda^3(1-\frac{1}{2\lambda})^2(1-\frac{1}{\lambda})} & \frac{1}{2\lambda^2(1-\frac{1}{2\lambda})^2} & \frac{1}{\lambda(1-\frac{1}{2\lambda})} & 0 & \cdots \\ \frac{1}{2^3\lambda^4(1-\frac{1}{2\lambda})^3(1-\frac{1}{\lambda})} & \frac{1}{2^2\lambda^3(1-\frac{1}{2\lambda})^3} & \frac{1}{2\lambda^2(1-\frac{1}{2\lambda})^2} & \frac{1}{\lambda(1-\frac{1}{2\lambda})} & \cdots \\ & & \cdots & & \end{pmatrix} \tag{4.2}$$

Note that: Since B is a triangle and the entries in the leading diagonal are none zero, B^{-1} exists. Moreover

$$(\lambda I - Q) = B^{-1} = \begin{pmatrix} \lambda - 1 & 0 & 0 & 0 & \cdots \\ -\frac{1}{2} & \lambda - \frac{1}{2} & 0 & 0 & \cdots \\ 0 & -\frac{1}{2} & \lambda - \frac{1}{2} & 0 & \cdots \\ 0 & 0 & -\frac{1}{2} & \lambda - \frac{1}{2} & \cdots \\ 0 & 0 & 0 & -\frac{1}{2} & \cdots \\ & & \cdots & & \end{pmatrix} \tag{4.3}$$

calculations show that

$$BB^{-1} = B^{-1}B = I \tag{4.4}$$

Matrix B can be compactly represented by the formula

$$B = (b_{nk}) = \begin{cases} \frac{1}{2^n\lambda^{n+1}(1-\frac{1}{2\lambda})^n(1-\frac{1}{\lambda})}, & k = 0 \\ \frac{1}{2^{n-k}\lambda^{n-k+1}(1-\frac{1}{2\lambda})^{n-k+1}}, & 1 \le k \le n \\ 0, & k > n \end{cases} \tag{4.5}$$

We now check that $B \in B(c)$, i.e., we checkout the conditions stated in theorem (1.2). From the proof of theorem (3.3) in chapter 3, we see that

$$\sum_{k=0}^{\infty} b_{nk} = \frac{1}{\lambda - 1} \text{ for each } n = 0, 1, 2, \ldots$$

see equality (3.31).

It is trivial that

$$\sum_{k=0}^{\infty} b_{0,k} = \frac{1}{\lambda - 1} \tag{4.6}$$

Hence

$$\lim_{n} \sum_{k=0}^{\infty} b_{nk} = \lim_{n} \frac{1}{\lambda - 1} \tag{4.7}$$

exists for all $\lambda \in \mathbb{C}$ such that $\lambda \neq 1$. Which establishes condition (ii) of theorem (1.2).

Since by hypothesis $Re(\frac{1}{\lambda}) < 1$, then

$$\left|1 - \frac{1}{\lambda}\right| \geq Re(1 - \frac{1}{\lambda}) = 1 - Re(\frac{1}{\lambda}) > 0 \tag{4.8}$$

Let $\beta = 1 - Re(\frac{1}{\lambda})$, then

$$\left|1 - \frac{1}{n\lambda}\right| \geq \frac{\beta + (n-1)}{n},\ n = 1, 2, 3, \ldots \tag{4.9}$$

And this implies that

$$\left|1 - \frac{1}{\lambda}\right| \geq \beta \text{ and } \left|1 - \frac{1}{2\lambda}\right| \geq \frac{\beta + 1}{2} \tag{4.10}$$

Using formula (4.5) for the matrix B and relation (4.10), we have for $k = 0$;

$$\left|\frac{1}{2^n \lambda^{n+1} (1 - \frac{1}{2\lambda})^n (1 - \frac{1}{\lambda})}\right| \leq \left|\frac{1}{2^n \lambda^{n+1} (\frac{\beta+1}{2})^n \beta}\right| \tag{4.11}$$

But

$$\lim_{n \longrightarrow \infty} \frac{1}{2^n \lambda^{n+1} (\frac{\beta+1}{2})^n \beta} = 0 \tag{4.12}$$

Which implies that,

$$\lim_{n \longrightarrow \infty} \frac{1}{2^n \lambda^{n+1} (1 - \frac{1}{2\lambda})^n (1 - \frac{1}{\lambda})} = 0 \tag{4.13}$$

Therefore, $\lim_{n\longrightarrow\infty} b_{nk} = 0,\ k = 0$

For $1 \leq k \leq n$, we have,

$$b_{nk} = \frac{1}{2^{n-k}\lambda^{n-k+1}(1-\frac{1}{2\lambda})^{n-k+1}} \tag{4.14}$$

and

$$\left|\frac{1}{2^{n-k}\lambda^{n-k+1}(1-\frac{1}{2\lambda})^{n-k+1}}\right| \leq \left|\frac{1}{2^{n-k}\lambda^{n-k+1}(\frac{\beta+1}{2})^{n-k+1}}\right| \tag{4.15}$$

But

$$\lim_{n\longrightarrow\infty} \frac{1}{2^{n-k}\lambda^{n-k+1}(\frac{\beta+1}{2})^{n-k+1}} = 0 \text{ for all } k = 1, 2, \ldots \tag{4.16}$$

Which implies that

$$\lim_{n\longrightarrow\infty} \frac{1}{2^{n-k}\lambda^{n-k+1}(1-\frac{1}{2\lambda})^{n-k+1}} = 0 \tag{4.17}$$

for all $k = 1, 2, 3, \ldots$

So that $\lim_{n\longrightarrow\infty} b_{nk} = 0, \forall k = 0, 1, 2, \ldots$ provided $Re(\frac{1}{\lambda}) < 1$. Which establishes condition (i) of theorem (1.2).

Now, summing absolutely along the n^{th} row of the matrix B, we get

$$\sum_{k=0}^{\infty} |b_{nk}| = \left|\frac{1}{2^{n}\lambda^{n+1}(1-\frac{1}{2\lambda})^{n}(1-\frac{1}{\lambda})}\right| + \sum_{k=1}^{\infty}\left|\frac{1}{2^{n-k}\lambda^{n-k+1}(1-\frac{1}{2\lambda})^{n-k+1}}\right| \tag{4.18}$$

$$\leq \left|\frac{1}{2^{n}\lambda^{n+1}(\frac{\beta+1}{2})^{n}\beta}\right| + \sum_{k=1}^{\infty}\left|\frac{1}{2^{n-k}\lambda^{n-k+1}(\frac{\beta+1}{2})^{n-k+1}}\right| \tag{4.19}$$

$$= \frac{1}{\beta(\beta+1)^{n}}\left|\frac{1}{\lambda^{n+1}}\right| + \sum_{k=1}^{n}\left|\frac{2}{\lambda^{n-k+1}(\beta+1)^{n-k+1}}\right| = \epsilon_n, \text{ say} \tag{4.20}$$

i.e.

$$\sum_{k=0}^{\infty} |b_{nk}| = \sum_{k=0}^{n} |b_{nk}| \leq \epsilon_n \tag{4.21}$$

It is clear from equality (4.20) that when n is large enough (ϵ_n) is a decreasing sequence of positive numbers which is bounded above. So that

$$\sup_n \sum_{k=0}^{n} |b_{nk}| \leq K, \text{ for some } K \in \mathbb{R} \text{ such that } K > 0. \tag{4.22}$$

Hence condition (iii) of theorem (1.2) is satisfied. Therefore, the matrix $B = (I\lambda - Q)^{-1}$ is conservative for all $\lambda \in \mathbb{C}$ such that $Re(\frac{1}{\lambda}) < 1$.
And this shows that B is surjective. Therefore $(I\lambda - Q) \in I$. Since c is a Banach space, then by the Bounded Inverse Theorem, see (Taylor, 1958), page 234,theorem 4.7B or(Limaye, 1996), pp.182 - 183; state I_2 is impossible. Upon consulting the state diagram, (Taylor, 1958) pp.235 - 238, we see that $\lambda I - Q \in I_1$. That is $\lambda \in \rho(Q)$.

Theorem 4.2 *If* $Re(\frac{1}{\lambda}) > 1, \lambda \neq \frac{1}{2}$ *then* $\lambda \in III_1\sigma(Q)$.

Proof: The matrix $I\lambda - Q$ is a triangle and therefore as an operator, $I\lambda - Q$ is one-one. We now consider the adjoint operator $\lambda I - Q^*$ such that $(\lambda I - Q^*)(x) = \theta$. Since Q^* for Q on c is given by a matrix (2.24), which acts on ℓ_1, we have that

$$\begin{pmatrix} (\lambda-1) & 0 & 0 & 0 & \cdots \\ 0 & (\lambda-1) & -\frac{1}{2} & 0 & \cdots \\ 0 & 0 & (\lambda-\frac{1}{2}) & -\frac{1}{2} & \cdots \\ 0 & 0 & 0 & (\lambda-\frac{1}{2}) & \cdots \\ & & \cdots & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \tag{4.23}$$

From which, we have

$$(\lambda - 1)x_0 = 0 \Longrightarrow x_0 = 0 \tag{4.24}$$

$$(\lambda - 1)x_1 - \frac{1}{2}x_2 = 0 \Longrightarrow x_2 = 2\lambda(1 - \frac{1}{\lambda})x_1 \tag{4.25}$$

Similarly,

$$x_3 = (2\lambda)^2(1 - \frac{1}{2\lambda})(1 - \frac{1}{\lambda})x_1 \tag{4.26}$$

$$x_4 = (2\lambda)^3(1 - \frac{1}{2\lambda})^2(1 - \frac{1}{\lambda})x_1 \tag{4.27}$$

$$\cdots$$

$$x_n = (2\lambda)^{n-1}(1 - \frac{1}{2\lambda})^{n-2}(1 - \frac{1}{\lambda})x_1, \tag{4.28}$$

$$n = 2, 3, 4 \ldots$$

Series,

$$\sum_{n=2}^{\infty}(2\lambda)^{n-1}(1 - \frac{1}{2\lambda})^{n-2}(1 - \frac{1}{\lambda})x_1 \tag{4.29}$$

Converges absolutely for $x_1 \neq 0$, when

$$\lim_{n\longrightarrow\infty}\left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\longrightarrow\infty}\left|\frac{(2\lambda)^n(1-\frac{1}{2\lambda})^{n-1}(1-\frac{1}{\lambda})x_1}{(2\lambda)^{n-1}(1-\frac{1}{2\lambda})^{n-2}(1-\frac{1}{\lambda})x_1}\right| \tag{4.30}$$

$$= \lim_{n\longrightarrow\infty}|2\lambda(1 - \frac{1}{2\lambda})| < 1 \tag{4.31}$$

i.e. when $|\lambda - \frac{1}{2}| < \frac{1}{2}$.

But $Re(\frac{1}{\lambda}) > 1$ iff $|\lambda - \frac{1}{2}| < \frac{1}{2}$, (Wenger, 1975) page 706. Hence x_1 need not be zero for x to be in ℓ_1.

Therefore $\lambda I - Q^*$ is not one to one and all these implies that $R(\lambda I - Q)$ is not dense in c, (Reade, 1985) page 265. Upon consulting the state diagram, we see that $\lambda I - Q \in III_1 \cup III_2$. To prove that $\lambda I - Q \in III_1$, it is enough to show that $\lambda I - Q^*$ is surjective, see (Taylor, 1958) pp.234 - 235. To this end, we set $(\lambda I - Q^*)x = y$, where y is an arbitrary element in ℓ_1

That is,

$$\begin{pmatrix} \lambda - 1 & 0 & 0 & 0 & \ldots \\ 0 & \lambda - 1 & -\frac{1}{2} & 0 & \ldots \\ 0 & 0 & (\lambda - \frac{1}{2}) & -\frac{1}{2} & \ldots \\ & \ldots & & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{pmatrix} \tag{4.32}$$

So that,

$$(\lambda-1)x_0 = y_0 \tag{4.33}$$

$$(\lambda-1)x_1 - \frac{1}{2}x_2 = y_1 \tag{4.34}$$

$$(\lambda-\frac{1}{2})x_2 - \frac{1}{2}x_3 = y_2 \tag{4.35}$$

$$(\lambda-\frac{1}{2})x_3 - \frac{1}{2}x_4 = y_3 \tag{4.36}$$

$$\ldots$$

In general,

$$(\lambda-\frac{1}{2})x_n - \frac{1}{2}x_{n+1} = y_n,\ n = 2,3,4,\ldots \tag{4.37}$$

If we choose $x_1 = 0$ and solve for the remaining $x_0, x_2, x_3, \ldots$, in terms of $y_0, y_1, y_2, y_3, \ldots$. We obtain,

$$\begin{aligned}
x_0 &= \frac{1}{\lambda-1}y_0 \\
x_2 &= -2y_1 \\
x_3 &= -2^2\lambda(1-\frac{1}{2\lambda})y_1 - 2y_2 \\
x_4 &= -2^3\lambda^2(1-\frac{1}{2\lambda})^2y_1 - 2^2\lambda(1-\frac{1}{2\lambda})y_2 - 2y_3 \\
x_5 &= -2^4\lambda^3(1-\frac{1}{2\lambda})^3y_1 - 2^3\lambda^2(1-\frac{1}{2\lambda})^2y_2 - 2^2\lambda(1-\frac{1}{2\lambda})y_3 - 2y_4 \qquad (4.38) \\
x_6 &= -2^5\lambda^4(1-\frac{1}{2\lambda})^4y_1 - 2^4\lambda^3(1-\frac{1}{2\lambda})^3y_2 - 2^3\lambda^2(1-\frac{1}{2\lambda})^2y_3 - 2^2\lambda(1-\frac{1}{2\lambda})y_4 - 2y_5
\end{aligned}$$

$$\ldots$$

And in general for $n = 3,4,5,\ldots$

$$\begin{aligned}
x_n &= -2^{n-1}\lambda^{n-2}(1-\frac{1}{2\lambda})^{n-2}y_1 - 2^{n-2}\lambda^{n-3}(1-\frac{1}{2\lambda})^{n-3}y_2 \\
&- 2^{n-3}\lambda^{n-4}(1-\frac{1}{2\lambda})^{n-4}y_3 - \cdots - 2^2\lambda(1-\frac{1}{2\lambda})y_{n-2} - 2y_{n-1} \qquad (4.39)
\end{aligned}$$

These equations define a matrix transformation. We denote the matrix by

$$H = (h_{nk}) = \begin{pmatrix} \frac{1}{\lambda(1-\frac{1}{\lambda})} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & -2 & 0 & 0 & 0 & \dots \\ 0 & -2^2\lambda(1-\frac{1}{2\lambda}) & -2 & 0 & 0 & \dots \\ 0 & -2^3\lambda^2(1-\frac{1}{2\lambda})^2 & -2^2\lambda(1-\frac{1}{2\lambda}) & -2 & 0 & \dots \\ 0 & -2^4\lambda^3(1-\frac{1}{2\lambda})^3 & -2^3\lambda^2(1-\frac{1}{2\lambda})^2 & -2^2\lambda(1-\frac{1}{2\lambda}) & -2 & \dots \\ & & \dots & & & \end{pmatrix} \quad (4.40)$$

matrix $H = (h_{nk})$ may be given by the formula

$$H = (h_{nk}) = \begin{cases} \frac{1}{\lambda(1-\frac{1}{\lambda})}, & k = n = 0 \\ -2^{n-k}\lambda^{n-k-1}(1-\frac{1}{2\lambda})^{n-k-1}, & 1 \le k \le n \\ 0 & \text{elsewhere} \end{cases} \quad (4.41)$$

Note: By definition

$$\lambda^{s-i} = (1-\frac{1}{2\lambda})^{s-i} = 0, \text{if } s < i \quad (4.42)$$

Obviously,

$$\sum_{n=0}^{\infty} |h_{nk}| = \left| \frac{1}{\lambda(1-\frac{1}{\lambda})} \right| < \infty \quad (4.43)$$

when $k = 0$,provided $\lambda \neq 1$

$$\sum_{n=0}^{\infty} |h_{nk}| = \sum_{n=0}^{\infty} \left| -2^{n-k}\lambda^{n-k-1}\left(1-\frac{1}{2\lambda}\right)^{n-k-1} \right| < \infty \quad (4.44)$$

for $k = 1, 2 \dots$ provided,

$$\lim_{n\longrightarrow\infty} \left| \frac{(n+1)^{th}\text{term}}{n^{th}\text{term}} \right| < 1 \quad (4.45)$$

That is, provided,

$$\lim_{n\longrightarrow\infty} \left| \frac{-2^{n-k+1}\lambda^{n-k}\left(1-\frac{1}{2\lambda}\right)^{n-k}}{-2^{n-k}\lambda^{n-k-1}\left(1-\frac{1}{2\lambda}\right)^{n-k-1}} \right| < 1 \quad (4.46)$$

So that $\left|2\lambda\left(1-\frac{1}{2\lambda}\right)\right| < 1$ or $\left|\lambda - \frac{1}{2}\right| < \frac{1}{2}$ or equivalently, provided $Re(\frac{1}{\lambda}) > 1$.
Which establishes condition (i) of theorem (1.4). Now summing absolutely along the columns, we have that when $k = 0$.

$$\sum_{n=0}^{\infty} |h_{nk}| = \sum_{n=0}^{\infty} \frac{1}{\lambda(1-\frac{1}{\lambda})} = \left|\frac{1}{\lambda(1-\frac{1}{\lambda})}\right| < \infty, \tag{4.47}$$

provided, $Re(\lambda) \neq 1$. For $k = 1, 2, 3, \ldots$, we have

$$\sum_{n=0}^{\infty} |h_{nk}| = \sum_{n=k+1}^{\infty} \left| -2^{n-k}\lambda^{n-k-1}\left(1-\frac{1}{2\lambda}\right)^{n-k-1} \right| \tag{4.48}$$

$$= \sum_{n=k+1}^{\infty} \left|-2(2\lambda-1)^{n-k-1}\right| \tag{4.49}$$

By hypothesis $Re(\frac{1}{\lambda} - 1) > 0$ or $\left|\lambda - \frac{1}{2}\right| < \frac{1}{2} \Longrightarrow |2\lambda - 1| < 1$.
Hence there exists a $\beta \in \mathbb{R}$ with $\beta > 0$ such that

$$|2\lambda - 1| < \beta < 1 \tag{4.50}$$

This implies for series (4.49) that

$$\sum_{n=0}^{\infty} |h_{nk}| = 2 \sum_{n=k+1}^{\infty} |2\lambda - 1|^{n-k-1} \leq 2 \sum_{n=k+1}^{\infty} \beta^{n-k-1}$$

$$= 2(1 + \beta + \beta^2 + \beta^3 + \ldots) = \frac{2}{1-\beta} \tag{4.51}$$

Since $0 < \beta < 1$. i.e.,

$$\sum_{n=0}^{\infty} |h_{nk}| \leq \frac{2}{1-\beta}, \ \forall k = 1, 2, \ldots \tag{4.52}$$

Let $M = \max. \left\{\left|\frac{1}{\lambda-1}\right|, \frac{2}{1-\beta}\right\}$
Then,

$$\sum_{n=0}^{\infty} |h_{nk}| \leq M, \text{ independent of } k. \tag{4.53}$$

This establishes condition (ii) of theorem (1.4) for matrix $H = (h_{nk})$. All these implies that $x \in \ell_1$. Therefore $\lambda I - Q^*$ is onto and therefore $\lambda I - Q$ has a bounded inverse. See (Taylor, 1958) pp. 233 - 234. So that $\lambda I - Q \in III_1$.

Theorem 4.3 *If* $\lambda = \frac{1}{2}$, *then* $\lambda \in III_1\sigma(Q)$

Proof: Let $x = (x_0, x_1, x_2, \dots) \in c$ and consider $(\frac{1}{2}I - Q)(x) = \theta$.
This implies that,

$$\begin{aligned} -\frac{1}{2}x_0 &= 0 \\ -\frac{1}{2}x_1 &= 0 \\ -\frac{1}{2}x_2 &= 0 \\ &\dots \end{aligned} \tag{4.54}$$

and in general,

$$-\frac{1}{2}x_n = 0, \ \forall n = 0, 1, 2, \dots \tag{4.55}$$

That is $x = \theta$. Hence the operator $\frac{1}{2}I - Q$ is one to one. So that $\frac{1}{2}I - Q \in 1 \cup 2$. Turning to the conjugate operator $\frac{1}{2}I - Q^*$, we set $(\frac{1}{2}I - Q^*)(x) = \theta, x \in \ell_1$. Where Q^* has the matrix (2.24). This implies that,

$$\begin{aligned} -\frac{1}{2}x_0 &= 0 \\ -\frac{1}{2}x_1 - \frac{1}{2}x_2 &= 0 \\ -\frac{1}{2}x_3 &= 0 \\ -\frac{1}{2}x_4 &= 0 \\ &\dots \end{aligned} \tag{4.56}$$

In general,

$$-\frac{1}{2}x_n = 0, \ n = 3, 4, 5, \dots \tag{4.57}$$

Which implies that $x_n = 0,\ n = 0, 3, 4, 5, \ldots$ or $x_1 = -x_2$. And these implies that either $x_1 = x_2 = 0$ or $x_1 = -a$ when $x_2 = a, a \in \mathbb{C}$ or $x_1 = b$, when $x_2 = -b, b \in \mathbb{C}$. All these implies that non-zero sequences are mapped into the zero sequence. Thus the operator $\frac{1}{2}I - Q^*$ has a non trivial kernel. And this implies that $\overline{R(\frac{1}{2}I - Q)} \neq X$. See (Reade, 1985) page 265. Thus $\frac{1}{2}I - Q \in III$. To obtain the conclusion, it is sufficient to prove that $\frac{1}{2}I - Q^*$ is surjective. Accordingly, let $y = (y_0, y_1, y_2, \ldots)$ be an arbitrary element in ℓ_1. If $x = (x_0, x_1, x_2, \ldots)$ exists such that $(\frac{1}{2}I - Q^*)(x) = y$, then as in system (4.57), we have,

$$\begin{aligned} -\frac{1}{2}x_0 &= y_0 \\ -\frac{1}{2}x_1 - \frac{1}{2}x_2 &= y_1 \\ -\frac{1}{2}x_3 &= y_2 \\ -\frac{1}{2}x_4 &= y_3 \\ &\cdots \end{aligned} \tag{4.58}$$

Solving the above system for x in terms of y we obtain

$$\begin{aligned} x_0 &= -2y_0 \\ x_1 &= -x_2 - 2y_1 \\ x_3 &= -2y_2 \\ x_4 &= -2y_3 \\ &\cdots \end{aligned} \tag{4.59}$$

Choosing $x_1 = 0$, we have

$$\begin{aligned} x_0 &= -2y_0 \\ x_2 &= -2y_1 \\ x_3 &= -2y_2 \\ &\cdots \end{aligned} \tag{4.60}$$

So that $x \in \ell_1$. Therefore $\frac{1}{2}I - Q^*$ is onto. Thus $\frac{1}{2}I - Q \in 1$. Hence $\frac{1}{2}I - Q \in III_1$.

Theorem 4.4 *If* $Re(\frac{1}{\lambda}) = 1,\ \lambda \neq 1$, *then* $\lambda \in II_2\sigma(Q)$

Proof: Let λ be a complex number such that $Re(\frac{1}{\lambda}) = 1,\ \lambda \neq 1$. Since $\lambda I - Q$ is a triangle, it is one-one and so $\lambda I - Q \in 1 \cup 2$. We must now consider the conjugate or the adjoint operator $\lambda I - Q^*$. Setting $(\lambda I - Q^*)x = \theta$ and solving for x_n in terms of $x_1, n = 2, 3, 4, \dots$ as in the proof of theorem (4.2) we find that:

$$x_0 = 0 \tag{4.61}$$

$$x_n = (2\lambda)^{n-1}(1 - \frac{1}{2\lambda})^{n-2}(1 - \frac{1}{\lambda})x_1,\ n = 2, 3, 4, \dots \tag{4.62}$$

It follows that ,

$$\begin{aligned} \sum_{n=0}^{\infty} |x_n| &= x_0 + |x_1| + \sum_{n=2}^{\infty} |x_n| \\ &= |x_1| + \sum_{n=2}^{\infty} \left| (2\lambda)^{n-1} \left(1 - \frac{1}{2\lambda}\right)^{n-2} \left(1 - \frac{1}{\lambda}\right) x_1 \right| = \infty \end{aligned} \tag{4.63}$$

when $Re(\frac{1}{\lambda}) = 1$, unless $x = \theta$. Hence the kernel of the operator $\lambda I - Q^*$ is trivial. Which implies that $\overline{R(\lambda I - Q)} = X$.

Thus $\lambda I - Q \in II$. On consulting the state diagram, we have that $\lambda I - Q \in I_1 \cup II_2$. To check the surjectivity of $\lambda I - Q^*$, we set $(\lambda I - Q^*)x = y$, where y is an arbitrary element in ℓ_1. Now solving the equation for x in terms of y as in the proof of theorem (4.2) gives, on setting $x_1 = 0$, the system

$$\begin{aligned} x_0 &= \frac{1}{\lambda(1 - \frac{1}{\lambda})} y_0 \\ x_2 &= -2y_1 \\ x_3 &= -2^2\lambda(1 - \frac{1}{2\lambda})y_1 - 2y_2 \\ x_4 &= -2^3\lambda^2(1 - \frac{1}{2\lambda})^2 y_1 - 2^2\lambda(1 - \frac{1}{2\lambda})y_2 - 2y_3 \dots \\ x_n &= -2^{n-1}\lambda^{n-2}\left(1 - \frac{1}{2\lambda}\right)^{n-2} y_1 - 2^{n-2}\lambda^{n-3}\left(1 - \frac{1}{2\lambda}\right)^{n-3} y_2 \\ &- 2^{n-3}\lambda^{n-4}\left(1 - \frac{1}{2\lambda}\right)^{n-4} y_3 - \dots - 2^2\lambda\left(1 - \frac{1}{2\lambda}\right) y_{n-2} - 2y_{n-1}, n \geq 2. \end{aligned} \tag{4.64}$$

It is clear from equation 4.41 that if $Re(\frac{1}{\lambda}) = 1$, then $x \notin \ell_1$. This implies that $\lambda I - Q^*$ is not surjective. On consulting the state diagram, state I_1 is ruled out. So that $\lambda I - Q \in II_2$. Hence the result.

Theorem 4.5 $1 \in III_3\sigma(Q)$

Proof: Let $x = (x_0, x_1, x_3, \dots) \in c$.
Consider the matrix equation

$$(I - Q)(x) = \theta \tag{4.65}$$

That is,

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \dots \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \dots \\ & \dots & & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \tag{4.66}$$

Which leads to the system,

$$-\frac{1}{2}x_0 + \frac{1}{2}x_1 = 0 \tag{4.67}$$

$$-\frac{1}{2}x_1 + \frac{1}{2}x_2 = 0 \tag{4.68}$$

$$-\frac{1}{2}x_2 + \frac{1}{2}x_3 = 0 \tag{4.69}$$

$$\dots$$

This implies that,

$$x = k \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}, \tag{4.70}$$

k is a scalar constant.

Thus non-zero vectors are mapped onto the zero vector. So that $I - Q$ is not one-one. And $I - Q \in 3$. Considering the adjoint operator $I - Q^*$, we set $(I - Q^*)x = \theta$ which leads to the equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \dots \\ & & \dots & & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \dots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \tag{4.71}$$

Which implies that,

$$-\frac{1}{2}x_2 = 0 \tag{4.72}$$

$$\frac{1}{2}x_2 - \frac{1}{2}x_3 = 0 \tag{4.73}$$

$$\frac{1}{2}x_3 - \frac{1}{2}x_4 = 0 \tag{4.74}$$

$$\dots$$

So that,

$$x = \begin{pmatrix} a_0 \\ a_1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \tag{4.75}$$

where $x_0 = a_0, x_1 = a_1, a_0, a_1$ some scalar constants. Hence $I - Q^*$ is not one-one. So that $I - Q^*$ has non-trivial kernel, which leads to the conclusion that

$$\overline{I - Q} \neq X \tag{4.76}$$

Therefore $I - Q \in III$. Upon consulting the state diagram, we have that $1 \in III_3\sigma(Q)$. The fine spectrum of the operator Q on c can be summarized by means of the circle centered at the point $(\frac{1}{2}, 0)$ of radius $\frac{1}{2}$.

The points in the interior of the circle make up $III_1\sigma(Q)$. Those on the circumference of the circle except the point $(1, 0)$ form $II_2\sigma(Q)$.
The set $III_3\sigma(Q) = \{1\}$ and $\rho(Q)$ consists of all points exterior to the circle. Hence the fine spectrum of $Q \in B(c)$ is constructed. In the next chapter we construct the spectrum of the almost N$\ddot{o}$rlund Q matrix on c_0 and c.
In the next chapter the spectrum of almost Norlund Q operator on c_0 and c is determined.

Chapter 5

The Spectrum of almost Nörlund Q operator on c_0, c

In this chapter we construct the spectrum of an almost Nörlund Q matrix as a bounded operator on c_0 and c spaces. The chapter is divided into two sections. Section one deals with the spectrum of an almost Nörlund Q matrix on c_0. Where as section two deals with its spectrum on c.

5.1 The Spectrum of almost triangular Q_1 matrix on c_0

In this section we determine the spectrum of $Q_1 \in B(c_0)$. We first of all give examples of almost triangular matrices. See definition (1.3.4) of almost triangular matrices. For example, the matrix A such that,

$$A = (a_{nk}) = \begin{pmatrix} a_{00} & a_{01} & a_{02} & 0 & 0 & 0 & \dots \\ a_{10} & a_{11} & a_{12} & a_{13} & 0 & 0 & \dots \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & 0 & \dots \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & \dots \\ & & & \dots & & & \end{pmatrix} \tag{5.1}$$

where $m = 2$ is an almost triangular infinite matrix.

Remark 5.1.1 *Some concrete examples of almost triangular matrices are:*

$$A = (a_{nk}) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \dots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \dots \\ & & \dots & & & \end{pmatrix} \tag{5.2}$$

$$A = (a_{nk}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \dots \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \dots \\ & & \dots & & & \end{pmatrix} \tag{5.3}$$

$$A = (a_{nk}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \dots \\ & & \dots & & & \end{pmatrix} \tag{5.4}$$

$$A = (a_{nk}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \dots \\ & & \dots & & & \end{pmatrix} \tag{5.5}$$

Matrix (5.5) is the matrix of our interest in this chapter. We call it almost Nörlund Q matrix and denote it by Q_1. That is

$$Q_1 = (q^1_{nk}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \dots \\ & & \dots & & & \end{pmatrix}$$

Corollary 5.1.1 $Q_1 \in B(c_0)$

This follows from theorem (1.3) and matrix (5.5). Since

$$\lim_{n \longrightarrow \infty} q^1_{nk} = 0, \text{for each k} \geq 0 \tag{5.6}$$

Also

$$\|Q_1\| = \sup_n \sum_{k=0}^{\infty} q^1_{nk} \tag{5.7}$$

$$= \sup_n \{1, 1, 1, \dots\} \tag{5.8}$$

$$= 1 \tag{5.9}$$

By theorem(1.7)

$$\|Q_1^*\| = \|Q_1\| = 1 \tag{5.10}$$

Corollary 5.1.2 *Let $Q_1 : c_0 \longrightarrow c_0$, then $Q_1^* \in B(\ell_1)$, where Q_1^* is the transposed matrix of Q_1*

Proof: All these clearly follow from lemmas (2.1.1) and (2.1.2) with T replaced by Q_1.

Theorem 5.1 *$Q_1 \in B(c_0)$ has the set of eigenvalues given as*

$$\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| < \frac{1}{2}\}$$

Proof: Suppose $Q_1x = \lambda x$, $\lambda \in \mathbb{C}$ and $x \in c_0$ is such that $x \neq \theta$. Then equation

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \dots \\ & & \dots & & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} \tag{5.11}$$

implies that,

$$x_0 = \lambda x_0 \tag{5.12}$$

$$\frac{1}{2}x_1 + \frac{1}{2}x_2 = \lambda x_1 \tag{5.13}$$

$$\frac{1}{2}x_2 + \frac{1}{2}x_3 = \lambda x_2 \tag{5.14}$$

$$\dots$$

$$\frac{1}{2}(x_n + x_{n+1}) = \lambda x_n, n \geq 1 \tag{5.15}$$

On solving the system, we have that if x_0 is the first non-zero entry of x, then $\lambda = 1$.
But $\lambda = 1 \Longrightarrow x_1 = x_2 = x_3 = \dots = x_n = x_{n+1} = \dots, n \geq 1$
So that $x \nrightarrow 0$. Hence $\lambda = 1$ is not an eigenvalue of $Q_1 \in B(c_0)$.
If x_1 is the first non-zero entry of x, then solving the system for x_n in terms of $x_1, n \geq 2$; gives,

$$x_2 = 2(\lambda - \frac{1}{2})x_1 \tag{5.16}$$

$$x_3 = 2^2(\lambda - \frac{1}{2})^2 x_1 \tag{5.17}$$

$$x_4 = 2^3(\lambda - \frac{1}{2})^3 x_1 \tag{5.18}$$

$$\cdots$$

In general,$x_n = 2^{n-1}(\lambda - \frac{1}{2})^{n-1}x_1, n \geq 2$. For $x_n \longrightarrow 0$ as $n \longrightarrow \infty$, we must have

$$\left|\frac{x_{n+1}}{x_n}\right| < 1, \forall n \geq 1 \tag{5.19}$$

$$\text{i.e,} \left|\frac{2^n(\lambda - \frac{1}{2})^n x_1}{2^{n-1}(\lambda - \frac{1}{2})^{n-1}x_1}\right| < 1, \forall n \geq 1 \tag{5.20}$$

Which implies that,

$$\left|2(\lambda - \frac{1}{2})\right| < 1 \tag{5.21}$$

$$\text{or} \left|\lambda - \frac{1}{2}\right| < \frac{1}{2} \tag{5.22}$$

Theorem 5.2 *The set of eigenvalues of $Q_1^* \in B(\ell_1)$ is the singleton set*

$$\{1\}$$

Proof: Suppose $Q_1^* x = \lambda x$ for $x \neq \theta$ in ℓ_1 and $\lambda \in \mathbb{C}$. Since

$$Q_1^* = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & \cdots \\ & & \cdots & & \end{pmatrix} \tag{5.23}$$

we have,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & \cdots \\ & & \cdots & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{pmatrix} \tag{5.24}$$

From which we have the system,

$$\begin{aligned} x_0 &= \lambda x_0 \\ \frac{1}{2}x_1 &= \lambda x_1 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 &= \lambda x_2 \\ \frac{1}{2}x_2 + \frac{1}{2}x_3 &= \lambda x_3 \\ \frac{1}{2}x_3 + \frac{1}{2}x_4 &= \lambda x_4 \\ &\cdots \end{aligned}$$

$$\frac{1}{2}x_{n-1} + \frac{1}{2}x_n = \lambda x_n, n \geq 2 \tag{5.25}$$

If x_0 is the first non-zero entry of x, then from system (5.25), $\lambda = 1$. But $\lambda = 1$ implies that $x_n = x_0, 0, 0 \ldots$ which obviously is in ℓ_1. If x_1 is the first non-zero entry of x, then $\lambda = \frac{1}{2}$.
But $\lambda = \frac{1}{2}$ implies from the same system that $x_1 = 0$ a contradiction. Hence $\lambda = \frac{1}{2}$ cannot be an eigenvalue of $Q_1^* \in B(\ell_1)$. Similarly, suppose x_2 is the first non-zero entry of x. Then $\lambda = \frac{1}{2}$; but from $\frac{1}{2}x_2 + \frac{1}{2}x_3 = \frac{1}{2}x_3$, we have $x_2 = 0$ a contradiction. By induction $x_n, n \geq 1$ cannot be the first non-zero entry of x. So that x_0 is the only first non-zero entry of x. So that $\lambda = 1$ is the only eigenvalue of $Q_1^* \in B(\ell_1)$.

Theorem 5.3 *The spectrum $\sigma(Q_1)$ of $Q_1 \in B(c_0)$ comprises the set*

$$\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\}$$

Proof: By virtue of theorems (5.1) and (5.2), it is enough to show that $(Q_1 - \lambda I)^{-1} \in B(c_0)$ for all $\lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$.
Let $A = (a_{nk}), n, k \geq 0$. Consider the matrix equation

$$(Q_1 - \lambda I)(a_{nk}) = I, \tag{5.26}$$

here I is the identity matrix. That is

$$\begin{pmatrix} (1-\lambda & 0 & 0 & 0 & \dots \\ 0 & (\frac{1}{2}-\lambda) & \frac{1}{2} & 0 & \dots \\ 0 & 0 & (\frac{1}{2}-\lambda) & \frac{1}{2} & \dots \\ 0 & 0 & 0 & (\frac{1}{2}-\lambda) & \dots \\ & & \dots & & \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & \dots \\ a_{10} & a_{11} & a_{12} & a_{13} & \dots \\ a_{20} & a_{21} & a_{22} & a_{23} & \dots \\ a_{30} & a_{31} & a_{32} & a_{33} & \dots \\ & & \dots & & \end{pmatrix} \tag{5.27}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ & & \dots & & \end{pmatrix}$$

Solving the matrix equation for the entries of the matrix $A = (a_{nk})$, we obtain the following:

For $k = 0$

$$a_{00} = \frac{1}{1-\lambda} \tag{5.28}$$

$$a_{10} = (-1)^{n-1} \frac{1}{(2(\frac{1}{2}-\lambda))^{n-1}} a_{n0}, n \geq 2 \tag{5.29}$$

$$a_{20} = (-1)^{n-2} \frac{1}{(2(\frac{1}{2}-\lambda))^{n-2}} a_{n0}, n \geq 3 \tag{5.30}$$

$$a_{30} = (-1)^{n-3} \frac{1}{(2(\frac{1}{2}-\lambda))^{n-3}} a_{n0}, n \geq 4 \tag{5.31}$$

$$\dots$$

$$a_{N,0} = (-1)^{n-N} \frac{1}{(2(\frac{1}{2}-\lambda))^{n-N}} a_{n0}, n \geq 2, N \geq 1 \tag{5.32}$$

For $k = 1$,

$$a_{01} = 0 \tag{5.33}$$

$$a_{11} = \frac{2 - a_{21}}{2(\frac{1}{2} - \lambda)} \tag{5.34}$$

$$a_{21} = \frac{-1}{2(\frac{1}{2} - \lambda)} a_{31} \tag{5.35}$$

$$a_{31} = \frac{-1}{2(\frac{1}{2} - \lambda)} a_{41} \tag{5.36}$$

$$\dots$$

This implies that,

$$a_{11} = \frac{2^{n-1}(\frac{1}{2} - \lambda)^{n-2} + (-1)^{n-1}}{(2(\frac{1}{2} - \lambda))^{n-1}} a_{n1}, n \geq 2 \tag{5.37}$$

$$a_{21} = (-1)^{n-2} \frac{1}{(2(\frac{1}{2} - \lambda))^{n-2}} a_{n1}, n \geq 3 \tag{5.38}$$

$$a_{31} = (-1)^{n-3} \frac{1}{(2(\frac{1}{2} - \lambda))^{n-3}} a_{n1}, n \geq 4 \tag{5.39}$$

$$\dots$$

In general,

$$a_{N1} = (-1)^{n-N} \frac{1}{(2(\frac{1}{2} - \lambda))^{n-N}} a_{n1}, n \geq N+1, N \geq 2 \tag{5.40}$$

For $k = 2$

$$a_{02} = 0 \tag{5.41}$$

$$a_{12} = \frac{-1}{2(\frac{1}{2} - \lambda)} a_{22} = \frac{-2^{n-2}(\frac{1}{2} - \lambda)^{n-3} + (-1)^{n-1}}{(2(\frac{1}{2} - \lambda))^{n-1}} a_{n2}, n \geq 3 \tag{5.42}$$

$$a_{22} = \frac{2^{n-2}(\frac{1}{2}-\lambda)^{n-3} + (-1)^{n-2}}{(2(\frac{1}{2}-\lambda))^{n-2}} a_{n2}, n \geq 3 \tag{5.43}$$

$$a_{32} = (-1)^{n-3} \frac{1}{(2(\frac{1}{2}-\lambda))^{n-3}} a_{n2}, n \geq 4 \tag{5.44}$$

$$a_{42} = (-1)^{n-4} \frac{1}{(2(\frac{1}{2}-\lambda))^{n-4}} a_{n2}, n \geq 5 \tag{5.45}$$

$$\dots$$

$$a_{N2} = (-1)^{n-N} \frac{1}{(2(\frac{1}{2}-\lambda))^{n-N}} a_{n2}, n \geq N+1, N \geq 3 \tag{5.46}$$

For $k = 3$,

$$a_{03} = 0 \tag{5.47}$$

$$a_{13} = \frac{-1}{2(\frac{1}{2}-\lambda)} a_{23} \tag{5.48}$$

$$a_{23} = \frac{-1}{2(\frac{1}{2}-\lambda)} a_{33} \tag{5.49}$$

$$a_{33} = \frac{2 - a_{43}}{2(\frac{1}{2}-\lambda)} \tag{5.50}$$

$$a_{43} = \frac{-1}{2(\frac{1}{2}-\lambda)} a_{53} \tag{5.51}$$

$$a_{53} = \frac{-1}{2(\frac{1}{2}-\lambda)} a_{63} \tag{5.52}$$

$$\dots$$

Which implies that,

$$a_{43} = (-1)^{n-4} \frac{1}{(2(\frac{1}{2}-\lambda))^{n-4}} a_{n3}, n \geq 5 \tag{5.53}$$

$$a_{53} = (-1)^{n-5}\frac{1}{(2(\frac{1}{2}-\lambda))^{n-5}}a_{n3}, n \geq 6 \tag{5.54}$$

$$\dots$$

The process is carried for $k = 4, 5, 6, \dots$

From these, matrix $A = (a_{nk})$ becomes matrix G, where

$$G = (g_{nk}) = \begin{pmatrix} \frac{1}{1-\lambda} & 0 & 0 & \dots \\ \frac{1}{(2(\frac{1}{2}-\lambda))^4}a_{50} & \frac{2^4\left(\frac{1}{2}-\lambda\right)^3+1}{\left(2\left(\frac{1}{2}-\lambda\right)\right)^4}a_{51} & \frac{-2^3\left(\frac{1}{2}-\lambda\right)^2+1}{\left(2\left(\frac{1}{2}-\lambda\right)\right)^4}a_{52} & \dots \\ \frac{-1}{(2(\frac{1}{2}-\lambda))^3}a_{50} & \frac{-1}{(2(\frac{1}{2}-\lambda))^3}a_{51} & \frac{2^3\left(\frac{1}{2}-\lambda\right)^2-1}{\left(2\left(\frac{1}{2}-\lambda\right)\right)^3}a_{52} & \dots \\ \frac{1}{(2(\frac{1}{2}-\lambda))^2}a_{50} & \frac{1}{(2(\frac{1}{2}-\lambda))^2}a_{51} & \frac{1}{(2(\frac{1}{2}-\lambda))^2}a_{52} & \dots \\ \frac{-1}{2(\frac{1}{2}-\lambda)}a_{50} & \frac{-1}{2(\frac{1}{2}-\lambda)}a_{51} & \frac{-1}{2(\frac{1}{2}-\lambda)}a_{52} & \dots \\ a_{50} & a_{51} & a_{52} & \dots \\ & \dots & & \end{pmatrix} \tag{5.55}$$

We now use matrix equations $(Q_1 - \lambda I)(g_{nk}) = I$, and $(g_{nk})(Q_1 - \lambda I) = I$ to obtain the values of entries $a_{50}, a_{51}, a_{52}, \dots,$ in terms of λ. The process is repeated, when the entries of matrix (5.55) are now expressed in terms of the numbers $a_{60}, a_{61}, a_{62}, \dots$ e.t.c. Using induction, the matrix $R = (r_{nk})$ below obtains. That is

$$R = (r_{nk}) = \begin{pmatrix} \frac{1}{1-\lambda} & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{\frac{1}{2}-\lambda} & \frac{-1}{2(\frac{1}{2}-\lambda)^2} & \frac{1}{2^2(\frac{1}{2}-\lambda)^3} & \frac{-1}{2^3(\frac{1}{2}-\lambda)^4} & \frac{1}{2^4(\frac{1}{2}-\lambda)^5} & \dots \\ 0 & 0 & \frac{1}{\frac{1}{2}-\lambda} & \frac{-1}{2(\frac{1}{2}-\lambda)^2} & \frac{1}{2^2(\frac{1}{2}-\lambda)^3} & \frac{-1}{2^3(\frac{1}{2}-\lambda)^4} & \dots \\ 0 & 0 & 0 & \frac{1}{\frac{1}{2}-\lambda} & \frac{-1}{2(\frac{1}{2}-\lambda)^2} & \frac{1}{2^2(\frac{1}{2}-\lambda)^3} & \dots \\ 0 & 0 & 0 & 0 & \frac{1}{\frac{1}{2}-\lambda} & \frac{-1}{2(\frac{1}{2}-\lambda)^2} & \dots \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{1-\lambda} & \dots \\ & & & \dots & & & \end{pmatrix} \tag{5.56}$$

Note that,

$$(Q_1 - \lambda I) = R^{-1} = \begin{pmatrix} 1-\lambda & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{2}-\lambda & \frac{1}{2} & 0 & \dots \\ 0 & 0 & \frac{1}{2}-\lambda & \frac{1}{2} & \dots \\ 0 & 0 & 0 & \frac{1}{2}-\lambda & \dots \\ & & \dots & & \end{pmatrix} \tag{5.57}$$

A simple calculation shows that $RR^{-1} = R^{-1}R = I$

Matrix (5.56) is also given by the formula

$$R = (r_{nk}) = \begin{cases} \frac{-1}{\lambda\left(1-\frac{1}{\lambda}\right)}, & n = k = 0 \\ \frac{-1}{2^{k-n}\lambda^{k-n+1}\left(1-\frac{1}{2\lambda}\right)^{k-n+1}}, & k \geq n \geq 1 \\ 0, & otherwise \end{cases} \tag{5.58}$$

It is clear that $r_{nk} \longrightarrow 0$ as $n \longrightarrow \infty$ for all $k \geq 0$, provided $\lambda \neq 1, \frac{1}{2}$.

By the ratio test, we see that

$$\sum_{k=0}^{\infty} \left| \frac{-1}{2^{k-n}\lambda^{k-n+1}\left(1-\frac{1}{2\lambda}\right)^{k-n+1}} \right| < \infty, n = 1, 2, 3 \dots \tag{5.59}$$

provided $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$.

When $n = 0$,

$$\sum_{k=0}^{\infty} |r_{nk}| = \left| \frac{-1}{\lambda(1-\frac{1}{\lambda})} \right| < \infty, \tag{5.60}$$

provided $\lambda \neq 1$.

Hence,

$$\sum_{k=n}^{\infty} |r_{nk}| < \infty, n = 0, 1, 2, \dots \tag{5.61}$$

provided $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$.

Moreover,

$$\sup_n \sum_{k=n}^{\infty} |r_{nk}| < \infty, \tag{5.62}$$

provided that $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$.

Which checks out all conditions specified in theorem (1.3) for matrix R.

Thus $R = (Q_1 - \lambda I)^{-1} \in B(c_0), \forall \lambda \in \mathbb{C}$ such that

$$\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}. \tag{5.63}$$

Which implies that $(Q_1 - \lambda I)^{-1} \notin B(c_0), \forall \lambda \in \mathbb{C}$, such that

$$\left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}. \tag{5.64}$$

5.2 The Spectrum of Almost Triangular Q_1 matrix on c

In this section the spectrum of $Q_1 \in B(c)$ is obtained. The folowing corollary arises from theorem (1.2) in chapter one .

Corollary 5.2.1 $Q_1 \in B(c)$

Proof: Conditions (i) and (ii) of theorem (1.2) easily follows from corollary (5.1.1). It is also clear from matrix (5.5), that

$$\sum_{k=0}^{\infty} q^1_{nk} = 1, \forall n \geq 0. \tag{5.65}$$

So that,

$$\lim_n \sum_{k=0}^{\infty} q^1_{nk} = 1 \tag{5.66}$$

Moreover,

$$\|Q_1\|_c = \sup_n \{\sum_{k=0}^{\infty} \left|q^1_{nk}\right|\} \tag{5.67}$$

$$= \sup_n \{1, 1, 1, \dots\} \tag{5.68}$$

$$= 1 \tag{5.69}$$

Which deals with part (iii) of theorem (1.2).

Corollary 5.2.2 *Let $Q_1 : c \longrightarrow c$, where Q_1 is the almost Nörlund Q matrix. Then $Q_1^* \in B(\ell_1)$ and*

$$Q_1^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \dots \\ & & \dots & & & \end{pmatrix} \tag{5.70}$$

Moreover,

$$\|Q_1\|_c = \|Q_1^*\|_{\ell_1} = 1 \tag{5.71}$$

Proof: By theorem (2.4)

$$Q_1^* = \begin{pmatrix} \chi(\lim_A) & (v_n)_0^\infty \\ & \\ (a_k)_0^\infty & Q_1^t \end{pmatrix} \tag{5.72}$$

For Q_1,

$$v_n = 0, n = 0, 1, 2, \dots \tag{5.73}$$

Since $\lim_n(q_{nk}^1) = 0, \forall k \geq 0$, it implies that $(a_k)_0^\infty = \theta$,

$$\chi = (\lim o\; Q_1)\delta - \sum_{k=0}^{\infty}(\lim o\; Q_1)\delta^k, \lim \in c^* \tag{5.74}$$

That is,

$$\chi = \lim \delta - \sum_{k=0}^{\infty} a_k = 1 - 0 = 1 \tag{5.75}$$

Therefore matrix (5.72) reduces to

$$Q_1^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \dots \\ & & \dots & & & \end{pmatrix} \tag{5.76}$$

By theorem (1.7),

$$\|Q_1\|_c = \|Q_1^*\|_{\ell_1} = 1 \tag{5.77}$$

Theorem 5.4 *The set of eigenvalues of* $Q_1 \in B(c)$ *is*

$$\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\}$$

Proof: Suppose $Q_1 x = \lambda x, for\ x \neq \theta$ in c and $\lambda \in \mathbb{C}$.

Then as in the proof of theorem (5.1), we have that if x_0 is the first non-zero entry of vector x, then $\lambda = 1$. But $\lambda = 1$ implies that $x_1 = x_2 = x_3 = \cdots = x_n = \dots, n \geq 1$ which implies that,

$$x = x_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} + x_1 \begin{pmatrix} 0 \\ 1 \\ 1 \\ \vdots \end{pmatrix} \tag{5.78}$$

So that, x is in the span of the vectors

$$\vec{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \text{and } \vec{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ \vdots \end{pmatrix} \tag{5.79}$$

But $\vec{a}$ and $\vec{b}$ are in c. So that x is in c. Hence $\lambda = 1$, is an eigenvalue of $Q_1 \in B(c)$ corresponding to the eigenvectors $\vec{a}$ and $\vec{b}$.

If x_1, is the the first non-zero entry of x, then solving for x_n in terms of x_1 as in the proof of theorem (5.1), gives

$$x_n = 2^{n-1}(\lambda - \frac{1}{2})^{n-1}x_1, \geq 2. \tag{5.80}$$

For x to be in c, we must have

$$\left|\frac{x_{n+1}}{x_n}\right| \leq 1, n \geq 1 \tag{5.81}$$

$$\text{i.e,} \left|\frac{2^n(\lambda - \frac{1}{2})^n x_1}{2^{n-1}(\lambda - \frac{1}{2})^{n-1}x_1}\right| \leq 1 \tag{5.82}$$

or $\left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}$.

Theorem 5.5 *The set of eigenvalues of $Q_1^* \in B(\ell_1)$ is the singleton set $\{1\}$.*

Proof: Suppose $Q_1^* x = \lambda x, x \neq \theta$ in ℓ_1 and $\lambda \in \mathbb{C}$. Then, the matrix equation $Q_1^* x = \lambda x$ implies that,

$$x_0 = \lambda x_0 \tag{5.83}$$

$$x_1 = \lambda x_1 \tag{5.84}$$

$$\frac{1}{2}x_2 = \lambda x_2 \tag{5.85}$$

$$\frac{1}{2}(x_2 + x_3) = \lambda x_3 \tag{5.86}$$

$$\frac{1}{2}(x_3 + x_4) = \lambda x_4 \tag{5.87}$$

$$\cdots$$

And in general,

$$\frac{1}{2}(x_n + x_{n+1}) = \lambda x_{n+1}, n \geq 2 \tag{5.88}$$

On solving the system for λ we have that, if x_0 is the first non-zero entry of x, then $\lambda = 1$. But $\lambda = 1$ implies that $x_2 = x_3 = x_4 = \cdots = x_n = 0, n \geq 2$. Which results in the vector $(x_0, x_1, 0, 0, \dots)^t$, $x_1, x_2 \in \mathbb{C}$. This vector is clearly in ℓ_1. If x_1 is the first non-zero entry of x, then $\lambda = 1$. But $\lambda = 1$ implies that $x_n = 0, n \geq 2$. This results in the vector $(0, x_1, 0, 0, \dots)^t \in \ell_1$. Which confirms that $\lambda = 1$, is an eigenvalue of $Q_1^* \in B(\ell_1)$. If x_2 is the first non-zero entry of x, then $\lambda = \frac{1}{2}$. But $\lambda = \frac{1}{2}$ gives $x_2 = 0$,which is a contradiction. Hence $\lambda = \frac{1}{2}$ cannot be an eigenvalue of $Q_1^* \in B(\ell_1)$. It is readily seen that $x_n, n \geq 2$ cannot be the first non-zero entry of x.
All these implies that $\lambda = 1$ is the only eigenvalue of $Q_1^* \in B(\ell_1)$.

Theorem 5.6 *The Spectrum $\sigma(Q_1)$ of $Q_1 \in B(c)$ forms the set*

$$\{\lambda \in \mathbb{C} : \Big|\lambda - \frac{1}{2}| \leq \frac{1}{2}\}$$

Proof: As in the proof of theorem (5.3), $\lim_{n \longrightarrow \infty} r_{nk}$ exist for $\lambda \in \mathbb{C}$, such that $\lambda \neq 1, \frac{1}{2}$. And $\sup_n \sum_{k=0}^{\infty} |r_{nk}| < \infty$ for $\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$. Which deals with part (i) and (ii) of theorem (1.2) for matrix $R = (Q_1 - \lambda I)^{-1}$. Also observe that

$$\left(\left|\sum_{k=0}^{\infty} r_{nk}\right|\right)_{n=0}^{\infty} \leq \left(\sum_{k=0}^{\infty} |r_{nk}|\right)_{n=0}^{\infty} \tag{5.89}$$

So that,

$$\sup_n \left|\sum_{k=0}^{\infty} r_{nk}\right| \leq \sup_n \sum_{k=0}^{\infty} |r_{nk}| \leq K \tag{5.90}$$

for some $K \in \mathbb{R}$, provided $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$. Hence,

$$\left(\sum_{k=0}^{\infty} r_{nk}\right)_{n=0}^{\infty} \tag{5.91}$$

is a a monotonic sequence which is bounded.
Therefore $\lim_{n \longrightarrow \infty} \sum_{k=0}^{\infty} r_{nk}$ exists provided, $\lambda \in \mathbb{C}$ is such that $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$. And this deals with part (iii) of theorem (1.2) for $R = (Q_1 - \lambda I)^{-1}$.
All these implies that $(Q_1 - \lambda I)^{-1} \notin B(c), \forall \lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}$. This

ends the proof of theorem (5.6) and it also ends the chapter. In the next chapter we summarize the results obtained in the thesis. We also point the way forward for future research.

Chapter 6

Overview of Results Obtained and Future Work

6.1 Introduction

In this chapter the results obtained in the thesis are summarised. We do the summary chapter by chapter. Finally we point the way forward for future research.

6.1.1 Summary of results obtained

In chapter two the following results are obtained:

i. $Q \in B(c_0)$ has no eigenvalues.

ii. The set of eigenvalues for $Q^* \in B(\ell_1)$ is

$$\left\{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| < \frac{1}{2}\right\} \cup \{1\}$$

iii. The spectrum of $Q \in B(c_0)$ is the set

$$\left\{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2}\right\}$$

iv. the set of eigenvalues of $Q \in B(c)$ is the singleton set $\{1\}$.

v. The eigenvalues of $Q^* \in B(\ell_1)$ form the set

$$\left\{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| < \frac{1}{2}\right\} \cup \{1\}$$

vi. The spectrum of $Q \in B(c)$ is the set

$$\left\{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2}\right\}$$

In chapter three, we obtain the following results

i. $Q \in B(bv_0)$ has no eigenvalues.

ii. The eigenvalues of $Q^* \in B(bv_0^*)$ are all $\lambda \in \mathbb{C}$ satisfying the inequality $|\lambda - \frac{1}{2}| < \frac{1}{2}$.

iii. The spectrum of $Q \in B(bv_0)$ is the set

$$\left\{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2}\right\}$$

.

iv. The only eigenvalue of $Q \in B(bv)$ is $\lambda = 1$.

v. The eigenvalue of $Q^* \in B(bv^*)$ are all $\lambda \in \mathbb{C}$ satisfying the inequality $|\lambda - \frac{1}{2}| < \frac{1}{2}$.

vi. The spectrum of $Q \in B(bv)$ comprises the set

$$\left\{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2}\right\}$$

In chapter four we construct the fine spectrum of $Q \in B(c)$. We summarize the results using a disc in the complex plane centered at the point $(\frac{1}{2}, 0)$ of radius $\frac{1}{2}$. The interior of the disc make-up $III_1\sigma(Q)$; those on the circumference except the point $(1, 0)$ form $II_2\sigma(Q)$; the point $(1, 0)$ is the set $III_3\sigma(Q)$ and all the pints exterior to the circle or disc form the set $\rho(Q)$.

In chapter five we find the spectrum of the almost Nörlund Q operator, the Q_1 operator. The following results are obtained:

i. $Q_1 \in B(c_0)$ has the set of eigenvalues as

$$\left\{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| < \frac{1}{2}\right\}$$

ii. The set of eigenvalues of $Q_1^* \in B(\ell_1)$ is the singleton set $\{1\}$.

iii. The spectrum of $Q_1 \in B(c_0)$ comprises the set

$$\left\{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2}\right\}$$

iv. The set of eigenvalues of $Q_1 \in B(c)$ is

$$\left\{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2}\right\}$$

v. The set of eigenvalues of $Q_1^* \in B(\ell_1)$ is the singleton set $\{1\}$.

vi. The spectrum of $Q_1 \in B(c)$ is the set

$$\left\{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2}\right\}$$

In conclusion, it is seen that the spectrum is the same in all the cases; although their sets of eigenvalues differ.

6.2 Future research

We intend to extend the results obtained in this thesis by:

(a) Investigating the spectra of a general Nörlund means.

(b) Investigating the spectrum of the almost Nörlund Q operator on bv_0 and bv spaces.

(c) Constructing the fine spectrum of the Q operator on bv_0 and bv.

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